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Essays on Real Life Assignment Problems

by

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Stability and Efficiency in the General Priority-based Assignment

1.1 Introduction

A popular and widely used school admissions practice is to allocate school seats taking into account student preferences. Though such policies are often called *school choice*, obvious scarcity constraints arise due to some schools being more demanded than others. Therefore a well-defined procedure is necessary to decide how the over-demanded schools are assigned. Abdulkadiroğlu and Sönmez (2003) formulated these concerns in a natural and appealing way, and their approach has since been at the heart of various school choice programs. Each school is endowed with a certain number of seats and an exogenous priority order over the set of students. For a matching of students to schools to be *stable*, a student i must not be left envying another student j at school x, while i has higher priority for x than j. If these priorities are strict orders and interpreted as schools' preferences over students, then we are back in Gale and Shapley's (1962) college admissions model.

For each school, however, there are several concerns that guide such decisions: (1) Can *siblings* attend the same school? (2) Do pupils get schools within their *walk-zone*? (3) Does the procedure *treat otherwise-equal students equally*? (4) Is the student body at a given school *diverse* (gender equality or racial balance)? For socio-economic perspectives, the Office of Educational Research and Improvement explicitly suggests that an assignment should be based on those concerns, and in the same time the question arises what kind of priorities can reflect such concerns. To our best knowledge, concerns only for each (1), (2) or (3) can be captured by the priority rankings over the set of students which allow ties among them, but a concern for diversity is different from the other three in that it cannot be necessarily generated by a ranking over the set of students, and it inherently exhibits indifference relations between some of the subsets of the set of students. Our challenge is to deal with all those concerns in an unified framework.

Let's consider the following example: there are six students who want to enter a school but it has just two seats. Six students are consisted of two white, two black, and two asian students, and the school wants to make a class as racially diverse as possible. In this case, every pair of students with different race should be the most preferred, and any of those pairs is equally preferable. It is clear that the ranking of the school inherently exhibits indifference relations among every pair of students with different race, and is over the subsets of the set of students. Note that this kind of priority ranking cannot be generated by a priority ranking over the students because the school's priority ranking is affected by whom you make a pair with.[1]

[1] Typically, the priorities are far from strict. For a given school many students are of *equal priority*, which

In this paper, we introduce a general class of priority rankings over the subsets of the set of students, *substitutable priorities with ties*, which captures indifferences and substitutability. We are able to study natural and appealing priority structures which take into account both *diversity* and *equal treatment*. We show the existence of stable matchings via a modification of the deferred acceptance algorithm. As in Erdil and Ergin (2008), there is a multiplicity of constrained efficient assignments, and arbitrarily breaking the ties can lead to constrained inefficiency. If the latter happens, the stable improvement cycles would improve upon the assignment to finally return a constrained efficient matching.

While a priority order over the students does not automatically lead to a ranking over sets of students, there is a simple class of preferences over sets studied widely in the literature. In a large class of two-sided matching models (see, e.g., Roth and Sotomayor, 1990), a ranking over sets is assumed to be *responsive* to a strict preference ranking over individuals. This condition essentially says that for any two sets that differ in only one student, the set containing the student with higher priority is ranked higher.[2] Then, core-stability is equivalent to pairwise stability. The existence of a stable matching is ensured, and Gale and Shapley's deferred acceptance algorithm gives the student optimal stable assignment. Erdil and Ergin (2008) begins with a model in which priorities can have ties, because many school choice programs declare large classes of students to be of equal priority. Gale and Shapley's algorithm does not necessarily give a student optimal assignment, but the *stable improvement cycles* always takes us to a constrained efficient outcome. Though the model

is captured by a model which allows indifferences. See, e.g., Erdil and Ergin (2008).

[2]Formally, a ranking $\succsim$ over the sets of students is said to be *responsive* to a ranking $\succsim'$ over the set of students if whenever $i \succsim' j$, we have $\{i\} \cup S \succsim \{j\} \cup S$ for any S.

with ties captures “equal treatment”, it is not possible to “prioritize diversity” via a responsive priority order. If the priorities are responsive, then the ranking between sets $S \cup \{i\}$ and $S \cup \{j\}$ should not depend on what S is.[3]

Kelso and Crawford (1983) introduced a class of rankings over sets significantly larger than that of responsive rankings. Their generalization of the Gale-Shapley process, *the salary adjustment process* ensures that if firms’ preferences over sets of workers satisfy the *gross substitutes* condition, then the core of the matching market is non-empty. Related to a diversity concern, Abdulkadiroğlu (2005) formulates priority rankings in a controlled school choice problem. Roughly speaking, the part of a school’s seats are reserved for a specific type of students, and he shows that priority rankings respecting type specific quotas fall in substitutable priorities. We discuss his formulation and ours carefully in the next section.

The rest of the paper is organized as follows: we see the leading example in Section 2, Section 3 introduces a model, we discuss stability and efficiency in Section 4, we consider the way to find a constrained efficient assignment in Section 5, we demonstrate our class of priorities in a controlled school choice setting in Section 6.

1.2 Motivating Example

Prior to ours, Abdulkadiroğlu (2005) considers a school choice problem with a diversity concern in a priority-based assignment problem. In that model, each student is endowed

[3] An explicit example is in the next section.

with a certain type, there are type specific quotas for each school and it prioritizes the subsets of the set of students by two rules. If the subset of the set of students satisfies a type specific quota constraint, then they are considered to be acceptable, otherwise unacceptable. Among acceptable subsets of the set of students, they are ranked responsive to a ranking over the set of students.

Recall the example in the introduction: there are six students, $N = W \sqcup B \sqcup A$ where $W = \{w_1, w_2\}$, $B = \{b_1, b_2\}$ and $A = \{a_1, a_2\}$, and there is no exogenous priority order over them. Apart from a race equality policy, they are to be "treated equally". The approach of Abdulkadiroğlu (2005) leads to the following priority ranking:

$$
\begin{aligned}
\{w_1, b_1\} &\succ \{w_1, a_1\} \succ \{w_1, b_2\} \succ \{w_1, a_2\} \succ \{b_1, w_2\} \succ \cdots \\
&\succ \emptyset \\
&\succ \underbrace{\{w_1, w_2\}, \{b_1, b_2\}, \{a_1, a_2\}}_{\text{they do not satisfy the constraint}}
\end{aligned}
$$

Note that if all students are of equal priority, $\{w_1, b_1\}$ and $\{w_1, a_1\}$ should be treated equally in light of a diversity concern. However, the above formulation results in a biased assignment, that is, as long as $\{w_1, b_1\}$ are included in applicants a school never chooses any other pair of students with different race. In this case, white and black students are thought of as having a higher priority than any asian students. More importantly, an assignment produced by the above ranking may end up with being wasteful. Suppose only w_1 and w_2 apply for this school. Even though there are enough seats available, only one of them can be admitted.

On the other hand, a natural and desirable priority ranking may be as follows:

$$\{w_i, b_j\} \sim \{w_i, a_k\} \sim \{b_j, a_k\} \succ \{w_1, w_2\} \sim \{b_1, b_2\} \sim \{a_1, a_2\} \succ \cdots , \qquad (\star)$$

where $i, j, k \in \{1, 2\}$. As far as we know this priority order does not fit into any model previously studied in the literature.

First of all, it allows ties between sets of students. The earlier models which studied ties in priority orders begin with a weak order (i.e., a complete, transitive binary relation) on the set of students. Then they impose that the priority order on sets is responsive to that weak order on students. It is not hard not see that the above priority order $\succsim$ cannot be generated in that fashion. If $\succsim$ were responsive to an order $\succsim'$ on $\{w_1, w_2, b_1, b_2, a_1, a_2\}$, we would have

$$\{w_1, b_1\} \succ \{w_1, w_2\} \quad \Longrightarrow \quad b_1 \succ' w_2$$

and

$$\{w_2, b_2\} \succ \{b_1, b_2\} \quad \Longrightarrow \quad w_2 \succ' b_1$$

a contradiction.

In addition, when all students apply to a school, the allocations obtained by the above priority structure cannot be generated by splitting a school into sub-schools: each sub-school prioritizes students with one race over those with the other, since there are only two seats and the number of race is three, so that there should be students with some race who are always ranked lower. This kind of treatment is sometimes called a type specific quota in the previous literature.

1.3 Preliminaries

Let N be a set of students, and X a set of schools. There are q_x seats at school x, for $x \in X$. These schools are to be assigned to the students such that each student receives at most one seat, and the allocation has to *respect exogeneously given priorities*, a notion formalized below.

Each school has a priority ranking $\succsim_x$ over all subsets of N. Formally speaking, $\succsim_x$ is a complete, transitive binary relation over 2^N. A **priority structure** $\succsim$, is a vector of priority orders $(\succsim_x)_{x \in X}$. If $S \succsim_x T$ and $T \succsim_x S$, then we write $S \sim_x T$, and say S and T are of equal priority with respect to $\succsim_x$. Clearly, being of equal priority is an equivalence relation. If $S \succsim_x T$, but not $T \succsim_x S$, then we write $S \succ_x T$.

When the demand for a school exceeds the supply, one can appeal to priorities to decide which students are to be assigned to the school. In other words, given a school x with q_x seats and a priority order $\succsim_x$, we have a choice correspondence $\mathcal{C}_x : 2^N \rightrightarrows 2^N$ such that

$$S' \subseteq S \text{ and } |S'| \leq q_x \text{ for each } S' \in \mathcal{C}_x(S)$$

$$S' \in \mathcal{C}_x(S) \iff S' \succsim_x S'' \text{ for all } S'' \subseteq S \text{ with } |S''| \leq q_x$$

Given a priority rule $\mathcal{C}_x$ and a set S, we define the set of **definitely chosen students**

$$DC_x(S) \quad = \quad \bigcap_{S' \in \mathcal{C}_x(S)} S' \quad = \quad \{i \in S \mid i \in S' \text{ for all } S' \in \mathcal{C}_x(S)\}$$

Note that $DC_x(S)$ can be empty.

Definition 1 A priority structure is **substitutable** if for each $x \in X$, for each $S, T \subseteq N$ with $S \subseteq T$,

(a) for each $T' \in \mathcal{C}_x(T)$, we have $T' \cap S \subseteq S'$ for some $S' \in \mathcal{C}_x(S)$.

(b) for each $S' \in \mathcal{C}_x(S)$, we have $T' \cap S \subseteq S'$ for some $T' \in \mathcal{C}_x(T)$.[4]

This definition covers various environments studied in the literature. For example, responsive preferences over sets of doctors (Roth, 1984), the school choice formulation of Abdulkadiroğlu and Sönmez (2003), and the school priorities with ties (e.g., Erdil and Ergin, 2008) are all special cases.[5] One contribution of this paper is that our generalization goes beyond, and covers natural priority structures which are not captured by any of the aforementioned models:

[4]It is helpful to define the **rejection correspondence** $\mathcal{R}_x$, which associates to each $S \subseteq N$, the family of subsets of S which can be rejected from among S. That is,

$$\mathcal{R}_x(S) = \{T \subseteq S \mid T = S \backslash S' \text{ for some } S' \in \mathcal{C}_x(S)\}.$$

The condition (b) of Definition 1 can be rewritten as

(b') for each $S' \in \mathcal{R}_x(S)$, we have $S' \subseteq T'$ for some $T' \in \mathcal{R}_x(T)$.

[5]Our definition is easily extended to allow different contracts between a student and a school. Then each school has a *weak* ranking over sets of contracts, and we get a generalization of Kelso and Crawford (1982) and Hatfield and Milgrom (2005).

Note that if $\mathcal{C}_x$ is a function for each $x \in X$, then the conditions (a) and (b) in Definition 1 are equivalent. While Kelso and Crawford (1982) use the formulation (a), Hatfield and Milgrom (2005) use the formulation (b'). In our generalized environment, these conditions do not imply each other any more.

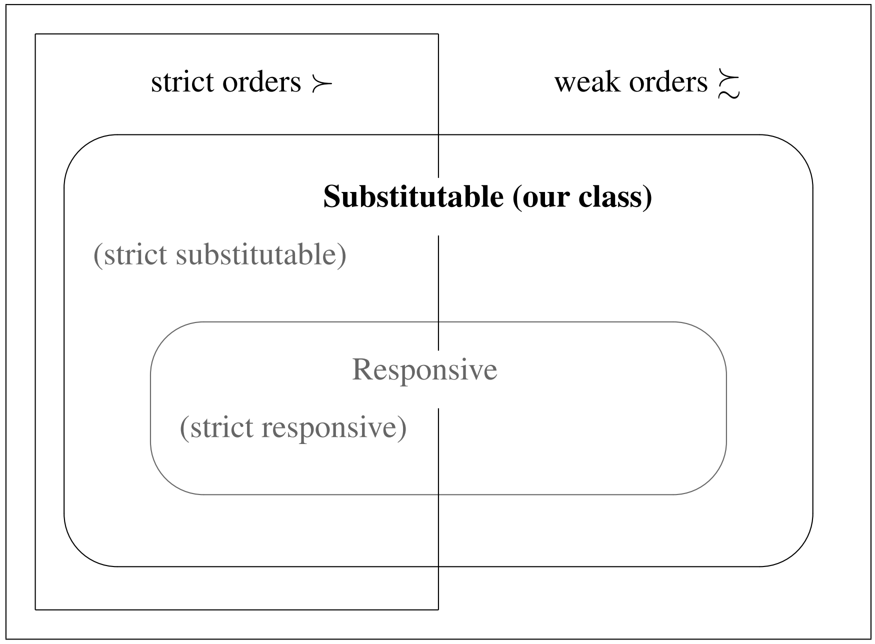

Figure 1.1: Substitutability with ties

Example 1 (Race equality) Remember the following priority structure is considered to be desirable in the previous sections. We see that it in fact falls in our class.

$$\{w_i, b_j\} \sim \{w_i, a_k\} \sim \{b_j, a_k\} \succ \{w_1, w_2\} \sim \{b_1, b_2\} \sim \{a_1, a_2\} \succ \cdots,$$

where $i, j, k \in \{1, 2\}$.

We only need to check cases where the size of a smaller set (a counterpart of S in the definition) is at least as large as two. If S consists of students with different race, then only pairs of students with different race are chosen in S, and in any larger set they should also be chosen. Note that no pair of students with the same race is chosen in both sets. Hence the conditions (a) and (b) hold. Otherwise, S consists of two students with the same race.

The condition (b) automatically holds because there is no student rejected. In a larger set, there should be a student with different race from S and any pair of students with different race will be chosen. Therefore, any intersection of a pair of students chosen in a larger set and S must be one of S and not both. Clearly, the condition (a) is also satisfied.

For the allocation problem, what we really need is a choice correspondence. In the above example, the ranking over singletons does not matter as long as they are acceptable.[6] $\Diamond$

Let μ be an assignment such that $\mu : N \to X \cup N$ with the following properties:

$$\forall i \in N, \qquad \mu(i) \in X \cup \{i\}$$

$$\forall x \in X, \qquad |\mu^{-1}(x)| \leq q_x$$

Each student i has a strict preference ranking R_i over $X \cup \{i\}$, where receiving i is interpreted as getting one's outside option. P_i denotes the strict part of R_i. Given a preference profile $R = (R_i)_{i \in N}$, we have a Pareto domination relation over all possible allocations.

Definition 2 Given students' preferences R, an assignment μ **respects priorities** $\succsim$, if for each $i \in N$, $\mu(i) R_i i$, and for each $x \in X$, we have $\mu^{-1}(x) \in \mathcal{C}_x(\{i \mid x R_i \mu(i)\})$.

The definition captures the idea that there should not be a set S more deserving of the

[6] If $q_x > 1$, then a priority order $\mathcal{C}_x$ does not necessarily provide a comparison between singletons. For example, let $q_x = 2$, and consider $\mathcal{C}_x$ such that

$$\mathcal{C}_x(\{1,2,3,4\}) = \{\{1,3\},\{2,3\},\{1,4\},\{2,4\}\},$$

whereas for any S with $|S| = 3$, all two-element-subsets of S are chosen. This priority order cannot be generated from a weak order over the set of students using responsiveness.

school x, than those students currently assigned.

Recall that a matching is called **pairwise stable** if it is not blocked by an individual student, or an individual school, or a student-school pair. That is, (1) each student i prefers her match to being unassigned; (2) each school prefers not to get rid of some of the assigned students; and (3) there is no student-school pair who are not matched, but would rather be matched. Formally,

(1) For all $i \in N, \quad \mu(i) R_i i$

(2) For all $x \in X, \quad \mu^{-1}(x) \in \mathcal{C}_x(\mu^{-1}(x))$

(3) There is no $(i, x) \in N \times X$ such that $x P_i \mu(i)$ and $i \in DC(\mu^{-1}(x) \cup \{i\})$

Clearly, respecting priorities implies pairwise stability. When does the converse hold? The next proposition gives an answer.

Proposition 1 *Let $\succsim$ be a substitutable priority structure.Then an assignment is pairwise stable if and only if it respects priorities. In other words, the set of pairwise stable assignments is the same as the weak core.*

From now on, we say an assignment which respects priorities a *stable* assignment. Does there always exist a stable assignment? A natural extension of Gale and Shapley's Deferred Acceptance Algorithm shows constructively that it does if the priority structure is substitutable.

Modified Deferred Acceptance Algorithm (MDA)

Round 1: All students apply to their favorite schools. For each school x, if A_x^1 is the set of applicants, *an* element S_x^1 in $\mathcal{C}_x(A_x^1)$ is declared the temporary winners, and the rest of the applicants, denoted $Z_x^1 = A_x^1 \backslash S_x^1$ are rejected.

