Pair-wise envy free and stable matchings for two-sided systems with techniques

by

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Introduction: The two-sided matching model of Gale and Shapley (1962) can be interpreted as one where a non-empty finite set of firms need to employ a non-empty finite set of workers. Further, each firm can employ at most one worker and each worker can be employed by at most one firm. Each worker has preferences over the set of firms and each firm has preferences over the set of workers. An assignment of workers to firms is said to be stable if there does not exist a firm and a worker who prefer each other to the ones they are associated with in the assignment. Gale and Shapley (1962) proved that every two-sided matching problem admits at least one stable matching.

In this paper we extend the above model by including a non-empty finite set of techniques. An assignment now comprises disjoint triplets, each triplet consisting of a firm, a worker and a technique. A technique can be likened to a machine that the firm and worker together use for production. Each firm has preferences over the set of ordered pairs of workers and techniques and each worker has preferences over the set of ordered pairs of firms and techniques. We call such models two-sided systems with techniques. There are two kinds of issues we address in the context of this model, now that concerns naturally extend beyond those of pair-wise stability as defined in Gale and Shapley (1962). The first issue is about the possibility of a pair of agents being better off than in their current assignment by perhaps using a different technique. The existence of such a possibility allows for a pair of agents to 'envy' the technique that may have been assigned to a different pair. It is natural to seek an assignment that excludes 'envy' and which may therefore be called 'pair-wise envy free'. The second issue that we address in this paper, pertains to a situation where each firm is initially endowed with a technique. In such a situation we are interested in proving the existence of an assignment such that no coalition can re-allocate the techniques that they have been endowed with, and consequently be better off. A matching which satisfies this property is called stable. Through out the paper, we assume as in Danilov (2003) (: though in a slightly different context) that the preferences of the workers are lexicographic, with firms enjoying priority over techniques. Modifying the analysis for three-sided systems as in Lahiri (2004), we show that a sufficient condition for a pair-wise envy free matching to exist is the satisfaction of a certain discrimination property. The discrimination property says: given two distinct firm-worker pairs, the technique that is best for the firm in one pair is

different from the technique that is best for the firm in the other. A modified version of an example in Lahiri (2004) is used to show that a pair-wise envy free matching may not exist if a two-sided system with techniques fails to satisfy the discrimination property. However, if we assume that the preferences of the firms are also lexicographic, with workers enjoying priority over techniques, then the discrimination property can be relaxed to obtain the desired result. The weaker version of the discrimination property requires that for every firm-worker pair there is a technique that is either best for the firm or for the worker and for no two distinct pair is such a technique identical. Such problems, which we call entirely lexicographic are the ones studied by Danilov (2003) in the context of three-sided systems. While Danilov (2003) proves the existence of a stable matching for an entirely lexicographic three-sided system, a pair-wise envy free matching may fail to exist in an entirely lexicographic two-sided system with techniques. The entire analysis concerning pair-wise envy free matchings makes essential use of the deferred acceptance procedure due to Gale and Shapley (1962).
Our subsequent result shows that a stable matching always exists for an entirely lexicographic two-sided system with techniques where each firm is initially endowed with a technique. The proof of this result uses both the Gale and Shapley (1962) theorem, as well as the theorem due Shapley and Scarf (1972) concerning the existence of a core allocation in a market where indivisible objects are traded. The proof of the relevant theorem in Shapley and Scarf (1972), uses Gale's Top Trading Cycle Algorithm.
The preference of a firm is separable if its preference over workers is independent of the technique and its preference over techniques is independent of the worker. The preference of a worker is separable if its preference over firms is independent of the technique and its preference over techniques is independent of the firm. A two-sided system with techniques is said to be separable if preferences of all firms and workers are separable. Replicating some of the proofs used earlier, we can show that if a two-sided system with techniques is separable, then the results that were established for entirely lexicographic two-sided systems with techniques, continue to remain valid.

