Twin Peaks: Expressive Voting and Soccer Hooliganism☆

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Abstract

We introduce a model of group behavior that combines expressive participation with strategic participation. Building on the idea that expressive voting in elections is much like rooting for a sports team we give applications to both sporting events and elections. In our model there is an expressive externality: we generally prefer watching and cheering a match at the pub with our friends to watching at home alone on television. We show that this results in the possibility of "tipping" - that participation may jump up when the externality becomes strong enough. We examine the implications for pricing by sports teams and for voter turnout. In particular we show under certain circumstances tipping may lead to twin peaked voter turnout in which low and high turnout are relatively more likely than intermediate turnout levels. We examine this empirically for both US Presidential and UK General elections and find substantial evidence for tipping in the UK.

Keywords: expressive voting, sports, peer punishment, group

☆First Version: June 2, 2013. We owe a particular debt of gratitude to Andrea Mattozzi. Financial support: MIUR PRIN 2017 n. 2017H5KPLL_01.

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1. Introduction

There are two models widely used to address the "paradox of voting" - the fact that the probability of being pivotal in large election is too small to justify individuals turning out to vote. One is the "ethical" voter model and the more recent incarnation as the peer pressure model,[4] the other is the model of expressive voting.[5] The peer pressure model incorporates a simple and not terribly satisfying form of expressive voting: it allows for a fixed fraction of committed voters. Here we integrate a more satisfactory model of expressive participation into the peer pressure model. Our basic premise is that (as others have noted) expression in voting is much like rooting for a sports team. Of particular importance in our view is that there is an externality. That is, there is a social component to expression: we generally prefer watching and cheering a match at the pub with our friends to watching at home alone on television.

We develop a model that combines peer pressure with expression. A key conclusion that follows from this model is the possibility of "tipping." That is, in peer pressure models the marginal cost of inducing additional participation is generally positive. When there is an expressive externality it may become negative, meaning that it is optimal for everyone to participate. Whether or not this is the case is endogenous, and we give two applications. The first is to sporting events where we study how ticket pricing should be designed to exploit the expressive externality, and in particular show that it may be desirable to keep ticket prices low to encourage the externality. This leads to counter-intuitive pricing conclusions for sports teams: for example, increasing stadium size may well cause tickets prices to rise.

Our second application is to voting where we demonstrate how voter turnout can change discontinuously even though the underlying stakes are drawn from a continuous distribution. This leads to a prediction that turnout should be twin-peaked. We conduct an empirical analysis to see whether this is in fact true. We gather evidence from the US and UK and show that there is substantial evidence that in the UK voter turnout does indeed have two peaks.

The core of our model is one of collective provision of incentives to participate. We know from the work of Ostrom (1990) and her successors how this can be achieved: groups can self-organize to overcome the free rider problem and provide public goods (such as participation) through peer monitoring and social punishments such as ostracism. In the context of sports the so-called "soccer hooligans" provide this service of disciplining those who fail to participate.

[4] See Levine and Mattozzi (2020).

[5] Recent and significant empirical evidence in favor of the long-standing literature on expressive voting can be found in Pons and Tricaud (2018).

Formal theories of peer enforcement originate in the work of Kandori (1992) on repeated games with many players and have been specialized to the study of organizations. The basic idea is that groups choose norms consisting of a target behavior for the group members and individual penalties for failing to meet the target; these norms are endogenously chosen in order to advance group interests. Specifically the group designs a mechanism to promote group interests subject to incentive constraints for individual group members, and it provides incentives in the form of punishments for group members who fail to adhere to the norm.[6]

2. The Model

The natural context for studying expression is that of a social network, a simple model of which is the following. A group k is composed of N_k members. Each group member faces a participation decision: to root for the team or not to root for the team, and if the group wins each group member has a utility v_k. The same goes for other participation decisions such as voting. We assume that members of group k derive utility $h_k > 0$ from expression, that is, participation. This is standard. In addition there is an externality: you benefit from the participation of your neighbors. Let us denote the strength of this externality by a non-negative parameter λ, which reflects both the number of neighbors whose participation you benefit from and how much you benefit from each. Suppose in fact that the fraction of the group that participates is ϕ_k and that each group member i independently draws a type y_i uniformly distributed on $[0, 1]$ with cost of participation $c(y_i) = c_0 + y_i$. Suppose that in addition to the direct benefit participation may result in the production of $p_k(\phi_k)$ of a public good with value v_k where $p_k(\phi_k)$ is non-decreasing. Then a non-participant receives utility

$$p_k(\phi_k)v_k + \lambda\phi_k h_k \tag{2.1}$$

while a participant receives utility

$$p_k(\phi_k)v_k + (1 + \lambda\phi_k)h_k - (c_0 + y_i) = p_k(\phi_k)v_k + \lambda\phi_k h_k - (c_0 + y_i - h_k). \tag{2.2}$$

Because of the expressive externality and possibly due to the dependence of $p_k(\phi_k)$ on ϕ_k the group has a collective interest in participation. It is useful to start by defining the *committed* members as those who have a net participation cost that is negative, that is, $c_0 - h_k + y_i \leq 0$. These members need no encouragement to participate and the fraction

[6] See for example Levine and Modica (2016) and Dutta, Levine and Modica (2021b).

who are committed is given by

$$\underline{\varphi} = \begin{cases} 0 & \text{if } h_k - c_0 < 0 \\ h_k - c_0 & \text{if } 0 \leq h_k - c_0 \leq 1 \, . \\ 1 & \text{if } h_k - c_0 > 1 \, . \end{cases} \tag{2.3}$$

Hereafter we will assume that $h_k - c_0 \leq 1$ to avoid the uninteresting case in which all group members are committed.

The group may also self-organize to encourage the participation of non-committed members through monitoring and punishment. It does so by establishing a social norm whereby members with relatively low costs are expected to participate. Specifically, a social norm is specified as a threshold $\varphi_k \in [0, 1]$ for participation: those types with $y_i < \varphi_k$ are expected to participate and those with $y_i > \varphi_k$ are not. If the social norm φ_k is followed, the expected fraction of the group that will participate is φ_k and in a large group we may assume that since we are averaging over many independent draws the realized participation is equal to the expected value. In particular, we may identify the social norm φ_k with the fraction that participates ϕ_k. The action of a member, whether she has participated or not, is observable by everyone, but for those who did not participate there is only a noisy signal of their type y_i. The signal is a binary signal $z_i \in \{0, 1\}$, where 0 means "good, followed the social norm" and 1 means "bad, did not follow the social norm," and it works as follows. If the social norm was violated, that is the member did not participate but $y_i < \varphi_k$, the bad signal is generated for sure; if i did not participate but $y_i > \varphi_k$ so that she did in fact follow the norm, there is nevertheless a chance π of the bad signal where π is a measure of the noise of the signal. If a member's behavior generates a bad signal she suffers an endogenous punishment P_k that the group applies through some form of ostracism. For the bulk of the paper we will focus on the benchmark case in which the net marginal cost of participation is constant: this corresponds to $\pi = 1/2$.

We model the behavior of the group as a mechanism design problem: to choose an incentive compatible φ_k, P_k to maximize the common *ex ante* utility of group members.

3. Cost Minimization

As a first step in solving the mechanism design problem we consider that maximizing utility requires that the cost of achieving a particular participation target φ_k be minimized. In other words, the group must choose a punishment scheme P_k so that compliance with the social norm is incentive compatible. If everyone complies with the social norm bad signals are still generated with probability π so πP_k is a cost to the group of inducing compliance. We define the *total cost* $C(\varphi_k)$ to the group of the target φ_k as the sum of the *direct cost of* *participation* $T(\varphi_k)$ plus the least cost of inducing compliance, which we call the *monitoring cost* $M(\varphi_k)$. Both these functions are defined for $\varphi_k \geq \underline{\varphi}$, because only these participation rates can be realized in the model. We now study the problem of minimizing the total cost for a given target φ_k.

Theorem 1. *The direct, monitoring, and total costs of inducing compliance with a social norm* $\varphi_k \geq \underline{\varphi}$ *are*

$$T(\varphi_k) = \varphi_k \left(\varphi_k + 2c_0\right)/2\,, \quad \varphi_k \geq \underline{\varphi} \tag{3.1}$$

$$M(\varphi_k) = \begin{cases} 0 & \textit{if} \;\; \varphi_k = \underline{\varphi} \\ \pi\left(1-\varphi_k\right)\left(c_0 - h_k + \varphi_k\right) & \textit{if} \;\; \varphi_k > \underline{\varphi} \end{cases} \tag{3.2}$$

$$C(\varphi_k) = \begin{cases} \underline{\varphi}\left(\underline{\varphi} + 2c_0\right)/2 & \textit{if} \;\; \varphi_k = \underline{\varphi} \\ \left(\frac{1}{2} - \pi\right)\varphi_k^2 + \left[\pi\left(1+h_k\right) + \left(1-\pi\right)c_0\right]\varphi_k - \pi\left(h_k - c_0\right) & \textit{if} \;\; \varphi_k > \underline{\varphi} \end{cases}. \tag{3.3}$$

Proof. By definition the direct cost of participation above the committed level $\underline{\varphi}$ is $T(\varphi_k) = \int_0^{\varphi_k} c(y)dy$ for $\varphi_k \geq \underline{\varphi}$, and direct computation gives

$$\int_0^{\varphi_k} c(y)dy = \varphi_k^2/2 + c_0\varphi_k = \varphi_k\left(\varphi_k + 2c_0\right)/2\,, \quad \varphi_k \geq \underline{\varphi}$$

Next we derive the monitoring cost. The incentive constraint is that members with $y_i \leq \varphi_k$ should be willing to participate, that is $c_0 - h_k + y_i \leq P_k$ - therefore it must be $P_k \geq c_0 - h_k + \varphi_k$; and members with $y_i > \varphi_k$ should not, that is $\pi P_k \leq c_0 - h_k + y_i$ or $\pi P_k \leq c_0 - h_k + \varphi_k$. Minimization of cost implies that the constraint should bind, that is, $\pi P_k = c_0 - h_k + y_i$. Notice that without monitoring (that is, no punishment or incentive provided not to participate) the participation rate is lowest and equal to $\underline{\varphi}$. Therefore, $M(\underline{\varphi}) = 0$. For $\varphi_k > \underline{\varphi}$, recalling that for these values $P_k = \varphi_k - h_k + c_0$, the monitoring cost is

$$\int_{\varphi_k}^{1} \pi P_k dy = \pi\left(1-\varphi_k\right)\left(c_0 - h_k + \varphi_k\right)$$

Taking these results together, we obtain (3.2). Hence the total cost $C(\varphi_k) \equiv T(\varphi_k) + M(\varphi_k)$ of inducing participation above the committed level of $\underline{\varphi}$ is easily verified to be as in the statement. □

To simplify the analysis in the remainder of the paper we assume $\pi = 1/2$ unless explicitly specified otherwise. In this case it follows directly from (3.3) that the cost function

$C(\varphi_k)$ is given by the following;

Corollary 1. *Suppose $\pi = 1/2$. Then*

$$C(\varphi_k) = \begin{cases} \underline{\varphi}\left(2c_0 + \underline{\varphi}\right)/2 & \text{if } \varphi_k = \underline{\varphi} \\ \frac{1+h_k+c_0}{2}\varphi_k - \frac{h_k-c_0}{2} & \text{if } \varphi_k > \underline{\varphi} \end{cases}. \tag{3.4}$$

4. No Public Good and Tipping

We now consider the optimal choice of φ_k in the special case in which $v_k = 0$, that is, the only benefit to the group from participation are the individual benefits. As indicated, we shall focus on the case $\pi = 1/2$. Our goal is to understand how the optimal φ_k depends upon the strength of the externality λ. In particular we will show that there is tipping in the sense that there is a critical value λ^* for which participation jumps discontinuously as this threshold is exceeded. It is this tipping phenomenon that is the main topic of the paper.

