ON NON-NASH EQUILIBRIA

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Abstract

I consider generalisations of the Nash equilibrium concept based on the idea that in equilibrium the players' beliefs should not be contradicted, even if they could possibly be incorrect. This possibility depends on the information about opponents' behaviour available to the players in equilibrium. Therefore the players' information is crucial for this notion of equilibrium, called Conjectural Equilibrium in general and Rationalizable Conjectural Equilibrium (Rubinstein-Wolinsky 1994) when the game and the players' Bayesian rationality are common knowledge. In this paper I argue for a refinement of Rationalizable Conjectural Equilibrium showing by propositions and by examples how this equilibrium notion works and how the suitable equilibrium concept depends on the players' information.

1 Introduction

It has always amused me that

what the economist calls an equilibrium of behavior,

psychologists tend to call frustration.

Kenneth E. Boulding, The Welfare Economics of Grants.

The literature on the foundations of solution concepts in game theory has extensively analysed the possible reasons for Bayesian rational players to play Nash equilibria. From this literature we have learned that the Nash equilibrium (or the Correlated equilibrium) behaviour is justified under restrictive common-knowledge assumptions (see e.g. Aumann-Brandenburger 1995). Alternative reasons to play Nash equilibria have then been searched for looking at Nash equilibria as possible steady states of plausible learning models. But if our attention is driven towards this equilibrium notion, then more general equilibrium concepts become relevant and the crucial aspect becomes the players' information about opponents' behaviour, since the characterisation of a strategy profile as an equilibrium depends on the information available to the players in that state. Therefore to study the notion of equilibrium in game theory, we should study the role of players' information in strategic situations.

This analysis can be approached from two point of view: the first is a dynamic approach studying how learning leads to different information patterns in equilibrium, the second is a static approach, which takes as given the information partition and characterises the possible steady states of generic learning processes, analysing how different information patterns lead to different equilibrium concepts. In this paper I follow this second strategy showing how the choice among different equilibrium concepts depends on the information the analyst thinks the players have. Similarly to what is done by standard economic theory with market equilibria, a dynamic model is informally in the background and I allude to it only to motivate definitions and results. The general notion of equilibrium I use in this analysis is the concept of Conjectural Equilibrium (CE) and of Rationalizable Conjectural Equilibrium (RCE) (Rubinstein-Wolinsky 1994). The aim of this paper is to argue for a refinement of RCE and to show the wide applicability of this approach to the analysis of strategic situation. This allows to show that Rubinstein-Wolinsky 1994 definition of RCE should be strengthen and that some of their statements should be fitted to the new context.

The general notion of CE has been defined in the seventies by Hahn for general equilibrium models (Hahn 1977 and 1978), but only very recently game theorists have rediscovered this idea in learning models (Fudenberg-Levine 1993 and Kalai-Lehrer 1993) or in the analysis of equilibrium concepts (Rubinstein-Wolinsky 1994). Earlier game theoretic definition can be found in Battigalli 1987, in Gilli 1987 and in Battigalli-Guaitoli 1988. After a first version of this paper was written, I became aware of Dekel-Fudenberg-Levine 1996 that studies a very similar equilibrium notion, even if their focus is on payoff uncertainty.

The rest of this paper is organised as follows. In section 2, I give the basic game theoretic notions used in section 3 to define Conjectural Equilibrium and its refinements. Section 4 provides an existence theorem. In section 5, I show the relations between (Strong Rationalizable) Conjectural Equilibrium and solutions concepts extensively used in game theory.

2 Imperfect Monitoring Games

First the notation: $\Delta(\cdot)$ is the set of all probability measures on $(\cdot)$, the subscript i means that the object refers to player i, if omitted it denotes a complete profile, $-i$ indicates the players different from i, and $-i, i$ denotes a complete profile with the i component stressed. I will remember the notation and the interpretations of the basic definitions for extensive games (EGs) as given by Osborne-Rubinstein 1994, which are slightly more general of that in Kreps-Wilson 1982. An **EXTENSIVE GAME** (EG) is defined by:

$$EG := (H; \quad \iota; \quad \mathcal{I}_i; \quad \bar{u}_i)_{i=1}^N \qquad \text{where}$$

- N denotes the set of players and the cardinality of the set itself,
- H is the set of histories h, which are sequences of players' actions: $h = (a^k)_{k=1}^K$, where a^k is the action taken after the history $(a^l)_{l=1}^{k-1}$.
- ι is the function that assign to each nonterminal history the player whose turn it is;

- $\mathcal{I}_i$ is a partition of $\{h \in H | \iota(h) = i\}$ and is the set of player i's information sets I_i;

- $\bar{u}_i$ is i's payoff function: $\bar{u}_i : \Delta(Z) \to \mathbf{R}$.

From the previous definitions, we can derive the following sets and functions:

- player i's set of pure strategies S_i, the set of player i's mixed strategies $\Sigma_i := \Delta(S_i)$, and the set of mixed strategy profiles $\Sigma := \otimes_{i \in N} \Sigma_i$, where $\otimes_i$ indicates the product of measures;

- the outcome function that identifies the terminal histories induced by an n-tuple of strategies $s \in S$: $\zeta : S \to Z$;

- i's payoff function directly defined of the set of players' possible strategies: $u_i := \bar{u}_i \circ \zeta$ i.e. $u_i : \Delta(S) \to \mathbf{R}$.

A **STRATEGIC GAME** (SG) is then defined by

$$SG := (S_i,\ \ u_i)_{i=1}^N.$$

The usual interpretation of SGs is as simultaneous moves games, i.e. games with a bijective outcome function (it is easy to see that an extensive game has simultaneous moves if and only if the outcome function is bijective). Rubinstein-Wolinsky 1994 enrich the environment defined by a SG by means of a **signal function**: $\eta_i : S \to M_i$, with the following interpretation: $\eta_i(s) = m_i \in M_i$ is the signal privately observed by $i \in N$ when $s \in S$ is played. In other words the mapping η_i represents player i's **information on opponents' behaviour**, which by definition could depend on player i's behaviour itself. Of course the functional form of η_i must be fitted to the case under analysis: the specification of η_i reflects player i's information partition as seen by the game theorist. An **IMPERFECT MONITORING GAME** (IMG) is defined by

$$G := (S_i,\ \ u_i,\ \ \eta_i)_{i=1}^N.$$

A general analysis of IMGs is in Gilli 1994, while a very similar model for repeated game is the object of many Lehrer's papers (see e.g. Lehrer 1989, 1990, 1991, 1992). In Gilli 1994 I have shown that it is possible to define an outcome function without any direct reference to an EG and that such function is originated by a "unique" EG (suitably defined). Therefore it is meaningful to consider the IMG (S_i, u_i, ζ), where ζ is the outcome function of the unique EG which gives rise to it.

