# A Central-Planning Approach to Dynamic Incomplete-Market Equilibrium*

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**Abstract**

We show that a central planner with *two selves,* or two "pseudo welfare functions", are sufficient to deliver the market equilibrium that prevails among any (finite) number of heterogeneous individual agents acting competitively in an incomplete financial market. Furthermore, we are able to exhibit a *recursive formulation* of the two-central planner problem. In that formulation, every aspect of the economy can be derived one step at a time, by a process of backward induction as in dynamic programming.

Dynamic asset pricing increasingly considers models with incomplete markets and heterogeneity in an attempt to improve over the empirical performance of benchmark complete-market representative-agent models. Numerous authors have pointed out the difficulties faced when solving these models.[1]

In this paper, we aim to find a technique for computing an equilibrium in an incomplete financial market, that is less onerous than the fixed-point tâtonnement process. The tâtonnement process presents the major drawback that a stochastic process for securities prices must be postulated *ab initio* to start the procedure of obtaining optimal portfolios. The trial-and-error procedure would wander in a vast space of stochastic processes. It is a hopeless undertaking. Direct calculation of equilibrium makes sense only in special cases in which the equilibrium has some properties that are known *a priori.*

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[1] Early examples includes Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1998) and Marcet and Singleton (1999). Levine and Zame (2001) show that market incompleteness is unimportant in the absence of aggregate risk. In the present paper we incorporate aggregate risk.

Like Cuoco and He (1994), our line of attack of this problem is to use a representative-agent concept, where the representative agent utility is defined as a stochastically weighted average of individual utilities. In Cuoco and He, the way in which the representative agent is defined over time is derived separately on the basis of individual financial choices, based on the dual approach of He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991). These individual financial choices involve as many value functions (interpreted as each person's financial wealth) as there are individuals in the economy.[2]

Below, we show that a central planner with *two selves,* or two "pseudo welfare functions", are sufficient to deliver the market equilibrium that prevails among any (finite) number of heterogeneous individual agents. The first self solves for individual consumption decisions and individual-specific components of state prices, taking the economy-wide components of state prices as given. Simultaneously, the second self chooses individual consumption rules and equilibrium state prices (i.e. such that the aggregate resource restriction is satisfied) taking as given the individual-specific components of the state prices. In an equilibrium of this game, the two selves agree and the competitive equilibrium is found.

In a complete-market setting, competitive equilibrium with heterogeneous agents is typically obtained by virtue of the Pareto optimality of the competitive equilibrium. Solving a Planner problem, which is the sum of individual utilities weighted by Pareto weights, gives the equilibrium allocation so that one can price assets off the marginal rate of substitution of this constructed representative agent. This approach dates back to Negishi (1960) and was used in, for instance, Constantinides (1982) and Dumas (1989). The definition of the two pseudo welfare functions we propose is, however, not based on a claim that the competitive equilibrium in an incomplete financial market is constrained Pareto optimal. Indeed, Magill and Quinzii (1996, Chapter IV) have a simple counter-example showing that this claim is not true. Nonetheless, our approach is reminiscent of the work of Grossmann (1977) who shows that the market equilibrium has some welfare properties from the vantage point of a central planner who would act as several incompletely coordinated selves.[3]

Taking our method one step further, we are able to exhibit a *recursive formulation* of the two-central planner problem. In that formulation, every aspect of the economy can be derived one step at a time, by a process of backward induction. Dynamic programming is used but, in principle, the dynamic program involves two value functions which are solved for simultaneously. The only state variables needed to summarize the distribution of resources across individuals are the current values of the individual-specific components of the Lagrange multipliers.

Many economists were tempted to believe that no no recursive formulation

[2] Barbachan (2001) extends Cuoco and He (1994) from the case of two individual agents to more than two.

[3] This was done in the context of a multi-good economy, whereas we employ here a multi-period economy. But it is well-known that analogous problems occur in both settings. In Grossmann (1977), the planner has as many selves as there are goods, whereas our planner has only two selves.

of an incomplete market equilibrium was possible.[4] Cuoco and He, however, write a system of partial differential equations which is applicable to the exchange economy that we consider here. That PDE could be solved backward in a recursive fashion. We expect, however, that our two-planner algorithm is general enough to be applied later to more complex settings.

From the technical point of view, the recent paper by Harris and Laibson (2001) is also related to our work in that it shows that some decision problems plagued by time inconsistency – considered before them as being outside the reach of recursive techniques – can be formulated recursively provided that the decision maker is split into several selves which play a Nash game with each other.

Throughout this paper, we assume that the incomplete-market equilibrium exists. Hart (1975) has exhibited a well-known counter-example showing that equilibrium may fail to exist. It involves a situation in which the rank of the rate-of-return matrix drops in some states of nature. Fortunately, Duffie and Shafer (1986) have shown that this occurs for a negligible subset of economies.

At any rate, existence is not the topic of our paper. Our paper is useful only to calculate equilibria after someone has shown that they exist.

Not only do we take it for granted that competitive-market equilibrium exists but, for most of the paper, and for the entire theoretical part of the paper, *we take as given the variance-covariance matrix of equilibrium rates of return*,[5] for which we assume that it remains of constant rank $N$ at all times with probability one. In a numerical illustration, however, we explain the way in which that matrix could be determined endogenously within an extended procedure, which would still be recursive. At this point, we cannot be sure that the extended procedure delivers the equilibrium when the variance-covariance matrix is endogenous. That issue is left for future research.

The balance of the paper is organized as follows. Section 1 describes the economy that we study. Section 2 reminds the reader of the dual formulation of the portfolio choice problem in He and Pearson (1991) and Karatzas et al. (1991), and gives the definition of the corresponding equilibrium. Section 3 presents a "simultaneous-game" formulation of the game played by the two selves of the central planner and shows that the extent to which the equilibrium of the game replicates the market equilibrium. Section 4 shows that the equilibrium in the Magill-Quinzii example is indeed a solution of the static game between the planners. Section 5 presents a recursive, or dynamic-game formulation of the same problem. This will be most useful for purposes of numerical

[4] See Judd, Kubler and Schmedders (2000) for the case of one (risky) asset. Kubler and Schmedders (2002) analyze the existence of recursive equilibria with minimal sufficient state spaces and construct a counter-example where the current exogenous state variables along with the wealth distribution across agents do not constitute sufficient state variables, even though the fundamentals of the economy are Markovian. Krebs (2001) shows the non-existence of recursive or Markov equilibria in infinite-horizon incomplete-market exchange economies without aggregate risk for general preferences (and with aggregate risk for homothetic preferences) when the wealth distribution is taken as a state variable. In this paper however, we suggest a different set of state variables and obtain recursivity.

[5] More precisely, we take as given the diffusion matrix of securities prices.

implementation, since dynamic programming can be used. Section 6 presents an example of an actual numerical implementation that illustrates also how the variance-covariance of returns could be endogenized. Section 7 demonstrates how the same approach can be generalized to the case of recursive utility, which is more general than that of time-additive utility, which is postulated in the rest of the paper. Section 8 contains the conclusion.

# 1 The economy

## 1.1 Information and technical assumptions

The economy that we consider evolves over a finite interval $[0, T]$ of the real line. $(\Omega, \mathcal{F}, P)$ is a probability space endowed with a filtration $\mathbf{F}$. $w(\omega, t)$ is a $K$-dimensional Wiener $((\Omega, \mathcal{F}, P) \times [0, T] \rightarrow \mathbb{R}^K)$ relative to the given filtration where the components are independent of each other.

We define a filtration $\mathbf{F}^w$ which is the filtration generated by the Wiener $w$.