<u>The Model</u>: We define a two-sided system with techniques in the following manner. Let W be a no-empty finite set denoting the set of workers, F a non-empty finite set denoting the set of firms and T a non-empty finite set denoting the set of techniques. Each $w \in W$ has preference over $(F \times T) \cup \{w\}$ defined by a weak order (: reflexive, complete, transitive binary relation) $\geq_w$ whose asymmetric part is denoted $>_w$. Each $f \in F$ has preference over $(W \times T) \cup \{f\}$ defined by a linear order (: anti-symmetric weak order) $\geq_f$ whose asymmetric part is denoted $>_f$.
Given $w \in W$ and $f \in F$, let $A(w) = \{(w,f,t) \in \{w\} \times F \times T / (f,t) >_w w\}$ and $A(f) = \{(w,f,t) \in W \times \{f\} \times T / (w,t) >_f f\}$. A(w) is called the acceptable set of w and A(f) is called the acceptable set of f.
Let $W^* = \{w \in W / A(w) \neq \phi\}$ and $F^* = \{f \in F / A(f) \neq \phi\}$.
For the sake of expositional simplicity, we assume the following:
(i) $W^* = W$, $F^* = F$;
(ii) For all $f \in F$ and $w \in W$: $A(f) = W \times T$ and $A(w) = F \times T$.

Any non-empty subset S of $F \cup W$ is called a coalition.

A one-one function η: $F \cup W \rightarrow (W \times F \times T) \cup (W \cup F)$, satisfying:
(i) for all $a \in W \cup F$: $\eta(a) \in A(a) \cup \{a\}$;
(ii)for all $w \in W$, $f \in F$ and $t \in T$ the following are equivalent: (a) $\eta(w) = (w,f,t)$; (b) $\eta(f) = (w,f,t)$;
is called a matching.

Given a matching η, $w \in W$, $f \in F$ and $t \in T$, let

$\eta^W(w) = w$ if $\eta(w) = w$,
$= (f,t)$ if $\eta(w) = (w, f, t)$;

$\eta^F(f) = f$ if $\eta(f) = f$.
$= (w,t)$ if $\eta(f) = (w, f, t)$.

Given a matching η and a coalition S, let $T(\eta,S) = \{t \in T / \eta(w) = (w,f,t)$ for some $w \in W \cap S$ and $f \in F \cap S\}$.

Given a matching μ a pair $(w,f) \in W \times F$ is said to envy a pair $(w',f') \in W \times F$ if $(f,t) >_w \mu^W(w)$ and $(w,t) >_f \mu^F(f)$ where $\mu^W(w') = (f',t) \in F \times T$.
A matching μ is said to be pair-wise envy-free if there does not exist $w \in W$, $f \in F$ and $t \in T$ such that: $(f,t) >_w \mu^W(w)$ and $(w,t) >_f \mu^F(f)$.

Note that in the definition of a pair-wise envy free matching if $\mu^W(w') = (f',t)$ for some $(w',f') \in W \times F$, then $(f,t) >_w \mu^W(w)$ and $(w,t) >_f \mu^F(f)$ implies that (w,f) envies (w',f'). However, if there does not exist $(w',f') \in W \times F$ such that $\mu^W(w') = (f',t)$, then although $(f,t) >_w \mu^W(w)$ and $(w,t) >_f \mu^F(f)$, it is not the case that (w,f) envies a pair in $W \times F$.

A matching μ is said to be blocked by a triplet $(w,f,t) \in W \times F \times T$ if $(f,t) >_w \mu^W(w)$ and $(w,t) >_f \mu^F(f)$.
Thus a matching μ is pair-wise envy free if and only if it is not blocked by any triplet in $W \times F \times T$.

Existence of pair-wise envy free matchings: A two-sided system with techniques is said to be lexicographic for workers if for all $w \in W$ there exists a linear order P_w on F such that for all $f,f' \in F$ with $f \neq f'$ and $t,t' \in T$: fP_wf' implies $(f,t) >_w (f',t')$.

A two-sided system with techniques is said to satisfy Discrimination Property (DP) if there exists a function β:$F \times W \rightarrow T$ such that (a) for all $w,w_1 \in W^*$ and $f,f_1 \in F$ with $w \neq w_1$ and $f \neq f_1$: $\beta(f,w) \neq \beta(f_1,w_1)$; (b) for all $w \in W$, $f \in F$ and $t \in T$, $(w,\beta(f,w)) \geq_f (w,t)$.

The following example shows that merely by assuming that a two sided system with techniques lexicographic workers does not guarantee the existence of a pair-wise envy free matching.