Let

$$\lambda^* \equiv \begin{cases} \frac{1-(h_k-c_0)}{2h_k} & \text{if } h_k - c_0 \geq 0 \\ \frac{1}{h_k}\left(\frac{1}{2} - (h_k - c_0)\right) & \text{if } h_k - c_0 < 0 \end{cases}$$

and φ_k^* be the optimal choice for group k. Note that λ^* is higher in the case $h_k - c_0 < 0$. In Appendix A we show that

Theorem 2 (Tipping Theorem). *Suppose $v_k = 0$ and $\pi = 1/2$. Then the optimal participation increases discontinuously in λ at threshold λ^*:*

$$\varphi_k^* = \begin{cases} \underline{\varphi} & \text{if } \lambda < \lambda^* \\ 1 & \text{if } \lambda > \lambda^* \end{cases} \tag{4.1}$$

The threshold λ^* is increasing in c_0 and decreasing in h_k.

For $0 \leq h_k - c_0$ we show in Appendix A that utility is $-\xi\varphi_k + (h_k - c_0)/2$ where $\xi \equiv (1/2) - (h_k - c_0)/2 - \lambda h_k$. This can be interpreted as a marginal cost of increasing participation net of the benefit of externality. The key point is that this is positive for $\lambda = 0$ (since we assume $h_k - c_0 \leq 1$) but becomes negative when the externality λ is sufficiently large. It is this switch from positive to negative marginal cost at λ^* that leads to tipping.

Robustness to $\pi \neq 1/2$: The Steep Slope Theorem

In what follows we assume $\pi \neq 1/2$ and examine whether tipping is robust to perturbations of π away from $1/2$. The result is the following:

Theorem 3 (The steep slope theorem). *Suppose $v_k = 0$. If $\pi > 1/2$, then φ_k^* is the same as with $\pi = 1/2$ (that is Theorem 2 still applies). If $\pi = 1/2 - \epsilon$ for $0 < \epsilon < 1/2$ and $h_k - c_0 \geq 0$, then*

$$\varphi_k^* = \begin{cases} \underline{\varphi} & \text{if } \lambda \leq (1-2\epsilon)\lambda^* \\ \frac{h_k}{2\epsilon}(\lambda - \lambda^*) + \frac{1+\underline{\varphi}}{2} & \text{if } \lambda \in ((1-2\epsilon)\lambda^*, (1+2\epsilon)\lambda^*) \\ 1 & \text{if } \lambda \geq (1+2\epsilon)\lambda^* \end{cases} \tag{4.2}$$

For $h_k - c_0 < 0$ and $\pi < 1/2$ the result is qualitatively the same, but the statement is more involved so we deal with it in Appendix B, where the above theorem is proven.

Observe that for $\pi = 1/2 - \epsilon$ and ϵ small the result is that φ_k^* as a function of λ increases steeply from $\underline{\varphi}$ to 1 in the interval $((1-2\epsilon)\lambda^*, (1+2\epsilon)\lambda^*)$ whose width tends to zero. We therefore conclude that tipping is a robust prediction when $\pi > 1/2$, and it is a good approximation if $\pi = 1/2 - \epsilon$ for ϵ sufficiently small.

5. Sports Team

We consider now the problem faced by a sports team whose fans are a self-organizing group. Here participation takes the form of attending a match, and the sports team charges a price r for attending. We assume in addition that the sports stadium has a maximum capacity and can accommodate only a fraction $Q < 1$ of fans. Subject to the capacity constraint the costs of the team are entirely sunk so it wishes to choose r to maximize revenue.

For clarity of exposure, we continue to assume $\pi = 1/2$. We also assume that $p_k(\phi_k)$ is constant, that is, that participation by the fans does not change the outcome of the match. Without loss of generality we may then assume that and $v_k = 0$, that is the only benefit to the group are the individual benefits which we now interpret as resulting from attending a match. Since the sports team charges a fee r for attendance, the constant c_0 in $c(y_i) = c_0 + y_i$ becomes $c_0 = \zeta_0 + r$, with $\zeta_0 < 0$. In effect the sports team controls the marginal cost

$$\xi \equiv \frac{1}{2} - \frac{h_k - \zeta_0 - r}{2} - \lambda h_k.$$

As in Theorem 2 demand will jump discontinuously from $\underline{\varphi}$ to Q when price is sufficiently low that marginal cost becomes negative. Hence the firm has a choice: set a price above the tipping price and sell just to committed fans $\underline{\varphi}$ or set the tipping price and fill the stadium. In Appendix B we fully characterize the solution. The most interesting result is:

Theorem 4. *Suppose that $\lambda \geq 1/(2h_k)$. Then it is optimal to sell at full capacity Q and* *set price*

$$r_Q^* = h_k - \zeta_0 + (2\lambda h_k - 1)\frac{Q}{1+Q}.$$

There are several observations about this result. First, as we show in Appendix C, in standard monopoly in the absence of tipping and with continuous demand it is also possible that the monopoly solution is to fill the stadium. However, increasing the stadium size will always lower the price charged by the team. Here the opposite is the case: a bigger stadium commands a higher price because it provides a greater benefit to the fans from the externality. Second, in neither case is there rationing *per se.* However, if there is a small amount of uncertainty about demand the solution is quite different in the continuous and in the tipping case. In the continuous case it will be optimal to set an intermediate price, sometimes filling the stadium and sometimes nearly filling the stadium. In the tipping case the discrete loss from tipping means that when the stadium is not filled attendance will drop dramatically, and consequently the team will generally avoid tipping by pricing sufficiently low that it will have to ration tickets.

6. Voting

We now suppose that there are two groups which are political parties $k \in \{L, S\}$ who participate in an election. The relative size of the two parties is $\eta_L > \eta_S > 0$ with $\eta_L + \eta_S = 1 - \eta_c$, where η_c is the size of the group of "civic voters" who split their votes equally to both parties and thus do not affect the outcome of the election. Here η_k is the fraction of people who would vote for $k = L, S$ with certainty *if* they decided to vote. The party that sends the most voters to the polls wins the election and receives a total prize of size V; the per capita value of the public good for each party k to win the election is thus $v_k = V/\eta_k$. We assume that the individual expressive payoff $h_k = hV$ is the same for both parties. We continue with the simplifying assumption $\pi = 1/2$ and in addition assume $0 < hV - c_0 < 1$ so that $0 < \underline{\varphi} = hV - c_0 < 1$ (cf. (2.3)). A bid b_k by group k is the number of voters mobilized to turnout by party k, that is $b_k = \eta_k \varphi_k$. For each party $k = L, S$, the set of feasible bids is given by $[\underline{\varphi}\eta_k, \eta_k]$.

Let $\Pi_k(b_k, b_{-k})$ denote the winning probability of party k as a function of bids b_k and b_{-k} submitted by both parties. Following Levine and Mattozzi (2020) we assume that the large party L wins the election in case of a tie.[7] Therefore,

[7] This is to simplify equilibrium and guarantee that equilibria always exist. See footnote 8 of Levine and Mattozzi (2020) for a more detailed discussion.

$$\Pi_L(b_L, b_S) = \begin{cases} 1 & \text{if } b_L \geq b_S \\ 0 & \text{if } b_L < b_S \end{cases} \qquad \Pi_S(b_S, b_L) = 1 - p_L(b_L, b_S).$$

Finally we assume that h is small and λ is large; specifically we consider the limit in which $h \to 0$ and $\lambda h \to \kappa$ for some $\kappa > 0$. Formally, we use

Assumption 1 (low expressive payoff and strong positive externality)**.** *$h \to 0$, $h\lambda \to \kappa > 0$ and $-c_0 \in (0, 1)$.*

Let

$$\overline{V} \equiv \frac{1 + c_0}{2\kappa}.$$

Our main result is that tipping takes place at $\overline{V}$: the details are in Appendix D.

Theorem 5. *Suppose Assumption 1 holds and $\eta_S > \eta_L \underline{\varphi}$. For $V > \overline{V}$ the aggregate bid by the two parties is $\overline{b} = \eta_S + \eta_L$; for $V < \overline{V}$ close to $\overline{V}$ aggregate bid is approximately $\underline{b} = \eta_S(1 + \underline{\varphi})$.*

For $V > \overline{V}$ all voters in both parties turn out and the aggregate bid $b = b_L + b_S$ by the two parties is $\overline{b} = \eta_S + \eta_L$, independently of the probability of winning. For $V < \overline{V}$ we have to study the game between the two parties because the probability of winning depends on bids. The analysis in Appendix D shows that for V close to $\overline{V}$ - which is the relevant case in general elections, where stakes are high - in equilibrium to a good approximation the large party bids η_S and the small party bids $\underline{\varphi}\eta_S$, so the aggregate bid is $\underline{b} = \eta_S(1 + \underline{\varphi})$. The main implication of this result is that the distribution of voter turnout is twin-peaked due to tipping: even if V has a continuous single-peaked distribution, as V crosses $\overline{V}$ we should observe a discontinuous upward jump in turnout.

To account for the fact that the actual voter turnout is certainly not Bernoulli, we have assumed a third group of civic voters, of size η_c, who split equally between the two parties (so that the strategic aspects of voting are unchanged) and who are not part of any social network. Rather they face the participation cost $c_c + y_i$ like other voters. The fraction of voters from this group is determined by $c_c + y_i \leq 0$ so it is $-c_c$. For the civic voters we assume that c_c is normally distributed with mean μ_c and standard error σ_c. Essentially, what civic voters do is to add additional terms in parties' bids: when parties bid b_L and b_S, the actual fraction of voters that cast votes for them are:

$$\tau_S = b_S - \frac{\eta_c}{2} c_c \qquad \tau_L = b_L - \frac{\eta_c}{2} c_c$$

where $1/2$ shows up because civic voters are divided equally to parties. The aggregate turnout is then

$$\tau := \tau_L + \tau_S = b - \eta_c c_c \tag{6.1}$$

where $c_c \sim \mathcal{N}(\mu_c, \sigma_c)$ is normal with mean μ_c and standard deviation σ_c, and b is Bernoulli with probability Q_1 of $\underline{b}$ (the probability that $V < \overline{V}$) and $1-Q_1$ of $\overline{b}$ (probability of $V > \overline{V}$). Equivalently, τ is a mixture of two normal distributions both with standard deviation $\sigma = \eta_c\sigma_c$, and one with mean $\mu_1 = \eta_c\mu_c + \underline{b}$ and the other with mean $\mu_1 + g = \eta_c\mu_c + \overline{b}$ so that $g = \overline{b} - \underline{b}$ is the change in turnout due to tipping, and where the first has probability Q_1 and the latter probability $1-Q_1$.[8] We call this a Bernoulli-Normal mixture. Our theory predicts voter turnout to follow such a Bernoulli-Normal mixture distribution rather than a single-peaked normal distribution (which would occur without tipping). In the remainder of this section we investigate this prediction using turnout data from presidential elections in the US and general elections in the UK.

