Reputational Wars of Attrition with Complex Bargaining Postures

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Abstract

Consider a two-person intertemporal bargaining problem in which players choose actions and collect payoffs while bargaining proceeds. Theory is silent regarding how the surplus is likely to be split, because a folk theorem applies. Perturbing such a game with a rich set of behavioral types for each player yields a specific asymptotic prediction for how the surplus will be divided, as the perturbation probabilities approach zero. Behavioral types may follow nonstationary strategies and respond to the opponent's play. How much should a player try to get, and how should she behave while waiting for the resolution of bargaining? In both respects she should build her strategy around the advice given by the 'Nash bargaining with threats' theory developed for two-stage games. The results suggest that there are forces at work in some dynamic games that favor certain payoffs over all others. This is in stark contrast to the classic folk theorems, to the further folk theorems established for repeated games with two-sided reputational perturbations, and to the permissive results obtained in the literature on bargaining with payoffs-as-you-go.

1 Introduction

What kind of reputation should a bargainer try to establish? Should she claim that her demand will never change, or that she will become more aggressive over time? Should improvements in her opponent's offer be punished as signs of weakness or should she promise to reward them with a softening of her own position? Is it useful to announce deadlines after which offers will be withdrawn? This paper addresses these questions in an essentially full-information two-person bargaining model in which there is a small possibility that each player might be one of a rich variety of behavioral types. For example, to use the terminology of Myerson (1991), rather than optimizing as a fully rational player would, the player might use an "r-insistent strategy" that always demands the amount r and never accepts anything less. But the player might instead employ a complex history-dependent strategy, a possibility not considered by previous papers in the behavioral bargaining literature.[1]

Now think about broader bargaining problems in which the players interact in payoff-significant ways BEFORE an agreement is reached. Such considerations were introduced by Fernandez and Glazer (1991) and Haller and Holden (1990).[2] For example, before two countries sign a treaty on trade or pollution abatement, their unilateral policies affect one another's payoffs. Here, possibilities for strategic posturing are even more interesting. Does each party maximize its immediate payoff before agreement, or is some degree of cooperation possible during negotiations? As time passes without agreement, do players treat one another more harshly? Is a player's behavior related to her demand, and to the opponent's demand?

Since our framework will generalize the model of Abreu and Gul (2000) in two ways, we pause now to summarize their work. An exogenous protocol specifies the times at which each of two impatient bargainers can make offers about how a fixed surplus will be divided. When an offer is made, the

[1] Adopting the idea of introducing behavioral perturbations from KMRW (1982), Myerson (1991) studied a two-person bargaining game with one-sided uncertainty, one-sided offers and a single type. Abreu and Gul (2000) performed a two-sided analysis with multiple types that we will summarize below, prompting Kambe (1999) to do a limit analysis of a related model as the probabilities of behavioral types approaches zero.

[2] Fernandez and Glazer (1991) is notable in particular for its demonstration that even in an alternating offers bargaining game with symmetric information, it is possible to have a multitude of subgame perfect equilibria, including many with substantial delay to agreement.

other party can accept (and the proposed division is implemented) or reject (and the bargaining continues). Payoffs of rational players are common knowledge, but for each player i, there are exogenous initial probabilities $\pi_i(k) > 0$ that player i is a k-insistent type (see footnote 2 above) who will never settle for any amount less than k. At the start of play normal players mimic behavioral types. Following the initial choice of types, in the limit as one looks at bargaining protocols allowing more and more frequent offers, a war of attrition ensues in which players either simply stick with their initial demands or concede to their opponent's. Equilibrium outcomes are essentially unique and do not depend on the fine details of the protocol. The way the surplus is divided, and the delay to agreement, depend on the set of behavioral types available for each player to imitate and their initial probabilities, and the discount factors of the players. If initial probabilities that players are behavioral are sufficiently low, there is usually almost no delay to agreement. In the limit as the $\pi_i(k)$'s approach zero, each player's expected payoff coincides with the payoff she would get if the Nash bargaining solution (Nash (1950)) were used to divide the surplus (with disagreement point zero). Kambe (1999) was the first to obtain this kind of Nash bargaining result, in his modification of the Abreu and Gul model.

Our paper considers two impatient players who are bargaining over the surplus generated by the "component game" G that they play in each period. After any history of play and of offers that have been made, the players have the option of entering into an enforceable Pareto efficient agreement governing play of both parties from that time on. There is some chance that either bargainer might be a behavioral player drawn from a rich finite set of behavioral types. Each of those types plays a particular dynamic strategy in the bargaining game. Its actions and demands might vary over time, and might respond in complicated ways to what the other side offers and does. Both the complexity of behaviors allowed in the sets of types and the fact that a game is played while bargaining proceeds make this a much more complicated model than that of Abreu and Gul.