**Definition 1** $\mathcal{L}^1$ *space: the set of adapted, measurable processes $b$ such that for every $T$:*[6]

$$\int_0^T \|b\| \, dt < \infty \qquad \textit{with probability one} \tag{1}$$

**Definition 2** $\mathcal{L}^2$ *is the space of adapted, measurable processes $b$ such that for every $T$:*[7]

$$\int_0^T \|b\|^2 \, dt < \infty \qquad \textit{with probability one} \tag{2}$$

**Assumption:** In what follows, all processes for which an Itô stochastic integral is written are assumed to belong to the space $\mathcal{L}^2$. All processes for which an integral over time is written are assumed to belong to the space $\mathcal{L}^1$.

## 1.2 Individuals and endowments

We consider an exchange economy with one good. There is a large but finite number $I$ of individuals who trade competitively in the financial market. They are indexed by $i$. They are endowed at time 0 with a stock of good $F_i(0)$ (initial wealth on hand or "financial wealth") and they receive over time a flow endowment $e_i(t)$ following a given Itô stochastic process. That process belongs to $\mathcal{L}^1_+$. Their consumption process is denoted $c_i$ and their utility functions are time additive:[8] $E_0 \left\{ \int_0^T u_i(c_i(s), s) \, ds \right\}$ with $u(.,.)$ satisfying the Inada conditions. As a result, the process $c_i$ belongs to $\mathcal{L}^1_+$. All individuals have the same information set, viz. the one provided by the filtration $\mathbf{F}^w$.

[6] If $a$ is a scalar, $\|a\|$ is absolute value. If $a$ is $N \times 1$ dimensional, then $a$ is in $\mathcal{L}^1$ if and only if each of its components is in $\mathcal{L}^1$.

[7] If $a$ is matrix-valued, then $\|a\|^2 = tr(aa^\intercal)$.

[8] For an extension, see Section 7 below.

The process $[c_i(t) - e_i(t)]$ is called "net consumption". We shall only be interested in equilibria in which $\sum_i F_i(0) = 0$.

**Definition 3** *The* aggregate resource restriction *is:*

$$\sum_i [c_i(t) - e_i(t)] = 0,\ \forall t \in [0,T],\ \textit{with probability 1} \tag{3}$$

### 1.3 Financial assets

There are $N+1$ securities one of which is instantaneously riskless. The $N$-dimensional Itô process for the "dividends" $\iota(s)$ is given. Individuals can choose to invest in these assets but, this being an exchange economy, the total net supply of each asset is equal to zero. Calling $[\alpha_i, \theta_i]$ the portfolio choice process of individual $i$, where $\alpha_i(t)$ is the number of units of the riskless asset and $\theta_i(t)$ is the vector containing the number of units of all the risky assets held by individual $i$ at time $t$,

**Definition 4** *The* market clearing condition *is:*

$$\sum_i \alpha_i(t) = 0,\ \forall t \in [0,T],\ \textit{with probability 1} \tag{4}$$

$$\sum_i \theta_i(t) = 0,\ \forall t \in [0,T],\ \textit{with probability 1} \tag{5}$$

## 2 The static formulation: equilibrium

We now write down the formulation of an equilibrium in the financial market. In order to reach equilibrium, individuals have to choose their portfolios $[\alpha_i(t), \theta_i(t)]$. In order to choose their portfolio, they have to postulate a stochastic process for financial market prices. The central planning formulation, which comes later, presents the major advantage that there is no need to postulate such a stochastic process.

### 2.1 Financial market prices

The stochastic process for price is assumed to be an Itô process, denoted as follows:

$$B(t) = B(0)\, e^{\int_0^t r(s)ds};\ B(0) = 1 \tag{6}$$

$$S(t) + \int_0^t \iota(s)\, ds = S(0) + \int_0^t \zeta(s)\, ds + \int_0^t \sigma(s)\, dw(s);\ S(0) = 1 \tag{7}$$

$S(\omega, t)$ is a process in $\mathbb{R}_+^N$ $(N < K)$. At the individual level, the optimization problem to be solved is:

$$\sup_{c_i(s), \alpha_i(s), \theta_i(s)} E_0 \left\{ \int_0^T u_i(c_i(s), s)\, ds \right\} \tag{8}$$

subject to:

$$\alpha_i(0)\,B(0)+\theta_i(0)^{\intercal}\,S(0)=F_i(0) \tag{9}$$

$$\begin{aligned}\alpha_i(t)\,B(t)+\theta_i(t)^{\intercal}\,S(t)+\int_0^t\left[c_i(s)-e_i(s)\right]ds=\\ \alpha_i(0)\,B(0)+\theta_i(0)^{\intercal}\,S(0)+\int_0^t\left[\alpha_i(s)\,B(s)\,r(s)+\theta_i(s)^{\intercal}\,\zeta(s)\right]ds\\ +\int_0^t\theta_i(s)^{\intercal}\,\sigma(s)\,dw(s)\,;\forall t\in\,]0,T[\text{ with probability 1}\end{aligned} \tag{10}$$

$$\alpha_i(T)\,B(T)+\theta_i(T)^{\intercal}\,S(T)=0 \tag{11}$$

and subject to (6, 7). As the market is incomplete, the matrix $\sigma$ has fewer rows than columns.

**Definition 5** *A net-consumption plan $[c_i(t)-e_i(t)]$ is said to be marketable from $F_i(0)$ if there exist stochastic processes $[\alpha_i(t),\theta_i(t)]$ such that Equations (9, 10, 11) are satisfied with probability one.*

Obviously, the sum of two marketable plans is a marketable plan.

**Lemma 6** *If $\sum_i F_i(0)=0$ and the market clearing condition is satisfied, then the aggregate resource restriction is satisfied.*

**Lemma 7** *If $\sum_i F_i(0)=0$, the aggregate resource restriction is satisfied and $[c_i(t)-e_i(t)]$ is marketable from $F_i(0)$ for all $i$, then the market clearing condition is satisfied.*

## 2.2 Minimax Individual Consumption Choice

We define an adapted process $\kappa$ in $\mathbb{R}^K$ such that:[9]

$$\sigma(t)\,\kappa(t)=\left[\zeta(t)-r(t)\times S(t)\right] \tag{13}$$

We define three scalar Itô processes $\xi$, $\eta_i$ and $Z_i^{-1}$ :

$$\xi(0,t)\triangleq\exp\left\{-\int_0^t r(s)\,ds-\frac{1}{2}\int_0^t\|\kappa(s)\|^2\,ds-\int_0^t\kappa(s)^{\intercal}\,dw(s)\right\} \tag{14}$$

$$\begin{aligned}&\eta_i^{\kappa^{\intercal}}(0,t)\\ \triangleq\ &\exp\left\{-\frac{1}{2}\int_0^t\|\nu_i(s)\|^2\,ds-\int_0^t\kappa(s)^{\intercal}\,\nu_i(s)\,ds-\int_0^t\nu_i(s)^{\intercal}\,dw(s)\right\}\\ i\ =\ &1,..I\end{aligned} \tag{15}$$

[9] There are many processes satisfying that restriction. One such process is:

$$\kappa(t)\triangleq\sigma(t)^{\intercal}\left[\sigma(t)\,\sigma(t)^{\intercal}\right]^{-1}\left[\zeta(t)-r(t)\times S(t)\right] \tag{12}$$

In that case, $\kappa$ is in the span of $\sigma^{\intercal}$ : $\kappa(t)=\sigma(t)^{\intercal}\,x(t)$ where $x(t)=\left[\sigma(t)\,\sigma(t)^{\intercal}\right]^{-1}\left[\zeta(t)-r(t)\times S(t)\right]$. This is the process selected by He and Pearson.

$$Z_i^{-1}(t) \triangleq Z_i^{-1}(0)\,\xi(0,t)\,\eta_i^{\kappa^\intercal}(0,t) \tag{16}$$