Example 1: Let $W = \{w_1,w_2\}$, $F = \{f_1,f_2\}$, $T = \{t_1, t_2\}$.
Suppose preferences are such that for all $w\in W$, $f\in F$ and $t\in T$: w prefers (t,f) to remaining single, f prefers (w,t) to remaining single and t prefers (w,f) to remaining single.
Further assume that the system is lexicographic for workers, with both w_1 and w_2 preferring t_1 to t_2 for any given firm f. Suppose both w_1 and w_2 prefer f_1 to f_2.
Suppose f_1 prefers (w_2,t_1) to (w_1, t_1) to (w_1,t_2) to (w_2,t_2) and f_2 prefers (w_1,t_1) to (w_2,t_1) to (w_2,t_2) to (w_1, t_2).
Let us consider the following four matchings:
(1) $\{(w_1 ,f_1 ,t_1), (w_2 ,f_2 ,t_2)\}$;
(2) $\{(w_1 ,f_1 ,t_2), (w_2 ,f_2 ,t_1)\}$;
(3) $\{(w_1 ,f_2 ,t_1), (w_2 ,f_1 ,t_2)\}$;
(4) $\{(w_1 ,f_2 ,t_2), (w_2 ,f_1 ,t_1)\}$.
Matching (1) is blocked by (w_2, f_2, t_1) since w_2 prefers (f_2,t_1) to (f_2,t_2) and f_2 prefers (w_2, t_1) to (w_2, t_2).
Matching (2) is blocked by (w_2, f_1,t_1) since w_2 prefers (f_1,t_1) to (f_2,t_1) and f_1 prefers (w_2,t_1) to (w_1,t_2).
Matching (3) is blocked by (w_1,f_1,t_2) since w_1 prefers (f_1,t_2) to (f_2,t_2) and f_1 prefers (w_1,t_2) to (w_2,t_2).
Matching (4) is blocked by (w_1,f_2,t_1) since w_1 prefers (f_2,t_1) to (f_2,t_2) and f_2 prefers (w_1,t_1) to (w_1,t_2).
Hence none of the four matchings above are stable. A matching where some agents are single can similarly be shown to be unstable, since a matching with any agent single must have either two agents each on a different side of the market or all four agents remaining single.
Further, $\beta(f_1,w_2) = \beta(f_2,w_1) = t_1$. This contradicts DP.

Theorem 1: Suppose a two-sided system with techniques which is lexicographic for workers satisfies DP. Then there exists a pair-wise envy-free matching.

Proof: Suppose
(i) For all $w\in W$, there exists a linear order P_w on F such that for all $f,f'\in F$ with $f \neq f'$ and $t,t'\in T$: vP_mv' implies $(f,t) >_w (f',t')$.
(ii) There exists a function $\beta:F\times W\rightarrow T$ such that (a) for all $w,m_1\in W$ and $f,v_1 \in F$ with $w\neq m_1$ and $f\neq v_1$: $\beta(f,w) \neq \beta(f_1,w_1)$; (b) for all $w\in W$, $f\in F$ and $t\in T$, $(w,\beta(f,w)) \geq_f (w,t)$.

For $f\in F$ let P_f be a linear order on W such that for all $w, w'\in W$: wP_fw' if and only if $(w, \beta(f,w)) \geq_f (w', \beta(f, w'))$
By the applying the algorithm called deferred acceptance procedure with firms proposing as in Gale and Shapley (1962) (see appendix for details), we get a function
ρ: $W\cup F\rightarrow W\cup F$ such that:
(i) for all $w\in W$, $f\in F$:$\rho(w)\in F\cup\{w\}$, $\rho(f)\in W\cup\{f\}$;
(ii) for all $a\in W\cup F$: $\rho(\rho(a)) = a$;
(iii) there does not exist $w\in W$ and $f\in F$ such that $w \neq\rho(f)$, $f\neq\rho(w)$, $wP_f\rho(f)$ and $fP_w\rho(w)$.
The matching μ is defined as follows:

If $w \in W$ and $f \in F$ are such that $\rho(w) = f \in F$, then let $\mu(w) = \mu(f) = \mu(\beta(f,w)) = (w,f,\beta(f,w))$. For any other 'a' belonging to $W \cup F \cup T$, let $\mu(a) = a$.
By DP, μ is well defined.
Suppose the matching so defined is not pair-wise envy free. Thus, there exists $w \in W$, $f \in F$ and $t \in T$ such that: $(f,t) >_w \mu^W(w)$ and $(w,t) >_f \mu^F(f)$.
Clearly $(w,\beta(f,w)) \geq_f (w,t) >_f \mu^F(f) = (w', \beta(f, w'))$ say. By the Gale-Shapley (1962) deferred acceptance procedure with firms proposing, firm f must have proposed to w and had subsequently been rejected by w in favor of some other firm f'. Since as the deferred acceptance procedure evolves, no worker descends down his preference scale, it must be the case that worker w prefers $\mu^W(w)$ to $(f, \beta(f,w))$. Since the system is lexicographic for workers, it must be the case that $\mu^W(w) >_w (f,t)$, leading to a contradiction. Q.E.D.