To simplify the analysis I make the following **STRUCTURAL ASSUMPTIONS**:

Assumption 1 *The analysis is restricted to the subclass of finite imperfect monitoring games, that is the sets N, S_i, M_i are finite.*

As usual, this assumption is useful to avoid further complication in the mathematical tools used for the analysis. A first consequence of assumption 1 is the following. Fix a finite number of player N and their pure strategies S. Then the space of IMGs over this form is given by $\mathcal{G}$, and I take $\mathcal{G} = \mathbf{R}^{N\times S} \times \mathbf{R}^{N\times S}$ where for $(x, y) \in \mathcal{G}$, $x(i, s)$ is the payoff to player i under strategy s and $y(i, s)$

is the signal to player i under strategy s. If a specific signal function η_i for each player is specified, then the space $\mathcal{G}(\eta_i)$ is obtained, and I take $\mathcal{G}(\eta_i) = \mathbf{R}^{N\times S}$ where for $x \in \mathcal{G}(\eta_i)$, $x(i,s)$ is the payoff to player i under strategy s.

Assumption 2 *The signal is defined to contain all of the information player i receives about opponents' strategic behaviour. Therefore*

$$\eta_i(s_i, s_{-i}) = \eta_i(s_i, s'_{-i}) \quad \Rightarrow \quad u_i(s_i, s_{-i}) = u_i(s_i, s'_{-i}).$$

This assumption means that after the game each player receive her payoff and hence the signal provides non less information than the payoff; each player recalls her own move in addition to her signal, and given her move and signal, player i can compute her payoff.

Using η_i it is possible to define an **INFORMATION PARTITION** for player i as follows: let be $\mathcal{S}_i(m_i, s_i|\eta_i) := \{s_{-i} \in S_{-i}|\eta_i(s_i, s_{-i}) = m_i\}$, then

$$\mathcal{S}_i(s_i|\eta_i) := (\mathcal{S}_i(m_i, s_i|\eta_i))_{m_i \in M_i}. \tag{1}$$

Thus $\mathcal{S}_i(m_i, s_i|\eta_i)$ is i's information about opponents' strategies when she receives the message m_i and plays the strategy s_i, $\mathcal{S}_i(s_i|\eta_i)$ is the information about opponents' behaviour that player i can possibly collect playing s_i. This expression depends on the strategy played by i, because she can collect different information playing different strategy. If (1) does not depend on s_i, the **information** is said to be **non manipulable**, e.g. this is the case of SGs with one-to-one signal functions, or of EGs with simultaneous moves and $\eta_i = \zeta$.

3 The Notion of Equilibrium in Imperfect Monitoring Games

$BR_i(\alpha_i)$ is player i's correspondence of best reply to $\alpha_i \in \Delta(S_{-i})$.

Definition 1 *The set of* **Conjectural Equilibrium in pure strategies** *(CE^P) for an IMG G is indicated by $CE^P(G)$. Then an n-tuple $s' \in S$ belongs to the set $CE^P(G)$ if and only if in the game G $\forall i \in N,\ \exists \alpha_i \in \Delta(S_{-i})$:*

$$s'_i \in BR_i(\alpha_i) \tag{2}$$

and

$$\alpha_i(\mathcal{S}_i(\eta_i(s'), s'_i | \eta_i)) = 1. \tag{3}$$

Remarks:

1. In a CE^P the players are Bayesian rational with respect to a conjecture α_i, which in equilibrium is not falsified by the player's information, i.e. each player's conjecture gives probability 1 to the event she actually observes. Therefore the outcome of strategic interaction does not provide any information that induces the players to change their behaviour: in this case I am actually assuming that for each player i all the relevant information is provided by $\eta_i(\cdot)$.

2. The concept of CE does not explain why the players choose a strategy profile s', but, as usual in static approaches, it says only that if for some

reason a strategy profile $s' \in CE^P$ is played, then the players' ex post information is such that there is no incentive to change behaviour.

3. No hypothesis about the players' knowledge of the strategic interaction's situation is specified by η_i; therefore this information should be considered as implicit in the definition of the equilibrium concept: it is possible to refine the equilibrium notion if common knowledge (CK) of the game and of rationality is assumed.

The notion of CE^P will be explained by means of an example. Consider the game of figure 1:

Fig. 1 about here

A possible story behind this game is the following: firm 1 may stay out (O) or enter in a market with either a small (S) or a big (B) investment, while firm 2 observes only if 1 enters or not and then decides whether to accommodate (A) or to fight (F). If 2 decides to accommodate, then it is plausible to suppose that it can not distinguish between S and B, although they are different outcomes. Therefore suppose that $\eta_i(s) = I_i(\zeta(s))$, where $I_i(\zeta(s))$ is i's information on the outcome $\zeta(s)$, i.e. it is the last information set for player i crossed by the outcome path when s is played. Then (O,A) is a CE^P: O is a best reply to $\alpha_1(\{F\}) = 1$, A is a best reply to $\alpha_2(\{O\}) = 1$ and

$\eta_i(O,F) = I_i(\zeta(O,F)) = \eta_i(O,A) = I_i(\zeta(O,A)) = z_1$. Note that (O,A) is not a Nash equilibrium (NE), but that $\zeta(O,A) = z_1$ is a NE outcome (the relation between CE and NE outcomes will be explained by theorem 4). This straightforward example shows an important characteristic of CE^P: it does not assume CK of rationality (and of the game). Indeed the conjecture of firm 1 in the previous example is not compatible with CK of rationality: since S is dominated by B, if firm 2's information set is reached, then 2 should conjecture that a rational firm 1 has played B and therefore accommodate. But then if there is CK of rationality and of the game, a rational firm 1 should know this reasoning and thus have $\alpha_1(\{A\}) = 1$: therefore 1 should play B and a rational firm 2 should answer with A. So the CE (O,F) is destroyed by CK of rationality, while (B,A) is a CE^P which satisfy CK of rationality. Of course (B,A) is also a NE. But there is another characteristic of CE^P: consider the version of the game of figure 1 reported in figure 2

Fig. 2 about here

In this case S is not dominated by B, so the previous argument does not work. With the same signal function as before (S,A) is a CE^P. In fact A is a best reply to $\alpha_2(\{S\}) = \alpha_2(\{B\}) = 1/2$, S is a best reply to $\alpha_1(\{A\}) = 1$ and $\eta_1(S,A) = I_1(\zeta(S,A)) = z_2$, $\eta_2(S,A) = \eta_2(B,A) = I_2(\zeta(S,A)) = \{z_2, z_4\}$. But if the game is CK, then 1 knows that 2 will observe $\{z_2, z_4\}$ and 2 knows that 1 will observe either z_2 or z_4 and all this is CK. Thus a rational firm 1, knowing that 2 can't distinguish between S and B, should play S, but then because of CK of rationality and of the game a rational firm 2 should have a conjecture concentrated on S, i.e. $\alpha_2(\{S\}) = 1$ and therefore 2 will choose F. So the CE^P (S, A) is destroyed by CK of rationality and of the game, in particular of opponent's signal function. Indeed it should be stressed that the CK of the game implies the CK of the signal functions, although their realisations can be private information. But the mere CK of the functions is enough to restrict players' possible conjectures.

The notion of Strong Rationalizable Conjectural Equilibrium modifies the definition of CE assuming CK of rationality and of the game. Define $M_j(\bar{s}_i, m_i|\eta) \subseteq M_j$ as follows: $M_j(\bar{s}_i, m_i|\eta) := \{m_j \in M_j|\eta_j(\bar{s}_i, s_{-i}) = m_j, s_{-i} \in \mathcal{S}_i(\bar{s}_i, m_i|\eta_i)\}$. Thus $M_j(\bar{s}_i, m_i|\eta)$ is the set of signals that j could receive according to i's information and given the CK of the game. Note that $M_i(\bar{s}_i, m_i|\eta) = \{m_i\}$.