$Z_i^{-1}(0)$ will be given a meaning very shortly. These processes satisfy the following stochastic differential equations:

$$\frac{d\xi(t)}{\xi(t)} = -r(t)\,dt - \kappa(t)^\intercal\,dw(t)\,;\xi(0)=1 \tag{17}$$

$$\frac{d\eta_i^{\kappa^\intercal}(t)}{\eta_i^{\kappa^\intercal}(t)} = -\kappa(t)^\intercal\,\nu_i(t)\,dt - \nu_i(t)^\intercal\,dw(t)\,;\eta_i^{\kappa^\intercal}(0)=1 \tag{18}$$

$$\frac{dZ_i^{-1}(t)}{Z_i^{-1}(t)} = -r(t)\,dt - [\kappa(t)+\nu_i(t)]^\intercal\,dw(t) \tag{19}$$

The restriction (13) on $\kappa$ guarantees that:

$$E_0\left[S(t)\,\xi(0,t) + \int_0^t \iota(s)\,\xi(0,s)\,ds\right] = S(0) = 1 \tag{20}$$

$$E_0\left[B(t)\,\xi(0,t)\right] = B(0) = 1 \tag{21}$$

**Lemma 8** *For as long as $\nu_i \in \ker\sigma$ (i.e. $\sigma\nu_i = 0$) and $c_i(t) - e_i(t)$ is marketable at all times with probability one, we have:*

$$E_0\left[\int_0^T [c_i(s) - e_i(s)]\,\xi(0,s)\,\eta_i^{\kappa^\intercal}(0,s)\,ds\right] = F_i(0) \tag{22}$$

**Proof.** This can be verified by direct application of Itô's lemma to (10), (14) and (15). ■

**Remark 9** *The discounted present value in (22) is invariant with respect to the choice of $\nu_i$ for as long as $\nu_i$ is in the kernel of $\sigma$.*

One might reformulate the optimization problem as one of maximizing:

$$\sup_{c_i(s)} E_0\left\{\int_0^T u_i(c_i(s), s)\,ds\right\} \tag{23}$$

subject to (22). However, that leaves the solution indeterminate for as long as we have not specified $\nu_i$. A duality reasoning would show that the choice of $\nu_i$ must be dictated by:[10]

$$\inf_{\nu_i(s)\in\ker\sigma(s)}\ \sup_{c_i(s)} E_0\left\{\int_0^T u_i(c_i(s), s)\,ds\right\} \tag{26}$$

[10] Starting at time $t$, the same sequence of decisions could have been obtained by solving the problem:

$$\inf_{\nu_i(s)\in\ker\sigma(s)}\ \sup_{c_i(s)} E_t\left\{\int_t^T u_i(c_i(s), s)\,ds\right\} \tag{24}$$

subject to:

$$E_t\left[\int_t^T [c_i(s) - e_i(s)]\,\xi(t,s)\,\eta_i(t,s)\,ds\right] = F(t) \triangleq \alpha_i(t)\,B(t) + \theta_i(t)^\intercal\,S(t) \tag{25}$$

while the Lagrange multiplier of that constraint would have been equal to $1/Z_i(t)$.

subject to (22). We call $Z_i^{-1}(0)$ the Lagrange multiplier of that constraint written at time 0. This is the main result of the dual approach of He and Pearson (1991) (Theorems 1, 2 and 3) and Karatzas et al. (1991), which is the extension to incomplete markets of the martingale methodology of Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986).

The legitimacy of this procedure is established by the following lemma which shows that the solution of the dual problem is, indeed, the solution of the primal problem.

**Lemma 10** *For any given $\xi$ process, if a solution to problem (26) exists and technical conditions are satisfied,*[11] *individual $i$ optimizing (26) subject to (22) chooses net trades $[c_i(s) - e_i(s)]$ that are marketable from $F_i(0)$.*

**Proof.** See He and Pearson (1991) proof of Theorem 2, pages 292-295, which is applicable in the absence of an endowment stream. He and Pearson had selected a process $\kappa$ in the span of $\sigma^\intercal$. That restriction is immaterial. If $\kappa$ is not in the span, it can always be decomposed: $\kappa = \widehat{\kappa} + \widehat{\nu}$, where $\widehat{\kappa}$ is in the span of $\sigma^\intercal$ and $\widehat{\nu}$ is in the kernel of $\sigma$. The restriction $\sigma\nu = 0$ is equivalent to the restriction $\sigma(\nu - \widehat{\nu}) = 0$.[12] Cuoco (1997) shows that technical conditions must be strengthened to generalize the result to the case in which individuals receive an endowment stream. ■

**Remark 11** *If we take for granted that the Lagrange multiplier at time 0 is $Z_i^{-1}(0)$, the time-0 problem (26) can be written equivalently:*

$$\inf_{\nu_i(s)\in\ker\sigma(s)} \sup_{c_i(s)} E_0\left\{\int_0^T u_i(c_i(s), s)\, ds\right\}$$
$$-Z_i^{-1}(0)\, E_0\left[\int_0^T [c_i(s) - e_i(s)]\,\xi(0,s)\,\eta_i^{\kappa^\intercal}(0,s)\, ds\right] \tag{27}$$

## 2.3 Market equilibrium

We continue to impose that $\sum_i F_i(0) = 0$.

**Definition 12** *A competitive market equilibrium is a set of decision processes $\{\{c_i\},\{\alpha_i\},\{\theta_i\}\}$ and price processes $\{B, S\}$ such that, for each individual $i$, $\{c_i\},\{\alpha_i\}$ and $\{\theta_i\}$ are the optimizing argument of (8) subject to (9) through (11) and such that the market clearing conditions (4) and (5) hold.*

[11] For the existence of a solution to the dual and primal problem and technical conditions guaranteeing the equivalence between the solution to the dual and the solution to the primal problem, see He and Pearson (1991), Karatzas et al. (1991) and Kramkov and Schachermayer (1999).

[12] We are not aware of technical conditions guaranteeing the validity of the dual approach for incomplete-market situations with random endowments and interim consumption when state prices are semimartingales. The economy-wide state price process $\xi(t)$ will turn out to be continuous in equilibrium. As we shall see, however, off equilibrium one has to allow for discontinuities in the state price process.

**Remark 13** *Since the aggregate resource restriction holds, the set of equilibrium initial Lagrange multipliers* $\{1/Z_i(0)\}$ *is not just any element in* $\mathbb{R}^I$. *It is truly an element of* $\mathbb{R}^{I-1}$ *because we must have:*

$$\sum_i \left[\frac{\partial}{\partial c_i} u_i\right]^{-1} \left(Z_i^{-1}(0), t\right) = \sum_i e_i(0) \tag{28}$$

*where* $\left[\frac{\partial}{\partial c_i} u_i\right]^{-1} (\cdot, t)$ *is the inverse marginal utility function of each individual with respect to consumption.*

Suppose that a competitive market equilibrium exists in which the initial Lagrange multipliers are equal to $\{1/Z_i(0)\}$ (satisfying (28)) and the diffusion matrix of traded asset prices is given by a $N \times K$ dimensional process $\sigma$. Then, we can define:

**Definition 14** *A competitive market "sub-equilibrium" is a set of processes* $\{\{c_i\}, \{\nu_i\}, \xi\}$, *in which, for each individual* $i$, $\{c_i\}, \{\nu_i\}$ *are the optimizing arguments of (27), and which are such that the aggregate resource restriction holds.*

# 3 The static formulation: central planning

We suppose that a market equilibrium exists in which the initial Lagrange multipliers are equal to $\left\{Z_i^{-1}(0)\right\}$ (satisfying (28)) and the diffusion matrix of traded asset prices is given by a $N \times K$ dimensional process $\sigma$.[13]

Our goal is now to define a central-planning problem that generates a sub-equilibrium.