Empirical estimation

We now examine the implication of our model that if V is either smaller than and close to $\overline{V}$ or above it the distribution of voter turnout has two peaks due to tipping. To examine whether this might be the case we gathered turnout data from US presidential elections (1920-2020) and UK general elections (1918-2019) beginning with the first election in women were permitted to vote.

Positive Serial Correlation and Stationarity

A crucial fact is that voter turnout has strong positive serial correlation. We model this by assuming that c_c follows an AR(1) process. If we assume that there is no tipping, that is $g = 0$ or $Q_1 \in \{0, 1\}$ this means that turnout is also an AR(1) process which we may write as

$$\tau_t = \rho_0 + \rho_1\tau_{t-1} + \varepsilon_t$$

with $\varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$ i.i.d and independent of τ_t. We can then estimate $\hat{\rho}_0$, $\hat{\rho}_1$ and $\hat{\sigma}_\varepsilon^2$ using standard OLS. The stationary distribution for τ_t is then a normal distribution with mean $\hat{\mu}_{stationary} = \frac{\hat{\rho}_0}{1-\hat{\rho}_1}$ and variance $\hat{\sigma}^2_{stationary} = \frac{\hat{\sigma}^2_\varepsilon}{1-\hat{\rho}_1^2}$. Estimation results are reported in Table 1. For both US and UK data, the estimated $\hat{\rho}_1$'s are positive and statistically significant.[9] Moreover, augmented Dickey-Fuller tests reject unit root hypotheses (i.e., $\rho_1 = 1$) for both US and UK turnout data.[10] These results indicate that the time series of voter turnout

[8] While the actual turnout is bounded between zero and one, a finite normal mixture model may well approximate it when the standard deviations are sufficiently small.

[9] p values of two-sided tests for $\rho_1 = 0$ are 0.014 for US and 0.007 for UK.

[10] MacKinnon approximate p values are 0.0383 for US presidential elections and 0.0192 for UK general elections. Both tests reject the null unit root hypotheses at 5% significance level.

Table 1: OLS estimation results for the AR(1) models

	$\hat{\rho}_0$	$\hat{\rho}_1$	$\hat{\sigma}_\varepsilon$	$\hat{\mu}_{stationary}$	$\hat{\sigma}_{stationary}$
US data	0.314	0.439	0.037	0.560	0.041
	(0.091)	(0.165)			
UK data	0.349	0.526	0.046	0.737	0.054
	(0.131)	(0.172)			

Note: Newey-West standard errors robust to heteroskedasticity and first-order autocorrelation are reported in parentheses below the estimates $\hat{\rho}_0$ and $\hat{\rho}_1$.

exhibit strong positive serial correlation and are stationary.

Partial Maximum Likelihood

With serial correlation the likelihood function is not tractable as it requires us to compute for each set of parameters a likelihood for each possible sequence of Bernoulli of which there are many. Instead we implement a partial maximum likelihood approach as described in Levine (1983). Here we obtain consistent estimates by maximizing the product of the stationary density functions, that is, proceeding "as if" the observations were independently drawn from its stationary distribution. The standard errors are then computed using both contemporaneous and lagged information matrices.

In the model turnout τ_t in each period t is a mixture of two normal distributions $\mathcal{N}(\mu_1, \sigma^2)$ and $\mathcal{N}(\mu_1 + g, \sigma^2)$. The probability that τ_t is drawn from $\mathcal{N}(\mu_1, \sigma^2)$ equals $Q_1 \in (0,1)$. Without loss of generality, we assume $g > 0$ to ensure identification. Let $\vartheta = (Q_1, \mu_1, g, \sigma)$ be the vector of parameters and $h(\cdot;\vartheta)$ denote the probability density function of τ_t. The stationary density function is then

$$f(\tau_t|\vartheta) = Q_1\phi(\tau_t; \mu_1, \sigma^2) + (1 - Q_1)\phi(\tau_t; \mu_1 + g, \sigma^2)$$

where $\phi(x; \mu, \sigma^2) = (1/\sqrt{2\pi\sigma^2})\exp(-(\tau_t - \mu)^2/2\sigma^2)$ denotes the density function for $\mathcal{N}(\mu, \sigma^2)$. Then, given the whole time series of turnout $\boldsymbol{\tau} := \{\tau_t\}_{t=1}^T$, the partial log-likelihood function is

$$\begin{aligned}\mathscr{L}(\vartheta; \boldsymbol{\tau}) &= \sum\nolimits_{t=1}^T \log f(\tau_t|\vartheta) \\ &= \sum\nolimits_{t=1}^T \log\left(Q_1 e^{-\frac{(\tau_t - \mu_1)^2}{2\sigma^2}} + (1 - Q_1)e^{-\frac{(\tau_t - \mu_1 - g)^2}{2\sigma^2}}\right) - \frac{T}{2}\log 2\pi\sigma^2 \qquad (6.2)\end{aligned}$$

We estimate $\vartheta = (Q_1, \mu_1, g, \sigma)$ using the partial maximum likelihood approach. The estimation results are presented in Table 2.

Table 2: ML Estimation results for US and UK

	U.S. presidential elections (1920-2020)		U.K. general elections (1918-2019)	
Parameters	Single-peaked Normal	Bernoulli-Normal mixture	Single-peaked Normal	Bernoulli-Normal mixture
$\hat{g}$	-	0.066	-	0.127
		(0.014)		(0.034)
$\hat{Q}$	-	0.626	-	0.177
		(0.233)		(0.159)
$\hat{g}\sqrt{\hat{Q}(1-\hat{Q})}$	-	0.032	-	0.049
		(0.006)		(0.006)
$\hat{\mu}_1$	0.554	0.529	0.727	0.622
	(0.014)	(0.009)	(0.023)	(0.031)
$\hat{\sigma}$	0.042	0.027	0.064	0.041
	(0.002)	(0.005)	(0.007)	(0.006)
Partial Log-likelihood	45.691	46.683	37.459	39.788
#.Observations	26	26	28	28

Note: Robust standard errors are reported in parentheses and they are computed following the method in Levine (1983) with lag $k = 4$. The choice $k = 4$ is made based on a tradeoff between bias and precision of estimates. As Table 1 shows, the serial correlation is around 0.5. It can be checked that for an AR(1) with a coefficient of 0.5 the contribution of lags after 4 to the stationary standard error is less than 1/10th of a percent. The choice of $k = 4$ also leaves us with 21 to 23 observations to use in the estimation.

Point Estimates

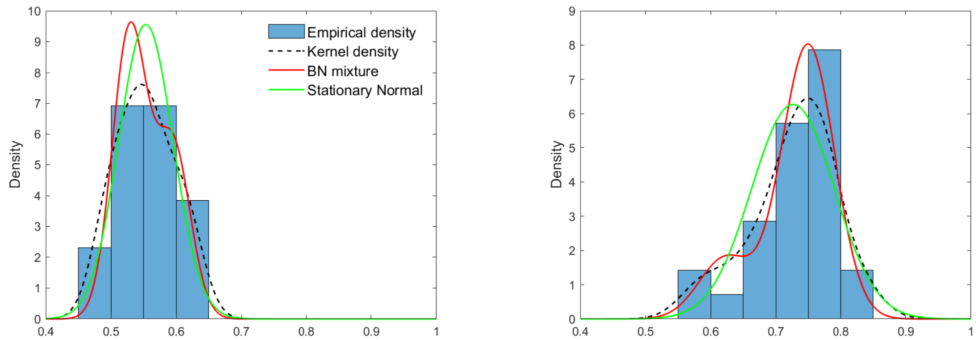
We estimate both the size of the gap $\hat{g}$ and the standard error of the binomial component $\hat{g}\sqrt{\hat{Q}(1-\hat{Q})}$. Both are large in economic terms: $\hat{g}$ indicates a 6.6% increment to turnout due to tipping in the US and 12.7% in the UK. Bearing in mind that at $Q = 1/2$ the standard error is half the gap, we find similarly large standard errors of the binomial: 3.2% in the US and 4.9% in the UK. These point estimates support the idea that tipping is real and important.

Finally, we plot the estimated probability densities from the point estimates in Figure 6.1. As can be seen tipping in the UK is substantial and the estimates indicate that most elections have high values in the sense that $V > \overline{V}$ while insofar as there is tipping in the US typically $V < \overline{V}$. This is consistent with the idea that elections have higher value in the UK than in the US, which is indicated as well by the fact that turnout is generally higher in the UK. In both cases the tipping point is similar - around 58-64%.

Sampling Error

It is important to understand whether these economically significant point estimates are simply due to sampling error in what is a relatively small sample. In particular, how likely is it that such large estimates could be generated from an underlying structure where

Figure 6.1: Estimated densities of turnout distribution for US (left panel) and UK (right panel)



Note: The red lines plot the densities implied the estimated Bernoulli-Normal mixture model. The green lines plot the densities implied by the stationary normal distribution under AR(1) estimated by OLS. The black dashed curves are the estimated kernel densities (the optimal bandwidth are 0.0254 for US and 0.0323 for UK). Blue bars are the empirical density of data.

there is no tipping, that is $g = 0$ or $Q \in \{0, 1\}$? We cannot simply apply asymptotic theory here because the hypothesis of no tipping is on the boundary of the parameter space and the distribution of the coefficient estimate $\hat{g}$ is a positive random variable, hence biased way from 0 even when the true value is 0, and does not converge in a large sample to an approximate normal. To understand better the role of sampling error we use a Monte Carlo experiment: we simulate $M = 10000$ samples drawn from the serially correlated model without tipping from above

$$\tau_t = \rho_0 + \rho_1 \tau_{t-1} + \varepsilon_t.$$

For each simulated sample m we estimate the parameter vector $\hat{\vartheta}_m$. This yields a collection of estimates $\left\{\hat{\vartheta}_m\right\}_{m=1}^{10000}$. The empirical cumulative distributions of $\{\hat{g}_m\}_{m=1}^{10000}$ and $\left\{\hat{g}_m \sqrt{\hat{Q_m}(1-\hat{Q_m})}\right\}_{m=1}^{10000}$ from Monte Carlo simulations for both UK and US data are presented in Figure 6.2.