$\vdots$

Round t: Those who were rejected in round $t-1$, apply to their next favorite school. For each school x, if A_x^t is the set of all students who have applied to x so far, *a* set of temporary winners $S_x^t \in \mathcal{C}_x(A_x^t)$ is chosen such that $Z_x^{t-1} \subseteq A_x^t \backslash S_x^t$.

[This ensures that those students that were rejected by x in a previous round are still rejected.]

When every student is either matched with a school or has been rejected by all schools in his list, the algorithm ends.

Proposition 2 *Given a substitutable priority structure $\succsim$, the Modified Deferred Acceptance Algorithm returns a stable assignment.*

The above algorithm is a generalization of the student-proposing deferred acceptance algorithm to an environment which allows school priority rankings over sets of students to be substitutable with ties. Note that "monotonicity of the rejection correspondence", i.e., the condition (b) of substitutability is enough for the proposition 2 if an assignment the MDA returns is pairwise stable, however, without the condition (a) of substitutability, we might have a pairwise stable outcome which does not respect priorities.

Example 2 Suppose there are two schools, $\{x, y\}$, and five students, $\{i_1, i_2, i_3, i_4, i_5\}$. Students' preferences are:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
x	y	x	y	x
		y	x	

And the priority structures are:

$\mathcal{C}_x$	$\mathcal{C}_y$
$\{i_1, i_2\}, \{i_3, i_4\}$	$\{i_2, i_3\}, \ldots$
$\{i_1, i_3\}, \{i_1, i_4\}, \{i_1, i_5\}, \{i_2, i_3\}, \{i_2, i_4\}$	$\{i_2, i_4\}, \ldots$
$\{i_2, i_5\}, \{i_3, i_5\}, \{i_4, i_5\}$	

This priority structure satisfies (b) but not (a). The MDA gives

1. Students i_1, i_3 and i_5 apply to a school x, and $\{i_1, i_5\}$ can be chosen. A student i_3 is rejected. A school y tentatively holds $\{i_2, i_4\}$.

2. A student i_3 applies to y, and she is tentatively accepted. A student i_4 is rejected.

3. A student i_4 applies to x, but a school x can hold $\{i_1, i_5\}$, and she is rejected. The algorithm ends.

The assignment is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & y & \emptyset & x \end{pmatrix}.$$

This is pairwise stable but does not respect priorities since both students 3 and 4 prefer x rather than their assigned schools, and the school x prefers $\{i_3, i_4\}$ to $\{i_1, i_5\}$. $\diamondsuit$

Definition 3 A priority structure $\succsim$ is **acceptant** if for each $x \in X$, for each $S \subseteq N$, and for each $S' \in \mathcal{C}_x(S)$ we have $|S'| = \min\{|S|, q_x\}$.

This captures the idea that an unused school seat cannot be denied to any student. No-blocking-pairs (NB) condition together with an acceptant priority structure means that stability implies **non-wastefulness**, i.e., if there exists $i \in N$ such that $aP_i\mu(i)$, then $|\mu^{-1}(x)| = q_x$.

As our model generalizes the one studied in Erdil and Ergin (2008), it follows that: (1) The outcome of the generalized deferred acceptance algorithm is not necessarily constrained efficient. (2) The constrained efficient set is not necessarily a singleton.

Given $\succsim$, define the **constrained efficient correspondence** $f^{\succsim}$, which assigns to each preference profile R, the set of stable assignments which are not Pareto dominated by another stable assignment.

1.4 Stability vs. efficiency

Let us call a priority structure $\succsim$ **efficient** if $f^{\succsim}$ is Pareto efficient. Ergin (2002) characterizes the efficient priority structures under the assumption of responsive priorities without ties. Ehlers and Erdil (2010) give a more general characterization allowing for ties in priority orders. Below, we will let the priorities be acceptant substitutable with ties, providing the most general statement in a much larger environment. This characterization result, as the ones before, confirms that $f^{\succsim}$ is Pareto efficient under very restrictive conditions.

Definition 4 Given a priority structure $\succsim$, a **weak cycle** is constituted of distinct $i, j, k \in N$, and $x, y \in X$ such that there exist $S_x, S_y \subseteq N \backslash \{i, j, k\}$ with $S_x \cap S_y = \varnothing$ satisfying

$$\textbf{(C)} \qquad \begin{array}{l} j \notin DC_x(S_x \cup \{i, j\}) \\ j \in DC_x(S_x \cup \{k, j\}) \\ k \notin DC_x(S_x \cup \{i, k\}) \\ i \notin DC_y(S_y \cup \{k, i\}) \end{array}$$

$$\textbf{(S)} \qquad |S_x| = q_x - 1 \text{ and } |S_y| = q_y - 1.$$

If $\succsim$ does not have any weak cycle, then it is called **strongly acyclic**.

Proposition 3 *Let $\succsim$ be an acceptant substitutable priority structure. $f^{\succsim}$ is efficient if and only if $\succsim$ is strongly acyclic.*

Ergin (2002) characterizes acyclicity as similarity of priorities. However, our acyclicity requires a much tighter interpretation of priorities.

Example 3 Recall the example of race equality. Students are $N = \{w_1, w_2, b_1, b_2, a_1, a_2\}$. Suppose there are two schools x, y, each with two seats. Consider both schools x and y

have exactly the same priority structure as in the example.

$$\{w_i, b_j\} \sim \{w_i, a_k\} \sim \{b_j, a_k\} \succ \{w_1, w_2\} \sim \{b_1, b_2\} \sim \{a_1, a_2\} \succ \cdots,$$

We see that it is weakly cyclic. Let $S_x = \{a_2\}$ and $S_y = \{b_2\}$. (S) holds.

$$\begin{aligned}
\mathcal{C}_x(\{a_2, \underbrace{w_1}_{i}, \underbrace{b_1}_{j}\}) &= \{\{a_2, w_1\}, \{a_2, b_1\}, \{w_1, b_1\}\} \\
\mathcal{C}_x(\{a_2, b_1, \underbrace{a_1}_{k}\}) &= \{\{b_1, a_1\}, \{b_1, a_2\}\} \\
\mathcal{C}_x(\{a_2, w_1, a_1\}) &= \{\{w_1, a_1\}, \{w_1, a_2\}\} \\
\mathcal{C}_y(\{b_2, w_1, a_1\}) &= \{\{w_1, b_2\}, \{w_1, a_1\}, \{b_2, a_1\}\}
\end{aligned}$$

which implies

$$b_1 \notin DC_x(S_x \cup \{w_1, b_1\})$$

$$b_1 \in DC_x(S_x \cup \{a_1, b_1\})$$

$$a_1 \notin DC_x(S_x \cup \{w_1, a_1\})$$

$$w_1 \notin DC_y(S_y \cup \{a_1, w_1\}).$$

Note that when preferences are

R_{w_1}	R_{w_2}	R_{b_1}	R_{b_2}	R_{a_1}	R_{a_2}
y		x	y	x	x
x				y	

$$\mu = \begin{pmatrix} w_1 & w_2 & b_1 & b_2 & a_1 & a_2 \\ x & w_2 & b_1 & y & y & x \end{pmatrix}$$ is constrained efficient, but not efficient. $\Diamond$

1.5 Stability Preserving Pareto Improvement

In the absence of ties, we know that $f^{\succsim}$ is singleton valued and reached simply by the deferred acceptance algorithm. On the other hand, when there are ties, the constrained efficient correspondence is not necessarily singleton-valued. Moreover, arbitrarily breaking the ties as we execute the deferred acceptance algorithm may lead to constrained inefficiency. In the case of responsive priorities the *stable improvement cycles algorithm* by Erdil and Ergin (2008) reaches a constrained efficient assignment. We explore whether such cycles can be used to solve the similar problem when priorities are acceptant and substitutable.

A special case of our environment is that of responsive priorities with ties. Motivated by the fact that an arbitrary resolution of ties in implementing the DA algorithm may lead to an assignment which is not constrained efficient, Erdil and Ergin (2008) explored stability preserving Pareto improvements. A *stable improvement cycle* is a cycle of distinct schools such that for any edge $x \to y$, there is a student i_x matched with x, who would like to be matched with y instead, and is one of the highest y-priority students among those who would like to move to y. They show that if a stable assignment is not constrained efficient, then it must admit a stable improvement cycle, and therefore by simply searching for stable improving cycles and implementing them successively, one can reach a constrained efficient assignment.

In our more general environment, Erdil and Ergin's definition does *not* capture all the improvement cycles that preserve stability. That is, it is possible that a stable matching μ is Pareto dominated by another stable matching ν, and μ does not admit a stable improvement cycle in the sense of Erdil and Ergin (2008).[7] This is because in our environment, the priority of a student is not absolute but relative in a sense that it depends on whom his colleagues are. Hence, we need to take it into account.

Given a stable assignment μ, who could be replacing, without violating stability, j's position at $\mu(j)$ if j were to disappear? It must be that when such a student ℓ replaces j, the new set of students must be chosen set in the face of those who would like to be replacing j at $\mu(j)$. To formalize this idea in general, let $\mathcal{E}_j^\mu$ stand for the set of students who envy j at assignment μ:

$$\mathcal{E}_j^\mu = \{i \mid \mu(j) P_i \mu(i)\}.$$

Then, the set of **students who can replace student j at μ** is

$$E_j^\mu = \{\ell \mid \{\ell\} \cup \mu^{-1}(\mu(j)) \backslash \{j\} \in \mathcal{C}_{\mu(j)}(\mathcal{E}_j^\mu \cup \mu^{-1}(\mu(j)) \backslash \{j\})\}$$

Note that each E_j^μ is not necessarily singleton and any two E_j^μ and $E_{j'}^\mu$ are not necessarily distinct. If we find a cycle of students, then it is feasible to exchange their assignments, and formally,

[7] An explicit example is in Remarks.

Definition 5 Given a priority structure $\succsim$, a preference profile R and a stable assignment μ, a **stable student improving cycle** consists of distinct students $i_0, i_1, \ldots, i_{n-1}, i_n = i_0$ such that $i_\ell \in E^{\mu}_{i_{\ell+1}}$ for all $\ell = 0, \ldots, n-1$.

The first relation between a constrained efficient assignment and a stable student improving cycle is the following:

Proposition 4 *Given an acceptant and substitutable priority structure $\succsim$, if a stable assignment does not admit a stable student improving cycle then it is constrained efficient.*

Unfortunately, the converse does not hold in general.[8] We further impose a weak assumption on the priority structure to hold the converse.

Suppose that a priority structure $\succsim$ is acceptant substitutable. We will define a weak form of "equal treatment of equals" as follows:

Definition 6 Let us say that $\succsim$ satisfies **equal treatment of equal students** if given $\{i, j\} \cup S \subseteq T$, and $\{i, j\} \cup S' \subseteq T'$ such that $T \subseteq T'$, $|S| = q_x - 1$ and $|S'| = q_x - 1$,

$$S \cup \{i\}, S \cup \{j\} \in \mathcal{C}_x(T) \text{ and } S' \cup \{i\} \in \mathcal{C}_x(T') \Longrightarrow S' \cup \{j\} \in \mathcal{C}_x(T'). \qquad (\textbf{ETE})$$

Which students are to be treated equally can change from one school to the other.[9] However, the critical requirement is that if two students are of equal priority in applicants at some school, then one can be replaced with the other in any larger applicants.

[8] An explicit example is in Remarks

[9] Note that our prioritizing diversity as in Section 2 satisfies the above property of equal treatment.

Proposition 5 *Assume an acceptant and substitutable priority structure $\succsim$ satisfies ETE. Suppose that the stable assignment μ admits a SIC, and let the shortest SIC be*

$$i_0 \to i_1 \to \cdots \to i_{n-1} \to i_0.$$

If the assignment ν is obtained by carrying out this cycle, i.e., if

$$\nu(i) = \begin{cases} \mu(i_{\ell+1}) & \textit{if } i = i_\ell \\ \mu(i) & \textit{otherwise} \end{cases},$$

then ν is stable.

Corollary 1 *Whenever an acceptant and substitutable priority structure $\succsim$ satisfies ETE, a stable assignment is constrained efficient if and only if it does not admit a stable student improving cycle.*

1.5.1 An algorithm

The above proposition leads to an algorithm whose outcome is a constrained efficient assignment. Starting from a stable assignment μ, one needs to construct a graph whose set of vertices is the set of students. For any pair (i, j) of vertices, there will be an edge from i to j if and only if $i \in E_j^\mu$. If this graph does not have a cycle which preserves stability, then μ is constrained efficient. Otherwise, we can let the cycle lead to a Pareto improving cyclic trade which would preserve stability.

Step 0:

Run the *Modified Deferred Acceptance Mechanism* to obtain an initial matching μ^0.

Step t $\geq$ 1:

$(t.a)$ Given μ^{t-1}, let the students stand for the vertices of a directed graph, where for each pair of students i and j, there is an edge $i \longrightarrow j$ if and only if $i \in E_j^{\mu^{t-1}}$.

$(t.b)$ If there are any stable student improving cycles in this directed graph, *select* a shortest one, and carry out this cycle to obtain μ^t, and go to step $(t + 1.a)$. If there is no such cycle, then return μ^{t-1} as the outcome of the algorithm.

This algorithm will return a student optimal stable assignment, but when there are more than one such assignments, the particular outcome will depend on the selections in running the MDA in Step 0, and the specification of the cycle search in later steps.

On the one hand, the algorithm ensures a constrained efficient outcome. On other hand, we know from Proposition 3 that if $\succsim$ is acyclical, then any constrained efficient assignment is Pareto efficient. Thus we have

Corollary 2 *If $\succsim$ is acyclical, then the above algorithm Pareto efficient.*

1.6 Application

We demonstrate a natural and practical subclass of our substitutability, called *a priority respecting type-specific quotas*. For a socio-economic purpose, a school usually sets a part of its seats for a specific characteristic, such as students with disability or in minority. For example, when a school wants to make a class racially balanced, this concern is reflected by splitting a whole seats into each race. The question is whether such a socio-economic policy is effective or not, in the sense that each prioritized student is better off by a policy or not.

Kojima (2010) argues that such a policy does not necessarily achieves that purpose. He gives the case that even though some group is given a good treatment by setting a specific quota, every student in the group is worse off, compared to the assignment without treatment. However, we are anxious about the result since comparison seems to be made by two different situations, wasteful and non-wasteful priorities.

In this section, we assume that each school initially has a priority ranking over students, and if it does not employ any specific policy, we assume that its priority follows responsiveness. Otherwise, depending on its type-specific quotas, we offer a way of constructing a priority structure over the sets, which both reflects an initial ranking and respects type-specific quotas.

A pre-priority, which is a weak order over the students, is denoted by $\succsim^{pre} \in N \times N$. (A strict part is denoted by $\succ^{pre}$, and indifference is denoted by $\sim^{pre}$.) Let a type space $T = \{\tau_1, \ldots, \tau_n\}$. Each student is in one of types, and a type function, $\tau : N \to T$, indicates it. For every school x, there are type-specific quotas $q_x^T = (q_x^{\tau_1}, \dots, q_x^{\tau_n})$ such that $1 \leq q_x^\tau \leq q_x$, and $\sum_\tau q_x^\tau = q_x$. S_τ denotes $\{s \in S : \tau(s) = \tau\}$.

The set of students is classified into the following families. $\forall S \subseteq N$ with $|S| = q_x$,

$$
\begin{aligned}
D_0 &= \left\{ S \subseteq N : \sum_{\tau \in T} ||S_\tau| - q^\tau| = 0 \right\} \\
D_1 &= \left\{ S \subseteq N : \sum_{\tau \in T} ||S_\tau| - q^\tau| = 2 \right\} \\
&\vdots \\
D_a &= \left\{ S \subseteq N : \sum_{\tau \in T} ||S_\tau| - q^\tau| = 2a \right\} \\
&\vdots
\end{aligned}
$$

Definition 7 *$\succsim$ satisfies a Respecting Constraint (RC) if*

$$S' \in D_a \text{ and } S'' \in D_{a+1} \Rightarrow S' \succ S''$$

Definition 8 *$\succsim$ satisfies a Restricted Responsiveness (RR) if for every $T \cup \{s'\}$, $T \cup \{s''\} \in D_a$, we have*

$$T \cup \{s'\} \succsim T \cup \{s''\} \Leftrightarrow s' \succsim^{pre} s''$$

We say that the priority structure respects type-specific quotas if a priority structure constructed from a pre-priority satisfying (RC) and (RR). These notions are similar to Abdulkadiroğlu (2005) but not a generalization. Our definition not only captures indifferences but also is compatible with non-wastefulness, but Abdulkadiroğlu (2005) is not. The following claim shows that an acceptant priority satisfying (RC) and (RR) falls in our class.

Claim 1 *For every pre-priority $\succsim^{pre}$, a priority structure $\succsim$ constructed from $\succsim^{pre}$ satisfying (RC) and (RR) is acceptant and substitutable.*

Therefore, when schools employ a priority respecting type-specific quotas, we know that the MDA always returns the stable assignment. Furthermore, we can directly show the following:

Claim 2 *A stable assignment is constrained efficient if and only if it does not admit a stable student improving cycle.*

Remark that a SIC is not consisted of distinct schools, and we do not apply a stable improving cycle in Erdil and Ergin (2008).