In the market setting, $\xi$ has been implied from the behavior of market prices (6) and (7). In the central planning setting, however, $\xi$ is just an adapted process to be determined. In both contexts, $\nu_i$ is an adapted process, to be determined, which is in the kernel of $\sigma$.

The central planner that achieves our goal has two selves which operate jointly in a Nash game with each other. The two selves solve two interdependent allocation problems with two different objective functions and constraints:

Problem 1:

$$\inf_{\{\nu_i(s)\in\ker\sigma(s)\}} \sup_{\{c_i(s)\}} \left\{ \sum_i E_0 \left\{ \int_0^T u_i(c_i(s), s)\, ds \right\} - \sum_i Z_i^{-1}(0)\, E_0 \left[ \int_0^T [c_i(s) - e_i(s)]\, \xi(0,s)\, \eta_i^{\kappa^\intercal}(0,s)\, ds \right] \right\} \tag{29}$$

[13] In this section, therefore, $\sigma$ is given. In Section 6, we illustrate how $\sigma$ could be obtained from the recursive version of the central planning algorithm. This is done numerically only. We leave for future research the generalization of the theory for the case of endogenous $\sigma$.

Problem 2:

$$\sup_{\{c_i(s)\}} \left\{ \sum_i Z_i(0) E_0 \left\{ \int_0^T \frac{1}{\eta_i^{\kappa^\intercal}(0,s)} u_i(c_i(s), s)\, ds \right\} \right.$$
$$\left. + \inf_{\xi(0,s)\in\mathbb{R}^+} \left[ -E_0 \left[ \int_0^T \sum_i [c_i(s) - e_i(s)]\, \xi(0,s)\, ds \right] \right] \right\} \tag{30}$$

Self 1 makes sure that the budget constraints are satisfied. It takes $\xi$ as given and makes exactly the same decisions as in the partial-equilibrium dual approach of He and Pearson and Karatzas et al. Self 2 acts very much like the central planner in a complete market problem or like an auctioneer; it makes sure that the aggregate resource restriction is satisfied at all times.[14] It takes the $\eta_i$'s as given in constructing his objective function. Observe that the decisions of one player serve to define the objective function of the other. The two selves could not be reduced to one since they face different objective functions and discount utility of consumption at different rates, but they agree on the consumption allocation. Indeed the FOCs for consumption are the same in both cases:

$$\begin{aligned} \frac{\partial}{\partial c_i(s)} u_i(c_i(s), s) &= Z_i^{-1}(0)\, \xi(0,s)\, \eta_i^{\kappa^\intercal}(0,s) \\ &\equiv Z_i^{-1}(s) \end{aligned} \tag{31}$$

**Remark 15** *When markets are dynamically complete, i.e. when $\sigma$ is a square matrix ($N = K$) of full rank, the kernel of $\sigma$ is the singleton* $0$*: there is a unique equivalent martingale measure. Problems 1 and 2 become respectively:*

$$\sup_{\{c_i(s)\}} \left\{ \sum_i E_0 \left\{ \int_0^T u_i(c_i(s), s)\, ds \right\} \right.$$
$$\left. - \sum_i Z_i^{-1}(0) E_0 \left[ \int_0^T [c_i(s) - e_i(s)]\, \xi(0,s)\, ds \right] \right\} \tag{32}$$

$$\inf_{\xi(0,s)\in\mathbb{R}^+} \sup_{\{c_i(s)\}} \left\{ \sum_i Z_i(0) E_0 \left\{ \int_0^T u_i(c_i(s), s)\, ds \right\} \right.$$
$$\left. + \left[ -E_0 \left[ \int_0^T \sum_i [c_i(s) - e_i(s)]\, \xi(0,s)\, ds \right] \right] \right\} \tag{33}$$

*so that central planning can be achieved by a planner with a single self. Indeed Planner 2 in this case needs no input from Planner 1.*

[14] Although reminiscent of the 'auctioneer' algorithm of Lucas (1994) and Heaton and Lucas (1996), our auctioneer is different as he directly targets the aggregate resource restriction (by choosing aggregate state prices), rather than market clearing in financial markets (by searching for asset holdings), as in their approach.

**Definition 16** *A Nash equilibrium of the above game is a set of decision processes* $\{\{c_i\},\{\nu_i\}\}$ *that are optimal for Planner 1 (in particular, individually marketable) given the values* $\{\xi\}$ *of the decisions of Planner 2 and a set of decision processes* $\{\{c_i\},\xi\}$ *that are optimal for Planner 2 (in particular, marketable in the aggregate and satisfying the budget constraint) given the decisions* $\{\nu_i\}$ *of Planner 1.*

**Theorem 17** *Suppose that a competitive market equilibrium exists in which the set of initial Lagrange multipliers is equal to* $\left\{Z_i^{-1}(0)\right\}$ *(satisfying the aggregate resource restriction (28) at time* $0$*) and the diffusion matrix of traded asset prices is given by a* $N \times K$ *dimensional process* $\sigma$*. The Nash equilibrium of the above game is a market sub-equilibrium.*

**Proof.** *First, for any given* $\xi$ *processes, Planner 1 chooses net trades* $[c_i(s) - e_i(s)]$ *that are marketable.*
Indeed, since Problem 1 is nothing but the sum taken over all individuals of the individual problems of the form (27), lemma 10 above implies that individual net trades are marketable.
*Second, for any given set of processes* $\{\nu_i(s) \in \ker \sigma(s)\}$*, the choice of* $\xi$ *by Planner 2 guarantees that the aggregate resource restriction is satisfied with probability one.* Indeed, the planner's objective function is nothing but a Lagrangian objective function incorporating that constraint. ■

**Remark 18** *Given any solution* $\{\{c_i\},\{\nu_i\},\xi\}$ *of the above game, equivalently written as* $\left\{\{c_i\},\left\{\eta_i^{\kappa^\intercal}\right\},\xi\right\}$*, define a process* $\widehat{\eta}$ *:*

$$\widehat{\eta}^{\kappa^\intercal}(0,t) \triangleq \exp\left\{-\frac{1}{2}\int_0^t \|\widehat{\nu}(s)\|^2 ds - \int_0^t \kappa(s)^\intercal \widehat{\nu}(s)\, ds - \int_0^t \widehat{\nu}(s)^\intercal dw(s)\right\}; i = 1, ..., I \quad (34)$$

*where* $\widehat{\nu}$ *is any adapted process satisfying the kernel constraint:* $\sigma\widehat{\nu} = 0$*. Then it can be checked readily that* $\left\{\{c_i\},\left\{\eta_i^{\kappa^\intercal}\widehat{\eta}^{\kappa^\intercal}\right\},\xi/\widehat{\eta}^{\kappa^\intercal}\right\}$ *is another solution of the game. To understand this, observe that* $\kappa$ *could have been reset at* $\kappa + \widehat{\nu}$ *where* $\widehat{\nu}$ *is an arbitrary element of the kernel of* $\sigma$*. Then the kernel condition would really be* $\sigma(\nu_i - \widehat{\nu}) = 0$*. But this last is equivalent to* $\sigma\nu_i = 0$*.*
*Faced with this indeterminacy, one could impose the condition that* $\kappa$ *be in the span of* $\sigma^\intercal$ *or that, for the first individual,* $\nu_1 = 0$*, (the latter being a much more convenient restriction in general). Either restriction will pin down (standardize)* $\kappa$ *which otherwise would be indeterminate, with a cancelling indeterminacy in each of the* $\nu_i$*'s.*
*In terms of economics, the meaningful kernel condition is one that says only that* differences *in* $\nu$*'s between* any two *individuals should be in the kernel, not that each single* $\nu_i$ *should be in the kernel. This corresponds to the fact that we want any pair of individuals to agree on the prices of traded securities.*