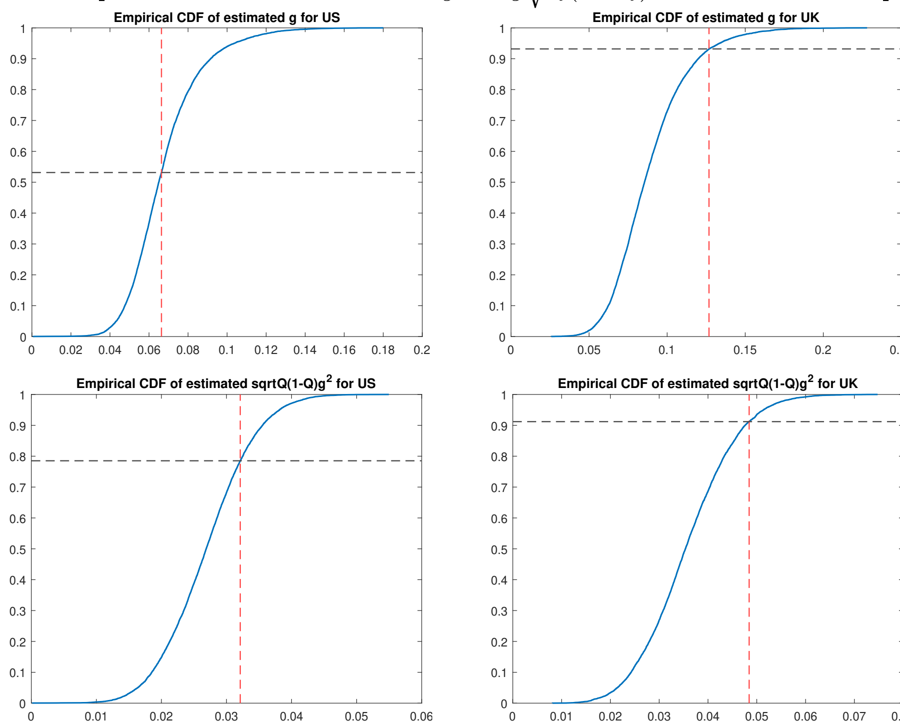
Table 3 reports the probability that $\hat{g}_m$ or $\hat{g}_m\sqrt{\hat{Q}_m(1-\hat{Q}_m)}$ obtained from the empirical distributions of these estimates would generate values as large as those observed in the actual data.

Table 3: Probabilities of data or higher from Monte Carlo experiments

	$\hat{g}$	$\sqrt{\hat{Q}(1-\hat{Q})\hat{g}^2}$
UK data	0.068	0.088
US data	0.469	0.215

The bottom line here is that is it relatively unlikely that the point estimates seen in the

Figure 6.2: Empirical distribution of estimates $\hat{g}$ and $\hat{g}\sqrt{\hat{Q}(1-\hat{Q})}$ from Monte Carlo experiments



Note: In all these figures, the red dashed lines denote the estimates obtained from the real data, and the black dashed lines denote probability of observing estimates that are equal or lower than the estimates obtained from real data.

UK arise without tipping: the probabilities are 6.8% for the gap and 8.8% for the standard error of the Bernoulli. On the other hand it may well be that the relatively large point estimates seen in the US arise without tipping as a statistical fluke: indeed there is nearly a 50% chance that a model without tipping would generate values of g as large as those estimated from the data.

We should note that the procedure here is conceptually the same as a randomization or permutation test (Young , 2019) in the sense that we ask how likely it is under the null hypothesis that we would see coefficient estimates as high as those we estimated: we do not ask the t-test question of how likely it is under the null hypothesis that we would see the ratio of coefficient estimates to standard errors that we see in the data. The reason for this is simple, the latter question is without economic interest: we are not concerned with whether the null hypothesis is exactly true, we know *a priori* it is not. In particular if we observe a low value of the gap, say 2% we would conclude that tipping was not an important phenomenon and would reject it as a useful model no matter the precision with which the coefficient of 2% was estimated. By contrast a t-test would not reject the null hypothesis if the standard error was sufficiently small. In other words we do not use a t-test approach because it is without useful economic meaning.

Bayes Factors

In the US the tipping we observe may be a statistical fluke. This is not the case in the UK, which leads us to examine more closely the ability of our estimation to discriminate between alternative values of g. We do so by reporting approximate Bayes factors for different values g_1 against a base value g_0. Note that these are not nested hypotheses.

To do so we must address two issues. First, we cannot compute the posterior conditional on the entire sample because we cannot compute the likelihood function, and second although it is natural to take $g_0 = 0$ we do not do so because at $g_0 = 0$ our estimator is not asymptotically normal.

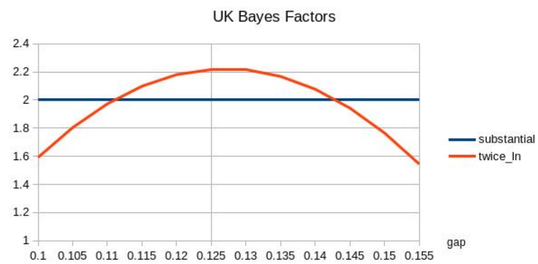
As we cannot compute the posterior conditional on the entire sample, we instead compute it conditional on the partial maximum likelihood estimate $\hat{\vartheta}$. If we were doing full maximum likelihood asymptotically this would be a sufficient statistic for the entire sample, but as we are doing partial maximum likelihood it is not: hence we are looking at the posterior conditional on a subset of the information available. In short, we use Bayes rule to compute not $f(\vartheta|Y)$ where Y is the entire sample, but rather $f(\vartheta|\hat{\vartheta}) = f(\hat{\vartheta}|\vartheta)f(\vartheta)/f(\hat{\vartheta})$. Provided both g_1 and g_0 are in the interior $f(\hat{\vartheta}|\vartheta)$ is asymptotically normal. We can then compute approximate Bayes factors using the procedure of Laplace, as outlined, for example, in Kass and Raftery (1995). For our partial maximum likelihood our estimates are consistent and asymptotically normal so that indeed the posterior will be concentrated near the partial maximum likelihood estimate. Hence when we integrate out the nuisance parameters we may treat the prior as approximately constant. If we additionally assume that in our prior g is independent of the nuisance parameters when we compute the ratio of marginal probabilities between g_1 and g_0 we find that the prior cancels out in the numerator and denominator and the Bayes factor is equal to the ratio of two normal densities. Letting $\hat{s} = 0.034$ be the asymptotic standard error of the partial maximum likelihood estimate $\hat{g}$ (cf. Table 2 for UK). The approximate Bayes factor is then given by

$$B_{10} = \frac{(1/(\hat{s}\sqrt{2\pi}) \exp\left(-(g_1 - \hat{g})^2/(2\hat{s}^2)\right)}{(1/(\hat{s}\sqrt{2\pi}) \exp\left(-(g_0 - \hat{g})^2/(2\hat{s}^2)\right)}.$$

Second, as indicated we cannot take $g_0 = 0$ because our estimator is not asymptotically normal there. Instead we use a value of g_0 sufficiently far in the interior that asymptotic theory makes sense. In particular for g_0 close to zero the asymptotic standard errors give a high probability of a negative draw of $\hat{g}$ and a much lower probability of the estimated value of $\hat{g}$ (or higher) than seen in the Monte Carlo. For this reason we chose g_0 so that the probability of observing $\hat{g}$ (or higher) based on the asymptotic distribution is exactly that from the Monte Carlo, that is, 6.8%. This is $g_0 = 7.6\%$ which is comfortably lower than the point estimate of 12.7% and well more than two standard deviations (coincidentally also 6.8%) above zero.

In Figure 6.3 as suggested by Kass and Raftery (1995) we report twice the natural logarithm of our approximate Bayes factor, with the understanding that greater than 2 represents substantial evidence in favor of g_1: in particular it means that the prior odds ratio relative to g_0 is raised by a factor of three or more.

Figure 6.3: Bayes Factors for the UK



As can be seen our estimation procedure is adequate to narrow down the range of values of g which are substantially more likely than 7.6% to about the range from 11% to 14%.

7. Conclusion

We have developed a model that combines peer pressure with expression and shown how this can lead to "tipping." This occurs when the expressive externality is strong enough that in the group the marginal cost of participation becomes negative so that it is optimal for everyone to participate. We argued that this potentially explains why sporting teams ration tickets: they do not wish to take the chance of triggering "tipping in reverse" by setting the price so high that it does not pay the fans to self-organize. Notice that teams are quite aware of fan self-organization and work hard to encourage it. The same can be said of musical bands.

Our more substantive application was to voting where we showed how voter turnout can change discontinuously even though the underlying stakes are drawn from a continuous distribution. We examined the prediction that turnout should be twin-peaked and found that there is substantial evidence that in the UK voter turnout does indeed have two peaks.

References

Dutta, R., D. K. Levine and S. Modica (2021b): "The Whip and the Bible: Punishment Versus Internalization," forthcoming *Journal of Public Economic Theory.*

Kandori, M. (1992): "Social Norms and Community Enforcement," *The Review of Economic Studies* 59: 63-80.

Kass, R. E., and Raftery, A. E. (1995): "Bayes factors", *Journal of the American Statistical Association*, 90(430): 773-795.

Levine, David (1983): "A remark on serial correlation in maximum likelihood," *Journal of Econometrics* 23: 337-342.

Levine, David and Salvatore Modica (2016): "Peer Discipline and Incentives within Groups", *Journal of Economic Behavior and Organization,* 123: 19-30

Levine, D. K. and A. Mattozzi (2020): "Voter Turnout with Peer Punishment," *American Economic Review* 110: 3298-3314.

Ostrom, E. (1990): *Governing the commons: the evolution of institutions for collective action*, Cambridge university press.

Pons, Vincent, and Clémence Tricaud (2018): "Expressive voting and its cost: Evidence from runoffs with two or three candidates," *Econometrica* 86: 1621-1649.

Young, Alwyn. "Channeling fisher: Randomization tests and the statistical insignificance of seemingly significant experimental results." *The Quarterly Journal of Economics* 134.2 (2019): 557-598.

Appendix A: Optimal Participation with Individual Benefits Only

We now consider the optimal choice of φ_k in the special case in which $v_k = 0$, that is, the only benefit to the group from participation are the individual benefits. As indictated, we shall focus on the case $\pi = 1/2$. Observe that $\phi_k = \varphi_k$ must hold to be in compliance with any social norm $\varphi_k \geq \underline{\varphi}$. Using (2.1), (2.2) and the simplifying assumption $v_k = 0$, we get group utility per capita:[11]

$$\begin{aligned} \mathcal{U}_k(\varphi_k) &\equiv (1-\varphi_k)\,\lambda\varphi_k h_k + \varphi_k\,(1+\lambda\varphi_k) - C\,(\varphi_k) \\ &= (1+\lambda)\,\varphi_k h_k - C\,(\varphi_k) \end{aligned} \tag{7.1}$$

[11]If $v_k > 0$ the utility $\mathcal{U}_k(\varphi_k)$ can be obtained by simply adding $p_k(\phi_k)v_k$ to the expression below.

for all $\varphi_k \in [\underline{\varphi}, 1]$. The following lemma characterizes $\mathcal{U}_k(\varphi_k)$ when $\pi = 1/2$.