We obtain strong characterizations of equilibria in the limit analysis as the probabilities of behavioral types approach zero. In particular, the "Nash bargaining with threats" concept (Nash (1953)) describes the equilibrium behavior and expected payoffs in a manner analogous to how the simpler Nash bargaining solution describes the asymptotic equilibria in Kambe (1999) and Abreu and Gul (2000). Thus, perturbing the full-information, simultaneous offers, play-as-you-bargain game with the slight possibility of behavioral types replaces a vast multiplicity of equilibria with a strong prediction about outcomes. This strong prediction is more striking when one views the model as a repeated game in which players can sign binding contracts. When those contracts are unavailable, the problem of multiple sustainable expectations about future play is so powerful that folk theorems persist even in the face of reputational perturbations (see especially Chan (2000) and Dekel and Pesendorfer (2003)). The contractual option provides enough stability to allow reputational perturbations to resolve the issue of how surplus is divided.

Section 2 introduces the model. Section 3 establishes the result for the special case of stationary postures. In Section 4 we provide the general characterization result. Section 5 establishes existence of equilibrium and Section 6 concludes.

2 The Model

In each round $\ell = 0, 1, 2, \ldots$,the actions chosen in a finite game $G = (S_i, U_i)_{i=1}^2$ determine the flow payoffs of players 1 and 2. Payoff streams are discounted according to the common interest rate $r > 0$. Thus, when players use actions $(s_1, s_2) \in (S_1, S_2)$, player i's payoff in that round is $U_i(s_1, s_2) \int_0^1 e^{-rt} dt$. If at any time players agree on a payoff pair in the feasible set F of G, that payoff is realized in all subsequent rounds: players sign an enforceable contract and there are no further strategic decisions. In any round before agreement is reached, each player chooses a demand and action pair $(u_i, m_i) \in (F_i, M_i)$,where F_i is the set of feasible payoffs of i (the i^{th} coordinate projection of F) and M_i is the set of mixed strategies in G.

While actions and demands can be changed only at integer times, one player's demands can be agreed to at any time $t \geq 0$ by the other player[3]. A demand u_i by player i can be interpreted as an offer to $j \neq i$ of the best payoff[4] for j consistent with i receiving u_i, which we denote by $h_j(u_i)$. Thus, an offer made at integer time l is valid ("stands") until it is replaced by another offer (possibly the same) at $l + 1$; a standing offer may be accepted at any time. Bargaining terminates at the first instant that offers made are

[3] This mixture of discrete and continous time simplifies the analysis of the "war of attrition" that arises, without causing problems with the definition of strategies and outcomes.

[4] This is well-defined because F is a compact set.

mutually compatible or that a standing offer is accepted. If two standing offers are accepted at the same instant, the final agreement is taken to be either of the standing offers with equal probability. A similar tie-breaking rule applies when players make mutually compatible offers. Until agreement is reached, a player's choice of a (demand, action) pair at any $l > 1$ can depend on the entire past history of (demand, action) pairs.

Each player is either "normal" (an optimizer) or with initial probability z_i,"behavioral". A behavioral player i may be one of a finite set of types $\gamma_i \in \Gamma_i$. Each type is a strategy in the dynamic bargaining game. At the start of play behavioral players simultaneously *announce* their type $\gamma_i \in \Gamma_i$. We interpret this as an announcement of a bargaining *posture*. Each $\gamma_i \in \Gamma_i$ is a *machine* defined by a *finite* set of states Q_i, an initial state $q_i^0 \in Q_i$, an output function $\xi_i : Q_i \to (F_i \times M_i)$, and a transition function $\psi_i : Q_i \times F_j \times M_j \to Q_i$. Denote by $\pi_i(\gamma_i)$ the conditional probability that a behavioral player i adopts the posture/machine γ_i. The set of postures and these conditional probabilities are held fixed throughout.

A normal player i also announces a machine in Γ_i as play begins, but of course she need not subsequently conform to her announcement. More generally, we could allow her to announce something outside Γ_i or to keep quiet altogether. Under our assumptions, however, in equilibrium she never benefits from exercising these additional options. A normal player can condition her choice of demand and mixed action in the n^{th} stage game on the full history of play in the preceding rounds, including both players' earlier mixed actions (assumed observable) and initial announcements.

Interpreting the interval over which players can concede as the limit of a sequence of increasingly fine discrete divisions of time, we assume that if players adopt a pair of mixed actions (m_1, m_2) in the n^{th} round, as round n progresses they experience the flow payoffs $(U_1(m_1, m_2)), (U_2(m_1, m_2))$, rather than payoffs associated with the realization of a particular pure strategy pair. It is as if randomization were done not once at the beginning of the round, but over and over again[5].

[5] There is a sense in which this continual randomization makes it redundant to assume observable randomizing devices. For generic games, realized payoffs reveal in any time interval, no mater how short, what mixed strategy pair is governing play.