**Proposition 19** *Since the Nash equilibrium of the game has the property that the aggregate resource restriction is satisfied at all times, we have, for an ex-*

*change economy:*

$$\sum_i \left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(0)\,\xi(0,t)\,\eta_i^{\kappa^\intercal}(0,t),t\right)=\sum_i e_i(t)\ \textit{with probability 1} \tag{35}$$

*Differentiating, it follows that the equilibrium choices of the $r$, $\xi$ and $\{\nu_i\}$ processes satisfy:*[15]

$$-\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)}Z_i^{-1}(t)\left[\kappa(t)+\nu_i(t)\right]^\intercal=\sum_i \sigma_i^e(t)\ \textit{with probability 1,} \tag{37}$$

*where $\sigma_i^e$ is the diffusion row vector process of the endowment of individual $i$, as well as:*

$$\begin{aligned}&\sum_i \frac{\partial}{\partial t}\left\{\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right)\right\}-\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)}Z_i^{-1}(t)\,r(t)\\&+\frac{1}{2}\sum_i \frac{1}{\left[\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)\right]^3}\\&\times\frac{\partial^3}{\partial c_i^3}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)\left[Z_i^{-1}(t)\right]^2\left[\kappa(t)+\nu_i(t)\right]^\intercal\left[\kappa(t)+\nu_i(t)\right]\\&=\sum_i \mu_i^e(t)\ \textit{with probability 1}\end{aligned} \tag{38}$$

*where $\mu_i^e$ is the drift process of the endowment of individual $i$.*

In Section 5, we provide a recursive formulation of the same central planning problem. But, first, we look at an illustration of the static method.

# 4 The Magill-Quinzii example solved by the static central plan

In their textbook, Magill and Quinzii (1996, Chapter IV) construct an example that purports to show that an equilibrium in an incomplete market is not Pareto optimal, not even under the constraint of marketability of consumption plans.

[15] The solution for $\kappa$ in the span ($\kappa=\sigma^\intercal x$ for some $x$) is:

$$\left[-\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)}Z_i^{-1}(t)\right]x(t)=\left[\sigma(t)\,\sigma(t)^\intercal\right]^{-1}\sum_i \sigma_i^e(t)\,\sigma(t)^\intercal \tag{36}$$

Indeed, we have been careful here not to give our algorithm involving two central planners any welfare interpretation.

We now use that same example to illustrate the way in which the two central planners would arrive at the incomplete market equilibrium.

Magill and Quinzii's example is set in a three-date (indexed by $t$), two-agent (indexed by $i$), two-state (indexed by $s$) environment. All uncertainty is resolved at $t = 1$ and both states have equal probability. The event-tree for the aggregate state $Y_{t,s}$ is:

| $t = 0$ | $t = 1$ | $t = 2$ |
|---|---|---|
| | $Y_{11}$ | $Y_{21}$ |
| $Y_0$ | | |
| | $Y_{12}$ | $Y_{22}$ |

The endowment processes $e_i$ as a function of the aggregate state $Y_{t,s}$ are as follows for the two agents:

$$
\begin{aligned}
e_1 &= \left(e_1\left(Y_0\right), e_1\left(Y_{11}\right), e_1\left(Y_{12}\right), e_1\left(Y_{21}\right), e_1\left(Y_{22}\right)\right) = (4, 0, 6, 6, 6) \\
e_2 &= \left(e_2\left(Y_0\right), e_2\left(Y_{11}\right), e_2\left(Y_{12}\right), e_2\left(Y_{21}\right), e_2\left(Y_{22}\right)\right) = (9, 8, 0, 8, 8)
\end{aligned}
$$

If state $Y_{11}$ occurs, agent 1 temporarily has a zero endowment, and similarly for agent 2 in state $Y_{12}$.

Agents have time-separable logarithmic utility, but are heterogeneous in terms of time preference:

$$
\begin{aligned}
E_0\left\{\sum_{t=0}^{2} u_i\left(c_i\left(Y_{t,s}\right), t\right)\right\} &= \log\left(c_i\left(Y_0\right)\right) + \beta_i\left[\frac{1}{2}\log\left(c_i\left(Y_{11}\right)\right) + \frac{1}{2}\log\left(c_i\left(Y_{12}\right)\right)\right] \\
&\quad + \beta_i^2\left[\frac{1}{2}\log\left(c_i\left(Y_{22}\right)\right) + \frac{1}{2}\log\left(c_i\left(Y_{22}\right)\right)\right]
\end{aligned}
$$

where the discount factors are given by $(\beta_1, \beta_2) = \left(\frac{1}{2}, \frac{1}{3}\right)$.

In each period, there is only one financial asset, a short-lived bond that permits lending and borrowing. Markets are incomplete, as there is no risky or state-contingent asset that would allow agents to hedge their endowment risk.

The spanning and kernel restrictions are now written with respect to the payoff matrix instead of the diffusion matrix. As there is no risky asset, the spanning condition implies that $\kappa\left(Y_{t,s}\right) = 0$. Also, the kernel restriction is vacuous, which means that $\nu_i$ can be chosen freely.

However, we also need to impose that $\eta_i\left(Y_{t,s}\right) = \exp\left(-\nu_i\left(Y_{t,s}\right)\right)$ be a martingale. This implies $\nu_i\left(Y_{t,s}\right) = 0$ for $t = 2$. For $t = 1$, we obtain

$$
\frac{1}{2} \times e^{-\nu(Y_{11})} + \frac{1}{2} \times e^{-\nu(Y_{12})} = 1
$$

It is straightforward to verify that the equilibrium consumption allocation described by Magill and Quinzii,

$$\begin{aligned} c_1 &= (c_1(Y_0), c_1(Y_{11}), c_1(Y_{12}), c_1(Y_{21}), c_1(Y_{22})) = (4, 0.8, 4.8, 2, 12) \\ c_2 &= (c_2(Y_0), c_2(Y_{11}), c_2(Y_{12}), c_2(Y_{21}), c_2(Y_{22})) = (9, 7.2, 1.2, 12, 2) \end{aligned}$$

along with the values

$$r = (r(Y_0), r(Y_{11}), r(Y_{12})) = (-38\%, 161\%, 161\%)$$

$$\begin{aligned} \nu_1 &= (\nu_1(Y_{11}), \nu_1(Y_{12}), \nu_1(Y_{21}), \nu_1(Y_{22})) = (-54\%, 125\%, 0, 0) \\ \nu_2 &= (\nu_2(Y_{11}), \nu_2(Y_{12}), \nu_2(Y_{21}), \nu_2(Y_{22})) = (125\%, -54\%, 0, 0) \end{aligned}$$

solves the following planning problems, given values for $\left\{Z_i^{-1}(0)\right\}$: $(Z_1(0), Z_2(0)) = (4, 9)$ :

Problem 1:

$$\begin{aligned} &\inf_{\{\nu_i(Y_{t,s})\}} \sup_{\{c_i(Y_{t,s})\}} \left\{ \sum_{i=1}^{2} E_0 \left[ \sum_{t=0}^{2} u_i(c_i(Y_{t,s}), t) \right] \right. \\ &\left. - \sum_{i=1}^{2} \frac{1}{Z_i(0)} E_0 \left[ \sum_{t=0}^{2} [c_i(Y_{t,s}) - e_i(Y_{t,s})] \xi(Y_{t,s}) \eta_i(Y_{t,s}) \right] \right\} \end{aligned} \tag{39}$$

Problem 2:

$$\begin{aligned} &\inf_{\xi(Y_{t,s})} \sup_{\{c_i(Y_{t,s})\}} \left\{ \sum_{i=1}^{2} Z_i(0) E_0 \left[ \sum_{t=0}^{2} \frac{1}{\eta_i(Y_{t,s})} u_i(c_i(Y_{t,s}), t) \right] \right. \\ &\left. + \left[ -E_0 \left[ \sum_{t=0}^{2} \sum_{i=1}^{2} [c_i(Y_{t,s}) - e_i(Y_{t,s})] \xi(Y_{t,s}) \right] \right] \right\} \end{aligned} \tag{40}$$

# 5 Recursive formulation

We now show that it is possible to develop a recursive (dynamic-programming) formulation of the static central-planner problem. This should be useful for numerical implementations.