It is worth noting that Theorem 1 is not valid for the kind of environment considered in Danilov (2003), after it has been adapted to the framework of two-sided matching with techniques. The kind of environment considered by Danilov (2003), which we refer to here as entirely lexicographic is the following:
A two-sided system with techniques is said to be entirely lexicographic if: (a) For all $w \in W$ there exists a linear order P_w on F such that for all $f,f' \in F$ with $f \neq f'$ and $t,t' \in T$: fP_wf' implies $(f,t) >_w (f',t')$; (t) For all $f \in F$ there exists a linear order P_f on W such that for all $w,w' \in W$ with $w \neq w'$ and $t,t' \in T$: wP_fw' implies $(w,t) >_f (w',t')$.

For instance, if each of W,F,T has at least two elements and there exists $t^* \in T$ such that for all $w \in W, f \in F$ and $t \in T \setminus \{t^*\}$: $(f,t^*) >_w (f,t)$ and $(w,t^*) >_f (w,t)$, then there does not exist any pair-wise envy free matching.

However, if we invoke the following Weak Discrimination Property (WDP) for an entirely lexicographic two-sided system with techniques, then we can prove the existence of a pair-wise envy free matching.
A two-sided system with techniques is said to satisfy Weak Discrimination Property (WDP) if there exists a function $\beta: F \times W \to T$ such that (a) for all $w,w_1 \in W$ and $f,f_1 \in F$ with $w \neq w_1$ and $f \neq f_1$: $\beta(f,w) \neq \beta(f_1,w_1)$; (b) for all $w \in W$ and $f \in F$: either $[(w,\beta(f,w)) \geq_f (w,t)$ for all $t \in T]$ or $[(f,\beta(f,w)) \geq_w (f,t)$ for all $t \in T]$.

Theorem 2: Suppose an entirely lexicographic two-sided system with techniques satisfies WDP. Then there exists a pair-wise envy-free matching.

Proof: As in Gale and Shapley (1962), we obtain a $\rho: W \cup F \to W \cup F$ such that:
(i) for all $w \in W$, $f \in F$: $\rho(w) \in F \cup \{w\}$, $\rho(f) \in W \cup \{f\}$;
(ii) for all $a \in W \cup F$: $\rho(\rho(a)) = a$;
(iii)there does not exist $w \in W$ and $f \in F$ such that $w \neq \rho(f)$, $f \neq \rho(w)$, $wP_f\rho(f)$ and $fP_w\rho(w)$.

The matching μ is defined as follows:
If $w \in W$ and $f \in F$ are such that $\rho(w) = f \in F$, then let $\mu(w) = \mu(f) = (w,f,\beta(f,w))$. For any other 'a' belonging to $W \cup F$, let $\mu(a) = a$.

It is easily verified that μ is pair-wise envy free. Q.E.D.

Stable Matchings: Let τ: $F \to T$ be a one-one function. For $f \in F$, $\tau(f)$ denotes the technique that f has been initially endowed with.
A two-sided system with techniques along with a one-one function τ from F to T is called a private ownership two-sided system with techniques.
A matching μ for such a system is said to be blocked by a coalition S if there exists a matching η on S such that (i) $\mu^W(W \cap S) = (F \cap S) \times T(\mu, S)$, $\eta^W(W \cap S) = (F \cap S) \times \tau(F \cap S)$; $\mu^F(F \cap S) = (W \cap S) \times T(\mu, S)$, $\eta^F(F \cap S) = (W \cap S) \times \tau(F \cap S)$; (ii) for all $f \in F \cap S$ and $w \in W \cap S$, $\eta^F(f) >_f \mu^F(f)$ and $\eta^W(w) >_w \mu^W(w)$.