Lemma 1. *Assume $\pi = 1/2$, $v_k = 0$ and let*

$$\xi \equiv \frac{1}{2} - \frac{h_k - c_0}{2} - \lambda h_k. \tag{7.2}$$

Then the per capita group utility is equal to

$$\mathcal{U}_k(\varphi_k) = -\xi\varphi_k + \frac{h_k - c_0}{2} \cdot \mathbb{1}\{\varphi_k > 0\} \quad \text{for} \quad \varphi_k \in [\underline{\varphi}, 1] \tag{7.3}$$

Here $\mathbb{1}\{\cdot\}$ is the indicator function. Note that ξ is the constant marginal cost of inducing participation rate φ_k. Without the externality, that is with $\lambda = 0$, the marginal cost is positive and the optimal participation is unambiguously equal to $\underline{\varphi}$. Thus without the externality we have a pretty standard model - self-organization never creates a discontinuity: only the committed members participate and participation is a continuous function of parameters (Levine and Modica (2016) and Levine and Mattozzi (2020)). When $h_k - c_0 < 0$ - so that $\underline{\varphi} = 0$ and participation incurs strictly positive costs for all member types - it follows from (7.3) that $\mathcal{U}_k(0) = p_k v_k$ and $\lim_{\varphi_k \downarrow 0} \mathcal{U}_k(\varphi_k) = (h_k - c_0)/2 < 0$. That is, $\mathcal{U}_k(\varphi_k)$ has a downward jump at zero. Recall that $\pi P_k = (c_0 - h_k + \varphi_k)/2 \to (c_0 - h_k)/2$ as $\varphi_k \to 0$, so the downward jump is exactly the monitoring cost of inducing the lowest type to participate.

Proof of Lemma 1. By (3.4), (7.1) and (7.2), for $\varphi_k > \underline{\varphi}$ we have

$$\begin{aligned}
\mathcal{U}_k(\varphi_k) &= (1+\lambda) h_k \varphi_k - C(\varphi_k) \\
&= (1+\lambda) h_k \varphi_k - \frac{1 + h_k + c_0}{2}\varphi_k + \frac{h_k - c_0}{2} \\
&= \left[\lambda h_k - \frac{1 - (h_k - c_0)}{2}\right]\varphi_k + \frac{h_k - c_0}{2} = -\xi\varphi_k + \frac{h_k - c_0}{2}
\end{aligned}$$

For $\varphi_k = \underline{\varphi}$, we shall establish that

$$\mathcal{U}_k(\underline{\varphi}) = \begin{cases} 0 & \text{if } h_k - c_0 < 0 \\ -\xi\underline{\varphi} + \frac{h_k - c_0}{2} & \text{if } h_k - c_0 \geq 0 \end{cases} \tag{7.4}$$

These two expressions together imply (7.3). By (7.1), $\mathcal{U}_k(\underline{\varphi}) = (1+\lambda) h_k \underline{\varphi} - C(\underline{\varphi})$. If $h_k - c_0 < 0$, then $\underline{\varphi} = 0$ and $C(\underline{\varphi}) = 0$ so that $\mathcal{U}_k(\underline{\varphi}) = 0$. If $h_k - c_0 \in [0, 1]$, then $\underline{\varphi} = h_k - c_0 \geq 0$ and $C(\underline{\varphi}) = \left(h_k^2 - c_0^2\right)/2$. Therefore

$$\mathcal{U}_k(\underline{\varphi}) = (1+\lambda)\, h_k\, (h_k - c_0) - \frac{1}{2}\left(h_k^2 - c_0^2\right)$$
$$= \left[\lambda h_k + \frac{h_k - c_0}{2}\right](h_k - c_0) = -\xi\underline{\varphi} + \frac{h_k - c_0}{2}$$

Finally, if $h_k - c_0 > 1$, then $\underline{\varphi} = 1$ and $C(\underline{\varphi}) = c_0 + 1/2$ so that $\mathcal{U}_k(\underline{\varphi}) = (1+\lambda)\, h_k - c_0 - 1/2 = -\xi + (h_k - c_0)/2$. These together establish (7.4). □

Recall that

$$\lambda^* \equiv \begin{cases} \frac{1-(h_k - c_0)}{2h_k} & \text{if } \; h_k - c_0 \geq 0 \\ \frac{1}{h_k}\left(\frac{1}{2} - (h_k - c_0)\right) & \text{if } \; h_k - c_0 < 0 \end{cases}$$

and $\varphi_k^* = \arg\max_{\varphi_k \in [\underline{\varphi},1]} \{\mathcal{U}_k(\varphi_k)\}$. We now prove Theorem 2, that tipping takes place at λ^*.

Proof. First observe that if $h_k - c_0 > 1$ then $\underline{\varphi} = 1$ (cf. (2.3)) so that $\varphi_k^* = 1$ holds trivially. Assume then $h_k - c_0 \leq 1$. It follows from Lemma 1 that $\mathcal{U}_k(\varphi_k)$ is linear in φ_k for $\varphi_k > \underline{\varphi}$ so that φ_k^* must be either $\underline{\varphi}$ or 1 whenever $\xi \neq 0$. Therefore, $\varphi_k^* = 1$ if $\mathcal{U}_k(1) > \mathcal{U}_k(\underline{\varphi})$ and $\varphi_k^* = \underline{\varphi}$ if $\mathcal{U}_k(1) < \mathcal{U}_k(\underline{\varphi})$. By (7.3) and (7.2), we have

$$\mathcal{U}_k(1) - \mathcal{U}_k(\underline{\varphi}) = \begin{cases} -\xi\left(1 - \underline{\varphi}\right) & \text{if } \; h_k - c_0 \geq 0 \\ (1+\lambda)\, h_k - c_0 - 1/2 & \text{if } \; h_k - c_0 < 0 \end{cases}$$

Therefore when $h_k - c_0 \geq 0$ we have $\varphi_k^* = \underline{\varphi}$ if $\xi > 0$ and $\varphi_k^* = 1$ if $\xi < 0$. By (7.2) ξ is strictly decreasing in λ and it is straightforward to verify that $\xi = 0$ if and only if

$$\lambda = \frac{1 - (h_k - c_0)}{2h_k}.$$

This proves the result for the case $h_k - c_0 \geq 0$. Now consider $h_k - c_0 < 0$ (where $\mathcal{U}_k$ has a downward jump at zero). In this case we have $\mathcal{U}_k(1) - \mathcal{U}_k(0) = (1+\lambda)\, h_k - c_0 - 1/2$, which is increasing in λ and it equals 0 for

$$\lambda = \frac{1}{h_k}\left(\frac{1}{2} - (h_k - c_0)\right)$$

This proves (4.1) for the case $h_k - c_0 < 0$. That λ^* is increasing in c_0 and decreasing in h_k follows immediately from its definition. □

Appendix B: Proof of Theorem 3

Let $\varphi_k^* = \arg\max_{\varphi_k \in [\underline{\varphi},1]} \{\mathcal{U}_k(\varphi_k)\}$ and observe that $\varphi_k^* = 1$ must hold when $h_k - c_0 \geq 1$ because $\underline{\varphi} = 1$ (cf. (2.3)), so we focus on $h_k - c_0 < 1$ (whence $\underline{\varphi} < 1$) from now on. Using the formulas of $C(\varphi_k)$ and $\mathcal{U}_k(\varphi_k)$ (cf. (3.3) and (7.1)), for $\varphi_k > \underline{\varphi}$ we have

$$C'(\varphi_k) = (1-2\pi)\,\varphi_k + \pi\,(1+h_k) + (1-\pi)\,c_0 \tag{7.5}$$

$$\mathcal{U}_k'(\varphi_k) = (1+\lambda)\,h_k - C'(\varphi_k) = (1+\lambda)\,h_k - (1-2\pi)\,\varphi_k - \pi\,(1+h_k) - (1-\pi)\,c_0 \tag{7.6}$$

Suppose φ_k^* is interior so that $\varphi_k^* \in (\underline{\varphi},1)$; then $\mathcal{U}_k'(\varphi_k^*) = 0$ and $\mathcal{U}_k''(\varphi_k^*) \leq 0$ must hold. For $\pi > 1/2$, however, $\mathcal{U}_k''(\varphi_k) = 2\pi - 1 > 0$ and thus $\varphi_k^* \in (\underline{\varphi},1)$ cannot hold. Therefore, φ_k^* must be a corner solution and it equals 1 or $\underline{\varphi}$ if $\mathcal{U}_k(1)$ is respectively larger or smaller than $\mathcal{U}_k(\underline{\varphi})$. From equations (3.3) and (7.1) simple algebra shows that $\mathcal{U}_k(1) - \mathcal{U}_k(\underline{\varphi}) > 0$ if and only if $\lambda > \lambda^*$, the same threshold as in Theorem 2. So for $\pi > 1/2$ the result is the same as in the case of $\pi = 1/2$.

Now consider $\pi < 1/2$. We start by assuming $h_k - c_0 \geq 0$ to prove the statement in the text. In this case $\mathcal{U}_k$ is continuous on $[\underline{\varphi},1]$ and strictly concave so the optimum φ_k^* is equal to $\underline{\varphi}$ if $\mathcal{U}_k'(\underline{\varphi}) \leq 0$, interior if $\mathcal{U}_k'(\underline{\varphi}) > 0 > \mathcal{U}_k'(1)$, and equal to 1 if $\mathcal{U}_k'(1) \geq 0$.