Example 4 (Racial Balance & Walk Zone) Three quotas, $(q^w, q^b, q^a) = (1, 1, 1)$. Students are pre-prioritized by within walk zone and out of walk zone, as follows:

$$\{w_1\} \sim^{pre} \{w_2\} \sim^{pre} \{b_1\} \succ^{pre} \{b_2\} \sim^{pre} \{a_1\}$$

A priority respecting type-specific quotas is

$$
\begin{aligned}
&\{w_1, b_1, a_1\} \sim \{w_2, b_1, a_1\} \succ \\
&\{w_1, b_2, a_1\} \sim \{w_2, b_2, a_1\} \succ \\
&\{w_1, w_2, b_1\} \succ \\
&\{w_1, w_2, b_2\} \sim \{w_1, b_1, b_2\} \sim \{w_1, w_2, a_1\} \succ \\
&\{b_1, b_2, a_1\} \succ \ldots
\end{aligned}
$$

Clearly this is not responsive. This example is different from one about gender equality in that an underlining priority is not necessarily indifferent. $\Diamond$

In Kojima (2010), the implication of introducing the affirmative action policy is discussed. He changes the notion of stability in that an assignment is "stable" if it is feasible, individually rational, and there is no blocking pair within feasible assignments. The main result (Theorem 1) is briefly that there is no stable mechanism that an increase of the number of some type-specific quota makes agents in that type weakly better off. The proof is by example, and the following example grasps the point.

Example 5 (Compared to the previous literature) Suppose there are two schools $\{x, y\}$ and both have 2 seats. There are 4 students, denoted by m_1, m_2, w_1, w_2. Preferences and pre-priorities are as follows:

R_{m_1}	R_{m_2}	R_{w_1}	R_{w_2}
x	x	y	y
y		x	

and

$\succsim_x^{pre}$	$\succsim_y^{pre}$
m_1	m_1
m_2	m_2
w_1	w_1
w_2	w_2

(1) Suppose there is no policy. Then $\succsim_x$ and $\succsim_y$ are responsive to pre-priorities, and the stable outcome by the DA is just

$$\mu = \begin{pmatrix} m_1 & m_2 & w_1 & w_2 \\ x & x & y & y \end{pmatrix}$$

(2) Suppose both schools have a gender equal policy. Then $\{m_1, m_2\}$ and $\{w_1, w_2\}$ are unacceptable in Abdulkadiroğlu (2005) or infeasible in Kojima (2010).

Then the DA outcome is

$$\mu' = \begin{pmatrix} m_1 & m_2 & w_1 & w_2 \\ x & \emptyset & y & \emptyset \end{pmatrix}$$

If stability is re-defined as the outcome which does not allow individual or a pairwise deviation within the feasible outcomes, then the above assignment is "stable" in this sense. You can see that compared with μ, m_2, w_2 are worse off, even though a gender equal policy is employed.

We want to point out here that it comes from wastefulness, but from any specific policy. Even though there is a vacancy in both schools, they leave it vacant in this formulation. Even though no woman wants a position at a school x, a gender equal constraint prohibits m_2 from attending a school x.

(3) Alternatively, if both schools construct a priority respecting type-specific quotas $(q_x^m, q_x^w) = (q_y^m, q_y^w) = (1, 1)$, and run the DA, then the outcome is trivially

$$\mu'' = \begin{pmatrix} m_1 & m_2 & w_1 & w_2 \\ x & x & y & y \end{pmatrix} = \mu$$

Note that both schools respect gender equality, but if there is no man-woman pair in applicants, then they accept the second best applicants. Furthermore, the notion of stability does not change. $\diamondsuit$

Remark that non-wastefulness and the formulation of Abdulkadiroğlu (2005) or Kojima (2010) are incompatible.

1.7 Discussion

In this paper, we develop a general class of priority rankings which captures indifferences and substitutability. As our example shows, a complicated but practical concern is well captured. In this section, we discuss how appropriate our class of priorities is and its limitation.

As we restrict attention to substitutability, it is because when priority structures are strict, Hatfield and Kojima (2008) show that substitutability is a maximal domain for the existence of stable assignments in a sense that

$$\succ \text{ is substitutable } \Leftrightarrow \forall R,\ S^{\succ}(R) \neq \emptyset,$$

where $S^{\succ}(\ \cdot\ )$ is the set of stable assignment under $\succ$. Since our class includes strict substitutable priorities as a special case, we do not go beyond substitutability.

This restriction excludes some interesting priority structure as follows:

Example 6 Suppose a school has two seats and there are four students, $\{s_1, s_2, s_3, s_4\}$, with the following attributes:

	s_1	s_2	s_3	s_4
race	black	black	white	white
gender	male	female	male	female

When a school respects both race and gender equality, a natural way of prioritizing them is

$$\{s_1, s_4\} \sim \{s_2, s_3\} \succ \{s_1, s_2\} \sim \{s_3, s_4\} \sim \{s_1, s_3\} \sim \{s_2, s_4\} \succ \cdots$$

This priority ranking does not satisfy the condition (a) in the definition 1, whereas the condition (b) holds. Suppose $S = \{s_1, s_2, s_3\}$ and $T = \{s_1, s_2, s_3, s_4\}$. Then $\mathcal{C}(T) = \{\{s_1, s_4\}, \{s_2, s_3\}\}$ but

$$\{s_1, s_4\} \cap S = \{s_1\} \not\subseteq \{s_2, s_3\} = \mathcal{C}(S).$$

Therefore, the condition (a) in the definition 1 does not hold. $\Diamond$

A critical difference from ours involves two or more dimensions of students' attributes. We may conclude that there is another conflict with stability, that is, stability and a general diversity concern.

1.8 Appendix

1.8.1 Proof of Proposition 1

Denote $U = \mu^{-1}(x)$. If μ is pairwise stable, then it must be that for any ℓ with $xR_\ell\mu(\ell)$, we have $U \in \mathcal{C}_x(U \cup \{\ell\})$. Since students' preferences over schools are strict, we can write this in a seemingly stronger way: for any ℓ with $aR_\ell\mu(\ell)$, we have $U \in \mathcal{C}_x(U \cup \{\ell\})$. We would like to show that if $S \subseteq N$ such that $aR_i\mu(i)$ for all $i \in S$, then $U \in \mathcal{C}_x(U \cup S)$. In order to conclude via induction on $|S|$, it is sufficient to show that

$$[U \in \mathcal{C}_x(U \cup S) \text{ and } U \in \mathcal{C}_x(U \cup \{k\})] \quad \Rightarrow \quad U \in \mathcal{C}_x(U \cup S \cup \{k\}).$$

First, note that $\mathcal{C}_x$ being consistent with $\succsim_x$ implies

$$U \succsim_x V \quad \text{for all } V \subseteq U \cup S \text{ such that } |V| \leq q_x, \tag{$\star$}$$

and

$$U \succsim_x V \quad \text{for all } V \subseteq U \cup \{k\} \text{ such that } |V| \leq q_x,$$

Now, suppose for a contradiction that $U \notin \mathcal{C}_x(U \cup S \cup \{k\})$. Then for any $T \in \mathcal{C}_x(U \cup S \cup \{k\})$, we have

$$T \succ_x U.$$

Combining this with the relationship ($\star$) above, we get $T \succ_x V$ for all $V \subseteq U \cup S$ such that

$|V| \leq q_x$, and we conclude that $T \nsubseteq U \cup S$. Therefore, for any $T \in \mathcal{C}_x(U \cup S \cup \{k\})$, we must have $k \in T$. On the other hand, $U \in \mathcal{C}_x(U \cup \{k\})$ implies that $\{k\} \in \mathcal{R}_x(U \cup \{k\})$, which implies, due to substitutability, $\{k\} \subseteq (U \cup S \cup \{k\}) \backslash T$ for some $T \in \mathcal{C}_x(U \cup S \cup \{k\})$, yielding the desired contradiction. □

1.8.2 Proof of Proposition 2

A_x^t is the set of students who have applied to school x in some round $k \leq t$. Hence

$$A_x^1 \subseteq A_x^2 \subseteq \cdots$$

The algorithm requires that those students rejected in rounds $k \leq t-1$ would still be rejected if they were considered to be among the applicant in round t. This can be ensured thanks to $\mathcal{C}_x$ being substitutable, because $Z_x^{t-1} = A_x^{t-1} \backslash S'_x$ for some $S'_x \in \mathcal{C}_x(A_x^{t-1})$ and $A_x^{t-1} \subseteq A_x^t$ together imply that there exist $Z_x^t = A_x^t \backslash S''_x$ such that $Z_x^t \supseteq Z_x^{t-1}$ for some $S''_x \in \mathcal{C}_x(A_x^t)$.

In order to see that the algorithm indeed ends, note that at any round if a student is not matched, then she applies to her next favorite school in the following round. Therefore, she either exhausts all her acceptable schools by going down all the way to the end of her preference list, or ends up being matched with some school.

Suppose that μ is the matching obtained as a result of the algorithm which ends in round

m. Stability of μ basically means

$$\mu^{-1}(x) \in \mathcal{C}_x(\{i \mid x R_i \mu(i)\})$$

But of course those who weakly prefer x to their match under μ are either matched with x, or have applied to x at some round of the algorithm. Thus, we need

$$\mu^{-1}(x) \in \mathcal{C}_x(A_x^m),$$

which clearly holds by the fact that $A_x^m \backslash \mu^{-1}(x) = Z_x^m \in \mathcal{R}_x(A_x^m)$. □

1.8.3 Proof of Proposition 3

The main part in proving the proposition is ($\Leftarrow$), i.e., showing that a strongly acyclic $\succsim$ leads to efficient $f^{\succsim}$. We will prove this part in two steps.

Given a priority structure $\succsim$, a **generalized weak cycle of size** n is constituted of distinct schools $x_0, x_1, \ldots, x_{n-1} \in X$ and distinct students $j, i_0, i_1, \ldots, i_{n-1} \in N$ with $n \geq 2$ such that

(1) $x_\ell \neq x_{\ell+1}$ for $\ell \in \{0, 1, \ldots, n-1\}$ (with $x_n = x_0$),

(2) there exist mutually disjoint sets of students $S_{x_0}, \ldots, S_{x_{n-1}} \subseteq N \backslash \{j, i_0, i_1, \ldots, i_{n-1}\}$

such that

$$
\textbf{(C)}\quad
\begin{aligned}
& j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\
& j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\
& i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\
& i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\
& \quad \vdots \\
& i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\
& i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\})
\end{aligned}
$$

$$
\textbf{(S)}\quad |S_{x_\ell}| = q_{x_\ell} - 1 \text{ for } \ell = 0, 1, \ldots, n-1.
$$

Step 1: If there exists a Pareto inefficient assignment $\mu \in f^{\succsim}(R)$, then $\succsim$ has a generalized weak cycle.

Proof of Step 1: Suppose that $\mu \in f^{\succsim}(R)$ is not Pareto efficient. Of all the Pareto improvements over μ, let ν be one which has the least number of students improving over μ. Denote by N' the set of students who are better off under ν compared with μ:

$$N' = \{i \mid \nu(i) P_i \mu(i)\}.$$

Denote by $\mathcal{E}_j^\mu$ the set of students who envy the student j under μ:

$$\mathcal{E}_j^\mu = \{\ell \in N \mid \mu(j) P_\ell \mu(\ell)\}.$$

Set $\mathcal{E}'_j$ to be the set of students in N' who envy j. That is,

$$\mathcal{E}'_j = \mathcal{E}^\mu_j \cap N' = \{\ell \in N' \mid \mu(j) P_\ell \mu(\ell)\}.$$

If $j \in N'$, then by the reshuffling lemma[10],we have $\mu(j) \in \nu(N')$. In particular, $\mu(j)$ is desired by some student in N' under μ, and hence $\mathcal{E}'_j$ is nonempty. Because μ respects priorities, we have

$$\mu^{-1}(\mu(j)) \in \mathcal{C}_{\mu(j)}(\mathcal{E}^\mu_j \cup \mu^{-1}(\mu(j))).$$

Furthermore, $\mathcal{E}'_j \subseteq \mathcal{E}^\mu_j$ and $\succsim$ being substitutable imply that

$$\mu^{-1}(\mu(j)) \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))).$$

Removing j from the choice set, we conclude, again using substitutability, that $\mu^{-1}(\mu(j))\backslash\{j\}$ is a subset of a chosen element from $\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}$. In other words

$$\mu^{-1}(\mu(j))\backslash\{j\} \subseteq S' \quad \text{for some} \quad S' \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}).$$

Any such S' has exactly one element from $\mathcal{E}'_j$, and let E'_j be the set of those elements:

$$E'_j = \left\{ \ell \;\middle|\; \begin{array}{c} \ell \in \mathcal{E}'_j, \text{ and } (\mu^{-1}(\mu(j))\backslash\{j\}) \cup \{\ell\} = S' \\ \text{for some } S' \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}) \end{array} \right\}$$

[10] Lemma 1 of Erdil and Ergin (2008)

Thus, E'_j is a nonempty subset of N' for each $j \in N'$. Consider a directed graph whose set of vertices is N'. For each $i \in E'_j$, let there be a directed edge from i to j. Therefore, every vertex in this graph has an incoming edge, and since it is a finite graph, there must be a cycle.

Let the shortest cycle in this graph consist of students $i_0, i_1, \ldots, i_{n-1}, i_n = i_0$, where $n \geq 2$, and there is an edge from i_ℓ to $i_{\ell+1}$ for $\ell = 0, 1, \ldots, n-1$. Denoting $\mu(i_\ell) = x_\ell$, since i_ℓ envy $i_{\ell+1}$, we have $x_\ell \neq x_{\ell+1}$ for each ℓ. In fact, these schools $x_0, \ldots, x_{n-1}$ must be distinct, for otherwise we would have a shorter cycle, which would give a Pareto improvement over μ, involving a smaller number of students improving. To be more precise, if $x_0 = x_k$ for some $k \leq n-1$, then the cyclic trade which allows i_ℓ take $x_{\ell+1}$ for $\ell = 0, \ldots, k-1$, and letting i_k take x_0 would lead to a Pareto improvement over μ. Since $k < n$, this would contradict with the assumption that ν was the "smallest" improvement over μ. Since $\mu(i_\ell) = x_\ell$, the students $i_0, \ldots, i_{n-1}$ are necessarily distinct.

The fact that μ respects priorities implies that it is non-wasteful. Since each x_ℓ is desired by some student at assignment μ, all seats at these schools must be assigned under μ. Denoting $S_{x_\ell} = \mu^{-1}(x_\ell) \backslash \{i_\ell\}$, we know that $S_{x_0}, \ldots, S_{x_{n-1}}$ are mutually disjoint subsets of $N \setminus \{i_0, i_1, \ldots, i_{n-1}\}$, because $x_0, x_1, \ldots, x_{n-1}$ are distinct schools. Moreover we have

(1) $|S_{x_\ell}| = q_{x_\ell} - 1$,

(2) $i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\})$

$$i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\})$$

$$\vdots \qquad (*)$$

$$i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\})$$

$$i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\})$$

because otherwise, if student i_ℓ were to be in $DC_{x_{\ell+1}}(S_{x_{\ell+1}} \cup \{i_{\ell+1}, i_\ell\})$ for some ℓ, then we would have $S_{x_{\ell+1}} \cup \{i_{\ell+1}\} \notin \mathcal{C}_{x_{\ell+1}}(S_{x_{\ell+1}} \cup \{i_{\ell+1}, i_\ell\})$, contradicting stability of μ.

Let ω be the assignment derived from μ by letting the students $i_0, i_1, \ldots, i_{n-1}$ exchange their schools along the improvement cycle suggested above. In other words,

$$\omega(i) = \begin{cases} \mu(i) & i \neq i_\ell \\ \mu(i_{\ell+1}) & i = i_\ell \end{cases}$$

ω Pareto dominates μ, whereas μ is constrained efficient, so ω must not be stable. Therefore the cyclic trade letting i_ℓ take $\mu(i_{\ell+1})$ for $\ell = 0, 1, \ldots, n-1, n \equiv 0$ cannot be respecting priorities. Then we know from Proposition 1 that there must be a blocking pair involving one of these schools. Suppose that j and x_0 form a blocking pair for ω, i.e., $\omega^{-1}(x_0) \notin \mathcal{C}_x(\omega^{-1}(x_0) \cup \{j\})$. Then $x_0 P_j \omega(j)$ and

$$j \in DC_{x_0}(\omega^{-1}(x_0) \cup \{j\}) = DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}). \tag{**}$$

First, note that $j \neq i_{n-1}$, because $\omega(i_{n-1}) = x_0 P_j \omega(j)$. Secondly, $j \neq i_0$, because $\omega(i_0) P_{i_0} \mu(i_0) = x_0$, while $x_0 P_j \omega(j)$. And lastly if $j = i_k$ for some $1 \leq k \leq n-2\}$, then we have an envy cycle

$$i_0 \to i_1 \to \cdots \to i_k \to i_0$$

which would allow a Pareto improvement involving only $k+1 \leq n-1$ students, contradicting our earlier choice of a smallest Pareto improvement over μ. Thus $j \notin \{i_0, \ldots, i_{n-1}\}$.

Furthermore, stability of μ implies

$$j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}). \tag{***}$$

Thus, combining (*), (**), and (***), we have a generalized weak cycle

$$\begin{aligned}
&j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\
&j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\
&i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\
&i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\
&\quad \vdots \\
&i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\
&i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\})
\end{aligned}$$

with $|S_{x_\ell}| = q_{x_\ell} - 1$ for $\ell = 0, 1, \ldots, n-1$.