Definition 2 *Fix a game $G \in \mathcal{G}$. The set of* **Strong Rationalizable Conjectural Equilibria in pure strategies** *($SRCE^P$) of game G is denoted by $SRCE^P(G)$. For any $i \in N$, let $B_i^P(m_i, G) \subseteq S_i$ be defined as follows: $\bar{s}_i \in B_i^P(m_i, G)$ if and only if*

$$\exists \alpha_i[\bar{s}_i] \in \Delta(\cup_{m_{-i} \in M_{-i}(\bar{s}_i, m_i|\eta)} B_{-i}^P(m_{-i}, G)) : \tag{4}$$

$$\bar{s}_i \in BR_i(\alpha_i[\bar{s}_i]) \tag{5}$$

$$\alpha_i[\bar{s}_i](\mathcal{S}_i(\bar{s}_i, m_i|\eta_i)) = 1. \tag{6}$$

Then an n-tuple $s' \in S$ *belongs to the set* $SRCE^P(G)$ *if and only if in the game* G $\forall i \in N \quad \exists B_i^P(m_i, G)$ *such that:*

$$s_i' \in B_i^P(\eta_i(s'), G) \quad \textit{and} \quad m_i = \eta_i(s'). \tag{7}$$

Remarks:

1. The expressions (4) and (5) are conditions of Bayesian rationality under the assumption of CK of the game and of Bayesian rationality, while expressions (6) and (7) are the usual equilibrium condition.

2. Note that in some sense there are three fixed points involved in definition 2: two are explicit in conditions (5) and (7), but since the sets B_i are defined through the sets B_{-i}, there is a circularity to be solved either by means of a fixed point argument or through an equivalent iterative definition (see definition 6 and theorem 1). Consequently the sets B_i^P and thus $SRCE^P$ may easily be empty: existence is proved for mixed SRCE in theorem 2.

The notion of Rationalizable Conjectural Equilibrium (RCE) has been first proposed by Rubinstein-Wolinsky 1994, but in their paper they do not stress that under the assumption of CK of the game and of rationality, the players' conjectures should be functions of the strategy the players themselves choose. RCE can be defined as follows[1]:

[1]The definition given by Rubinstein-Wolinsky 1994 is different, but equivalent.

Definition 3 *Fix a game* $G \in \mathcal{G}$. *The set of* **Rationalizable Conjectural Equilibria in pure strategies** *(*RCE^P*) of game* G *is denoted by* $RCE^P(G)$. *For any* $i \in N$, *let* $Q_i^P(m_i, G) \subseteq S_i$ *be defined as follows:* $\bar{s}_i \in Q_i^P(m_i, G)$ *if and only if*

$$\exists \alpha_i \in \Delta(\cup_{m_{-i} \in M_{-i}} Q_{-i}^P(m_{-i}, G)) : \tag{8}$$

$$\bar{s}_i \in BR_i(\alpha_i) \tag{9}$$

$$\alpha_i(\mathcal{S}_i(\bar{s}_i, m_i|\eta_i)) = 1. \tag{10}$$

Then an n-tuple $s' \in S$ *belongs to the set* $RCE^P(G)$ *if and only if in the game* G $\forall i \in N \quad \exists Q_i^P(m_i, G)$ *such that:*

$$s'_i \in Q_i^P(\eta_i(s'), G) \quad \text{and} \quad m_i = \eta_i(s'). \tag{11}$$

Therefore RCE^P are defined as $SRCE^P$ where the players' conjectures are restricted to be constant, i.e. $\forall i, \ \forall s_i \ \alpha_i[s_i] = \alpha_i$. In general however the rational behaviour of i's opponents depends on the signals they receive which in turn depend on the strategy i has chosen. In other words in general $M_{-i}(\bar{s}_i, m_i|\eta)$ depends on $\bar{s}_i$ and it is a strict subset of M_{-i}: when G is CK this information is available to i and can be used to refine i's conjectures, thus even player i's conjecture should depend on i's strategy. If this fact is taken into account, then the definition of RCE should be refined as in definition 2. Note that $SRCE^P$ coincides with RCE^P when a non manipulable information partition is defined, but in general $SRCE^P \subseteq RCE^P$. Therefore when SRCE instead of RCE is considered, the following Rubinstein-Wolinsky 1994 statement should be changed:

"when the signal is the actual path being played, in any game any CE outcome is also RCE outcome ... The reason again is that RCE does not impose any restrictions off the equilibrium path and so there is nothing to distinguish it from a CE when the path is known" (Rubinstein-Wolinsky 1994, p. 306). Actually according to definition 2 the assumption of CK of the game and of rationality imposes some restriction even off the equilibrium path, as the following example shows. Consider as signal function the outcome function of the well known Selten game, which is pictured in figure 3.

Fig. 3 about here

If player 1 chooses L, then the outcome for both player is L and thus every strategy of player 2 is a rational choice and thus $\alpha_1[L] \in \Delta(\{U, D\})$. If instead player 1 chooses R, then player 2 observes it and her rational choice is U, and this is CK. Therefore $\alpha_1[R] \in \Delta(\{U\})$. Consequently $L \notin BR_1(\alpha_1[L])$ since $u_1(L, \alpha_1[L]) = 2 < u_1(R, \alpha_1[R]) = 3$. Thus (L, D) is not a $SRCE^P$, while it is obviously a CE^P and even a NE. The reason for this result is that in many cases the assumption of CK of game and rationality is sufficient to refine the set of possible conjectures even off the equilibrium path, while CE and NE do not fully exploit this assumption, CE because it does not assume CK of anything and NE because it is defined on the strategic form and therefore consider any strategic situation as if it had simultaneous moves and perfect monitoring and thus a non manipulable information partition.

The extension of these definitions to the case of mixed strategies (and thus of distributions over signals) is easy. Define as follows the probability distribution induced over M_i by a probability measure $\alpha \in \Delta(S)$: $\forall m_i \in M_i \;\; p_i(m_i; \alpha) := \sum_{\{s|\eta_i(s)=m_i\}} \alpha(s)$. Denote a generic distribution of probability on M_i by $\rho_i \in \Delta(M_i)$. Then it is possible to define i's information partition derived from ρ_i and $\bar{\sigma}_i$:

$$\mathcal{S}_i^m(\bar{\sigma}_i, \rho_i|\eta_i) := \{\sigma_{-i} \in \Sigma_{-i}|p_i(\bar{\sigma}_i, \sigma_{-i}) = \rho_i\} \subseteq \Sigma_{-i}.$$

Note that player i's partition is defined through the probability distribution $\rho_i \in \Delta(M_i)$, but this does not mean that the players necessarily observe the distribution of signals induced by the players' mixed strategy: the fact is that if the players conjectured distribution is different from the actual one, then in the long run the players will find that they are wrong adjusting consequently their conjectures. Moreover note that Σ_{-i} is a simplex in $\mathbf{R}^k$ for some finite k, and that $\mathcal{S}_i^m(\bar{\sigma}_i, \rho_i|\eta_i)$ belongs to the Borel sigma-algebra of $\mathbf{R}^k$ since $p_i(\sigma_i, \sigma_{-i})$ is continuous in σ_{-i}: therefore it is meaningful to write about the probability of $\mathcal{S}_i^m(\bar{\sigma}_i, \rho_i|\eta_i)$. Define $M_j^m(\bar{\sigma}_i, \rho_i|\eta) \subseteq \Delta(M_j)$ as follows: $M_j^m(\bar{\sigma}_i, \rho_i|\eta) := \{\rho_j \in \Delta(M_j)|p_j(\bar{\sigma}_i, \sigma_{-i}) = \rho_j, \sigma_{-i} \in \mathcal{S}_i^m(\bar{\sigma}_i, \rho_i|\eta_i)\}$. Thus $M_j^m(\bar{\sigma}_i, \rho_i|\eta)$ is the information that j could receive according to i's information and given the CK of the game.