For the purpose, adopt a Markovian setting:[16]

$$e_i(t) = e_i(0) + \int_0^t \mu_i^e(\{e_i(s)\}, Y(s), s) ds + \int_0^t \sigma_i^e(\{e_i(s)\}, Y(s), s) dw(s) \tag{41}$$

where $Y$ is an Itô process in $\mathbb{R}^K$:

$$Y(t) = Y(0) + \int_0^t \mu(\{e_i(s)\}, Y(s), s) ds + \int_0^t \rho(\{e_i(s)\}, Y(s), s) dw(s) \tag{42}$$

[16] We assume that the functions $\mu_i^e, \sigma_i^e, \mu$ and $\rho$ satisfy growth and Lipschitz conditions.

The aim in the recursive setting is to derive the competitive market equilibrium one step at a time. We show that this can be done by having the two Planners play a dynamic game. Given the Markovian setting, it is natural to focus on Markov strategies and Markov Perfect equilibria. That is, we restrict attention to Nash equilibria that are subgame perfect, and only consider Markov strategies.[17]

It is important in that context to keep two basic ideas in mind. First, even though the process $\xi$ in equilibrium is a continuous process, since it solves Equation (35), it should nevertheless generally be conceived of as a jump process, or an instantaneous control, which accommodates the current values of $\{\eta_i\}$ to satisfy that equation.

Second, the timing of the game needs to be specified carefully:

- The state variables of the game are $\{\eta_i(0,t)\}$. The two players arrive at time $t$ with given values $\{\eta_i(0,t)\}$ for these state variables.

- At time $t$, they play simultaneously a Nash game in which Planner 1 chooses $\{\nu_i(t)\}$, Planner 2 chooses $\xi(0,t)$ to satisfy (35) and they both agree on the choice of $\{c_i(t)\}$. Note that the choice of $\xi(0,t)$ by Planner 2 only depends on the values of the state variables $\{\eta_i(0,t)\}$, and $\{e_i(t)\}$, not on the choice of $\{\nu_i(t)\}$ made by Planner 1. Hence, Planner 1 can take into account that choice of $\xi(0,t)$ by Planner 2 and still be playing Nash.

- As they move to time $t+dt$, a realization $dw(t)$ of the Wiener occurs. This leads to a realization of $Z_i^{-1}(t+dt) = Z_i^{-1}(0)\,\xi(0,t+dt)\,\eta_i(0,t+dt)$. At that time Planner 2 will instantaneously accommodate the aggregate resource constraint by adjusting $\xi(0,t+dt)$. That behavior can be anticipated at time $t$ by Planner 1, by virtue of the subgame perfection assumption.

In short, we have the following definition:

**Definition 20** *A Markov Perfect equilibrium of the above game is a set of admissible, measurable functions*

$$c_i^*(\{\eta_i\},\{e_i\},t)$$
$$\nu_i^*(\{\eta_i\},\{e_i\},Y,t)$$

*and*

$$\xi^*(\{\eta_i\},\{e_i\},t)$$

*such that the decisions* $\{\{c_i^*\},\{\nu_i^*\}\}$ *are optimal for Planner 1 given the values* $\{\xi^*\}$ *of the decisions of Planner 2 and given the current state variables*

[17] See Maskin and Tirole (2000) for a rigorous treatment of Markov Perfect Equilibria in discrete-time. Borkar and Ghosh (1992) prove existence in continuous-time stochastic settings. Continuous-time applications include for instance Budd, Harris and Vickers (1993) in industrial organization and Harris and Laibson (2001) for the case of an intrapersonal game.

*$\{\eta_i(t)\}, \{e_i(t)\}, Y(t), t$ and such that the decisions $\{\{c_i^*\}, \xi^*\}$ are optimal for Planner 2 given the decisions $\{\nu_i^*\}$ of Planner 1 and given the current state variables $\{\eta_i(t)\}, \{e_i(t)\}, Y(t), t,$ where the processes $(\{c_i^*\}, \{\nu_i^*\}, r^*, \kappa^*)$ are defined by:*

$$c_{i,t}^* = c^*(\{\eta_i(0,t)\}, \{e_i(t)\}, t) \tag{43}$$

$$\nu_{i,t}^* = \nu^*(\{\eta_i(0,t)\}, \{e_i(t)\}, Y(t), t) \tag{44}$$

*and:*

$$\xi_t^* = \xi^*(\{\eta_i(0,t)\}, \{e_i(t)\}, t). \tag{45}$$

**Theorem 21** *Suppose that a competitive market equilibrium exists in which the initial Lagrange multipliers are equal to $\left\{Z_i^{-1}(0)\right\}$ and the diffusion matrix of traded asset prices is given by a $N \times K$ dimensional process $\sigma$. The Markov Perfect equilibrium of the above dynamic game is a market sub-equilibrium.*

The task is now to prove this claim and to characterize the Markov Perfect equilibrium. The remainder of this section is devoted to this. We define the value function(s) and construct the corresponding Hamilton-Jacobi-Bellman PDE(s). The first-order conditions are then shown to generate a competitive market sub-equilibrium.

## 5.1 Value functions

Define the value function of Planner 1:

$$\begin{aligned} & J\left(\left\{Z_i^{-1}(0) \times \xi(0,t) \times \eta_i(0,t)\right\}, \{e_i(t)\}, Y(t), t\right) \\ & \triangleq \inf_{\{\nu_i(s) \in \ker \sigma(s)\}} \sup_{\{c_i(s)\}} \sum_i E_t \left\{ \int_t^T u_i(c_i(s), s)\, ds \right\} \\ & - \sum_i Z_i^{-1}(0)\, \xi(0,t)\, \eta_i(0,t)\, E_t\left[\int_t^T [c_i(s) - e_i(s)]\, \xi(t,s)\, \eta_i(t,s)\, ds\right] \end{aligned} \tag{46}$$

In this definition, $\{\eta_i(0,t)\}$ is an argument because it is a state variable but $\xi(0,t)$ is an argument because Planner 1 plays Nash and takes the action of Planner 2 as a given.

**Assumption**: We assume that $J$ is $C^{2,2,2,1}$.

Given the functional form of the argument, we have:

$$\frac{\partial J}{\partial \eta_i} = \frac{\partial J}{\partial Z_i^{-1}} Z_i^{-1}(0)\, \xi \tag{47}$$

$$\frac{\partial J}{\partial \xi} = \sum_i \frac{\partial J}{\partial Z_i^{-1}} Z_i^{-1}(0)\, \eta_i \tag{48}$$

and therefore:

$$\xi \frac{\partial}{\partial \xi} J\left(\left\{Z_i^{-1}(0)\, \eta_i \xi\right\}, \{e_i\}, Y, t\right) = \sum_i \eta_i \frac{\partial}{\partial \eta_i} J\left(\left\{Z_i^{-1}(0)\, \eta_i \xi\right\}, \{e_i\}, Y, t\right) \tag{49}$$

**Lemma 22** *The envelope theorem implies:*

$$\frac{\partial}{\partial Z_i^{-1}} J\left(\left\{Z_i^{-1}(0)\,\xi\eta_i\right\},\{e_i\},Y,t\right) = -E_t\left[\int_t^T [c_i(s)-e_i(s)]\xi(t,s)\,\eta_i(t,s)\,ds\right] \tag{50}$$

*so that this partial derivative is equal to minus the financial wealth of investor $i$.*

We need to incorporate a constraint that the individual specific diffusion $\nu_i$ is in the kernel. "$\nu_i$ is in the kernel" imposes the constraint: $\sigma\nu_i = 0$. We assign Lagrange multiplier $Z_i^{-1}(0)\,\xi(0,t)\,\eta_i(0,t)\,\theta_i^{\intercal}$ to that last constraint.