Step 2: If $\succsim$ has a generalized weak cycle, then it has a weak cycle.

Proof of Step 2: Suppose that $\succsim$ has a generalized weak cycle and let the size of its shortest generalized weak cycle be n. We will show that $n = 2$, which will prove step 2, since a weak cycle is a generalized weak cycle of size 2. Suppose that $x_0, x_1, \ldots, x_{n-1} \in X$; $j, i_0, i_1, \ldots, i_{n-1} \in N$ and $S_{x_0}, \ldots, S_{x_{n-1}} \subseteq N \setminus \{j, i_0, \ldots, i_{n-1}\}$ form a shortest generalized weak cycle. We will assume that it is of size $n \geq 3$, and reach a contradiction.

Let us look at the the set of definitely chosen students from $S_{x_1} \cup \{i_0, i_2\}$ according to the priorities of x_1. Is i_0 in this set or not?

If so, i.e., if $i_0 \in DC_{x_1}(S_{x_1} \cup \{i_0, i_2\})$, then

$$
\begin{aligned}
&i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\})\\
&i_0 \in DC_{x_1}(S_{x_1} \cup \{i_0, i_2\})\\
&i_2 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_2\})\\
&i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}),
\end{aligned}
$$

which is a weak cycle, i.e., a generalized weak cycle of length 2, contradicting with our assumption of shortest cycle being of length at least 3.

If on the other hand, $i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_0, i_2\})$, then we get the following generalized weak cycle

$$
\begin{aligned}
&j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\})\\
&j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\})\\
&i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\})\\
&i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\})\\
&\quad\vdots\\
&i_2 \notin DC_{x_3}(S_{x_3} \cup \{i_3, i_2\})\\
&i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_2, i_0\})
\end{aligned}
$$

with $|S_{x_\ell}| = q_{x_\ell} - 1$ for $\ell = 0, 1, 3, \ldots, n-1$. This cycle is shorter than the one we started with, because it does not have x_2, hence yields the desired contradiction to our original cycle being the shortest.

$\Longrightarrow$**:** Let N, X, and q and $\succsim$ be given. Assume that $\succsim$ has a weak cycle. Let $i, j, k \in N$, and $x, y \in X$ such that there exist $S_x, S_y \subseteq N \backslash \{i, j, k\}$ with $S_x \cap S_y = \varnothing$ satisfying

$$j \notin DC_x(S_x \cup \{i, j\})$$

$$j \in DC_x(S_x \cup \{k, j\})$$

$$k \notin DC_x(S_x \cup \{k, i\})$$

$$i \notin DC_y(S_y \cup \{k, i\})$$

with $|S_x| = q_x - 1$ and $|S_y| = q_y - 1$.

Consider the preference profile R where students in S_x and S_y, respectively, rank x and y as their top choice, and the preferences of i, j, and k are such that $yP_ixP_iiP_i\cdots$, $xP_jjP_j\cdots$, and $xP_kyP_kkP_k\cdots$. Finally, let students outside $S_x \cup S_y \cup \{i, j, k\}$ prefer not to be assigned to any school. Consider the assignment μ such that for each $\ell \in S_x \cup \{i\}$ one has $\mu(\ell) = x$, and for each $\ell \in S_y \cup \{k\}$ one has $\mu(\ell) = y$. Now the only candidates for blocking pairs are (j, x), (k, x), and (i, y). However, the weak cycle conditions are such that $j \notin DC_x(S_x \cup \{i, j\})$ $k \notin DC_x(S_x \cup \{k, i\})$, and $i \notin DC_y(S_y \cup \{k, i\})$, ensuring that μ respects priorities $\succsim$. Moreover, there is only one assignment that Pareto dominates μ, namely the assignment ν obtained from μ by letting i and k trade their assigned schools. Since $j \in DC_x(S_x \cup \{j, k\})$, $xP_j\nu(j)$ and $\nu^{-1}(x) = S_x \cup \{k\}$, the assignment ν does not respect $\succsim$. Thus μ is constrained efficient, but not Pareto efficient. $\square$

1.8.4 Proof of Proposition 4

We will now show that if a stable assignment μ is Pareto dominated by another stable assignment ν, then μ must admit a SIC. From this, it will follow that if μ does not admit a SIC, then it must be constrained efficient.

Let $N' = \{i \in N \mid \mu(i) \neq \nu(i)\}$ and $X' = \{\nu(i) \mid i \in N'\}$. For any $i \in N'$, we know by the reshuffling lemma that $\mu(i) \in X'$.

Let $\mu(i) = x$. Denote

$$D_x^\mu = \{j \in N \mid xP_j\mu(j)\}, \quad D_x' = \{j \in N' \mid xP_j\mu(j)\}, \quad D_x'' = \{j \in N \backslash N' \mid xP_j\mu(j)\}$$

and set

$$\bar{D}_x = D_x' \sqcup D_x'' \sqcup \mu^{-1}(x) = D_x^\mu \sqcup \mu^{-1}(x).$$

Stability of μ implies that

$$\mu^{-1}(x) \in \mathcal{C}_x(\bar{D}_x)$$

Moreover, stability of ν implies that

$$D_x'' \subseteq T'' \text{ for some } T'' \in \mathcal{R}_x(D_x^\nu \sqcup \nu^{-1}(x)) \qquad (\star).$$

ν Pareto dominates μ, so those who desire x at ν, desire x at μ as well. Therefore $D_x^\nu \subseteq D_x^\mu$. Moreover, if $j \in \nu^{-1}(x)$, then either $j \in \mu^{-1}(x)$ or $j \in D_x'$. And finally, since

$\mu(i) = x$ and $i \in N'$, we know that $i \notin \nu^{-1}(x)$, and $\nu(i) P_i x$. Therefore $i \notin D_x^\nu$. Thus

$$D_x^\nu \sqcup \nu^{-1}(x) \subseteq D_x^\mu \cup \nu^{-1}(x) \subseteq \bar{D}_x \backslash \{i\}. \qquad (\star\star)$$

Now we conclude by using $(\star)$, $(\star\star)$, and substitutability that

$$D_x'' \subseteq T' \text{ for some } T' \in \mathcal{R}_x(\bar{D}_x \backslash \{i\}).$$

Denoting

$$S' = (\bar{D}_x \backslash \{i\}) \backslash T',$$

we have

$$S' \in \mathcal{C}_x(\bar{D}_x \backslash \{i\}) \quad \text{and} \quad S' \cap D_x'' = \varnothing.$$

Note that

$$\bar{D}_x \backslash \{i\} = D_x' \sqcup D_x'' \sqcup \mu^{-1}(x) \backslash \{i\},$$

and $|\mu^{-1}(x) \backslash \{i\}| \leq q_x - 1$. Since $\succsim$ is acceptant, and $|\bar{D}_x \backslash \{i\}| \geq q_x$, we must have $|S'| \geq q_x$. Because of $|\mu^{-1}(x) \backslash \{i\}| \leq q_x - 1$ and that $S' \cap D_x'' = \varnothing$, we have

$$S' \cap D_x' \neq \varnothing.$$

Hence, there exists $i' \in D'_x$ such that $\{i'\} \cup \mu_x^{-1} \backslash \{i\} \in \mathcal{C}_x(\bar{D}_x \backslash \{i\})$, i.e.,

$$i' \in E_i^{\mu}.$$

Now construct a directed graph with N' being its set of vertices. For any $i \in N'$, the above argument shows that there is $i' \in N'$ such that $i \in E_i^{\mu}$, so draw an edge $i' \rightarrow i$. Since this is a finite graph with every vertex having an incoming edge, there must be cycle. By construction, this is a SIC. $\square$

1.8.5 Proof of Proposition 5

Denote the assignment obtained by carrying out this SIC by ν, i.e., define matching ν as

$$\nu(j) = \begin{cases} \mu(i_{\ell+1}) & \text{if } j = i_\ell \\ \mu(j) & \text{otherwise} \end{cases}$$

Case 1: If the schools $\mu(i_0), \mu(i_1), \ldots, \mu(i_{n-1})$ are distinct, then it is "straightforwardly verified" that ν is stable.

Case 2: Now consider the case in which the schools $\mu(i_0), \mu(i_1), \ldots, \mu(i_{n-1})$ are not distinct. Suppose for a contradiction that ν is not stable. So by Proposition 1 it must admit a blocking pair (j, x), with $j \in N$, and $x \in X$. That is,

$$j \in DC_x(\nu^{-1}(x) \cup \{j\}).$$

Note that such a school x must appear more than once in the SIC, for otherwise $\nu^{-1}(x) = \mu^{-1}(x) \backslash \{i_{\ell+1}\} \cup \{i_\ell\}$ and $i_\ell \in E_{i_{\ell+1}}$, and hence $j \notin DC_x((\mu^{-1}(x) \backslash \{i_{\ell+1}\}) \cup \{i_\ell, j\})$, contradicting with (j, x) being a blocking pair.

Suppose that the school x is involved in moves $i_{k^t} \to i_{k^t+1}$ for $t = 1, \ldots, m$, so the SIC looks like:

$$i_0 \to \cdots \to i_{k^1} \to i_{k^1+1} \to \cdots \to i_{k^2} \to i_{k^2+1} \to \cdots \to i_{k^m} \to i_{k^m+1} \to \cdots \to i_{n-1},$$

where $k^t \in \{0, \ldots, n-1\}$ and $\mu(i_{k^t+1}) = x$ for all $t \in \{1, 2, \ldots, m\}$.

Since (j, x) is a blocking pair for ν, we have $x P_j \nu(j)$ and $j \in DC_x(\nu^{-1}(x) \cup \{j\})$. Thus $x P_j \nu(j) R_j \mu(j)$, and $j \in \mathcal{E}^\mu_{k^t+1}$ for all t.

The definition of SIC and substitutability implies that for each $t \in \{1, \ldots, m\}$ there exists A_t such that

$$[\mu^{-1}(x) \backslash \{i_{k^1+1}, \ldots i_{k^m+1}\}] \cup \{i_{k^t}\} \ \subseteq \ A_t \in \mathcal{C}_x(\nu^{-1}(x) \cup \{j\}),$$

because $i_{k^t} \in \mathcal{E}^\mu_{i_{k^t+1}}$ for all $t \in \{1, 2, \ldots, m\}$.

Note that j is in A_t, because $j \in DC_x(\nu^{-1}(x) \cup \{j\})$.

Thus we get

$$[\mu^{-1}(x) \backslash \{i_{k^1+1}, \ldots, i_{k^m+1}\}] \cup \{i_{k^t}\} \cup \{j\} \ \subseteq \ A_t \in \mathcal{C}_x(\nu^{-1}(x) \cup \{j\}).$$

Let us write A_t as the disjoint union

$$A_t = B_t \sqcup \mu^{-1}(x) \backslash \{i_{k^1+1}, \ldots, i_{k^m+1}\}$$

Note that, for all $t \in \{1, \ldots, m\}$:

$$\{i_{k^t}, j\} \subseteq B_t \subseteq \{i_{k^1}, \ldots, i_{k^m}, j\} \quad \text{and } |B_t| = m.$$

There must exist t, t' such that $B_t \neq B_{t'}$, for otherwise $\{i_{k^1}, \ldots, i_{k^m}, j\} \subseteq B_t$ contradicting with $|B_t| = m$. Let the symmetric difference of B_t and $B_{t'}$ be $\{i_{k^r}, i_{k^s}\}$, where $r < s$, so that

$$B_t = \tilde{B} \cup \{i_{k^r}\} \quad \text{and} \quad B_{t'} = \tilde{B} \cup \{i_{k^s}\},$$

and hence

$$A_t = \tilde{A} \cup \{i_{k^r}\} \quad \text{and} \quad A_{t'} = \tilde{A} \cup \{i_{k^s}\},$$

where $\tilde{A} = \tilde{B} \sqcup (\mu^{-1}(x) \backslash \{i_{k^1}, \ldots, i_{k^m}\})$.

Since

$$A_t, A_{t'} \in \mathcal{C}_x(\nu^{-1}(x) \cup \{j\})$$

$$\nu^{-1}(x) \cup \{j\} \subseteq \mathcal{E}^{\mu}_{i_{k^s+1}} \cup \mu^{-1}(x) \backslash \{i_{k^s+1}\} \text{ and}$$

$$\mu^{-1}(x) \backslash \{i_{k^s+1}\} \cup \{i_{k^s}\} \in \mathcal{C}_x(\mathcal{E}^{\mu}_{i_{k^s+1}} \cup \mu^{-1}(x) \backslash \{i_{k^s+1}\}),$$

ETE[11] implies that $\mu^{-1}(x)\backslash\{i_{k^s+1}\} \cup \{i_{k^r}\} \in \mathcal{C}_x(\mathcal{E}^{\mu}_{i_{k^s+1}} \cup \mu^{-1}(x)\backslash\{i_{k^s+1}\})$, and therefore

$$i_{k^r} \in E^{\mu}_{i_{k^s+1}}.$$

Hence there is a shorter SIC which looks like

$$i_0 \to \cdots \to i_{k^1} \to i_{k^1+1} \to \cdots \to i_{k^r} \to i_{k^s+1} \to \cdots \to i_{k^m} \to i_{k^m+1} \to \cdots \to i_{n-1},$$

contradicting with the initial assumption that the original SIC was the shortest such cycle.

$\square$

1.8.6 Proof of Claim 1

It is obvious that $\succsim$ is acceptant by (RR) condition and $\succsim^{pre}$ which is defined over the set of students. We need to show that $\succsim$ is substitutable.

Lemma 1 *For every $\succsim$ constructed from $\succsim^{pre}$ satisfying (RC) and (RR) conditions, if $|S| \geq q_x$, then $S' \in \mathcal{C}(S)$ if and only if S' has the following properties:*[12]

(1) $|S'| = q_x$

(2) $|S_\tau| \leq q_\tau \Rightarrow S_\tau \subseteq S'$

(3) $|S_\tau| > q_\tau \Rightarrow |S'_\tau| \geq q_\tau$ *and for all* $s_i \in S'_\tau$, $s_i \succsim \hat{s}$ *for all* $\hat{s} \in S_\tau \backslash S'_\tau$

[11] Recall that ETE requires that if i_{k^r} can substitute i_{k^s} to complement some set A, then she can substitute him to complement any other set B in any larger applicants. See Footnote **??**.

[12] By construction, a priority ordering over single elements are the same between $\succsim$ and $\succsim^{pre}$, so we use $\succsim$ for the comparison among single elements.

(4) $|S'_\tau| > q_\tau \Rightarrow$ *for all* $s_i \in S'_\tau$, $s_i \succsim \hat{s}$ *for all* $\hat{s} \in S \backslash S'$.

Proof:

(⇒) Let $S' \in \mathcal{C}(S)$ and fix arbitrary. Since $\succsim$ is acceptant and $|S| > q_x$, $|S'| = q_x$ for any $S' \in \mathcal{C}(S)$. Suppose $S' \in D_a$.

For (2), if $S' \in D_0$ then the condition trivially holds, so we assume $a \geq 1$ for D_a and suppose for a contradiction. Then there is $S_{\tau'}$ such that $S_{\tau'} \nsubseteq S'$. Pick an agent in $S_{\tau'} \backslash S'$, denoted by s'. Since $|S'| = q_x$ and $\sum_\tau q_\tau = q_x$, there must be τ'' such that $|S'_{\tau''}| > q_{\tau''}$. Let $s'' \in S'_{\tau''}$.

Consider $S'' = S' \backslash \{s''\} \sqcup \{s'\}$. Since $S' \in D_a$,

$$\sum_\tau ||S'_\tau| - q_\tau| = \sum_{\tau \neq \tau', \tau''} ||S'_\tau| - q_\tau| + ||S'_{\tau'}| - q_{\tau'}| + ||S'_{\tau''}| - q_{\tau''}| = 2i.$$

On the other hand,

$$\sum_\tau ||S''_\tau| - q_\tau| = \sum_{\tau \neq \tau', \tau''} ||S''_\tau| - q_\tau| + ||S''_{\tau'}| - q_{\tau'}| + ||S''_{\tau''}| - q_{\tau''}|,$$

and

$$\begin{aligned} ||S'_{\tau'}| - q_{\tau'}| &= ||S''_{\tau'}| - q_{\tau'}| + 1 \\ ||S'_{\tau''}| - q_{\tau''}| &= ||S''_{\tau''}| - q_{\tau''}| + 1. \end{aligned}$$

Hence, $S'' \in D_{a-1}$ and by (RC), $S'' \succ S'$, but $S' \in \mathcal{C}(S)$, a contradiction.

For (3), suppose that $|S_\tau| \geq q_\tau$ and the condition does not hold. Then there is $\hat{s} \in S_\tau \backslash S'_\tau$ such that $\hat{s} \succ s'$ for some $s' \in S'_\tau$. Since they are in the same type, $S' \backslash \{s'\} \sqcup \{\hat{s}\} \in D_i$. From (RR), $S' \backslash \{s'\} \sqcup \{\hat{s}\} \succ S'$, a contradiction. Suppose $|S'_\tau| < q_\tau$, then by the similar argument to the proof of (2), we conclude that it never happens.