It is easy to to verify that

$$\mathcal{U}_k'(\underline{\varphi}) \leq 0 \Longleftrightarrow \lambda \leq \pi \frac{1-(h_k-c_0)}{h_k} = 2\pi\lambda^*$$

$$\mathcal{U}_k'(1) \geq 0 \Longleftrightarrow \lambda \geq (1-\pi)\,\frac{1-(h_k-c_0)}{h_k} = 2\,(1-\pi)\,\lambda^*.$$

Observe that $\pi < 1/2$ implies $2\pi\lambda^* \leq \lambda^* \leq 2(1-\pi)\lambda^*$, and both boundaries converge to λ^* as $\pi \to 1/2$ from below. The stationary point $\mathcal{U}_k'(\varphi_k^o) = 0$ being

$$\varphi_k^o(\lambda) = \frac{\lambda h_k + (1-\pi)\,(h_k-c_0) - \pi}{1-2\pi}, \tag{7.7}$$

the optimal solution is as follows:

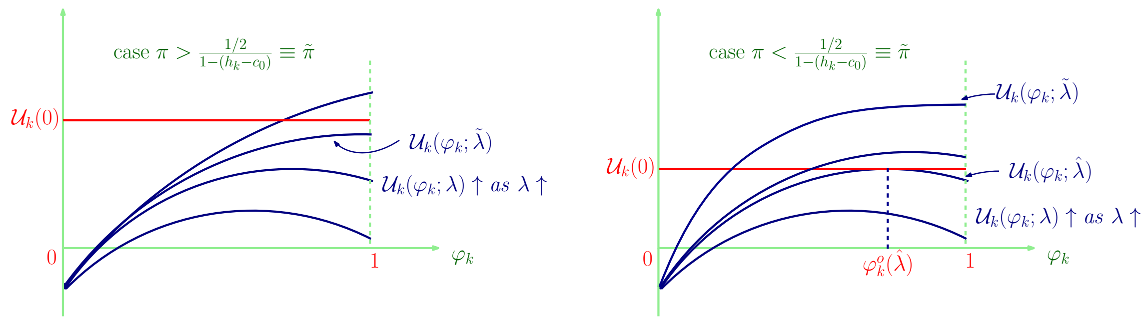
$$\varphi_k^* = \begin{cases} \underline{\varphi} & \text{if } \lambda \leq 2\pi\lambda^* \\ \frac{\lambda h_k + (1-\pi)\underline{\varphi} - \pi}{1-2\pi} & \text{if } \lambda \in (2\pi\lambda^*, 2(1-\pi)\lambda^*) \\ 1 & \text{if } \lambda \geq 2(1-\pi)\lambda^* \end{cases} \tag{7.8}$$

Plugging $\pi = 1/2 - \epsilon$ for $\epsilon \in (0,1/2)$ in to (7.8) yields (4.2). This ends the proof of the statements in Theorem 3 in the text.

We turn now to the case $h_k - c_0 < 0$ and $\pi < 1/2$ which we mentioned just after the

Theorem. In this case we have $\underline{\varphi} = 0$, $\mathcal{U}_k(0) = 0$ and $\mathcal{U}_k(\varphi_k)$ has a downward jump at zero (because $C(\varphi_k)$ has an upward jump of the same size there). Before formally stating the results, it is helpful to visualize the situation first:

Figure 7.1: $\mathcal{U}_k(\varphi_k;\lambda)$



In what follows, we write $\mathcal{U}_k(\varphi_k)$ as $\mathcal{U}_k(\varphi_k;\lambda)$ for all $\varphi_k > 0$ to make explicit its dependence on λ. Figure 7.1 depicts the family of functions $\mathcal{U}_k(\varphi_k;\lambda)$ depending on whether π lies above or below a threshold

$$\tilde{\pi} \equiv \frac{1/2}{1-(h_k-c_0)}\ .$$

Two observations are critical to the analysis. First, $\mathcal{U}_k(\varphi_k;\lambda)$ strictly increases in λ for all $\varphi_k > 0$ because $\partial\mathcal{U}_k(\varphi_k;\lambda)/\partial\lambda = h_k\varphi_k > 0$. Second, as is clear from (7.7), the stationary point $\varphi_k^o(\lambda)$ is continuously increasing in λ and there exists a unique threshold

$$\tilde{\lambda} \equiv \frac{(1-\pi)\,(1+c_0-h_k)}{h_k}$$

such that $\varphi_k^o(\tilde{\lambda}) = 1$. As the left panel of Figure 7.1 shows, for $\pi > \tilde{\pi}$ it holds that $\mathcal{U}_k(1;\tilde{\lambda}) < \mathcal{U}_k(0)$ so that $\varphi_k^* = 0$ for λ close to $\hat{\lambda}$; as λ grows further $\mathcal{U}_k(1;\lambda)$ crosses $\mathcal{U}_k(0)$ at some threshold and φ_k^* jumps to 1. We will show that this threshold coincides with $\lambda^* = \frac{1}{h_k}\left(\frac{1}{2}-(h_k-c_0)\right)$defined in the main text so that the tipping result in Theorem 2 still applies. The right panel suggests that for $\pi < \tilde{\pi}$ we have $\mathcal{U}_k(1;\tilde{\lambda}) > \mathcal{U}_k(0)$ so that for λ slightly below $\tilde{\lambda}$ the optimal solution φ_k^* is given by $\varphi_k^o(\lambda) \in (0,1)$ and the value equals $\mathcal{U}_k(\varphi_k^o(\lambda);\lambda)$; as λ decreases further $\mathcal{U}_k(\varphi_k^o(\lambda);\lambda)$ strictly decreases and there exists a threshold $\hat{\lambda}$ (derived below) such that $\mathcal{U}_k(\varphi_k^o(\hat{\lambda});\hat{\lambda}) = 0$. Hence, φ_k^* drops discontinuously from $\varphi_k^o(\hat{\lambda})$ to 0 as λ crosses $\hat{\lambda}$ from above; whence the tipping result holds.

In the remainder of this appendix we formally derive φ_k^* for the two cases: $\pi \geq \tilde{\pi}$ and $\pi < \tilde{\pi}$. To begin with, it is useful to observe that $\tilde{\lambda} < \lambda^*$ if and only if $\pi > \tilde{\pi}$ and that $\mathcal{U}_k(1;\lambda) < \mathcal{U}_k(0)$ if and only if $\lambda < \lambda^*$.

Case 1: $\pi > \tilde{\pi}$ so that $\tilde{\lambda} < \lambda^*$. For $\lambda > \lambda^*$, it follows from previous arguments that

$\mathcal{U}_k(\varphi_k;\lambda)$ is increasing in φ_k^* on $(0,1]$ and $\mathcal{U}_k(1;\lambda) > \mathcal{U}_k(0)$. Therefore, $\varphi_k^* = 1$. For $\lambda < \lambda^*$, we argue that $\mathcal{U}_k(0) > \mathcal{U}_k(\varphi_k;\lambda)$ for all $\varphi_k \in (0,1]$ and therefore $\varphi_k^* = 0$. To see this, notice that $\mathcal{U}_k(\varphi_k;\lambda)$ is strictly increasing in λ for all $\varphi_k > 0$ and it is strictly increasing in φ_k for $\lambda = \lambda^* > \tilde{\lambda}$. Hence, for all $\varphi_k \in (0,1]$ and $\lambda < \lambda^*$ we have

$$\mathcal{U}_k(\varphi_k;\lambda) < \mathcal{U}_k(\varphi_k;\lambda^*) \le \mathcal{U}_k(1;\lambda^*) = 0 = \mathcal{U}_k(0)\,.$$

Taken together, when $\pi > \tilde{\pi}$ it holds that $\varphi_k^* = 1$ for $\lambda > \lambda^*$ and $\varphi_k^* = 0$ for $\lambda < \lambda^*$, which coincides with 4.1 in Theorem 2.

Case 2: $\pi < \tilde{\pi}$ so that $\tilde{\lambda} > \lambda^*$. For this case, we shall establish that

$$\varphi_k^* = \begin{cases} 0 & \text{if } \lambda < \hat{\lambda} \\ \frac{\lambda h_k - (1-\pi)(c_0 - h_k) - \pi}{1-2\pi} & \text{if } \lambda \in \left[\hat{\lambda}, \tilde{\lambda}\right] \\ 1 & \text{if } \lambda > \tilde{\lambda} \end{cases} \tag{7.9}$$

where

$$\hat{\lambda} \equiv \frac{1}{h_k}\left[\pi + (1-\pi)(c_0 - h_k) + 2\sqrt{\left(\frac{1}{2} - \pi\right)\pi(c_0 - h_k)}\right].$$

It can be verified $\hat{\lambda} < \lambda^* < \tilde{\lambda}$ when $\pi < \tilde{\pi}$. Moreover, for $\lambda = \hat{\lambda}$, we have

$$\varphi_k^* = \frac{\hat{\lambda} h_k - (1-\pi)(c_0 - h_k) - \pi}{1 - 2\pi} = \sqrt{\frac{\pi (c_0 - h_k)}{1/2 - \pi}} \in (0,1)$$

for $\pi \in (0, \tilde{\pi})$. Therefore, in the same spirit of Theorem 2 for the case of $\pi = 1/2$, φ_k^* increases discontinuously as λ crosses a threshold $\hat{\lambda}$. To show (7.9), first consider $\lambda \ge \tilde{\lambda}$. In this case, $\mathcal{U}_k(\varphi_k;\lambda)$ is strictly increasing on $(0,1]$ and $\mathcal{U}_k(1;\lambda) > \mathcal{U}_k(0)$ because $\lambda \ge \tilde{\lambda} > \lambda^*$. Hence $\varphi_k^* = 1$. Below we assume $\lambda < \tilde{\lambda}$ such that φ_k^* can only be $\varphi_k^o(\lambda)$ (if positive) or 0. Notice that $\varphi_k^o(\lambda) \in (0,1)$ if and only if $\mathcal{U}_k'(1;\lambda) < 0 < \lim_{\varphi_k \downarrow 0} \mathcal{U}_k'(\varphi_k;\lambda)$. $\mathcal{U}_k'(1;\lambda) < 0$ is equivalent to $\lambda < \tilde{\lambda}$. $0 < \lim_{\varphi_k \downarrow 0} \mathcal{U}_k'(\varphi_k;\lambda)$ holds if $(1+\lambda) h_k - \pi (1 + h_k) - (1-\pi) c_0 > 0$,or equivalently,

$$\lambda > \frac{\pi + (1-\pi)(c_0 - h_k)}{h_k} \equiv \underline{\lambda}\,.$$

Hence, $\varphi_k^o(\lambda) \in (0,1)$ if and only if $\underline{\lambda} < \lambda < \tilde{\lambda}$, and it holds that $\varphi_k^o(\lambda) = 1$ for $\lambda = \tilde{\lambda}$ and $\varphi_k^o(\lambda) \downarrow 0$ for $\lambda \downarrow \underline{\lambda}$. Assume $\underline{\lambda} < \lambda < \tilde{\lambda}$. For $\varphi_k^o(\lambda)$ to be globally optimal, it must hold that $\mathcal{U}_k(\varphi_k^o(\lambda);\lambda) \ge \mathcal{U}_k(0)$. Because $\mathcal{U}_k(\varphi_k^o(\lambda);\lambda)$ is continuous and strictly increasing in λ and it satisfies $\mathcal{U}_k(\varphi_k^o(\tilde{\lambda});\tilde{\lambda}) = \mathcal{U}_k(1;\tilde{\lambda}) > \mathcal{U}_k(0)$ and $\lim_{\lambda \downarrow \underline{\lambda}} \mathcal{U}_k(\varphi_k^o(\lambda);\lambda) = (h_k - c_0)/2 < 0$, there exists a unique threshold $\lambda \in \left(\underline{\lambda}, \tilde{\lambda}\right)$ such that $\mathcal{U}_k(\varphi_k^o(\lambda);\lambda) = 0$. It can be verified that this threshold is precisely $\hat{\lambda}$ defined above and it indeed satisfies $\underline{\lambda} < \hat{\lambda} < \tilde{\lambda}$. Therefore,

$\mathcal{U}_k(\varphi_k^o(\lambda);\lambda) < 0$ and thus $\varphi_k^* = 0$ for $\lambda \in \left(\underline{\lambda}, \hat{\lambda}\right)$, while $\mathcal{U}_k(\varphi_k^o(\lambda);\lambda) > 0$ and $\varphi_k^* = \varphi_k^o(\lambda)$ for $\lambda < \left(\hat{\lambda}, \tilde{\lambda}\right)$. Finally, for $\lambda \le \underline{\lambda}$, it holds that $\mathcal{U}_k(\varphi_k;\lambda)$ is strictly decreasing in φ_k and hence $\varphi_k^* = 0$. Combining these together, we obtain (7.9).