One could define an analogous value function for Planner 2 but in the present exchange-economy setting, Planner 2 has no intertemporal decisions to make. Hence, that is really pointless.

## 5.2 Conditions of optimality

The Hamilton-Jacobi-Bellmann PDE for $J$ is:[18]

$$0 = \sup_{\{\theta_i(t)\}}\inf_{\{\nu_i(t)\}}\sup_{\{c_i(t)\}}\sum_i\left[u_i(c_i,t)-\frac{1}{Z_i}(c_i-e_i(t))\right] + \mathcal{D}^{r(t),\kappa(t)^{\intercal},\nu_i^{\intercal}} J\left(\left\{Z_i^{-1}\right\},\{e_i\},Y,t\right) - \sum_i Z_i^{-1}\theta_i^{\intercal}\sigma\nu_i \tag{51}$$

where the operator $\mathcal{D}^{r,\kappa^{\intercal},\nu_i^{\intercal}}$ is defined as:

$$\begin{aligned}
\mathcal{D}^{r,\kappa^{\intercal},\nu_i^{\intercal}} J \triangleq{}& \frac{\partial J}{\partial t} - r\sum_i \frac{\partial J}{\partial Z_i^{-1}} Z_i^{-1} + \sum_i \frac{\partial J}{\partial e_i}\mu_i^e + \left(\frac{\partial J}{\partial Y}\right)^{\intercal}\mu \\
&+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial Z_i^{-1}\partial Z_j^{-1}}(\kappa+\nu_i)^{\intercal}(\kappa+\nu_j)\,Z_i^{-1}Z_j^{-1} \\
&+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial e_i\partial e_j}(\sigma_i^e)(\sigma_j^e)^{\intercal} + \frac{1}{2}tr\left(\frac{\partial^2 J}{\partial Y\partial Y}\rho\rho^{\intercal}\right) \\
&-\sum_i\sum_j \frac{\partial^2 J}{\partial Z_i^{-1}\partial e_j}(\kappa+\nu_i)^{\intercal} Z_i^{-1}(\sigma_j^e)^{\intercal} - tr\left(\sum_i \frac{\partial^2 J}{\partial Z_i^{-1}\partial Y}(\kappa+\nu_i)^{\intercal} Z_i^{-1}\rho^{\intercal}\right) \\
&+tr\left(\sum_i \frac{\partial^2 J}{\partial e_i\partial Y}\sigma_i^e\rho^{\intercal}\right)
\end{aligned} \tag{52}$$

and where $\kappa$ and $r$ are given by (37) and (38) respectively, but are taken as given by Planner 1 in his choice of $\{\nu_i\}$.[19] The four first-order conditions associated with the game are:

[18] We omit verification of the fact that the solution of this PDE is identically equal to the value of problem (46), given the standard nature of the program faced by Planner 1.

[19] Recall that (37) and (38) follow from (54) by differentiation. They are therefore, properly part of the system made up of the PDE (52) and the first-order conditions (53 to 56).

- Joint condition

$$\frac{\partial}{\partial c_i} u_i(c_i, t) = Z_i^{-1} \tag{53}$$

- Planner 2's condition:

$$\sum_i [c_i - e_i(t)] = 0 \tag{54}$$

- Planner 1's conditions:

$$\sum_j \frac{\partial^2 J}{\partial Z_i^{-1} \partial Z_j^{-1}} (\kappa + \nu_j)^\intercal Z_i^{-1} Z_j^{-1}$$

$$- \sum_j \frac{\partial^2 J}{\partial Z_i^{-1} \partial e_j} Z_i^{-1} \sigma_j^e - \sum_k \frac{\partial^2 J}{\partial Z_i^{-1} \partial Y_k} Z_i^{-1} \rho_k = Z_i^{-1} \theta_i^\intercal \sigma; \qquad 1 \times K \times I \tag{55}$$

$$\sigma \nu_i = 0; \qquad N \times 1 \times I \tag{56}$$

The first condition is the usual condition of optimality of consumption. The interpretation (50) of the derivative $\frac{\partial}{\partial Z_i^{-1}} J$ as (minus) the financial wealth of individual $i$ shows that the last two first-order conditions are, by construction, identical to He and Pearson's conditions (20, 21) but generalized to incorporate cross-derivatives across individuals. This shows that the $\theta_i^\intercal$s are interpretable as portfolios.

The last three conditions are linear and can be organized into a large partitioned system that can be solved easily for all the $\theta_i^\intercal$s, and $\nu_i$s.

# 6 Numerical implementation

We choose to illustrate our method on the example of the limited-participation equilibrium of Basak and Cuoco (1998). In this section, we adopt their notation. Even though we do not know yet how to handle the general case of limited participation – in which each investor is assigned a list of securities to which he/she has access, – we can handle the specific case situation analyzed by Basak and Cuoco, in which there is only one Wiener shock in the economy, one risky asset and one instantaneously riskless asset and just two (or two categories of) finite-lives agents. Agent 1 has access to both securities, whereas Agent 2 has access to the riskless security only. Basak and Cuoco calculate the equilibrium for the case in which Agent 2 has logarithmic utility. We show here how this can be generalized to any utility function.

In this setup, the risky security is effectively redundant since a group of identical agents (those of Category 1) are the only ones having access to it. No trading of it actually takes place at any time. In Basak and Cuoco, the security is nonetheless "held", but only because agents of Category 1 are endowed with it. The cash-flow process for the risky security, which we prefer to view as the process for the flow endowment of agents of Category 1 is:

$$\delta(t) = \delta(0) + \int_0^t \overline{\mu}\delta(s)\,ds + \int_0^t \overline{\sigma}\delta(s)\,dw(s) \tag{57}$$

The remainder of the agents' endowments is unconventional in the sense that they are not defined by exogenous cash flows. Instead, agents of the two categories share the total zero net supply of the riskless security, which yields an endogenous rate of interest. Agents of Category 1 are endowed with a short position in $\beta$ shares of the bond and agents of Category 2 are endowed with a long position in the same $\beta$ shares of the bond ($\beta > 0$). Agents of Category 1 are the only ones receiving a flow endowment, of the kind we consider in this article. On an average, Category 1 agents consume less than their endowment because they start out with a short position of the bond. This allows Category 2 agents to consume something out of Category 1's flow endowment $\delta$.

Since the riskless security is tradable by all, the initial endowment of bonds only serves to specify the initial distribution of financial wealth. This is not important for our procedure since we are only interested in generating subequilibria, i.e., equilibria with given initial Lagrange multipliers, as opposed to equilibria with given initial wealth distribution.[20]

Concerning the diffusion matrix of security prices, we need to distinguish between securities to which both classes of investors have access and securities that are exclusively available to Category 1 agents. It is easiest to view this market as an incomplete market in which the only traded security is the instantaneously riskless one, the risky security being absent. In our approach, that pins down the diffusion matrix of traded securities: $\sigma = 0$. Note that here $\sigma$ does not contain the volatility of the risky security. The risky security is redundant. Its volatility $\sigma_1$ can be determined separately from the determination of the equilibrium. We first treat the case of exogenous $\sigma_1$ and then show how to find $\sigma_1$ endogenously in our recursive numerical procedure. If we maintain that $\kappa$ should be in the span of $\sigma^{\intercal}$, it follows that $\kappa = 0$, (as in the example of Magill and Quinzii).[21] Furthermore, there is no kernel restriction on the choice of $\nu_1$ and $\nu_2$.