For (4), suppose not. Then there is $\hat{s} \in S \backslash S'$ such that $\hat{s} \succ s'$ for some $s' \in S'_\tau$. Since $\hat{s} \in S \backslash S'$ and (1) – (3), there is $S_{\tau'}$ such that $|S_{\tau'}| > q_{\tau'}$ and $\tau(\hat{s}) = \tau'$. This fact and $|S'_\tau \backslash \{s'\}| \geq q_\tau$ imply $S' \backslash \{s'\} \sqcup \{\hat{s}\} \in D_a$. Then (RR) implies $S' \backslash \{s'\} \sqcup \{\hat{s}\} \succ S'$, a contradiction. $\square$

<u>($\Leftarrow$)</u> Suppose S' satisfies (1) – (4) but $S' \notin \mathcal{C}(S)$. Then there is $S'' \in \mathcal{C}(S)$ such that $S'' \succ S'$. Note that S'' satisfies (1) – (4) by ($\Rightarrow$) and S' is in D_a if and only if S'' is in D_a. By induction on $|S' \cap S''|$.

(step 1) $|S' \cap S''| = q_x - 1$. Then let $s'_1 \in S' \backslash S''$ and $s''_1 \in S'' \backslash S'$. Since $S', S'' \in D_a$ and by supposition,

$$S'' \succ S' \Rightarrow s''_1 \succ s'_1,$$

by (RR). Since $s''_1 \in S \backslash S'$ and S' satisfies (4),

$$|S_{\tau(s'_1)} \cap S'| = q_{\tau(s'_1)}.$$

Then $|S_{\tau(s'_1)} \cap S''| < q_{\tau(s'_1)}$, even though $|S_{\tau(s'_1)}| > q_{\tau(s'_1)}$. A contradiction.

(step n) Assume the conclusion holds for the case that $|S' \cap S''| \leq q_x - (n-1)$, and consider $|S' \cap S''| = q_x - n$. Let $s'_1, \ldots, s'_n \in S' \backslash S''$ and $s''_1, \ldots, s''_n \in S'' \backslash S'$. Without loss of generality, we assume that $s'_1 \succsim s'_2 \succsim \cdots \succsim s'_n$ and $s''_1 \succsim s''_2 \succsim \cdots \succsim s''_n$.

Case 1 ($s'_1 \succ s''_1$). Then $S'' \backslash \{s''_1\} \sqcup \{s'_1\} \in D_a$ and $S'' \backslash \{s''_1\} \sqcup \{s'_1\} \succ S''$, a contradiction.

Case 2 ($s'_1 \sim s''_1$). Then $S'' \sim S'' \backslash \{s''_1\} \sqcup \{s'_1\}$. This means that $S'' \backslash \{s''_1\} \sqcup \{s'_1\} \succ S'$ and $|[S'' \backslash \{s''_1\} \sqcup \{s'_1\}] \cap S'| = q_x - (n-1)$. This case reduces to $n-1$, and by assumption, the conclusion holds.

Case 3 ($s''_1 \succ s'_1$). Then $s''_1 \in S \backslash S'$ and (4) imply that $|S'_{\tau(s'_1)}| = q_{\tau(s'_1)}$. Note that $\tau(s'_1) \neq \tau(s''_1)$. Then

$$
\begin{aligned}
|S_{\tau(s'_1)} \cap S''| < q_{\tau(s'_1)} \quad if \quad \tau(s''_i) \neq \tau(s'_1) \quad \forall i \in \{2, \ldots, n\} \\
|S_{\tau(s'_1)} \cap S''| = q_{\tau(s'_1)} \quad if \quad \tau(s''_i) = \tau(s'_1) \quad \exists i \in \{2, \ldots, n\}
\end{aligned}
$$

Clearly, a case that $|S_{\tau(s'_1)} \cap S''| < q_{\tau(s'_1)}$ leads to a contradiction. When $|S_{\tau(s'_1)} \cap S''| = q_{\tau(s'_1)}$, since $|S_{\tau(s'_1)}| > q_{\tau(s'_1)}$, $s'_1 \succsim s''_i$. Then $S'' \backslash \{s''_i\} \sqcup \{s'_1\} \in D_a$, and

$$S'' \backslash \{s''_i\} \sqcup \{s'_1\} \succsim S''.$$

Since $S'' \in \mathcal{C}(S)$, it must be

$$S''\backslash\{s''_i\} \sqcup \{s'_1\} \sim S''.$$

Then $s'_1 \sim s''_i$. Therefore $S''\backslash\{s''_i\} \sqcup \{s'_1\} \in \mathcal{C}(S)$ and $S''\backslash\{s''_i\} \sqcup \{s'_1\} \succ S'$. Notice that $|[S''\backslash\{s''_i\} \sqcup \{s'_1\}] \cap S'| = q_x - (n-1)$, which reduces to $n-1$. $\square$

Proof of Claim 1:

Without loss of generality, we just focus on $S \sqcup \{s''\}$ and S. Since $\succsim$ is acceptant, it suffices to show a case that $|S| \geq q_x + 1$.

Proof of the condition (a) Suppose $S' \in \mathcal{C}(S \sqcup \{s''\})$ and $S' \in D_a$. If $s'' \notin S'$, then $S' \in \mathcal{C}(S)$ and the condition (a) trivially holds. So we assume that $s'' \in S'$.

Case 1 $|S_{\tau(s'')}| \leq q_{\tau(s'')}$. Then since $S_{\tau(s'')} \subseteq S'$, for all $\hat{s} \in S\backslash S'$,

$$S'\backslash\{s''\} \sqcup \{\hat{s}\} \in D_{a+1}.$$

Then we have $\hat{s}_1 \succsim \hat{s}_2 \succsim \cdots \succsim \hat{s}_n$ for $S\backslash S'$ by (RR).

We claim that $S'' = S'\backslash\{s''\} \sqcup \{\hat{s}_1\} \in \mathcal{C}(S)$. Clearly $|S''| = q_x$.

For all S''_τ with $|S''_\tau| \leq q_\tau$, $S''_\tau = S'_\tau \subseteq S''$ if $\tau \neq \tau(s'')$. $S''_{\tau(s'')} = S'_{\tau(s'')}\backslash\{s''\} \subseteq S''$.

For all S''_τ with $|S''_\tau| > q_\tau$, since $\hat{s}_1 \succsim \hat{s}$ for all $\hat{s} \in S \backslash S'$ and

$$q_{\tau(\hat{s}_1)} \leq |S'_{\tau(\hat{s}_1)}| < |S''_{\tau(\hat{s}_1)}|,$$

(3) and (4) holds. By the lemma 1, $S'' \in \mathcal{C}(S)$.

Case 2 $|S_{\tau(s'')}| > q_\tau$. If $|S'_{\tau(s'')}| = q_{\tau(s'')}$, then there is $\hat{s} \in S \backslash S'$ such that $\tau(\hat{s}) = \tau(s'')$. For these $\hat{s}$, $S' \backslash \{s''\} \sqcup \{\hat{s}\} \in D_a$. By (RR), we can find $\hat{s}_1$ who is at least as good as any other agent who is in $S_{\tau(s'')}$. It is easy to see that $S' \backslash \{s''\} \sqcup \{\hat{s}_1\}$ satisfies (1) – (4).

Otherwise, $|S'_{\tau(s'')}| > q_{\tau(s'')}$. Then for all $\hat{s} \in S \backslash S'$,

$$S' \backslash \{s''\} \sqcup \{\hat{s}\} \in D_a.$$

Then we have $\hat{s}_1 \succsim \hat{s}_2 \succsim \cdots \succsim \hat{s}_n$ for $S \backslash S'$ by (RR). It is analogous to see that $S' \backslash \{s''\} \sqcup \{\hat{s}_1\}$ satisfies (1) – (4).

Hence, the condition (a) holds. $\square$

Proof of the condition (b) Suppose $R' \in \mathcal{R}(S)$. By definition, $R' = S \backslash S'$ for some $S' \in \mathcal{C}(S)$. We claim that there is $\hat{s} \in (S \sqcup \{s''\}) \backslash R'$ such that $R' \sqcup \{\hat{s}\} \in \mathcal{R}(S \sqcup s'')$.

Case 1 $|(S \sqcup \{s''\})_{\tau(s'')}| \leq q_{\tau(s'')}$. Then $|S_{\tau(s'')}| < q_{\tau(s'')}$. This implies that there is τ' such that $|S'_{\tau'}| > q_{\tau'}$. Then for all $s_i \in R'_{\tau'}$, $\hat{s} \succsim s_i$ for all $\hat{s} \in S'_{\tau'}$. Consider such types

$\{\tau'_1, \cdots, \tau'_m\}$. Take $s^* \in \bigcup_{i\in\{1,\ldots,m\}} S_{\tau'_i}$ in a way that $\hat{s} \succsim s^*$ for all $\hat{s} \in \bigcup_{i\in\{1,\ldots,m\}} S_{\tau'_i}$. We see that $R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{s''\})$. Let $S'' = (S \sqcup \{s''\}) \backslash (R' \sqcup \{s^*\})$. Since $|(S \sqcup \{s''\})_{\tau(s'')}| \leq q_{\tau(s'')}$, $(S \sqcup \{s''\})_{\tau(s'')} \subseteq S''$. For $\tau(s^*)$, $|S''_{\tau(s^*)}| \geq q_{\tau(s^*)}$ and by construction, S'' satisfies other properties. Hence, $S'' \in \mathcal{C}(S \sqcup \{s''\}) \Leftrightarrow R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{s''\})$.

Case 2 $|(S \sqcup s'')_{\tau(s'')}| > q_{\tau(s'')}$. Then we can find $\hat{s} \in (S \sqcup \{s''\})_{\tau(s'')}$ such that $s_i \succsim \hat{s}$ for all $s_i \in (S \sqcup \{s''\})_{\tau(s'')}$. Consider τ' such that $|S'_{\tau'}| > q_{\tau'}$ and let them be $\{\tau'_1, \ldots, \tau'_m\}$ (possibly empty). If for all such τ'_i, $s \succsim \hat{s}$, for all $s \in S'_{\tau'_i}$ or there is no such τ_i, then $S \backslash (R' \sqcup \{\hat{s}\})$ satisfies (1) – (4) and we are done. Otherwise there is $s_i \in S'_{\tau'_i}$ such that $\hat{s} \succ s_i$ for some $i \in \{1, \ldots, m\}$. Then we can find $\hat{s}_i$ such that $s_i \succsim \hat{s}_i$ for all $s_i \in S'_{\tau'_i}$. Let s^* be such that $\hat{s}_i \succsim s^*$ for any i. Then it also easy to see that $S'' = S \backslash (R' \sqcup \{s^*\})$ satisfies (1) – (4). Therefore, $R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{s''\})$.

Hence the condition 2 holds. □

1.8.7 Proof of Claim 2

We only need to show that if a stable assignment admits a SIC, then a new assignment followed by the SIC is stable.

As the same as the proof of Proposition 4, consider the shortest SIC and denote the assignment obtained by carrying out the SIC by ν. First of all, if the schools involved in the SIC, $\mu(i_0), \ldots, \mu(i_{n-1})$, are distinct, it is obvious that ν is stable.

We assume that the schools are not distinct. Suppose for a contradiction, that ν is not stable. Suppose that there is a blocking pair (j, x) and the school x is involved in moves $i_{k^t} \to i_{k^t+1}$ for $t = 1, \ldots, m$.

Note that $\tau(i_{k^t}) \neq \tau(i_{k^u})$, for any $t, u \in \{1, \ldots, m\}$ and $t \neq u$. Otherwise $\tau = \tau(i_{k^t}) = \tau(i_{k^u})$ for some t and $u \neq t$. Then we can see that $i_{k^u} \in E^{\mu}_{i_{k^t+1}}$.

Since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$ and $i_{k^u} \in \mathcal{E}^{\mu}_{i_{k^t+1}}$,

$$\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^t}\} \succsim \mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^u}\}.$$

Since i_{k^t} and i_{k^u} are in the same type,

$$\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^t}\} \in D_a$$

$$\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^u}\} \in D_a,$$

which implies by (RR) that

$$i_{k^t} \succsim i_{k^u}.$$

Similarly, $i_{k^u} \succsim i_{k^t}$, and therefore

$$i_{k^t} \sim i_{k^u}.$$

This implies $i_{k^u} \in E^{\mu}_{k^u+1}$, which is contradicting with the supposition that the SIC is the shortest.

Consider a blocking pair (j, x) such that $j \in DC_x(\nu^{-1} \cup \{j\})$.

Case 1 $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| \leq q_{\tau(j)}$. Since μ is stable and j is not in $\mu^{-1}(x)$, $|(\mu^{-1}(x))_{\tau(j)}| \geq q_{\tau(j)}$. If $|(\mu^{-1}(x))_{\tau(j)}| = q_{\tau(j)}$, then there is i_{k^t+1} such that $\tau(i_{k^t+1}) = \tau(j)$. Then $|\mu^{-1}(x) \backslash \{i_{k^t+1}\}| = q_{\tau(j)} - 1$. Since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$, $j \in E^{\mu}_{i_{k^t+1}}$ and $\tau(i_{k^t}) = \tau(j)$. Since $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| \leq q_{\tau(j)}$, there must be another i_{k^u+1} such that $\tau(i_{k^u+1}) = \tau(j)$. If $|(\mu^{-1}(x))_{\tau(j)}| > q_{\tau(j)}$, then we can easily see that there are distinct i_{k^t+1}, i_{k^u+1} such that $\tau(i_{k^t+1}) = \tau(i_{k^u+1}) = \tau(j)$.

Therefore, there are at least two agents i_{k^t+1}, i_{k^u+1} such that $\tau(i_{k^t+1}) = \tau(i_{k^u+1}) = \tau(j)$ in the SIC. Then

$$E^{\mu}_{i_{k^t+1}} = E^{\mu}_{i_{k^u+1}},$$

which contradicts that the SIC is the shortest.

Case 2 $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| > q_{\tau(j)}$. Since $j \in DC_x(\nu^{-1}(x) \sqcup \{j\})$ and $|\nu^{-1}(x) \sqcup \{j\}| = q_x + 1$, there is i_{k^t} such that $i_{k^t} \notin A \in \mathcal{C}_x(\nu^{-1}(x) \sqcup \{j\})$. Then $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(i_{k^t})}| > q_{\tau(i_{k^t})}$. Note that

$$j \succsim i_{k^t}$$

because if $\tau(i_{k^t}) = \tau(j)$, then $j \succsim i_{k^t}$ by (RR), and otherwise since $\mathcal{C}_x$ satisfies substitutability and $\nu^{-1}(x) \backslash \{i_{k^t}\} \sqcup \{j\} \in \mathcal{C}_x(\nu^{-1}(x) \sqcup \{j\})$, (4) of the lemma 1 implies $j \succsim i_{k^t}$.

Suppose $j \sim i_{k^t}$. Since $\nu^{-1}(x) \backslash \{i_{k^t}\} \sqcup \{j\}, \nu^{-1}(x) \in D_a$ for some a,

$$\nu^{-1}(x) \backslash \{i_{k^t}\} \sqcup \{j\} \sim \nu^{-1}(x),$$

implying that $j \notin DC_x(\nu^{-1} \sqcup \{j\})$.

Suppose $j \succ i_{k^t}$. We know that $\tau(i_{k^t}) \neq \tau(i_{k^u})$ for all $t, u \neq t \in \{1, \ldots, m\}$.

$$\begin{aligned} \left|(\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^t}, j\})_{\tau(i_{k^t})}\right| &> \left|(\mu^{-1} \backslash \{i_{k^1+1}, \ldots, i_{k^m+1}\} \sqcup \{i_{k^t}, j\})_{\tau(i_{k^t})}\right| \\ &= \left|(\nu^{-1}(x) \sqcup \{j\})_{\tau(i_{k^t})}\right| > q_{\tau(i_{k^t})} \end{aligned}$$

Since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$, $i_{k^t} \succsim j$, a contradiction. $\square$

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Chapter 2

Strategy-proofness and Stability of the Boston Mechanism: An Almost Impossibility Results

2.1 Introduction

In many school choice districts in the world, student and school matchings are determined by using a central clearinghouse. Each student is asked to submit their preference ranking of schools and each school sets a priority ranking of students, and *some* mechanism calculates which student is assigned to which school. There are three competing mechanisms, the Boston mechanism, the deferred acceptance mechanism and the top trading cycle mechanism, which are mainly used for solving school choice problems. In this paper, we focus on the Boston mechanism, which is used now in Denver and Minneapolis[1].

We consider three desiderata for a mechanism in a school choice problem, *Pareto efficiency*, *strategy-proofness*, and *stability*. A mechanism is Pareto efficient if it returns an

[1]As we will discuss in Section 1.1, though the Boston mechanism, as named, was used in Boston, the Boston public school districts abandoned the Boston mechanism since 2005.

assignment which is not Pareto dominated by any other feasible assignment. A mechanism is strategy-proof if no student is better off by misrepresenting her true preference. Hence, there is no room for gaming a mechanism. A mechanism is said to be stable if no student nor pair of a student and a school has incentive to deviate from an assignment. Each of the three mechanisms has one or two of those properties, but none of them satisfies all the three properties simultaneously. The following table summarizes three properties of the three mechanisms.

	DA	TTC	Boston
Pareto efficiency	$\times$	✓	✓
Strategy-proofness	✓	✓	$\times$
Stability	✓	$\times$	$\times$

Figure 2.1: Properties of three competing mechanisms.