Definition 4 *The set of* **Conjectural Equilibria in mixed strategies** *(CE) of game G is denoted by $CE(G)$. Then an n-tuple $\bar{\sigma} \in \Sigma$ belongs to the set*

$CE(G)$ if and only if in the game G $\forall i \in N$, $\quad \exists \mu_i \in \Delta(\Sigma_{-i})$:

$$\bar{\sigma}_i \in BR_i(\mu_i) \tag{12}$$

$$\mu_i(\mathcal{S}_i^m(\bar{\sigma}_i, p_i(\bar{\sigma})|\eta)) = 1, \tag{13}$$

with the obvious meaning of the abuse of notation in $BR_i(\cdot)$.

Definition 5 *Fix a game $G \in \mathcal{G}$. The set of* **Strong Rationalizable Conjectural Equilibria in mixed strategies** *(SRCE) of game G is denoted by $SRCE(G)$. For any $i \in N$, let $\quad B_i(\rho_i, G) \subseteq \Sigma_i \quad$ be defined as follows: $\bar{\sigma}_i \in B_i(\rho_i, G)$ $\quad$ if and only if*

$$\exists \mu_i[\bar{\sigma}_i] \in \Delta(\cup_{\rho_{-i} \in M_{-i}^m(\bar{\sigma}_i, \rho_i|\eta)} B_{-i}(\rho_{-i}, G)) : \tag{14}$$

$$\bar{\sigma}_i \in BR_i(\mu_i[\bar{\sigma}_i]) \tag{15}$$

$$\mu_i[\bar{\sigma}_i](\mathcal{S}_i^m(\bar{\sigma}_i, \rho_i|\eta_i)) = 1. \tag{16}$$

Then an n-tuple $\sigma' \in \Sigma$ belongs to the set $SRCE(G)$ if and only if in the game G $\forall i \in N \quad \exists B_i(\rho_i, G) \quad$ such that:

$$\sigma_i' \in B_i(p_i(\sigma'), G) \quad \textit{and } \rho_i = p_i(\sigma'). \tag{17}$$

As in definition 2, SRCE are defined through three fixed points, therefore non emptiness is not trivial to prove, as theorem 2 will show.

4 Existence of Strong Rationalizable Conjectural Equilibria

As shown by the example of figure 3 it is not true that in general $NE \subseteq SRCE$, while $NE \subseteq RCE$. Therefore it is not possible to use the Nash existence theorem to prove that SRCE is not empty and I should follow an alternative route. First I prove that the definition of SRCE is equivalent to the requirement that each player can construct a hierarchy of strategy sets such that each strategy is rationalised by the next level beliefs, given the signal received and given CK of the game and of Bayesian rationality. Then I use this alternative definition of SRCE to show that $SRCE \neq \emptyset$.

Definition 6 *Define:* $B_i'(\rho_i, G) := \bigcap_{t\geq 1} B_i'(t, \rho_i, G)$ *where* $B_i'(0, \rho_i, G) := \Sigma_i$ *and* $\forall t \geq 1$ $\bar{\sigma}_i \in B_i'(t, \rho_i, G)$ *if and only if*

$$\bar{\sigma}_i \in B_i'(t-1, \rho_i, G) \quad and \quad \exists \mu_i[\bar{\sigma}_i] \in \Delta(\cup_{\rho_{-i} \in M_{-i}^m(\bar{\sigma}_i, \rho_i|\eta)} B'_{-i}(t-1, \rho_{-i}, G)):$$

$$\bar{\sigma}_i \in BR_i(\mu_i[\bar{\sigma}_i]) \quad and \quad \mu_i[\bar{\sigma}_i](\mathcal{S}_i^m(\bar{\sigma}_i, \rho_i|\eta_i)) = 1.$$

Theorem 1

$$\forall G \in \mathcal{G}, \quad \forall i \in N \quad \forall \rho_i \in \Delta(M_i) \qquad B_i(\rho_i, G) = B_i'(\rho_i, G).$$

PROOF: fix a $G \in \mathcal{G}$ and, for every $i \in N$, a $\rho_i \in \Delta(M_i)$. To simplify the notation omit G.

First I prove that $B_i(\rho) \subseteq B_i'(1, \rho)$. Suppose that $\sigma_i' \in B_i(\rho)$, then $\exists \mu_i[\sigma_i'] \in \Delta(\cup_{\rho_{-i} \in M_{-i}^m(\sigma_i', \rho_i|\eta)} B_{-i}(\rho_{-i}))$: $\sigma_i' \in BR_i(\mu_i[\sigma_i'])$. Since by definition $B_i(\rho_i) \subseteq$

Σ_i and $\Delta(\cup_{\rho_{-i}\in M^m_{-i}(\sigma'_i,\rho_i|\eta)}B_{-i}(\rho_{-i}))\subseteq\Delta(\Sigma_{-i})$, then definition 6 implies that $\sigma'_i\in B'_i(1,\rho_i)$. Now assume that for all i $B_i(\rho)\subseteq B'_i(t,\rho)$. Then $\sigma'_i\in B_i(\rho_i)$, implies that $\exists\mu_i[\sigma'_i]\in\Delta(\cup_{\rho_{-i}\in M^m_{-i}(\sigma'_i,\rho_i|\eta)}B_{-i}(\rho_{-i}))\subseteq\Delta(\cup_{\rho_{-i}\in M^m_{-i}(\sigma'_i,\rho_i|\eta)}B'_{-i}(t,\rho_{-i}))$ such that: $\sigma'_i\in BR_i(\mu_i[\sigma'_i])$ and $\mu_i[\sigma'_i](\mathcal{S}^m_{-i}(\sigma'_i,\rho_i|\eta_i))=1$, where the inclusion follows from the induction hypothesis. Therefore $\sigma'_i\in B'_i(t+1,\rho_i)$ and thus $B_i(\rho_i)\subseteq B'_i(\rho_i)$.