Note: The requirements for a matching to be blocked by a coalition are considerably different from the requirements for a matching to be blocked by a triplet, as defined earlier. First a blocking coalition must comprise of firms and workers, while a blocking triplet comprises of a firm, a worker and a technique. Second a blocking coalition is in the context of an initial endowment of techniques where as no initial endowment is involved in the case of a blocking triplet.

A matching μ is said to be stable if it is not blocked by any coalition.

Theorem 3: Every entirely lexicographic private ownership two-sided system with techniques has at least one stable matching.

Proof: As in Gale and Shapley (1962), we obtain a ρ: $W \cup F \to W \cup F$ such that:
(i) for all $w \in W$, $f \in F$: $\rho(w) \in F \cup \{w\}$, $\rho(f) \in W \cup \{f\}$;
(ii) for all $a \in W \cup F$: $\rho(\rho(a)) = a$;
(iii) there does not exist $w \in W$ and $f \in F$ such that $w \neq \rho(f)$, $f \neq \rho(w)$, $wP_f\,\rho(f)$ and $fP_w\rho(w)$.

Case 1: $\#W \leq \#F$: If $\{w \in W/\rho(w) = w\} \neq \phi$ then $\{f \in F/\rho(f) = f\} \neq \phi$. Thus, if $\{w \in W/\rho(w) = w\} \neq \phi$, then there exists $w \in W$ and $f \in F$ such that $w \neq \rho(f)$, $f \neq \rho(w)$, $wP_f\,\rho(f)$ and $fP_w\rho(w)$, leading to a contradiction. Hence $\{w \in W/\rho(w) = w\} = \phi$. Thus, $\{f \in F/\rho(f) \in W\} \neq \phi$.
Case 2: $\#W > \#F$: If $\{f \in F/\rho(f) = f\} \neq \phi$ then $\{w \in W/\rho(w) = w\} \neq \phi$. Thus, if $\{f \in F/\rho(f) = f\} \neq \phi$, then there exists $w \in W$ and $f \in F$ such that $w \neq \rho(f)$, $f \neq \rho(w)$, $wP_f\,\rho(f)$ and $fP_w\rho(w)$, leading to a contradiction. Hence suppose $\{f \in F/\rho(f) = f\} = \phi$. Thus, $\{f \in F/\rho(f) \in W\} \neq \phi$.
For $f \in F$, such that $\rho(f) \in W$, let R_f be the linear order on T such that for all $t, t' \in T$: tR_ft' if and only if $(\rho(f), t) >_f (\rho(f), t')$.
Applying Gale's Top Trading Cycle Algorithm as in Shapley and Scarf (1972), there exists a one-one function x: $\{f \in F/\rho(f) \in W\} \to T$ satisfying the following property: Given any non-empty subset F^0 of $\{f \in F/\rho(f) \in W\}$ and a one-one function y: $F^0 \to \{\tau(f)/f \in F^0\}$, $\{y(f) >_f x(f)$ for some $f \in F^0]$ implies $[x(f') >_{f'} y(f')$ for some $f' \in F^0 \backslash \{f\}]$.
The matching μ is defined as follows:
If $w \in W$ and $f \in F$ are such that $\rho(w) = f \in F$, then let $\mu(w) = \mu(f) = (w, f, x(f))$. For any other 'a' belonging to $W \cup F$, let $\mu(a) = a$.

Towards a contradiction suppose there exists a matching η on S such that (i) $\mu^W(W\cap S) = (F\cap S)\times T(\mu,S)$, $\eta^W(W\cap S) = (F\cap S)\times\tau(F\cap S)$; $\mu^F(F\cap S) = (W\cap S)\times T(\mu,S)$, $\eta^F(F\cap S) = (W\cap S)\times \tau(F\cap S)$; (ii) for all $f\in F\cap S$ and $w\in W\cap S$, $\eta^F(f) >_f \mu^F(f)$ and $\eta^W(w) >_w \mu^W(w)$. Let $f\in F\cap S$ and $w\in W\cap S$ be such that $\eta(w) = (w,f,t) = \eta(f)$. Thus, $\eta^F(f) = (w,t) >_f \mu^F(f) = (\rho(f), x(f))$ and $\eta^W(w) = (f,t) >_w \mu^W(w) = (\rho(w), x(\rho(w)))$. If $\rho(f) \neq w$, then $wP_f\rho(f)$ and $fP_w\rho(w)$ leading to a contradiction. Thus, $\rho(f) = w$ and $\rho(w) = f$. Thus, given $f\in F\cap S$ there exists $y(f)\in \tau(F\cap S)$ such that: $\eta^F(f) = (\rho(f), y(f))$, where $(\rho(f), y(f)) >_f (\rho(f), x(f))$. Since $f,f'\in F\cap S$ with $f \neq f'$ implies $y(f) \neq y(f')$, we are again lead to a contradiction. Thus μ is stable. Q.E.D.