This paper characterizes priority structures for which the Boston mechanism is Pareto efficient, strategy-proof, and stable. We show that the Boston mechanism is strategy-proof if and only if it is stable if and only if the priority structure is *strongly acyclic*. For practical purposes, it is important to see how frequent strongly acyclic priority structure emerges. The main contribution of this paper is that any practical priority structure is quasi-cyclic. More formally, if there are at least two schools whose total quota is less than the total number of students in the school choice problem, then any priority structure is quasi-cyclic. In words, the paper shows an impossibility result for the Boston mechanism being strategy-proof or stable in practice.

There are several prior works on the Boston mechanism. The most related is Hsu (2011). He proposes a sufficient condition for dominance solvability of the preference revelation game induced by the Boston mechanism. His sufficient condition is both on a preference profile and a priority structure, while here we consider only the priority structure. Since the Boston mechanism is not strategy-proof in general, one research direction is to explore stability of the outcomes in a weaker solution concept, Nash equilibrium. Though the Boston mechanism is not stable under the true preferences, Ergin and Sönmez (2006) show that the set of stable assignments is equivalent to the set of Nash equilibrium outcoems. That is, the set of stable assignments of the Boston mechanism is Nash implementable. Note that a stable assignment obtained by a Nash equilibrium is not necessarily efficient. Haeringer and Klijn (2009) further investigate when a stable assignment induced by a Nash equilibrium is efficient, and it is characterized by a condition on a priority structure, called X-acyclicity[2].

For other works on the Boston mechanism, with incomplete information on students' exact utilities, Abdulkadiroğlu, Che and Yasuda (2011) show that when all students' ordinal preferences are the same and all schools rank all students equally, the Boston mechanism ex ante Pareto dominates the deferred acceptance mechanism. As a characterization of the Boston mechanism, Kojima and Ünver (2011) provide axioms, respecting preference rankings, consistency, resource monotonicity, and an auxiliary invariance property. They

[2]We will discuss X-acyclicity in Section 4.

show that these axioms are equivalent to a mechanism that is the Boston mechanism for some priority.

2.1.1 The other two mechanisms

The deferred acceptance mechanism was introduced by Gale and Shapley (1962), and is used in Boston, New York and other cities. In 2005-06, the Boston public schools switched from a mechanism now called the Boston mechanism to the student-proposing deferred acceptance mechanism[3]. The deferred acceptance mechanism is stable, and furthermore is strategy-proof, as shown by Dubins and Freedman (1981). It is not efficient in general but is constrained efficient, that is, an assignment obtained by the deferred acceptance mechanism is not Pareto dominated by any other stable assignment. Ergin (2002) showed that the deferred acceptance mechanism is efficient if and only if a priority structure is Ergin-acyclic[4]. Roughly speaking, if all schools have similar priority rankings over students, then a priority structure is Ergin-acyclic.

The top trading cycle mechanism was introduced by Abdulkadiroğlu and Sönmez (2003), as an adaptation of a trading mechanism proposed by Shapley and Scarf (1974) into the school choice context. This mechanism is Pareto efficient and strategy-proof, but not stable. Kesten (2006) characterizes the stable top trading cycle mechanism by imposing a condition on a priority structure called Kesten-acyclicity[5]. A part of the main theorem in

[3]In this paper, we say the student-proposing deferred acceptance mechanism just the deferred acceptance mechanism.

[4]We will discuss Ergin-acyclicity in Section 4.

[5]We will discuss Kesten-acyclicity in Section 4.

Kesten (2006) says that the top trading cycle mechanism is stable if and only if a priority structure is Kesten-acyclic. However, Kesten-acyclicity implies Ergin-acyclcity, but not the converse.

2.2 Model

There are N students and X schools. N and X are finite. Each student i has a strict preference R_i over $X \cup \{i\}$. A student i ranks not only schools but also herself, $\{i\}$, because it is usual in the United States that a student prefers home-schooling rather than going to some schools. The strict part of R_i is written by P_i. Each school x is endowed with $q_x (\geq 1)$ seats and strict priorities $\succ_x$ over the set of students. We denote a preference profile by $R = (R_i)_{i \in N}$ and a priority structure by $\succ = (\succ_x)_{x \in X}$.

Assignment μ is a mapping from N to $N \cup X$ with (1) for all $i \in N$, $\mu(i) \in X \cup \{i\}$ and (2) for all $x \in X$, $|\mu^{-1}(x)| \leq q_x$. Denote the set of assignments by M. An assignment μ is **individually rational** if for all $i \in N$, $\mu(i) R_i i$. A **blocking pair** is defined by a pair of a student i and a school x such that $x P_i \mu(i)$ and $i \succ_x j$ for some $j \in \mu^{-1}(x)$. An assignment is said to be **non-wasteful** if there is no pair of a student, i, and a school x, such that $x P_i \mu(i)$ and $|\mu^{-1}(x)| < q_x$. An assignment is **stable** if it is individually rational, non-wasteful and there is no blocking pair. An assignment μ is said to be **Pareto efficient** if there is no $\nu \in M$ such that for all i, $\nu(i) R_i \mu(i)$ and for some j, $\nu(i) P_j \mu(j)$. Although school priority structures play a similar role to preferences, because they are thought of as a public good, we assume schools are not taken into consideration with respect to efficiency in this paper.

Given a priority structure, a mechanism f is a mapping from the set of preference profiles to an assignment. An assignment of a student i under R is denoted by $f_i(R)$. A mechanism is stable or Pareto efficient if each outcome is stable or Pareto efficient, respectively. A mechanism f is said to be **strategy-proof** if there is no R and R'_i such that $f_i(R'_i, R_{-i})P_i f_i(R)$. Throughout the paper, we write the Boston mechanism by f^B.

2.2.1 Description of the Boston mechanism

Given $\succ$, the Boston mechanism f^B works as follows:

Step 1: Each student i applies to the top ranked school, if any. Each school x accepts the most preferred students on the basis of its priority ranking until the positions are filled, and rejects the others.

$\vdots$

Step t: Each student who is rejected at the $t-1$ step applies to the next top ranked school, if any. Each school x accepts the most preferred students until the remaining positions are filled, and rejects the other.

The algorithm terminates when no student applies to a school.

Note that once you are accepted at some school, your assignment is finalized.

Observation 1 *Given any priority structure $\succ$, f^B is Pareto efficient.*

However, f^B is neither stable nor strategy-proof in general.

Example 7 Suppose there are three students, $\{i, j, k\}$, and two schools, $\{x, y\}$, with one seat. Consider the following preferences and priorities[6].

$$\begin{array}{llll} R_i: & x & y & \qquad \succ_x: & i & j & k \\ R_j: & y & x & \qquad \succ_y: & k & j & i \\ R_k: & x & y & & & & \end{array}$$

Then

$$f^B(R) = \begin{pmatrix} i & j & k \\ x & y & k \end{pmatrix}.$$

First of all, $f^B(R)$ is not stable, since $yP_k f^B_k(R) = k$ and $k \succ_y \left(f^B(R)\right)^{-1}(y) = j$. Secondly, a student k has an incentive to misreport her true preference. Consider $R'_k : y$. Then

$$f^B(R'_k, R_{-k}) = \begin{pmatrix} i & j & k \\ x & j & y \end{pmatrix},$$

and a student k becomes better off. $\diamondsuit$

The condition on a priority structure in the next section characterizes those properties of f^B hold.

[6]We abbreviate R_i or $\succ_x$ just by listing schools or students in its order. For each student, only schools which are preferred to being matched with herself are listed since the ranking of the other schools does not matter in a mechanism. For example, $R_i : xP_iyP_ii$ is written as $R_i : x\ y$.

2.3 Results

We first introduce our main condition on a priority structure.

Definition 9 We say that $\succ$ is **quasi-cyclic** if there are distinct $i, j, k \in N$ and $x, y \in X$ such that

(C) $i \succ_x j \succ_y k$,

(S) there are two distinct sets $S_x, S_y \subseteq N \backslash \{i, j, k\}$ such that $|S_x| = q_x - 1$ and $|S_y| = q_y - 1$ and $\forall \ell \in S_x$, $\ell \succ_x j$ and $\forall \ell \in S_y$, $\ell \succ_y k$.

We say that $\succ$ is **strongly acyclic** if it is not quasi-cyclic.

We are ready to state a characterization theorem.

Theorem 2.1 *The following are equivalent:*

(1) f^B *is strategy-proof*

(2) f^B *is stable*

(3) $\succ$ *is strongly acyclic*

Proof : In Appendix.

When a priority structure is strongly acyclic, then no student has incentive to misrepresent her preferences and furthermore the outcome by the Boston mechanism is Pareto efficient and stable. One surprising thing is that stability of the Boston mechanism is equivalent to strategy-proofness of it in our setting.

2.3.1 Impossibility Result

However, our strong acyclicity condition is extremely strong. In other words, there is almost no room for the Boston mechanism to be stable or strategy-proof. The following proposition tells why:

Proposition 6 *If* $|N| \geq 3$, $|X| \geq 2$ *and there are two schools,* x *and* y*, such that* $q_x + q_y \leq |N| - 1$*, then any* $\succ$ *is quasi-cyclic.*

Proof : By supposition, there are two schools x and y such that $q_x + q_y \leq |N| - 1$. Fix $\succ$ arbitrary. Rename students in each ranking of a school x and y as follows:

$$\begin{array}{ll} \succ_x & j_1 \succ_x \ldots \succ_x j_n \\ \succ_y & k_1 \succ_y \ldots \succ_y k_n \end{array}$$

where $|N| = n$. Note that the same student is named differently in those two schools. We will show that $\succ$ is quasi-cyclic.

$\underline{(\mathbf{Case 1})}$ $j_n \neq k_n$. Since j_n is the least ranked at x, k_n is ranked better than j_n at x. Then there are the other $|N| - 2$ students listed at x. Choose one student from $|N| - 2$ students, and let her be i. Since $q_y \geq 1$ and by supposition, $q_x - 1 \leq |N| - 3$ so that it is possible to find $q_x - 1$ students who are ranked higher than j_n from $N \backslash \{i, j_n, k_n\}$ (possibly empty). Let them be S_x. For a school y, it is possible to choose $q_y - 1$ students who are distinct from $S_x \cup \{i, j_n, k_n\}$ because

$$\begin{aligned} |N| - \underbrace{3}_{i,j_n,k_n} - \underbrace{(q_x - 1)}_{S_x} &\geq |N| - 3 - (|N| - q_y - 2) \\ &= q_y - 1 \end{aligned}$$

Therefore, since

$$i \succ_x j_n \succ_y k_n,$$

i, j_n, k_n satisfy the condition (C), and both S_x and S_y follows the condition (S) of quasi-cyclicity.

$\underline{(\textbf{Case2})}$ $j_n = k_n$. At x, there are $|N| - 2$ students who are ranked higher than j_{n-1}. Choose i and $q_x - 1$ students distinctly among them. Let the $q_x - 1$ students be S_x. It is possible because $q_x \leq |N| - 3$. At y, since $j_n = k_n$, i and j_{n-1} are ranked higher than k_n, furthermore, there are $|N| - 3$ students other than i and j_{n-1} who are ranked higher than k_n. As similar to Case 1, it is possible to find $q_y - 1$ students who are ranked higher than k_n and distinct from i, j_{n-1} and S_x. Let them be S_y. Now $i \succ_x j_{n-1} \succ_y k_n$ so that i, j_{n-1}, k_n satisfy the condition of (C), and S_x and S_y satisfy the condition of (S) of quasi-cyclicity. $\square$

Proposition 1 is a new impossibility result: $\succ$ is strongly acyclic and hence the Boston mechanism is stable and strategy-proof only if all schools have at least $|N|/2$ seats. In practice, no school has $|N|/2$ or more seats, and therefore the Boston mechanism is always neither stable nor strategy-proof[7].

2.4 Discussion

2.4.1 On the three mechanisms

We see the relationship among three competing mechanisms, the Boston mechanism, the deferred acceptance mechanism, the top trading cycle mechanism. As we noted earlier, none of them satisfies stability, Pareto efficiency, and strategy-proofness in the same time. We compare three mechanisms which meet three properties in terms of the flexibility of a priority structure.

The deferred acceptance mechanism, say f^{DA}, is stable and strategy-proof, but not Pareto efficient. Ergin (2002) characterizes its Pareto efficiency by the following acyclic condition on $\succ$:

Definition 10 *Ergin (2002)*

$\succ$ is Ergin-cyclic if there are distinct $i, j, k \in N$ and $x, y \in X$ such that

(C) $i \succ_x j \succ_x k \succ_y i$,

(S) there are two distinct sets $S_x, S_y \subseteq N \backslash \{i, j, k\}$ such that $|S_x| = q_x - 1$ and $|S_y| = q_y - 1$ and $\forall \ell \in S_x$, $\ell \succ_x j$ and $\forall \ell \in S_y$, $\ell \succ_y i$.

[7]Many papers characterize desirable properties of the deferred acceptance mechanism or the top trading cycle mechanism by Ergin-acyclicity or Kesten-acyclicity, respectively. For instance, Ergin-acyclicity is equivalent to that the deferred acceptance mechanism is group-strategy-proof, consistent (Ergin (2002)), Nash implementable (Haeringer and Klijn (2009)) or robustly stable (Kojima (2011)).

However, in case of the boston mechanism, as Proposition 1 states, we have no chance for $\succ$ being acyclic, and therefore to explore other additional properties does not convey fruitful insights.

$\succ$ is said to be Ergin-acyclic if it is not Ergin-cyclic.

The part of his main theorem tells that the deferred acceptance mechanism is stable, Pareto efficient, and strategy-proof if and only if $\succ$ is Ergin-acyclic.

Observation 2 If $\succ$ is Ergin-cyclic, then it is quasi-cyclic.

If a priority structure is Ergin-cyclic, then $j \succ_x k \succ_y i$ and there is S_y such that $|S_y| = q_y - 1$ and for all $\ell \in S_y$, $\ell \succ_y i$. Furthermore, there is S_x such that $|S_x| = q_x - 1$ and for all $\ell \in S_x$, $\ell \succ_x j$ so $\ell \succ_x k$. Hence, it is quasi-cyclic.

Example 8 Suppose there are three students i, j, k and two schools x, y with $q_x = 1$ and $q_y = 1$. Consider the following priority structure.

$$\succ_x\colon \quad i \succ_x j \succ_x k$$

$$\succ_y\colon \quad i \succ_y j \succ_y k$$

This is Ergin-acyclic but is quasi-cyclic by the proposition 1. $\Diamond$

Another mechanism is the top trading cycle mechanism, denoted by f^{TTC}. f^{TTC} is Pareto efficient and strategy-proof but not stable. Kesten (2006) characterizes when all three conditions meet by the following acyclic condition on $\succ$:

Definition 11 *Kesten (2006)*

$\succ$ is Kesten-cyclic if there are distinct $i, j, k \in N$ and $x, y \in X$ such that

(C) $i \succ_x j \succ_x k$ and $k \succ_y i \succ_y j$

(S) there is $S_x \subseteq N \backslash \{i, j, k\}$ such that $|S_x| = q_x - 1$ and $S_x \subseteq U_x(i) \cup [U_x(j) \backslash U_y(k)]$ where $U_z(\ell) = \{m \in N | m \succ_x \ell\}$.

$\succ$ is Kesten-acyclic if it is not Kesten-cyclic.

The part of his main theorem states that the top trading cycle mechanism is stable, Pareto efficient, and strategy-proof if and only if $\succ$ is Kesten-acyclic. The relation between Ergin- and Kesten-acyclicity is the following:

Observation 3 *Kesten (2006)*

If $\succ$ is Ergin-cyclic, then it is Kesten-cyclic.

The natural question arises whether there is any relation between Kesten- and our strong acyclicity. The answer is no.

Example 9 Suppose there are three students i, j, k and two schools x, y with $q_x = 1$ and $q_y = 2$. A priority structure is as follows:

$$\succ_x: \quad i \succ_x j \succ_x k$$

$$\succ_y: \quad k \succ_y j \succ_y i$$

This is Kesten-cyclic but is strongly acyclic because the condition (S) never holds. Suppose instead $q_y = 1$ and the following priority structure.

$$\succ_x: \quad i \succ_x j \succ_x k$$

$$\succ_y: \quad i \succ_y j \succ_y k$$

Then this is Kesten-acyclic but is quasi-cyclic as in the previous example. $\Diamond$

2.4.2 Relationship to X-acyclicity

The set of stable assignments is Nash implementable in the Boston mechanism (Ergin and Sönmez (2006)), but a stable assignment is not necessarily efficient. Haeringer and Klijn (2009) propose strong X-acyclicity and prove that it is equivalent to the stable assignment induced by some Nash equilibrium is efficient[8]. On the other hand, we characterize that an efficient stable assignment is a dominant strategy equilibrium outcome if and only if $\succ$ is strongly acyclic. We see the relationship between strong X-acyclicity and our strong acyclicity.

Definition 12 *Haeringer and Klijn (2009)*

$\succ$ is weakly X-cyclic if there are distinct $i, j \in N$ and $x, y \in X$ such that

(C) $i \succ_x j$ and $j \succ_y i$

(S) There are distinct $S_x \subseteq N \backslash \{i\}$ and $S_y \subseteq N \backslash \{j\}$ such that $|S_x| = q_x - 1$ and $|S_y| = q_y - 1$ and $\forall \ell \in S_x$, $\ell \succ_x j$ and $\forall \ell \in S_y$, $\ell \succ_y i$.

$\succ$ is strongly X-acyclic if it is not weakly X-cyclic.

Similar to Kesten-acyclicity, strong X-acyclicity implies Ergin-acyclicity.

Observation 4 *Haeringer and Klijn (2009)*

If $\succ$ is Ergin-cyclic, then it is weakly X-cyclic.