Now suppose that $\sigma'_i\in B'_i(\rho)$. Note that $\forall i,\ \ \forall t'\in\mathbf{N},\ \ \ B'_i(\rho)\subseteq B'_i(t'+1,\rho)\subseteq B'_i(t',\rho)$. Therefore $\sigma'_i\in B'_i(\rho)$ implies $\sigma'_i\in B'_i(t'+1,\rho_i)\subseteq B'_i(t',\rho_i)$. Then by definition $\exists\mu_i[\sigma'_i]\in\Delta(\cup_{\rho_{-i}\in M^m_{-i}(\sigma'_i,\rho_i|\eta)}B'_{-i}(t',\rho_{-i}))$ such that:

$$\sigma'_i\in BR_i(\mu_i[\sigma'_i])\quad\text{and}\quad\mu_i[\sigma'_i](\mathcal{S}^m_i(\sigma'_i,\rho_i|\eta_i))=1.$$

But this is equivalent to the definition of $B_i(\rho_i)$, with $B'_{-i}(t',\rho)$ as $B_{-i}(\rho)$. $\square$

Remark: from theorem 1 it is immediate that $\forall G\in\mathcal{G}\quad\sigma'\in SRCE(G)$ if and only if $\forall i\in N\ \ \exists B'_i(\rho_i,G):\quad\sigma'_i\in B'_i(p_i(\sigma'),G)$.

Theorem 2

$$\forall G\in\mathcal{G}\qquad SRCE(G)\neq\emptyset.$$

PROOF: fix a game $G\in\mathcal{G}$ and omit G in the notation. The proof is organised as follows: first I prove that $\forall i,\ \ \forall\rho_i\ \ B'_i(\rho_i)\neq\emptyset$, then I prove that $\exists\sigma'$ such that $\sigma'\in B'(p(\sigma'))$. Both results are based on fixed point theorems, therefore the proof depends on the properties of the opportune correspondences.

Note that $\forall i\ \ B'_i(t+1,\rho)\subseteq B'_i(t,\rho)$. Therefore if I prove (by induction) that $\forall t\ \ B'_i(t,\rho)$ is nonempty and compact, then $B'_i(\rho)\neq\emptyset$ would follows from the property of a decreasing succession of nonempty compact sets. By definition $B_i'(0,\rho) = \Sigma_i$ which is nonempty and compact. Suppose that $\forall i \;\; B_i'(t,\rho) \neq \emptyset$ and compact. Then $B_i'(t+1,\rho) \neq \emptyset$ iff $\exists \sigma_i' \in B_i'(t,\rho_i)$ (which is nonempty by the induction hypothesis) such that

$$\exists \mu_i[\sigma_i'] \in \Delta(\cup_{\rho_{-i} \in M_{-i}^m(\sigma_i',\rho_i|\eta)} B'_{-i}(t,\rho_{-i})) : \tag{18}$$

$$\mu_i[\sigma_i'](\mathcal{S}_i^m(\sigma_i',\rho_i|\eta_i)) = 1 \tag{19}$$

$$\sigma_i' \in BR_i(\mu_i[\sigma_i']). \tag{20}$$

Now if there exists a continuous function $\mu_i[\sigma_i]$ satisfying conditions (18) and (19), then the maximum theorem would imply that the correspondence

$$BR_i(\mu_i) : \Sigma_i \rightarrow\rightarrow \Sigma_i$$

is upper hemi-continuous with nonempty convex compact values. Thus Kakutani theorem would imply the existence of the fixed point (20) and thus that $B_i'(t+1,\rho)$ is nonempty (and compact). Therefore the crucial point is to prove that the correspondence $\mu_i : \Sigma_i \rightarrow\rightarrow \Delta(\Sigma_{-i})$ satisfying conditions (18) and (19) has a continuous selection. Now consider the following facts:

1. $\mathcal{S}_i^m(\sigma_i,\rho_i|\eta_i)$ is lower hemi-continuous (l.h.c.) in σ_i and in ρ_i since $p_i(\sigma_i,\sigma_{-i})$ is continuous in σ_i, σ_{-i};

2. $\{\rho_j | p_j(\sigma_i,\sigma_{-i}) = \rho_j\}$ is l.h.c. in σ_{-i} since $p_j(\sigma_i,\sigma_{-i})$ is continuous in σ_{-i};

3. $M_j^m(\sigma_i,\rho_i|\eta)$ can be written as follows:

$$M_j^m(\sigma_i,\rho_i|\eta) = \cup_{\sigma_{-i} \in \mathcal{S}_i^m(\sigma_i,\rho_i|\eta_i)} \{\rho_j | p_j(\sigma_i,\sigma_{-i}) = \rho_j\}.$$

Therefore facts 1 and 2 and proposition 11.23 in Border 1985 imply that $M_j^m(\sigma_i, \rho_i|\eta)$ is l.h.c. in σ_i and in ρ_i;

4. fact 3 and proposition 11.25 in Border 1985 imply that $M_{-i}^m(\sigma_i, \rho_i|\eta)$ is l.h.c. in σ_i and in ρ_i;

5. fact 4 and proposition 11.23 in Border 1985 imply that $B_i'(t, \rho_i)$ is l.h.c. in ρ_i;

6. facts 4, 5, proposition 11.23 in Border 1985 imply that $\cup_{\rho_{-i} \in M_{-i}^m(\sigma_i, \rho_i|\eta)} B_{-i}(t, \rho_{-i})$ is l.h.c. in σ_i.

Therefore the correspondence $\mu_i : \Sigma_i \rightarrow\rightarrow \Delta(\Sigma_{-i})$ satisfying conditions (18) and (19) is l.h.c. Moreover it has closed convex values: thus by theorem 14.7 in Border 1985 (see also Michael 1956) has a continuous selection. Therefore $B_i'(\rho_i)$ is nonempty and compact. Moreover $B'(\rho) := \times_{i \in N} B_i'(\rho)$ is obviously convex. Finally fact 5, propositions 11.23 and 11.25 in Border 1985 and the continuity of $p_i(\sigma)$ imply that the correspondence

$$B'(p) : \Sigma \rightarrow\rightarrow \Sigma$$

is l.h.c in σ and have closed convex values. Therefore theorem 15.4 in Border 1985 implies that it has a fixed point: $\exists \sigma' : \quad \sigma' \in B'(p(\sigma'))$ and therefore $SRCE \neq \emptyset$. $\square$

5 Information Partitions and Equilibrium Concepts

In this section I will consider the relationships between the notions of (Strong Rationalizable) Conjectural Equilibrium and the classic solution concepts, such as NE and Extensive Form Rationalizability (EFR). These propositions show the prominent role of players' information partition when we want to analyse a strategic situation. It is trivial to see that $\forall G \in \mathcal{G}, \quad NE(G) \subseteq RCE(G) \subseteq CE(G)$ (Rubinstein-Wolinsky 1994). More complex are the relationship between SRCE and NE. The previous example of figure 3 shows that in a subclass of $\mathcal{G}$ SRCE is actually a refinement of NE, while the following theorem 3 shows that when the information partition is non manipulable, e.g. when the signal function is one-to-one, SRCE and NE coincides.

For fixed N and S, denote the set of signal functions inducing a non manipulable information partition by $NM \subseteq \mathbf{R}^{N \times S}$.