Appendix C: Proof of Theorem 4

We first derive the optimal solution without tipping; that is, the sports team only sells to committed fans whose fraction equals $\underline{\varphi}$. By (2.3) and the fact that $c_0 = \zeta_0 + r$, we have

$$\underline{\varphi} = \begin{cases} 0 & \text{if } h_k - \zeta_0 - r < 0 \\ h_k - \zeta_0 - \rho & \text{if } 0 \le h_k - \zeta_0 - r \le 1 \\ 1 & \text{if } h_k - \zeta_0 - r > 1\,. \end{cases} \tag{7.10}$$

Then we have a standard monopoly pricing problem:

$$\max_r \{r \cdot \min \{h_k - \zeta_0 - r, Q\}\}$$

where r is the ticket price the sports team charges and $h_k - \zeta_0 - r \ge 0$ has to hold. The maximum of $r(h_k - \zeta_0 - r)$ is $r = (h_k - \zeta_0)/2$ giving $\underline{\varphi} = (h_k - \zeta_0)/2 > 0$. If this is less than the stadium capacity Q then this is the solution and profit is $(h_k - \zeta_0)^2/4$. If it is greater than Q then r should be chosen so that $h_k - \zeta_0 - r = Q$, that is $r = h_k - \zeta_0 - Q$. In other words the solution is sell to the fraction $(h_k - \zeta_0)/2$ if the constraint does not bind, otherwise sell at full capacity Q at price

$$r^* = \begin{cases} (h_k - \zeta_0)/2 & \text{if } \ (h_k - \zeta_0)/2 \le Q \\ h_k - \zeta_0 - Q & \text{if } \ (h_k - \zeta_0)/2 > Q \end{cases}$$

with profit

$$\Pi^* = \begin{cases} (h_k - \zeta_0)^2/4 & \text{if } \ (h_k - \zeta_0)/2 \le Q \\ (h_k - \zeta_0 - Q)\,Q & \text{if } \ (h_k - \zeta_0)/2 > Q \end{cases}.$$

Notice that the optimal solution r^* without tipping is non-increasing in capacity Q. Moreover, when $(h_k - \zeta_0)/2 > Q$, the sports team can optimally sell at full capacity Q by charging a sufficiently low price such that the fraction of committed fans just meets the stadium capacity Q.

The following theorem, which contains Theorem 4 in the main text as a special case, fully characterizes the optimal pricing strategy of the sports team.

Theorem 6. *If $2h_k\lambda \le 1 - Q$ the monopoly solution r^* is always optimal. If $2h_k\lambda > 1 - Q$*

then for $\lambda \geq \hat{\lambda}_Q$ it is optimal for the sports team to exploit self-organization of fans, sell at capacity Q and charge price r_Q^*; otherwise the optimum is given by the monopoly solution r^* in the text. The externality threshold is

$$\hat{\lambda}_Q = \begin{cases} \frac{1-Q}{2h_k} & \text{if } h_k - \zeta_0 \geq 2Q \\ \frac{1}{2h_k}\left[\left(\frac{h_k-\zeta_0}{2} - 1\right)^2 + \left(\frac{1}{Q} - 1\right)\left(\frac{h_k-\zeta_0}{2}\right)^2\right] & \text{if } h_k - \zeta_0 < 2Q \end{cases}$$

and

$$r_Q^* = \begin{cases} (1+2\lambda)h_k - \zeta_0 - 1 & \text{if } \hat{\lambda}_Q \leq \lambda < 1/(2h_k) \\ h_k - \zeta_0 + (2\lambda h_k - 1)\frac{Q}{1+Q} & \text{if } \lambda \geq 1/(2h_k) \end{cases}$$

r_Q^* is weakly increasing in Q for $\lambda \geq \hat{\lambda}_Q$.

Proof. The key observation is that to trigger self-organization so that all fans participate the following two conditions must be satisfied:

(i) *Incentive compatibility*: the marginal cost must be non-positive, that is $\xi \leq 0$.

(ii) *Group rationality*: the group payoff from participation at maximum capacity Q must be higher than without self-organization, that is $\mathcal{U}_k(Q) - \mathcal{U}_k(\underline{\varphi}) \geq 0$.

By (7.2) and $c_0 = \zeta_0 + r$ we have

$$\xi = \frac{1}{2} - \frac{h_k - c_0}{2} - \lambda h_k = \frac{1}{2} - \frac{h_k - \zeta_0 - r}{2} - \lambda h_k\,. \tag{7.11}$$

Therefore $\xi \leq 0$ if and only if

$$r \leq r^I \equiv (1+2\lambda)h_k - \zeta_0 - 1$$

There would be no point in using prices higher than this: only the committed fans would buy the ticket, and $r > r^I$ is equivalent to $h_k - \zeta_0 - r < 1 - 2\lambda h_k$ so if $2\lambda h_k \geq 1$ this means selling no tickets at all (not optimal), while if $2\lambda h_k < 1$ then profit would be $r(h_k - \zeta_0 - r)$ which is the same as the monopoly problem.

At price r^I we have $c_0 = \zeta_0 + r^I = (1+2\lambda)\,h_k - 1$, and $h_k - c_0 = 1 - 2\lambda h_k \geq 0$ if and only if $2\lambda h_k \leq 1$. Therefore, by (7.10), the fraction of committed voters under price r^I is given by

$$\underline{\varphi} = \begin{cases} 1 - 2\lambda h_k & \text{if } \ 2\lambda h_k \leq 1 \\ 0 & \text{if } \ 2\lambda h_k > 1 \end{cases}$$

Noe that if $1 - 2\lambda h_k \geq Q$ (or equivalently $2h_k\lambda \leq 1 - Q$), then $\underline{\varphi} \geq Q$ and $r^I \leq h_k - \zeta_0 - Q$ so that the profit must be bounded above by $r^I Q \leq (h_k - \zeta_0 - Q)\,Q \leq \Pi^*$. Therefore, for $2h_k\lambda \leq 1 - Q$ the monopoly solution is optimal. In what follows we then assume $2h_k\lambda > 1 - Q$

so that $\underline{\varphi} < Q$. From here on we distinguish two cases: $2\lambda h_k \leq 1$ and $2\lambda h_k > 1$.

Case 1: $2\lambda h_k \leq 1$.In this case it follows from Lemma 1 that $\mathcal{U}_k(\varphi_k)$ is linear in φ_k on $(\underline{\varphi}, 1]$ and is continuous at $\underline{\varphi}$. By Theorem 2 participation at maximal capacity Q occurs only if $\xi \leq 0$, or equivalently $r \leq r^I$. r^I is thus the highest price the seller can charge that induce participation rate Q under self-organization, in which case the profit is $r^I Q$. It remains to compare this profit with the monopoly profit Π^*. If $h_k - \zeta_0 \geq 2Q$, we have $\Pi^* = (h_k - \zeta_0 - Q)\, Q < r^I Q$ because $r^I = h_k - \zeta_0 + 2\lambda h_k - 1 > h_k - \zeta_0 - Q$. The optimal solution is then to sell at capacity Q with price r^I for all $\lambda > \frac{1-Q}{2h_k}$. If $h_k - \zeta_0 < 2Q$, then $\Pi^* = (h_k - \zeta_0)^2/4$. Simple algebra shows that $r^I Q \geq (h_k - \zeta_0)^2/4$ if and only if

$$\lambda > \frac{1}{2h_k}\left[\left(\frac{h_k - \zeta_0}{2} - 1\right)^2 + \left(\frac{1}{Q} - 1\right)\left(\frac{h_k - \zeta_0}{2}\right)^2\right] = \hat{\lambda}_Q$$

Moreover, for all $0 < h_k - \zeta_0 \leq 2Q$ it can be verified that

$$\frac{1-Q}{2h_k} < \frac{1}{2h_k}\left[\left(\frac{h_k - \zeta_0}{2} - 1\right)^2 + \left(\frac{1}{Q} - 1\right)\left(\frac{h_k - \zeta_0}{2}\right)^2\right] \leq \frac{1}{2h_k}$$

Hence, when $(h_k - \zeta_0)/2 \leq Q$, it is optimal to sell to capacity if $\lambda \in \left[\hat{\lambda}_Q, 1/(2h_k)\right]$.Below we will show that the same is also true for all $\lambda > 1/(2h_k)$. The conclusion will be that when $(h_k - \zeta_0)/2 \leq Q$ it is optimal to sell at full capacity for all $\lambda \geq \hat{\lambda}_Q$.

Case 2: $2\lambda h_k > 1$. In this case we have $h_k - c_0 < 0$, $\underline{\varphi} = 0$ and there is a discontinuous drop of $\mathcal{U}_k(\varphi_k)$ at 0. For selling at maximum capacity Q to be optimal, the price r must satisfy the group rationality constraint that $U_k(0) \leq \mathcal{U}_k(Q)$. Simple algebra shows that this condition is equivalent to

$$r \leq h_k - \zeta_0 + (2\lambda h_k - 1)\frac{Q}{1+Q} \equiv r^{II}$$

Notice that r^{II} is increasing in Q so that the capacity limit Q does restrict the price the firm can charge. Again, for r^{II} to be optimal, it must hold that $r^{II}Q > \Pi^*$. If $(h_k - \zeta_0)/2 > Q$, then $r^{II}Q > \Pi^*$ if and only if $r^{II} = h_k - \zeta_0 + (2\lambda h_k - 1)\frac{Q}{1+Q} > h_k - \zeta_0 - Q$, which always holds because $2\lambda h_k - 1 > 0 > -Q$. If $(h_k - \zeta_0)/2 \leq Q$, then $\Pi^* = (h_k - \zeta_0)^2/4$ and $r^{II}Q > \Pi^*$ holds if and only if

$$\lambda > \frac{1}{2h_k}\left\{(1+Q)\left[\left(\frac{h_k - \zeta_0}{2Q} - 1\right)^2 - 1\right] + 1\right\} \equiv \tilde{\lambda}_Q$$

Notice that $\tilde{\lambda}_Q < 1/(2h_k)$ always holds for $(h_k - \zeta_0)/2 < Q$, because $\left(\frac{h_k - \zeta_0}{2Q} - 1\right)^2 - 1 < 0$.