[8]They also show that strong X-acyclicity ensures the set of stable assignments to be singleton.

However, there is no inclusion relationship between strong X- and our strong acyclicity, as the example 3 applies: The first priority structure is weakly X-cyclic and the second one is strongly X-acyclic[9].

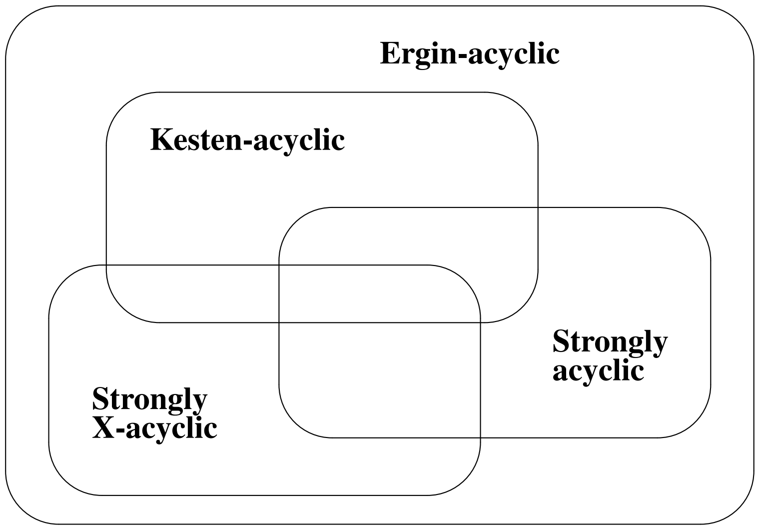

Figure 2.2: Configuration of four acyclic priority structures.

2.5 Appendix

2.5.1 Proof of Theorem 2.1

We prove $(1) \Rightarrow (3)$ and $(2) \Rightarrow (3)$, and then $(3) \Rightarrow (2)$ and $(3) \Rightarrow (1)$.

$\underline{(1) \Rightarrow (3) \& (2) \Rightarrow (3)}$ By way of contradiction. Suppose there is a quasi-cycle. Then consider the following preference profile R: for all $\ell \in S_x$, x is their top choice, for all $\ell \in S_y$,

[9]Although we use the same example for Kesten- and strong X-acyclicity, there is no inclusion relation between them.

y is their top choice,

$$\begin{aligned} R_i: &\quad x \quad i \\ R_j: &\quad x \quad y \quad j \quad , \\ R_k: &\quad y \quad k \end{aligned}$$

and for any others, $N\backslash[\{i,j,k\} \cup S_x \cup S_y]$, their top choice is not being matched. Then

$$f^B(R) = \left(\begin{array}{cccccc} i & j & k & \overbrace{\ell_1 \ldots \ell_{q_x - 1}}^{S_x} & \overbrace{\ell'_1 \ldots \ell'_{q_y - 1}}^{S_y} & \overbrace{\ell''_1 \ldots \ell''_n}^{N\backslash[\{i,j,k\}\cup S_x \cup S_y]} \\ & & & & & \\ x & j & y & x \ldots x & y \ldots y & \ell''_1 \ldots \ell''_n \end{array} \right)$$

Then $yP_j f^B_j(R)$ and $j \succ_y k$ so that a pair j and y blocks the assignment. Hence f^B is not stable.

If j misrepresents R'_j as y is his top choice. Then

$$f^B(R'_j, R_{-j}) = \left(\begin{array}{cccccc} i & j & k & \overbrace{\ell_1 \ldots \ell_{q_x - 1}}^{S_x} & \overbrace{\ell'_1 \ldots \ell'_{q_y - 1}}^{S_y} & \overbrace{\ell''_1 \ldots \ell''_n}^{N\backslash[\{i,j,k\}\cup S_x \cup S_y]} \\ & & & & & \\ x & y & k & x \ldots x & y \ldots y & \ell''_1 \ldots \ell''_n \end{array} \right)$$

Therefore,

$$y = f^B_j(R'_j, R_{-j}) P_j f^B_j(R) = j,$$

which contradicts the assumption that f^B is strategy-proof.

$\underline{(3) \Rightarrow (2)}$ Suppose f^B is unstable. Then there exists R such that $f^B(R)$ is unstable. Let $f^B(R)$ be μ. We will show that there is a quasi-cycle.

Since μ is unstable, Pareto efficiency of f^B implies that there is a blocking pair j and

y such that $yP_j\mu(j)$ and $j \succ_y \ell$ for some $\ell \in \mu^{-1}(y)$. Take k such that $\ell \succsim_y k$ for all $\ell \in \mu^{-1}(y)$. Then there are $q_y - 1$ students who have higher priority than k in $\mu^{-1}(y)$. Let them be S_y.

Since j is not assigned y under R, when j applies to y, there are already q_y students accepted at y and hence it is not the first step. Because j applies to y under R, j should be rejected in all the previous steps, especially in the first step. In the first step, j applies to a school different from y, say x, and j is rejected because there are at least q_x students who are of higher priority than j. Consider the top q_x higher ranked students within the applicants at x in the first step. Let one of them be i and the other of them be S_x. Note that they are accepted in the first step, so their assignment is x at μ.

Hence, i, j, k, S_x and S_y are distinct and, together with x and y, they consist a quasi-cycle.

$\underline{(3) \Rightarrow (1)}$ Suppose a priority structure is strongly acyclic, but f^B is not strategy-proof. Then there are k, R and R'_k such that

$$f^B_k(R'_k, R_{-k})P_k f^B_k(R).$$

Let $f^B_k(R'_k, R_{-k})$ and $f^B(R)$ be y and μ, respectively. From $(3) \Rightarrow (2)$, f^B is stable so that for all $\ell \in \mu^{-1}(y)$, $\ell \succ_y k$ under R.

Under (R'_k, R_{-k}), k is assigned y, which implies, together with the procedure of the Boston mechanism, that there is a step $t > 1$ such that a school y is not filled until step t under R. Otherwise, since R_{-k} is the same across R and (R'_k, R_{-k}), if all $\mu^{-1}(y)$ apply to y in the first step, then it contradicts that k is assigned y under (R'_k, R_{-k}). Note that applications of $N \backslash \{k\}$ in the first step is exactly the same both under R and (R'_k, R_{-k}).

Let j be the student who is assigned y under R and does not apply to y in the first step. Let $\mu^{-1}(y) \backslash \{j\}$ be S_y. Note that $j \succ_y k$ and $\forall \ell \in S_y$, $\ell \succ_y k$ by stability of $f^B(R)$. Then under R, j should be rejected by some school, say x, in the first step. It is because there are at least q_x students who is of higher priority than j and apply to x in the first step. Choose the top q_x higher ranked students within them, and split those q_x students into one student and the others, and let each of them be i and S_x. Clearly, $i \succ_x j$ and $|S_x| = q_x - 1$ and $\forall \ell \in S_x$, $\ell \succ_x j$. Overall, i, j, k, S_x and S_y are all distinct, and together with x and y, they consist a quasi-cycle. $\square$

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Chapter 3

Dominant Strategy Implementation of Stable Rules

3.1 Introduction

The priority-based assignment problem is the allocation problem in which agents are allocated at most one indivisible object. There are types on each object, and each object type is endowed with a strict priority ranking over subsets of the set of agents.[1] Each agent has strict preferences over object types and being unassigned. School Choice is the best example.[2]

The deferred acceptance algorithm introduced by Gale and Shapley (1962) is the most widely used method for obtaining the stable assignment in priority-based assignment problems. An assignment is *stable* if it is not blocked by any individual agent or any agent-object pair. Roth and Sotomayor (1990) show that the deferred acceptance algorithm finds

[1] Coarse priorities may capture real life well, but since there is no strategy-proof and constrained efficient stable rule under coarse priorities, we restrict our attention to the class of strict priorities. For more reference, see Abudulkdiroğlu et al. (2009) and Erdil and Ergin (2008).

[2] In case of school choice, if a school has two seats, then each seat is thought of as an object, and the school is as their type.

the *agent-optimal stable assignment* which is a stable assignment that any agent weakly prefers to any other stable assignment. We consider the class of priorities called *substitutable*, first introduced by Kelso and Crawford (1982), which ensures the existence of a stable assignment in our model.[3]

This paper discusses whether the agent-optimal stable assignment generated by some algorithm is implemented in dominant strategies in a preference revelation game. A (single-valued) algorithm is said to be implementable in dominant strategies if the outcome that the algorithm generates is the unique dominant strategy equilibrium outcome in the preference revelation game. Our main result is that when the priority structure satisfies substitutability and *cardinal monotonicity* the deferred acceptance algorithm is dominant strategy implementable, and furthermore it is the unique dominant strategy implementable stable algorithm.

As in a prior work by Kumano and Watabe (2011), they see how untruthful dominant strategies in the preference revelation game induced by the deferred acceptance algorithm look under substitutable and quota-filling priority structure that is a strict subclass of our model. They find that the first k-th ranked objects in any untruthful dominant strategy coincide with those in the truthful dominant strategy, where the k-th object is an assignment under true preferences. From that fact, they directly show dominant strategy implementability of the deferred acceptance algorithm.

Our results build on Mizukami and Wakayama (2007) and Saijo et al. (2007), who prove that dominant strategy implementability is equivalent to the combination of *strategy-*

[3] See Kelso and Crawford (1982) and Hatfield and Milgrom (2005).

proofness and *weak nonbossiness*. Strategy-proofness says that truthful revelation is a dominant strategy. Hatfield and Milgrom (2005) show that the deferred acceptance algorithm is strategy-proof when the priority structure satisfies substitutability and cardinal monotonicity.[4] In their paper, cardinal monotonicity is called the law of aggregate demand. In addition, Alcalde and Barbera (1994) and Sakai (2010) prove that the deferred acceptance algorithm is the unique strategy-proof algorithm among stable ones. However, Dasgupta et al. (1979) and Repullo (1985) provide examples that strategy-proofness does not imply dominant strategy implementation. The bottom line is, strategy-proofness merely says that "truthtelling is a dominant strategy" for every agent, and it is not enough to guarantee the uniqueness of the dominant strategy equilibrium outcome. If there were untruthful dominant strategies for an agent (we will provide an example in which there is an untruthful equilibrium in the preference revelation game induced by the deferred acceptance algorithm) and he follows his untruthful dominant strategy instead of his truthful one, then it may affect assignments of others even though there is no effect on his own assignment. Such an unintended action may make someone worse off in the sense that the corresponding outcome might end up with a different assignment than one obtained by truthful revelation.

As noted earlier, strategy-proofness implies dominant strategy implementation provided an additional condition, weak nonbossiness, holds. Standard nonbossiness, which is introduced by Satterthwaite and Sonnenschein (1981), says that any agent cannot change the entire assignment unless he changes his own assignment. Weak nonbossiness modifies

[4]Dubins and Freedman (1981) and Roth (1982) show that it is a dominant strategy for agents to list their preferred matchings in the deferred acceptance algorithm in a one-to-one assignment problem. Hatfield and Milgrom (2005) generalize this result to a many-to-one problem.

this by requiring that if an agent does not change his own assignment in any cases then the entire assignment stays the same. Kojima (2010) shows that stability and nonbossiness are incompatible; since the deferred acceptance algorithm is stable, this implies it violates standard nonbossiness. Our main contribution is that, nevertheless, the deferred acceptance algorithm is weakly nonbossy for every substitutable priority structure. Hence, we conclude that the deferred acceptance algorithm is dominant strategy implementable, and therefore it is immune to manipulation. This observation supports the use of the deferred acceptance algorithm as an appropriate candidate among stable assignment procedures.

3.2 Model

We denote by A the finite set of indivisible object types. Let $q = (q_a)_{a \in A}$, where $q_a \in \mathbb{Z}_{++}$, be the number of available objects of type a. Denote by N the finite set of agents. A preference profile is a vector of linear orders $R = (R_i)_{i \in N}$, where R_i denotes the preference of agent i defined over $\mathcal{X}_i = A \cup \{\emptyset\}$. The symbol $\emptyset$ stands for being assigned to oneself. The asymmetric part of R_i is denoted by P_i. An object a is *acceptable* to agent i if $a \, P_i \, \emptyset$. Let $\mathcal{R} = \prod_{i \in N} \mathcal{R}_i$ be the set of all preference profiles.

A *priority structure* is a vector of linear orders $\succeq = (\succeq_a)_{a \in A}$, where $\succeq_a$ is defined over the power set of N. The asymmetric part of $\succeq_a$ is denoted by $\succ_a$. For each object a, define $\mathcal{X}_a = \{S \subseteq N \mid \#S \leqslant q_a\}$. For each object a, denote a *choice function* of C_a of the power set of N into $\mathcal{X}_a$ such that for every $S \subseteq N$, $C_a(S) \subseteq S$ and $C_a(S) \succeq_a T$ for every $T \subseteq S$ with $T \in \mathcal{X}_a$. A choice function $C_a(\cdot)$ is *substitutable* if $C_a(T) \cap S \subseteq C_a(S)$ for every pair (S, T) of subsets of N with $S \subseteq T$.[5] A priority structure is substitutable is every object has a substitutable choice function. Substitutability is discussed in a labor market model by Kelso and Crawford (1982). This condition simply says that if the set of agents expands and an agent is admitted by an object from a larger set of agents, then he must be admitted by the same object from any subset of agents including him. The following notion is discussed in Alkan (2001) and Alkan and Gale (2003). A choice function $C_a(\cdot)$ is *cardinally monotonic* if $\#C_a(S) \leqslant \#C_a(T)$ for every pair (S, T) of subsets of N with $S \subseteq T$. A priority structure is cardinally monotonic if every object has a cardinally monotonic choice function.

3.2.1 The Deferred Acceptance Algorithm and its Strategy-Proofness

An *assignment* is a function $\mu : N \to A \cup \{\emptyset\}$ satisfying: (i) for every agent i, $\mu(i) \in \mathcal{X}_i$ and (ii) for every object a, $\#\{i \in N \mid \mu(i) = a\} \leqslant q_a$. We denote the set of assignments by X. A *rule* g is a function of $\mathcal{R}$ into X. If $g(R) = \mu$ for some $R \in \mathcal{R}$, then denote $g_i(R) = \mu(i)$ for every agent i. An assignment μ is *stable for* R if it satisfies the following conditions: (i) for every agent i, $\mu(i)\, R_i\, \emptyset$ and (ii) there does not exist $(i, a) \in N \times A$ such that $a\, P_i\, \mu(i)$ and $i \in C_a(\{k \in N \mid \mu(k) = a\} \cup \{i\})$.[6] We denote the set of stable assignments for R by $\varphi_S(R)$. A rule g is *stable* if $g(R) \in \varphi_S(R)$ for every $R \in \mathcal{R}$. The relation φ_S of $\mathcal{R}$ into X is referred to as the *stable correspondence*. In our model, the stable correspondence is non-empty valued for every substitutable priority structure. A rule g is *strategy-proof* if for

[5] A stronger notion of substitutability, called *responsiveness*, is commonly used in the existing literature. We focus on substitutable priorities because priorities may be non-responsive but substitutable in applications.

[6] Condition (i) is the *individual rationality* and condition (ii) is the *pairwise stability*.

every $R \in \mathcal{R}$, every agent i and every $R'_i \in \mathcal{R}_i$, we have $g_i(R) R_i g_i(R_{-i}, R'_i)$.

Gale and Shapley (1962) propose the following assignment procedure, called the *deferred acceptance algorithm*. At the first step, each agent applies to his most preferred acceptable object. The set of agents applying to object a at the first step is N_a^1. Object a tentatively accepts $C_a(N_a^1)$ and rejects the remaining. At the rth step, each agent who was rejected at step $r-1$ applies to his next preferred acceptable object. The set of agents applying to object a at step r is N_a^r. Object a tentatively accepts $C_a(C_a(N_a^{r-1}) \cup N_a^r)$ and rejects the remaining. The algorithm terminates when every agent is held tentatively by some object or has been rejected by every object that is acceptable for him. If an agent is tentatively held by an object at the last step, he is assigned that object. Otherwise he is assigned nothing.[7] It is known that under any substitutable priority structure, the deferred acceptance algorithm produces a unique stable assignment Pareto dominating any other stable assignment, called the *agent-optimal stable assignment*. We denote by f the deferred acceptance algorithm. Abudlkadiroğlu (2005) shows that substitutability of choice functions itself is not sufficient for the existence of a strategy-proof stable rule. Hatfield and Milgrom (2005) show that substitutability coupled with cardinal monotonicity (the law of aggregate demand) are sufficient for strategy-proofness of the deferred acceptance algorithm in our setting. Therefore, the deferred acceptance algorithm is strategy-proof in the paper.

Remark 1 For every substitutable and cardinally monotonic priority structure, the deferred

[7] The above explanation can be found in Roth and Sotomayor (1990).

acceptance algorithm is strategy-proof.

3.2.2 Weak Nonbossiness

The second desirable property of rules is *nonbossiness*, discussed by Satterthwaite and Sonnenschein (1981). Intuitively, non-bossiness implies that no agent can change the assignments of others without changing his own assignment.[8] Kojima (2010) shows that nonbossiness and stability are incompatible. Our finding is that a weaker notion of nonbossiness and stability are compatible as long as the agent-optimal stable assignment is well-defined.[9]

Definition 13 A rule g is *weakly nonbossy* if for every agent i, every $R \in \mathcal{R}$, and every $R'_i \in \mathcal{R}_i$, if $g_i(R''_{-i}, R_i) = g_i(R''_{-i}, R'_i)$ for every $R''_{-i} \in \mathcal{R}_{-i}$, then $g(R) = g(R_{-i}, R'_i)$.