[20] As Basak and Cuoco point out, the initial distribution of wealth, in their set up, also determines whether an equilibrium exists ($\beta$ must be positive, but not so large that agents of Category 1 could never repay their initial short position in the bond). But this is not an issue that we consider.

[21] Here the notation differs from that of Basak and Cuoco. They call $\kappa$ the price of risk in the market for the risky security. Our analog is denoted $\nu_1$. Indeed, the risky security is priced by agents of Category 1 only. The common component $\xi$ of state prices does not price the risky security. The riskless security is the only one the two categories of agents have to agree on.

## 6.1 Exogenous $\sigma_1$

Planner 2's problem is easily solved. Assuming isoelastic utility functions ($e^{-\rho t}\frac{c^{\gamma_i}-1}{\gamma_i}$), it chooses $\xi$ that solves:[22]

$$\sum_i \left\{ \left[e^{+\rho t} Z_1^{-1}(0)\,\xi\eta_i\right]^{\frac{1}{\gamma_i-1}} - e_i(t) \right\} = 0 \tag{59}$$

The value of $r$ follows from (38) and (53):

$$r\left(Z_1^{-1}, Z_2^{-1}, \delta, \{\nu_i\}, t\right) = \rho + \frac{1}{-\sum_{i=1,2}\frac{1}{\gamma_i-1}e^{\frac{\rho t}{\gamma_i-1}}\left(Z_i^{-1}\right)^{\frac{1}{\gamma_i-1}}} \tag{60}$$

$$\times\left\{\overline{\mu}\delta - \frac{1}{2}\sum_{i=1,2}\frac{1}{\gamma_i-1}\left(\frac{1}{\gamma_i-1}-1\right)e^{\frac{\rho t}{\gamma_i-1}}\left(Z_i^{-1}\right)^{\frac{1}{\gamma_i-1}}\nu_i^{\intercal}\nu_i\right\} \tag{61}$$

Let us now focus on Planner 1 who chooses $\nu_1$ and $\nu_2$.[23] Under the present circumstances, the PDE for its value function $J\left(Z_1^{-1}, Z_2^{-1}, \delta, t\right)$ is:

$$\begin{aligned} 0 = &\sum_{i=1,2}\left[\left(\frac{1}{\gamma_i}-1\right)e^{\frac{\rho t}{\gamma_i-1}}\left(Z_i^{-1}\right)^{\frac{\gamma_i}{\gamma_i-1}} - \frac{e^{-\rho t}}{\gamma_i}\right] + Z_1^{-1}\delta + \frac{\partial J}{\partial t} \\ &-r\left(Z_1^{-1}, Z_2^{-1}, \delta, \{\nu_i\}, t\right)\sum_{i=1,2}\frac{\partial J}{\partial Z_i^{-1}}Z_i^{-1} \\ &+\frac{\partial J}{\partial \delta}\overline{\mu}\delta - \frac{1}{2}\sum_{i=1,2}\sum_{j=1,2}\frac{\partial^2 J}{\partial Z_j^{-1}\partial\delta}\overline{\sigma}\delta\left[\frac{\partial^2 J}{\partial Z_i^{-1}\partial Z_j^{-1}}\right]^{-1}_{i,j}\frac{\partial^2 J}{\partial Z_i^{-1}\partial\delta}\overline{\sigma}\delta + \frac{1}{2}\frac{\partial^2 J}{\partial\delta^2}\overline{\sigma}^2\delta^2 \end{aligned} \tag{62}$$

while the first-order conditions are:

$$\sum_{j=1,2}\frac{\partial^2 J}{\partial Z_i^{-1}\partial Z_j^{-1}}\nu_j Z_j^{-1} - \frac{\partial^2 J}{\partial Z_i^{-1}\partial\delta}\overline{\sigma}\delta = 0;\, i = 1,2$$

and the solution is:

$$\nu_j Z_j^{-1} = \sum_{i=1,2}\left[\frac{\partial^2 J}{\partial Z_i^{-1}\partial Z_j^{-1}}\right]^{-1}_{i,j}\frac{\partial^2 J}{\partial Z_i^{-1}\partial\delta}\overline{\sigma}\delta;\, i = 1,2$$

where $\left[\frac{\partial^2 J}{\partial Z_i^{-1}\partial Z_j^{-1}}\right]^{-1}_{i,j}$ refers to the $(i,j)$th element of the matrix inverse of $\left[\frac{\partial^2 J}{\partial Z_i^{-1}\partial Z_j^{-1}}\right]$.

To be completed.

[22]
$$e^{-\rho t}c_i^{\gamma_i-1} = Z_1^{-1} \tag{58}$$

[23] The choice of $\{\nu_i\}$ does not depend on $r$.

### 6.2 Endogenizing $\sigma_1$

To be written.

# 7 Recursive utility

The method described above can be extended straighforwardly from the case of time-additive utility that we have considered so far, to the case of non time-additive, but recursive utility. Dumas, Uppal and Wang (2000), following Geoffard (1996), have shown that recursive utility can be rewritten in the quasi-addititive form of "variational utility". They have also shown that, in a complete market, variational utility permits straighforward welfare analysis and computation of Pareto optima, which are, in that case also, market equilibria.

The same method can be implemented here, not for purposes of welfare analysis, but as a way of computing the equilibrium within an incomplete market when individuals have recursive utility. The reader will note, however, that the variational formulation will introduce a set of individual agent weights in the objective function of Planner 1 (while there are none in the case of time-additive utility) and that another set of state variables $\left\{Z_i^{-1}\right\}$ will also, as above, have to be used. So, the total number of state variables will increase by the number of agents relative to the case of time-additive utility.

# 8 Conclusion

We present a methodology for solving the competitive equilibria of economies with dynamically incomplete markets and heterogeneous agents. The nature of the algorithmic device we propose, a central planner with two selves, provides new insights regarding the fundamental difference between economies with and without complete financial markets. The first central Planner essentially solves for individual consumptions, portfolios and investor-specific components of state prices in the sense of He and Pearson, given economy-wide state prices. In other words, he solves a partial equilibrium problem. Simultaneously, the second Planner chooses equilibrium state prices to satisfy the aggregate resource restriction, given the individual-specific choices of Planner 1. Planner 2 acts like an (intertemporal) auctioneer. It is crucial that Planner 2 internalizes the investor-specific components of the state prices of Planner 1 in his choices. This makes the two Planners agree on consumption and generates equilibrium.

Our analysis is reminiscent of the work of Grossman (1977) who studied equilibria in multi-good economies with incomplete markets. He analyzed the welfare properties of these equilibria by introducing the notion of a central planner with incomplete coordination. Instead of exploring welfare properties, we pursue a similar construction in order to solve for the competitive equilibrium in a multi-period economy.

In a Markovian setting, we establish a recursive formulation of the two-central planner problem. The equilibrium can, therefore, be constructed one time step at a time, using standard dynamic programming techniques.

We believe our methodology has numerous interesting applications in dynamic asset pricing and the analysis of risk-sharing, beyond the confines of the standard complete market paradigm. We plan to pursue these applications in future research. In future work we also aim to extend our methodology to handle the general case of limited participation, where asset markets are incomplete in different ways for different individuals.

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