Note: A characteristic feature of the Top-Trading Cycle Algorithm used to establish the theorem due to Shapley and Scarf (1972) that we invoke, is the following: there exists a partition $\{F_1,\ldots,F_k\}$ of $\{ f\in F/\rho(f) \in W\}$ such that (i) $x(F_j) = \tau(F_j)$ for $j = 1,\ldots,k$; (ii) if $y:\{ f\in F/\rho(f) \in W\}\rightarrow T$ is any function with $y(f) >_f x(f)$ for some $f \in F_i$ and $i\in\{1,..,k\}$, then there exists $j \in\{1,..,k\}$ with $j < i$, and $f'\in F_j$ such that $y(f) = x(f')$.
It follows as a direct consequence of this observation that there does not exist any non-empty subset F^0 of $\{ f\in F/\rho(f) \in W\}$ and a function $y: \{f\in F/\rho(f) \in W\}\rightarrow T$ with $y(F^0) = x(F^0)$ such that $y(f) >_f x(f)$ for all $f\in F^0$. For if there did exist such a y, then letting $i = \min \{j/ F_j \cap F^0 \neq \phi\}$, we require that for $f\in F_i$, $y(f) \in x(F_j)$, for some $j < i$. This would contradict the requirement that $y(F^0) = x(F^0)$.

Separable Preferences: A two-sided system with techniques is said to be separable for workers if for all $w\in W$ there exists linear orders P_w on F and Q_w on T such that for all $(f,t), (f',t') \in F\times T$: $(f,t) \geq_w (f',t)$ if and only if fP_wf' and $(f,t) \geq_w (f,t')$ if and only if tQ_wt'
A two-sided system with techniques is said to be separable for firms if for all $f\in F$ there exists linear orders P_f on W and Q_f on T such that for all $(w,t), (w',t') \in W\times T$: $(w,t) \geq_f (w',t)$ if and only if wP_fw' and $(w,t) \geq_f (w,t')$ if and only if tQ_ft'.
A two-sided system with techniques is said to be separable if it is separable for both firms and workers.
If a two-sided system with techniques is separable then WDP reduces to the following: There exists a function $\beta:F\times W\rightarrow T$ such that (a) for all $w,w_1\in W$ and $f,f_1 \in F$ with $w\neq w_1$ and $f\neq f_1$:$\beta(f,w) \neq \beta(f_1,w_1)$; (b) for all $w\in W$ and $f\in F$: either $[\beta(f,w))Q_f\, t$ for all $t\in T]$ or $[\beta(f,w))\, Q_w\, t$ for all $t\in T]$.

If a two-sided system with techniques is separable, then the equivalent versions of Theorems 2 and 3 continue to be valid.

Theorem 4: Suppose a separable two-sided system with techniques satisfies WDP. Then there exists a pair-wise envy-free matching.

Theorem 5: Every separable private ownership two-sided system with techniques has at least one stable matching.

The proofs are identical to the ones provided for Theorems 2 and 3 respectively.

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Appendix

Deferred Acceptance Procedure With Firms Proposing (due to Gale and Shapley (1962): To start each firm makes an offer to her favorite worker, i.e. to the worker ranked first according to her preferences. Each worker who receives one or more offers, rejects all but his most preferred of these. Any firm whose offer is not rejected at this point is kept "pending".

At any step any firm whose offer was rejected at the previous step, makes an offer to her next choice (i.e., to her most preferred worker, among those who have not rejected her offer), so long as there remains a worker to whom she has not yet made an offer. If at any step of the procedure, a firm has already made offers to, and been rejected by all workers, then she makes no further offers. Each worker receiving offers rejects all but his most preferred among the group consisting of the new offers together with any firm that he may have kept pending from the previous step.

The algorithm stops after any step in which no firm is rejected. At this point, every firm is either kept pending by some worker or has been rejected by every worker. The matching ρ that is defined now, associates to each firm the worker who has kept her pending, if there be any. Further, workers who did not receive any offers at all, and firms who have been rejected by all the workers, remain single.