Theorem 3

$$\forall G \in \mathcal{G}(\eta \in NM) \qquad NE(G) = SRCE(G) = CE(G).$$

PROOF: for this class of games it is immediate that $NE(G) \subseteq SRCE(G) \subseteq CE(G)$, since in this case $SRCE(G) = RCE(G)$. Therefore the result is proved showing that for this class of games $CE(G) \subseteq NE(G)$. Denote the support of a probability measure by $SUPP[\cdot]$. Suppose that $\sigma' \in CE(G)$. Then $\quad \forall i \in$

$N \quad \exists \mu_i \in \Delta(\Sigma_{-i})$ such that:

$$\sigma'_i \in BR_i(\mu_i) \tag{21}$$

$$\mu_i(\{\sigma_{-i} | p_i(\cdot; \sigma'_i, \sigma_{-i}) = p_i(\cdot; \sigma'_i, \sigma'_{-i})\}) = 1. \tag{22}$$

Consider condition (21), which implies that $\forall s'_i \in SUPP[\sigma'_i]$, $\forall s_i \in S_i$

$$\int [\sum_{s_{-i}} u_i(s'_i, s_{-i}) \sigma_{-i}(s_{-i})] \mu_i(d\sigma_{-i}) \geq \int [\sum_{s_{-i}} u_i(s_i, s_{-i}) \sigma_{-i}(s_{-i})] \mu_i(d\sigma_{-i}).$$

But $\int [\sum_{s_{-i}} u_i(s'_i, s_{-i}) \sigma_{-i}(s_{-i})] \mu_i(d\sigma_{-i}) = \sum_{s_{-i}} u_i(s'_i, s_{-i}) \int \sigma_{-i}(s_{-i}) \mu_i(d\sigma_{-i}) = \sum_{s_{-i}} u_i(s'_i, s_{-i}) \hat{\sigma}_{-i}(s_{-i})$ where

$$\hat{\sigma}_{-i}(s_{-i}) := \int \sigma_{-i}(s_{-i}) \mu_i(d\sigma_{-i}). \tag{23}$$

Therefore the previous inequality can be rewritten as follows: $\forall s'_i \in SUPP[\sigma'_i]$, $\forall s_i \in S_i$ $\sum_{s_{-i}} u_i(s'_i, s_{-i}) \hat{\sigma}_{-i}(s_{-i}) \geq \sum_{s_{-i}} u_i(s_i, s_{-i}) \hat{\sigma}_{-i}(s_{-i})$ i.e.

$$\forall \sigma_i \in \Sigma_i \quad u_i(\sigma'_i, \hat{\sigma}_{-i}) \geq u_i(\sigma_i, \hat{\sigma}_{-i}). \tag{24}$$

Note that relations (22) and (23) imply that $p_i(\cdot; \sigma'_i, \hat{\sigma}_{-i}) = p_i(\cdot; \sigma'_i, \sigma'_{-i})$. Therefore assumption 2 implies $u_i(\sigma'_i, \sigma'_{-i}) = u_i(\sigma'_i, \hat{\sigma}_{-i}) \geq u_i(\sigma_i, \hat{\sigma}_{-i}) \quad \forall \sigma_i \in \Sigma_i$ because of inequality (24). Moreover since the information partition is non manipulable and relations (22) and (23) hold, then $p_i(\cdot; \sigma_i, \hat{\sigma}_{-i}) = p_i(\cdot; \sigma_i, \sigma'_{-i})$. Therefore assumption 2 implies $u_i(\sigma_i, \hat{\sigma}_{-i}) = u_i(\sigma_i, \sigma'_{-i})$ and thus

$$\forall \sigma_i \in \Sigma_i \quad u_i(\sigma'_i, \sigma'_{-i}) \geq u_i(\sigma_i, \sigma'_{-i}) \quad \textit{i.e.} \quad \sigma' \in NE(G).$$

□

The notion of CE requires much less information about opponents' behaviour than the notion of NE. It is therefore surprising that both these concepts provide the same restrictions on observable behaviour for a quite comprehensive class of games. A version of this theorem is proved for EGs in Battigalli 1990 and in Fudenberg-Levine 1993, here we provide the opportune version for this more general setting.

Theorem 4 *Let $N = \{1, 2\}$. Then*

$$\forall G \in \mathcal{G}(\eta_i = \zeta) \quad \zeta(NE(G)) = \zeta(CE(G))$$

with the obvious meaning of the abuse of notation.

PROOF: since $\forall G \in \mathcal{G}: \ \ NE(G) \subseteq CE(G)$ then it is immediate that $\forall G \in \mathcal{G}$: $\zeta(NE(G)) \subseteq \zeta(CE(G))$. Therefore I need to prove that under the assumption of the theorem $\zeta(CE(G)) \subseteq \zeta(NE(G))$. Let $(\bar{\sigma}_1, \bar{\sigma}_2) \in CE(G)$. Then I should prove that $\exists(\hat{\sigma}_1, \hat{\sigma}_2) \in NE(G): \quad \zeta(\bar{\sigma}_1, \bar{\sigma}_2) = \zeta(\hat{\sigma}_1, \hat{\sigma}_2)$. By definition $(\bar{\sigma}_1, \bar{\sigma}_2) \in CE(G) \quad \Rightarrow \exists \mu_1 \in \Delta(\Sigma_2)$ and $\exists \mu_2 \in \Delta(\Sigma_1)$ such that:

$$\forall \bar{s}_1 \in SUPP[\bar{\sigma}_1] \quad \bar{s}_1 \in BR_1(\mu_1) \tag{25}$$

$$\mu_1(\{\sigma_2 | p_1(\cdot; \bar{\sigma}_1, \sigma_2) = p_1(\cdot; \bar{\sigma}_1, \bar{\sigma}_2)\}) = 1 \tag{26}$$

$$\forall \bar{s}_2 \in SUPP[\bar{\sigma}_2] \quad \bar{s}_2 \in BR_2(\mu_2) \tag{27}$$

$$\mu_2(\{\sigma_1 | p_2(\cdot; \bar{\sigma}_2, \sigma_1) = p_2(\cdot; \bar{\sigma}_2, \bar{\sigma}_1)\}) = 1. \tag{28}$$

Now define $\hat{\sigma}_2 := \int \sigma_2 d\mu_1$ and $\hat{\sigma}_1 := \int \sigma_1 d\mu_2$. Then (25) and (27) imply:

$\bar{\sigma}_1 \in BR_1(\hat{\sigma}_2)$ and $\bar{\sigma}_2 \in BR_2(\hat{\sigma}_1)$ because

$$\int_{\Sigma_{3-i}} \sum_{S_{3-i}} u_i(s_i, s_{3-i})\sigma_{3-i}(s_{3-i})d\mu_i = \sum_{S_{3-i}} u_i(s_i, s_{3-i}) \int_{\Sigma_{3-i}} \sigma_{3-i}(s_{3-i})d\mu_i =$$

by the previous definition $= \sum_{S_{3-i}} u_i(s_i, s_{3-i})\hat{\sigma}_{3-i}(s_{3-i}) \quad i \in \{1, 2\}$. But then $\hat{\sigma}_1 \in BR_1(\hat{\sigma}_2)$ and $\hat{\sigma}_2 \in BR_2(\hat{\sigma}_1)$ since according to (26), (28) and the hypothesis of the theorem $\hat{\sigma}_i$ differs from $\bar{\sigma}_i$ only out of the equilibrium path and the actions out of equilibrium do not affect the expected utility. Therefore $(\hat{\sigma}_1, \hat{\sigma}_2) \in NE(G)$. Moreover by construction $\hat{\sigma}_i$ and $\bar{\sigma}_i$ differ only out of the equilibrium path and therefore the outcomes should coincide: $\zeta(\hat{\sigma}_1, \hat{\sigma}_2) = \zeta(\bar{\sigma}_1, \bar{\sigma}_2)$. $\square$

Note that the conditions $N = \{1, 2\}$ and $\eta_i = \zeta$ are both relevant: the first guarantees that $\hat{\sigma}_{3-i}$ is well defined, while if we don't restrict the possible information partitions then the theorem is trivially false: think e.g. of $\eta_i = \bar{k}$. On the other hand these are not the most stringent conditions securing the realisation equivalence between CE and NE: see Fudenberg-Levine 1993 for results on EG with perfect monitoring of actions.