Theorem 3.1 *For every substitutable priority structure, the deferred acceptance algorithm is weakly nonbossy.*

Proof: Consider any preference profile $R \in \mathcal{R}$. Consider any agent i and $R'_i \in \mathcal{R}_i$. Suppose that $f_i(R''_{-i}, R_i) = f_i(R''_{-i}, R'_i)$ for every $R''_{-i} \in \mathcal{R}_{-i}$. We shall show that $f(R) = f(R_{-i}, R'_i)$.

We introduce some notation. Let $\mathcal{Y}_i = \{b \in \mathcal{X}_i \mid i \in C_b(\{i\})\}$. For each $a \in \mathcal{X}_i$ and each $\tilde{R}_i \in \mathcal{R}_i$, denote by $U_i(a, \tilde{R}_i) = \{b \in \mathcal{Y}_i \mid b \, \tilde{R}_i \, a\}$ the upper contour set of agent i at

[8]Formally, a rule g is nonbossy if for every agent i, every $R \in \mathcal{R}$, and every $R'_i \in \mathcal{R}_i$, if $g_i(R) = g_i(R_{-i}, R'_i)$, then $g(R) = g(R_{-i}, R'_i)$.

[9]Mizukami and Wakayama (2007) and Saijo et al. (2007) also discuss weak nonbossiness. Mizukami and Wakayama (2007) call this *quasi-strong nonbossiness*.

a under $\tilde{R}_i$ restricted to $\mathcal{Y}_i$. Since R_i is a linear order over $\mathcal{X}_i$, there exists an element of $R^*_{-i} \in \mathcal{R}_{-i}$ such that $f_i(R''_{-i}, R_i)\, R_i\, f_i(R^*_{-i}, R_i)$ for every $R''_{-i} \in \mathcal{R}_{-i}$. In other words, the preferences R^*_{-i} maximize the size of the upper contour set of agent i at $f_i(R''_{-i}, R_i)$ under R_i with respect to $R''_{-i} \in \mathcal{R}_{-i}$. By our hypothesis, $f_i(R^*_{-i}, R_i) = f_i(R^*_{-i}, R'_i)$. We denote that object by a.

Firstly, note that under any substitutable priority structure, every object $a \in \mathcal{X}_i \setminus \mathcal{Y}_i$ is *redundant* for agent i in the sense that he has no chance to be held tentatively by an object in $\mathcal{X}_i \setminus \mathcal{Y}_i$ at any step under the deferred acceptance algorithm.[10] This fact also yields that including any object in $\mathcal{X}_i \setminus \mathcal{Y}_i$ in R_i and R'_i does not affect the assignments of other agents.[11]

Case 1: $U_i(a, R_i) = U_i(a, R'_i)$, and the rank orders under R_i and R'_i are the same within the two sets.

It suffices to show that $U_i(f_i(R), R_i) = U_i(f_i(R_{-i}, R'_i), R'_i)$. By the construction of R^*_{-i}, $f_i(R) R_i f_i(R^*_{-i}, R_i) = a$, that is, $f_i(R) \in U_i(a, R_i)$. The hypothesis in this case yields that $f_i(R) \in U_i(a, R'_i)$. Then, $f_i(R_{-i}, R'_i) \in U_i(a, R'_i)$. We have seen that the two preferences R_i and R'_i have the same rank order, and result in the same object $f_i(R) = f_i(R_{-i}, R'_i)$. Therefore, due to the definition of the deferred acceptance algorithm, both R_i and R'_i produce the same entire assignment, that is, $f(R) = f(R_{-i}, R'_i)$. This establishes

[10] We shall show that for every substitutable priority structure, if $i \notin C_a(\{i\})$, then $i \notin C_a(S \cup \{i\})$ for every $S \subseteq N$. Consider any pair $(i, a) \in N \times A$ such that $i \notin C_a(\{i\})$. Suppose, by way of contradiction, that $i \in C_a(S \cup \{i\})$ for some $S \subseteq N$. Since the priority structure is substitutable, it follows from that $C_a(S \cup \{i\}) \cap \{i\} \subseteq C_a(\{i\})$, which implies that $i \in C_a(\{i\})$, a contradiction.

[11] An intuitive explanation is as follows. Consider any object $a \in \mathcal{X}_i \setminus \mathcal{Y}_i$ and assume that $i \in N^t_a$. Divide step t into two substeps: only agent i applies to object a first and then the remaining agents $N^t_a \setminus \{i\}$ apply to object a. At any rate, the deferred acceptance algorithm produces the same assignment.

the case.

Case 2: Otherwise.

Note that $U_i(a, R_i)$ is non-empty because $a \in U_i(a, R_i)$. If $U_i(a, R_i)$ is a proper subset of $U_i(a, R'_i)$, then set $R^a_j : b \, P^a_j \, \emptyset$ for every agent $j \in \{\mu_b \mid b \in U_i(a, R_i)\}$, and $R^a_j : \emptyset$ for every agent $j \in N \setminus \big(\{\mu_b \mid b \in U_i(a, R_i)\} \cup \{i\}\big)$. In this case, we have $f_i(R^a_{-i}, R_i) = a$ and $f_i(R^a_{-i}, R'_i)$ is the top object in $U_i(a, R'_i) \setminus U_i(a, R_i)$ with respect to R'_i, distinct from a. This is a contradiction.

It remains to consider the case that $U_i(a, R_i)$ is not a proper subset of $U_i(a, R'_i)$[12]. Denote $U_i(a, R_i) = \{b^1, \cdots, b^k, a\}$, where $b^1 \, P_i \, \cdots \, P_i \, b^k \, P_i \, a$. Then we take the smallest index $\ell \in \{1, \cdots, k\}$ such that $U_i(b^\ell, R_i) \neq U_i(b^\ell, R'_i)$. For every $b \in U_i(a, R_i)$ such that $b \, P_i \, b^\ell$, set $R^a_j : b \, P^a_j \, \emptyset$ for every agent $j \in \mu_b$. For every agent $j \in N \setminus \big(\{\mu_b \mid b \in U_i(a, R_i) \text{ and } b \, P_i \, b^\ell\} \cup \{i\}\big)$, set $R^a_j : \emptyset$. Then, since $i \notin C_b(\mu_b \cup \{i\})$ for every $b \in U_i(a, R_i)$ by stability for (R^*_{-i}, R_i), it follows from the fact $b^\ell \in \mathcal{Y}_i$ that $f_i(R^a_{-i}, R_i) = b^\ell$. On the other hand, it is the case that $f_i(R^a_{-i}, R'_i)$ is the ℓth element of $U_i(a, R'_i)$ from the top with respect to R'_i, distinct from b^ℓ. This is a contradiction. This establishes the case.

Cases 1 and 2 establish the theorem. $\square$

3.3 Preference Revelation Game

The mechanism designer aims to achieve socially desirable outcomes but does not know preferences that are private information of the agents. The task of the mechanism designer

[12] Note that since $a \in U_i(a, R'_i)$, there must be $b \in U_i(a, R_i)$, other than a. Otherwise, it either contradicts that $U_i(a, R_i)$ is not a proper subset or reduces to case 1.

is to construct a procedure independent of private information in order to achieve the prescribed desirable assignments. An ordered pair (M, h) is called a *mechanism* if h is a function of M into X, and $M = \prod_{i \in I} M_i$, where M_i is a non-empty set for each agent i. The Cartesian product M is called the *strategy space*. Each element $m \in M$ is called a *strategy profile*. A triplet (M, h, R) is called a *game* if (M, h) is a mechanism and $R \in \mathcal{R}$. We restrict our attention to the class of mechanisms, where $M_i = \mathcal{R}_i$ for every agent i. The resulting games are referred to as *preference revelation games*.

An element $m_i \in M_i$ is a *dominant strategy* for agent i of (M, h) at R_i if for every $m_{-i} \in M_{-i}$ and every $m'_i \in M_i$, $h_i(m_{-i}, m_i) R_i h_i(m_{-i}, m'_i)$. Denote by $\mathcal{D}^i_{(M,h)}(R_i)$ the set of dominant strategies for agent i of (M, h) at R_i. Let $\mathcal{D}_{(M,h)}(R) = \prod_{i \in N} \mathcal{D}^i_{(M,h)}(R_i)$ the set of dominant strategy equilibria of (M, h) at R.

Given a mechanism (M, h), we want to identify the composite correspondence $h \circ \mathcal{D}_{(M,h)}$ of $\mathcal{R}$ into M as the actual market outcomes, where the solution concept is a dominant strategy equilibrium:

$$(h \circ \mathcal{D}_{(M,h)})(R) = h(\mathcal{D}_{(M,h)}(R)) = \{h(m) \mid m \in \mathcal{D}_{(M,h)}(R)\}.$$

The following figure depicts this formulation.

Definition 14 A mechanism (M, h) *implements the relation φ of $\mathcal{R}$ into X in dominant strategy equilibria* if $h(\mathcal{D}_{(M,h)}(R)) = \varphi(R)$ for every $R \in \mathcal{R}$.

In particular, for each rule g, the ordered pair $(\mathcal{R}, g)$ is called the *associated direct mech-*

$$\begin{tikzcd}
R \in \mathcal{R} \arrow[rr, twoheadrightarrow, "\varphi(R) = (h \circ \mathcal{D}_{(M,h)})(R)"] \arrow[dr, "\mathcal{D}_{(M,h)}(R)"'] & & X \\
& m \in M \arrow[ur, "h(m)"'] &
\end{tikzcd}$$

Figure 3.1: Implementation of φ in dominant strategy equilibria

anism. Given a preference profile $R \in \mathcal{R}$, the ordered pair $(\mathcal{R}, g)$ induces a preference revelation game. If a rule g is dominant strategy implementable by $(\mathcal{R}, g)$, then we say that g is dominant strategy implementable by the associated direct mechanism. It is well-known that the concept of dominant strategy implementation does not preclude agents having untruthful dominant strategies. There is no need to rule out untruthful dominant strategies as long as those lead to the same assignment as the truthful one.

3.3.1 Multiple Equilibria in Dominant Strategies

Under any substitutable and cardinally monotonic priority structure, truth-telling is merely a dominant strategy equilibrium of the preference revelation game induced by the deferred acceptance algorithm in our setting, that is, $R \in \mathcal{D}_{(\mathcal{R},f)}(R)$. Hence $f(R) \in f(\mathcal{D}_{(\mathcal{R},f)}(R))$. The question raised here is whether truth-telling is the unique dominant strategy equilibrium. The answer is *negative*. Long ago, the literature on implementation theory argued that there is nothing that guarantees that agents always choose the truth-telling dominant strategies when they have alternative untruthful dominant strategies. Suppose that there are two agents, $N = \{1, 2\}$, and two objects, $A = \{a, b\}$. The true preferences and the priority structure are the following:

$R = (R_1, R_2)$
R_1: $a\, P_1\, b\, P_1\, \emptyset$
R_2: $\emptyset\, P_2\, a\, P_2\, b$

$\succeq = (\succeq_a, \succeq_b)$
$\succeq_a$: $\{1\} \succ_a \{2\} \succ_a \emptyset$
$\succeq_b$: $\{1\} \succ_b \{2\} \succ_b \emptyset$

Each agent has 5 possible strategies. We put $u_i(m_1, m_2) = r$ if $f_i(m_1, m_2)$ is the rth ranked assignment with respect to the true preference R_i. The payoffs $(u_1(m_1, m_2), u_2(m_1, m_2))$ of the preference revelation game induced by the deferred acceptance algorithm is given by the following:

	M_2				
M_1	$aP_2bP_2\emptyset$	$aP_2\emptyset$	$bP_2aP_2\emptyset$	$bP_2\emptyset$	$\emptyset$
$aP_1bP_1\emptyset$	$(2,0)$	$(2,2)$	$(2,0)$	$(2,0)$	$(2,2)$
$aP_1\emptyset$	$(2,0)$	$(2,2)$	$(2,0)$	$(2,0)$	$(2,2)$
$bP_1aP_1\emptyset$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,2)$	$(1,2)$
$bP_1\emptyset$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,2)$	$(1,2)$
$\emptyset$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,2)$

Figure 3.2: Preference revelation game.

Agent 2 does not care how he ranks unacceptable objects.[13] On the other hand, agent 1

[13]Precisely speaking, agent 2 has two dominant strategies: $\emptyset\ P_2\ a\ P_2\ b$ and $\emptyset\ P_2\ b\ P_2\ a$.

has two non-trivial dominant strategies at R_1, $a\,P_1\,b\,P_1\,\emptyset$ and $a\,P_1\,\emptyset$. In addition to the true preferences, there is another dominant strategy equilibrium. One remark is that even though there are multiple dominant strategy equilibria, any dominant strategy equilibrium leads to the unique stable assignment with respect to the true preferences in this example. In the next subsection, we will prove this observation. However, it must be emphasized that, in general, the implementability of strategy-proof rules in dominant strategies is not straightforward because it might end up with $f(R') \neq f(R)$ for some untruthful equilibrium $R' \in \mathcal{D}_{(\mathcal{R},f)}(R)$.[14]

3.3.2 Dominant Strategy Implementation of the Deferred Acceptance Algorithm

Restricting the class of mechanisms, Mizukami and Wakayama (2007) and Saijo et al. (2007) show that any rule or any social choice function is dominant strategy implementable by the associated direct mechanism if and only if it is strategy-proof and weakly nonbossy.

Remark 2 The deferred acceptance algorithm is dominant strategy implementable by the associated direct mechanism if and only if it is strategy-proof and weakly nonbossy.

We obtain the following observation.

Corollary 3 *For every substitutable and cardinally monotonic priority structure, the deferred acceptance algorithm f is dominant strategy implementable by the associated direct*

[14] It is possible to construct a strategy-proof rule that is not dominant strategy implementable by the associated direct mechanism in the context of social choice. An example is available upon request.

mechanism $(\mathcal{R}, f)$.

Proof: Immediate from Remarks 1 and 2, and Theorem 1. $\square$

Proposition 7 *For every substitutable priority structure, the deferred acceptance algorithm is dominant strategy implementable if and only if it is strategy-proof.*

Proof: Immediate from the fact that weak nonbossiness is automatically satisfied for every substitutable priority structure. $\square$

Let us go back to Figure 3.2. The possibility of multiple equilibria implies that the equilibrium correspondence $\mathcal{D}_{(\mathcal{R},f)}$ is *set-valued*. Strategy-proofness merely guarantees that the equilibrium correspondence is *non-empty-valued*. We have verified that the composite $f \circ \mathcal{D}_{(\mathcal{R},f)}$ is eventually *single-valued*, which is identical with the deferred acceptance algorithm itself. If the society wants to achieve the agent-optimal stable assignment with respect to the true preferences, it suffices to use the deferred acceptance algorithm as the actual matching procedure.

3.3.3 Uniqueness of Dominant Strategy Implementable Stable Rule

Among stable rules, Alcalde and Barbera (1994) show that only the (agent-proposing) deferred acceptance algorithm is strategy-proof for a special case of substitutable and cardinally monotonic priority structure. Sakai (2010) shows their result for substitutable and cardinally monotonic priority structures.

Remark 3 For every substitutable and cardinally monotonic priority structure, the deferred acceptance algorithm is the unique strategy-proof stable rule.

This observation, together with the fact that strategy-proofness is a necessary condition for dominant strategy implementation, yields the following.

Corollary 4 *For every substitutable and cardinally monotonic priority structure, the deferred acceptance algorithm is the unique dominant strategy implementable stable rule.*

Proof: By Remark 3, any other stable rules are not dominant strategy implementable by any mechanisms by Theorem 4.1.1 in Dasgupta et al (1979). The assertion is immediate from Corollary 1. □

3.4 Discussion

The literature has paid attention only to strategy-proofness or *dominant strategy incentive compatibility* of the deferred acceptance algorithm. Hatfield and Milgrom (2005) show that the combination of substitutability and cardinal monotonicity (the law of aggregate demand) is sufficient and almost necessary for strategy-proofness of the deferred acceptance algorithm. We can interpret strategy-proofness as the existence of a dominant strategy equilibrium in the preference revelation game induced by the deferred acceptance algorithm. We showed that it is possible to identify the set of equilibrium outcomes, without imposing any further restriction to priority structures. There is no need to eliminate untruthful equilibria in dominant strategies to implement the agent-optimal stable matching with respect to the true preferences.

In order to obtain our result, priorities of objects cannot be private information. In the context of two-sided matching problems, in which both preferences of agents and priorities of objects are private information, strategy-proof stable mechanisms do not exist due to the results by Roth (1982) and Sönmez (1999). Alcalde and Barbera (1994) study some preference domain restrictions to guarantee the existence of strategy-proof stable mechanism.

We argued the existence of multiple equilibria in dominant strategies. The literature on implementation theory often drops the requirement that agents adopt dominant strategies, as it seems natural that we need to check all equilibrium outcomes in Nash equilibria. Haeringer and Klijn (2009) show that a further restriction on priorities, the so-called *Ergin-acyclicity*, is necessary and sufficient for Nash implementation of the stable correspondence by the deferred acceptance algorithm in a special case of our model. Finally, Sönmez (1999) and Kara and Sönmez (1997) show that it is possible to implement the stable correspondence in Nash equilibria by some indirect mechanism.

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