Therefore, selling at the maximum capacity at price r^{II} is optimal for all $\lambda > 1/2h_k$.Taken together, if $\lambda > 1/(2h_k)$ then it is optimal to sell at the maximum capacity Q at price r^{II}.

Combining these together: it is optimal to exploit the self-organization of fans and sell at capacity Q if $\lambda > \hat{\lambda}_Q$. □

Appendix D: Proof of Theorem 5

The following lemma characterizes the per capita utility $\mathcal{U}_k(b_k, b_{-k}, V)$ for party $k = L, S$ as a function of both parties' bids and the total prize V.

Lemma 2. *Suppose* $\pi = 1/2$ *and* $\underline{\varphi} = hV - c_0 \in (0, 1)$. *Then*

$$\mathcal{U}_k(b_k, b_{-k}, V) = \Pi_k(b_k, b_{-k})\frac{V}{\eta_k} - \xi(V)\frac{b_k}{\eta_k} + \frac{hV - c_0}{2}$$

where

$$\xi(V) \equiv \frac{1}{2} - \frac{hV - c_0}{2} - \lambda hV$$

is the marginal cost of increasing turnout rate $\varphi_k = b_k/\eta_k$ *for party* k *and it is decreasing in total prize* V.

Proof. Consider any implementable turnout rate $\varphi_k \in [\underline{\varphi}, 1]$ for party k. By (2.1), (2.2) and (3.4), the per capita utility of party k is given by[12]

$$\begin{aligned}\mathcal{U}_k(\varphi_k) &= (1 - \varphi_k)\left[p_k(\varphi_k)v_k + \lambda\varphi_k h_k\right] + \varphi_k\left[p_k v_k + (1 + \lambda\varphi_k)\, h_k\right] - C\left(\varphi_k\right)\\ &= p_k(\varphi_k)v_k + (1 + \lambda)\,\varphi_k h_k - \frac{1 + h_k + c_0}{2}\varphi_k + \frac{h_k - c_0}{2}\\ &= p_k(\varphi_k)v_k - \left(\frac{1}{2} - \frac{h_k - c_0}{2} - \lambda h_k\right)\varphi_k + \frac{h_k - c_0}{2}\end{aligned}$$

In this voting context we have $v_k = V/\eta_k$, $h_k = hV$, $\varphi_k = b_k/\eta_k$ and $p_k(\varphi_k) = \Pi_k(\eta_k\varphi_k, \eta_{-k}\varphi_{-k})$ for $k = L, S$. Plugging these into $\mathcal{U}_k(\varphi_k)$ yields the statements in this lemma. □

In what follows we prove Theorem 5. Under Assumption 1 we have

$$\underline{\varphi} \to -c_0 \quad \text{and} \quad \xi(V) \to \frac{1 + c_0}{2} - \kappa V = \kappa\left(\overline{V} - V\right) \tag{7.12}$$

where recall that

$$\overline{V} \equiv \frac{1 + c_0}{2\kappa}$$

[12] Under assumptions $\pi = 1/2$ and $\underline{\varphi} = hV - c_0 \in (0, 1)$, the cost function $C\left(\varphi_k\right)$ is continuous at $\underline{\varphi}$.

In the following analysis we work in the limit so we take $\underline{\varphi} = -c_0$ and $\xi(V) = \kappa\left(\overline{V} - V\right)$. Therefore, the marginal cost $\xi(V)$ is decreasing in V and it becomes negative for V larger than $\overline{V}$.

We first establish that, under Assumption 1, for $V > \overline{V}$ there exists a unique equilibrium in dominant strategy in which both parties bid their maxima (i.e., $b_k = \eta_k$ for $k = L, S$) so that $\overline{b} = b_L + b_S = \eta_L + \eta_S$. By Lemma 2 and Assumption 1, we have

$$\begin{aligned}\mathcal{U}_k(b_k, b_{-k}, V) &= \Pi_k(b_k, b_{-k})\frac{V}{\eta_k} - \xi(V)\frac{b_k}{\eta_k} + \frac{\underline{\varphi}}{2} \\ &\to \Pi_k(b_k, b_{-k})\frac{V}{\eta_k} - \kappa\left(\overline{V} - V\right)\frac{b_k}{\eta_k} - \frac{c_0}{2}\end{aligned} \tag{7.13}$$

The second step follows from the fact that $\underline{\varphi} = hV - c_0 \to -c_0$ and $\xi(V) \to \kappa\left(\overline{V} - V\right)$ under Assumption 1 (cf.(7.12)). Observe for $V > \overline{V}$ that $\mathcal{U}_k(b_k, b_{-k}, V)$ is strictly increasing in b_k because $\Pi_k(b_k, b_{-k})$ is non-decreasing in b_k and $\kappa\left(\overline{V} - V\right)$ is strictly negative. Hence, it is a strictly dominant strategy for each party to bid is maximum and this yields the unique equilibrium.[13] This proves the statement in Theorem 5 for $V > \overline{V}$.

In what follows we assume $\eta_S > \eta_L\underline{\varphi}$ and $V < \overline{V}$ so that the marginal cost $\kappa\left(\overline{V} - V\right)$ is strictly positive. We will exploit results from Levine and Mattozzi (2020) to establish that for V smaller than but close to $\overline{V}$ there exists a unique equilibrium in mixed strategy. Moreover, in this equilibrium party L bids almost surely η_S while party S bids almost surely $\eta_S\underline{\varphi}$ so that the total bid is almost surely $\eta_S\left(1 + \underline{\varphi}\right)$. This then completes the proof for Theorem 5.

We introduce a few definitions and notations. For each party $k \in \{L, S\}$, we define its *desire to bid* $B_k(V)$ as the highest for which party k prefers to get the prize V for sure to bidding $\eta_k\underline{\varphi}$ and get no prize. By (7.13), $B_k(V)$ is given by the solution b_k to

$$\frac{V}{\eta_k} - \xi(V)\frac{b_k}{\eta_k} + \frac{\underline{\varphi}}{2} = -\xi(V)\underline{\varphi} + \frac{\underline{\varphi}}{2}\ .$$

This yields

$$B_k(V) = \frac{V}{\xi(V)} + \eta_k\underline{\varphi}\ .$$

We further define party k's *willingness-to-bid* as $W_k(V) = \min\{B_k(V), \eta_k\}$; this equals the the maximum bid party k is willing or afford to pay. Since $\underline{\varphi} > 0$ and $\eta_L > \eta_S$, we have $W_S(V) < W_L(V)$; that is, party S is the *disadvantage group* who has the lower willingness-to-bid. Finally, we let $\underline{V}$ denote the lowest level of prize V such that the disadvantage party

[13] If $V = \overline{V}$, then $\xi(V) = 0$ and $\mathcal{U}_k(b_k, b_{-k}, V)$ is weakly increasing in b_k. Hence, bidding the maximum is still the unique equilibrium in pure and weakly dominant strategies.

S is just indifferent between winning prize V for sure with bid $\eta_L \underline{\varphi}$ (i.e., the smallest bid of the advantaged party L) and bidding $\eta_L \underline{\varphi}$ and get no prize. Therefore, by (7.12), $\underline{V}$ is the solution V to

$$\frac{V}{\eta_S} - \xi(V) \frac{\eta_L}{\eta_S} \underline{\varphi} + \frac{\varphi}{2} = -\xi(V) \underline{\varphi} + \frac{\varphi}{2} \ .$$

Using the fact that $\xi(V) = \kappa \left(\overline{V} - V \right)$, we obtain

$$\underline{V} = \overline{V} \frac{\kappa \underline{\varphi} (\eta_L - \eta_S)}{1 + \kappa \underline{\varphi} (\eta_L - \eta_S) 1} < \overline{V} \ .$$

We denote each party k's (mixed) bidding strategy by F_k, a cdf on $\left[\eta_k \underline{\varphi}, \eta_k \right]$. The following lemma follows from Levine and Mattozzi (2020).

Lemma 3. *(Levine and Mattozzi, 2020) Suppose $\underline{V} < V < \overline{V}$ and $\eta_S > \eta_L \underline{\varphi}$. Then there is a unique equilibrium in which both parties play the mixed strategies given by*

$$F_L(x) = \begin{cases} 1 & \text{if } x \geq W_S(V) \\ \frac{\xi(V)}{V} \left(x - \underline{\varphi} \eta_S \right) & \text{if } x \in \left[\eta_L \underline{\varphi}, W_S(V) \right) \\ 0 & \text{if } x < \eta_L \underline{\varphi} \end{cases} \tag{7.14}$$

$$F_S(x) = \begin{cases} 1 & \text{if } x \geq W_S(V) \\ 1 - \frac{\xi(V)}{V} \left(W_S(V) - x \right) & \text{if } x \in \left[\underline{\varphi} \eta_L, W_S(V) \right) \\ 1 - \frac{\xi(V)}{V} \left(W_S(V) - \underline{\varphi} \eta_L \right) & \text{if } x \in \left[\underline{\varphi} \eta_S, \underline{\varphi} \eta_L \right) \\ 0 & \text{if } x < \underline{\varphi} \eta_S \end{cases} \tag{7.15}$$

The aggregate bid b is the sum of two independent random variables with $b_k \sim F_k$ for $k \in \{L, S\}$.

Proof. This lemma is a direct application of Theorem 1 in the Online Appendix of Levine and Mattozzi (2020) to our model. □

Since $\lim_{V \nearrow \overline{V}} \xi(V) = \lim_{V \nearrow \overline{V}} \kappa \left(\overline{V} - V \right) = 0$, so $\lim_{V \nearrow \overline{V}} V / \xi(V) = \infty$. Therefore for V sufficiently close to $\overline{V}$ we have $W_S(V) = \min \left\{ V/\xi(V) + \eta_S \underline{\varphi}, \eta_S \right\} = \eta_S$. Using (7.15), (7.15) and letting $F_k^-(x) = \lim_{y \nearrow x} F_k(y)$, we obtain

$$F_S(\underline{\varphi} \eta_S) = 1 - \frac{\xi(V)}{V} \left(\eta_S - \underline{\varphi} \eta_L \right) \to 1 \quad \textit{and} \quad F_S^-(\underline{\varphi} \eta_S) = 0 \ ,$$
$$F_L(\eta_S) = 1 \quad \textit{and} \quad F_L^-(\eta_S) = \frac{\xi(V)}{V} \left(1 - \underline{\varphi} \right) \eta_S \to 0 \ .$$

These together imply that the probabilities of $b_S = \underline{\varphi} \eta_S$ and $b_L = \eta_S$ tend to 1. Consequently, the probability $b_S + b_L = \eta_S \left(1 + \underline{\varphi} \right)$ tends to 1 as $V \to \overline{V}$ from below. This establishes the statement for V being smaller but sufficiently close to $\overline{V}$ for Theorem 5.