When the **information available during the game** is relevant for the outcome of strategic interaction situations, these contexts are usually modelled as EGs. One of the most important solution concept in this case is EFR (see Pearce 1984 and Battigalli 1997; the version of EFR I use is based on Battigalli's paper). I need some terminology:

- an information set I_i' precedes I_i'' iff $\forall h \in I_i'', \ \exists \hat{h} \in I_i' : \quad \hat{h} \subset h$. This relation is denoted by $I_i' \prec I_i''$.

- a **conjecture** of a player $i \in N$ is a probability measure $c_i \in \Delta(S_{-i})$,
- c_i **reaches** $I \in \mathcal{I}_i$ iff $\mathbf{P}\{I|s_i, c_i\} > 0$ for some $s_i \in S_i$,
- s_i' is a **I-replacement for** s_i iff $s_i(I') = s_i'(I') \;\; \forall I' \in \mathcal{I}_i \setminus \{I\} : \;\; I \not\prec I'$, i.e. s_i' differs from s_i only at I and at its followers,
- a **consistent updating system** for player i, denoted by CUS_i, is a mapping $c_i(\cdot) : \mathcal{I}_i \rightarrow \Delta(S_{-i})$ such that $\forall I \in \mathcal{I}_i \;\; c_i(I)$ reaches I and

$$\forall I', I'' \in \mathcal{I}_i, \;\; I' \prec I'' \;\; \& \;\; c_i(I') \;\; \text{reaches} \;\; I'' \;\; \Rightarrow \;\; c_i(I') = c_i(I'').$$

Definition 7 *The set of* **extensive form rationalizable pure strategies** *for an extensive game E, denoted by P(E), is defined inductively as follows:* $P(E) := \times_{i \in N} P_i(E), \quad P_i(E) := \bigcap_{t \geq 1} P_i(t, E) \quad P_i(0, E) := S_i$ *and* $s_i \in P_i(t+1, E)$ *iff:*

$$s_i \in P_i(t, E) \tag{29}$$

and

$$\exists c_i(\cdot) \in CUS_i : \quad \forall I \in \mathcal{I}_i(s_i, t) \tag{30}$$

$$(i) \qquad c_i(I) \in \Delta(P_{-i}(t, E))$$

$$(ii) \qquad u_i(s_i, c_i(I)) \geq u_i(s_i', c_i(I)) \quad \forall s_i' \in P_i(t, E) \;\; \text{and } I\text{-replacement for } s_i,$$

where $\mathcal{I}_i(s_i, t)$ *is the set of* $I \in \mathcal{I}_i$ *reached by* s_i, s_{-i} *for some* $s_{-i} \in P_{-i}(t, E)$.

Roughly, EFR implies weak sequential rationality since it requires that a strategy specifies a best reply at all information sets that can be reached by that strategy, given a conjecture consistent with the information sets on the equilibrium path. Moreover it is assumed that this procedure is CK and therefore it is recursively iterated.

Let P(G) be the set of EFR pure strategies of the extensive game E represented as an IMG G with information partition $\eta_i = \zeta_E$. I prove that for this class of games the strategies in $SRCE^P$ coincides with the EFR strategies which are also part of CE^P.

Theorem 5

$$\forall G \in \mathcal{G}(\eta_i = \zeta) \quad SRCE^P(G) = P(G) \bigcap CE^P(G).$$

PROOF: fix a game $G \in \mathcal{G}$ s.t. $\eta_i = \zeta$: this condition implies that the signal is public, thus if $s' \in SRCE^P$, then $\forall i, j \in N$, $\forall m_i$, $\quad M_j(s'_i, m_i|\zeta) = \zeta(s')$.

I will prove first that $SRCE^P \subseteq P \bigcap CE^P$. Since $SRCE^P \subseteq CE^P$, I need to prove only that $SRCE^P \subseteq P$, and this is done by induction. Therefore I want to show first that $SRCE^P \subseteq P(1)$ and then that

$$SRCE^P \subseteq P(t) \quad \Rightarrow \quad SRCE^P \subseteq P(t+1). \tag{31}$$

Let define $\hat{\mathcal{I}}_i(\zeta(s'))$ as the set of player i's information sets reached by the outcome path when s' is played. If $s' \in SRCE^P$, then $\forall i \in N \quad s'_i \in B^P_i(\zeta(s'))$. Thus $\exists \alpha_i[s'_i] \in \Delta(B^P_{-i}(\zeta(s')))$ such that $s'_i \in BR_i(\alpha_i[s'_i])$ and

$$\alpha_i[s'_i](\{s_{-i}|\eta_i(s'_i, s_{-i}) = \zeta(s')\}) = 1. \tag{32}$$

Now construct a consistent updating system for player i $c_i(\cdot)$ as follows:

$$c_i(\cdot) = \begin{cases} \alpha_i[s'_i] & \forall I \in \hat{\mathcal{I}}_i(\zeta(s')) \\ c'_i(\cdot) & \text{otherwise} \end{cases}$$

where $c'_i \in CUS_i$. Note that

1. $c_i(\cdot) \in CUS_i$ because $c'_i(\cdot)$ is a CUS by definition and $\alpha_i[s'_i]$ satisfies definition 7 $\forall I \in \hat{\mathcal{I}}_i(\zeta(s'))$, since condition (32) implies that $\forall I \in \hat{\mathcal{I}}_i(\zeta(s'))$ $\mathbf{P}\{I|s'_i, \alpha_i[s'_i]\} > 0$ and $\alpha_i[s'_i]$ is constant along the equilibrium path;

2. $\hat{\mathcal{I}}_i(\zeta(s')) \subseteq \mathcal{I}_i(s'_i, 1)$ because by definition

$$\mathcal{I}_i(s'_i, 1) := \{I \in \mathcal{I}_i | \exists s_{-i} \in S_{-i} : \mathbf{P}(I|s'_i, s_{-i}) > 0\}$$

3.

$$u_i(s'_i, c_i(I)) = \begin{cases} u_i(s'_i, \alpha_i[s'_i]) & I \in \hat{\mathcal{I}}_i(\zeta(s')) \\ u_i(s'_i, c'_i(I)) & \text{otherwise.} \end{cases}$$

Moreover out of the outcome path the conjectures do not matter for i's payoff maximisation: therefore $u_i(s'_i, c_i(I)) \geq u_i(s_i, c_i(I))$ $\forall s_i \in S_i$ and I-replacement for s'_i and thus $s'_i \in P(1)$, i.e. $SRCE^P \subseteq P(1)$.

To show implication (31) assume that $SRCE^P \subseteq P(t)$. Now define $B(\zeta)$ as the set of fixed point of $B(\zeta(\cdot))$: $B(\zeta) := \{s' \in S | s' \in B^P(\zeta(s'))\}$. Suppose $s' \in SRCE^P$, then $\forall i$ $s'_i \in B_i(\zeta)$. Consequently $\exists \alpha_i[s'_i] \in \Delta(B_{-i}(\zeta)) \subseteq \Delta(P_{-i}(t))$ such that $s'_i \in BR_i(\alpha_i[s'_i])$ and $\alpha_i[s'_i](\{s_{-i}|\eta_i(s'_i, s_{-i}) = \zeta(s')\}) = 1$, where the inclusion follows from the induction hypothesis. Now construct a $c_i(\cdot)$ in the following way:

$$c_i(\cdot) = \begin{cases} \alpha_i[s'_i] & \forall I \in \hat{\mathcal{I}}_i(\zeta(s')) \\ c'_i(\cdot) & \text{otherwise} \end{cases}$$

where $c'_i \in CUS_i$ is chosen in such a way that $s'_i \in P_i(t+1)$. This is possible because:

1. $c_i(\cdot) \in CUS_i$ because $c'_i(\cdot)$ is a CUS by definition and $\alpha_i[s'_i]$ satisfies definition 7 $\quad \forall I \in \hat{\mathcal{I}}_i(\zeta(s'))$, since $\quad \forall I \in \hat{\mathcal{I}}_i(\zeta(s')) \quad \mathbf{P}\{I|s'_i, \alpha_i[s'_i]\} > 0$ and $\alpha_i[s'_i]$ is constant along the equilibrium path;

2. $\hat{\mathcal{I}}_i(\zeta(s')) \subseteq \mathcal{I}_i(s'_i, t)$ because by definition

   $$\mathcal{I}_i(s'_i, t) := \{I \in \mathcal{I}_i | \exists s_{-i} \in P_{-i}(t) \supseteq B_{-i}(\zeta) \supseteq \{s'_{-i}\} : \mathbf{P}(I|s'_i, s_{-i}) > 0\}$$

   and thus $\quad \forall I \in \hat{\mathcal{I}}_i(\zeta(s')) \quad c_i(I) = \alpha_i[s'_i] \in \Delta(B_{-i}(\zeta)) \subseteq \Delta(P_{-i}(t))$. Moreover it is possible to choose $c'_i(\cdot)$ in such a way that $\quad \forall I \in \mathcal{I}_i(s'_i, t) \setminus \hat{\mathcal{I}}_i(\zeta(s')) \quad c'_i(I) \in \Delta(P_{-i}(t))$, since this choice is arbitrary;

3. $$u_i(s'_i, c_i(I)) = \begin{cases} u_i(s'_i, \alpha_i[s'_i]) & I \in \hat{\mathcal{I}}_i(\zeta(s')) \\ u_i(s'_i, c'_i(I)) & \text{otherwise.} \end{cases}$$

Therefore there exists a $c'_i(I)$ such that $\quad u_i(s'_i, c_i(I)) \geq u_i(s_i, c_i(I)) \quad \forall s_i \in P_i(t)$ and I-replacement for s'_i and thus $s'_i \in P_i(t+1)$.

Now I want to prove that $P \bigcap CE^P \subseteq SRCE^P$. Some preliminary remarks are necessary. First note that the iterative definition 6 of $SRCE$ can trivially be specialised to the case of pure strategies. Therefore by definition $s' \in SRCE^P$ iff $s' \in \cap_{t\geq 1} B'(t, \zeta(s'))$; since $B'(t, \zeta(\cdot)) : S \longrightarrow\!\!\!\!\!\!\rightarrow S$ is a continuous correspondence in the discrete topology, then $s' \in SRCE^P$ iff $\forall t \geq 1 \quad s' \in B'(t, \zeta(s'))$. Now define $B'(t,\zeta)$ as the set of fixed point of $B'(t, \zeta(\cdot))$: $B'(t,\zeta) := \{s' \in S | s' \in B'(t, \zeta(s'))\}$. Therefore the inclusion that I want to prove is equivalent to $\forall t \geq 1 \quad P(t) \cap CE^P \subseteq B'(t,\zeta)$ that I prove by induction.

Note that $P(1) \bigcap CE^P \subseteq B'(1,\zeta)$ is immediate since $CE^P = B'(1,\zeta)$. Moreover note that there are two possibilities: either $CE^P = SRCE^P$ or $CE^P \supset SRCE^P$. In the first case the result follows trivially, while in the second case the thesis is equivalent to prove $P \subseteq SRCE^P$. Therefore I need to prove that $\quad P(t) \subseteq B'(t,\zeta) \quad \Rightarrow \quad P(t+1) \subseteq B'(t+1,\zeta)$. Suppose that $s' \in P(t+1)$. Therefore $\forall i \in N \quad \exists c_i(.) \in CUS_i : \forall I \in \mathcal{I}_i(s'_i, t+1)$ the conditions of the definition of EFR hold. In particular $\quad \forall I \in \mathcal{I}_i(s'_i, t+1) \quad c_i(I) \in \Delta(P_{-i}(t)) \subseteq \Delta(B'_{-i}(t,\zeta)) \quad$ and s'_i is a best reply among I-replacement in $P(t) \subseteq B'_i(t,\zeta)$. Note that

1. by definition $\hat{\mathcal{I}}_i(\zeta(s')) \subseteq \mathcal{I}_i(s'_i, t+1)$;
2. $\forall I \in \hat{\mathcal{I}}_i(\zeta(s')) \quad SUPP[c_i(I)] \subseteq B'_{-i}(t,\zeta) \subseteq \mathcal{S}_i(s'_i, \zeta(s')|\zeta) \quad$ since by definition any $\hat{s}_j \in B'_j(t,\zeta)$ should be consistent with the public signal $\zeta(s')$;
3. $c_i(\cdot) \in CUS_i$ is constant along the equilibrium path.

Therefore it is possible to pose $\alpha_i[s'_i] := c_i(I) \quad \forall I \in \hat{\mathcal{I}}_i(\zeta(s'))$. Consider the first $I \in \hat{\mathcal{I}}_i(\zeta(s'))$, i.e. the first information set of player i along the outcome path: from the previous relations $s'_i \in BR_i(\alpha_i[s'_i])$ because information sets with probability zero are irrelevant for maximisation. Moreover by construction $\alpha_i[s'_i] \in \Delta(B'_{-i}(t,\zeta))$ and $\alpha_i[s'_i](\mathcal{S}_i(s'_i,\zeta(s')|\zeta)) = 1$. Therefore $s'_i \in B'_i(t+1,\zeta(s'))$ and thus $P_i(t+1) \subseteq B'_i(t+1,\zeta)$. $\square$

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