3 Stationary Postures

This section studies the case where each behavioral type $\gamma_i \in M_i$, $i = 1, 2$, is *stationary*, that is, γ_i demands the same amount in any period, regardless of the history of play (and never accepts less), and plays the same action in every period until settlement is reached. These are the natural generalizations of the behavioral types of Myerson (1991) and Abreu and Gul (2000), to settings in which bargainers make payoff-relevant strategic choices in each period before reaching agreement. Whereas Abreu and Gul (2000) do a stationary perturbation of a bargaining game similar to that of Rubinstein (1982), with many behavioral types on each side, this section does the same sort of perturbation of much more complex bargaining problems of the kind introduced by Fernandez and Glazer (1991) and Haller and Holden (1990) and generalized by Busch and Wen (1995).

The equilibrium existence result of Section 5 applies immediately to this setting; we do not duplicate it here. At the heart of our characterization of equilibrium payoffs is the idea of "Nash bargaining with threats" (Nash (1953)), which is summarized below:

Recall the (standard) Nash (1950) bargaining solution for a convex nonempty bargaining set $F \subseteq \mathbb{R}^2$, relative to a disagreement point $d \in \mathbb{R}^2$. The Nash bargaining solution, denoted $u^N(d)$, is the unique solution to the maximization problem

$$\max_{u \in F}(u_1 - d_1)(u_2 - d_2)$$

when there exists $u \in F$ *s.t.* $u \gg d$. If there does not, $u^N(d)$ is defined to be the strongly efficient point $u \in F$ which satisfies $u \geq d$.

In Nash (1953) the above solution is derived as the unique limit of solutions to the *non-cooperative* Nash demand game when F is perturbed slightly and the perturbations go to zero. Nash's paper also *endogenizes* the choice of threats, and consequently disagreement point, and this second contribution plays a central role here. Starting with a game G, the bargaining set F is taken to be the convex hull of feasible payoffs of G. The threat point d is determined as the non-cooperative (Nash) equilibrium of the following two 'stage' game:

Stage 1 The two players independently choose (possibly mixed) threats m_i, $i = 1, 2$. The expected payoff from (m_1, m_2) is the disagreement payoff denoted $d(m_1, m_2)$.

Stage 2 The player's final payoffs are given by the Nash bargaining solution relative to the disagreement point determined in Stage 1.

Thus players choose threats to maximize their Stage 2 payoffs given the threats chosen by their opponent. Note that the set of player i's *pure* strategies in the threat game are her set of *mixed* strategies in the game G. Since the Nash bargaining solution yields a strongly efficient feasible payoff as a function of the threat point, the Nash threat game is strictly competitive in the space of pure strategies (of the threat game). Nash shows that the threat game has an equilibrium in pure strategies (i.e., players do not mix over mixed strategy threats), and consequently that all equilibria of the threat game are equivalent and interchangeable. In particular the threat game has a unique equilibrium payoff (u_1^*, u_2^*) where $u^* = u^N(d(m_1^*, m_2^*))$ and m_i^* is an equilibrium threat for player i. To avoid distracting qualifications *we assume henceforth that the stage game is non-degenerate in the sense that it yields* $u^* > d(m_1^*, m_2^*)$. Our solution essentially yields (u_1^*, u_2^*) as the only equilibrium payoff which survives in the limit as the probability of behavioral types goes to zero.

We assume that one of the behavioral types on each side plays the "Nash bargaining with threats" (NBWT) strategy, demanding the Nash payoff and playing the Nash threat action. There are no restrictions on the demands and threats of all the other types that may be present; a clumsier assumption that would have essentially the same effect would be the requirement of a rich set of types on each side. The earliest analog of Assumption 1 in the reputational literature is the presence of a "Stackelberg leader" type in Fudenberg and Levine (1989).

Assumption 1 (NBWT) *: For each player i, there exists $\gamma_i^* \in \Gamma_i$, such that in each period γ_i^* demands u_i^* (and accepts nothing less) and takes action m_i^*.*

Why does this setting always lead to a war of attrition? Here is a preview of the argument. If player 1 is pretending to be some greedy type, but 2 admits to rationality, 2 should give up and accept 1's offer right away (for reasons akin to the Coase conjecture (Coase (1972))). Thus, there can be no equilibrium in which *neither* rational player imitates an advantageous type, or else player 1, say, could deviate and imitate his greediest behavioral type, and be conceded to immediately. Similarly, if only player 1 were imitating a behavioral type, it would have to be his greediest one; if it asks for more than his NBWT payoff, we will show that if 2 imitates her NBWT type, she will win the ensuing war of attrition with player 1 with high probability, making this a profitable deviation. The only remaining possibility (other than nonexistence, which is ruled out by Proposition 2) is that each rational player adopts a behavioral posture and hopes the other will concede soon.

Lemma 1 establishes that being the first to reveal rationality is tantamount to conceding to one's opponent. We remark that this will emphatically *not* be the case in Section 4, where nonstationary postures necessitate quite a different line of attack.

For a given stationary posture γ_i, let u_i denote player i's stationary demand, and m_i her stationary action. Recall that $h_j(u_i)$ is the corresponding offer to player j (that is, $(u_i, h_j(u_i))$ is an efficient feasible payoff in the stage game).

We will assume that (stationary) postures penalize non-acceptance. *No analogous assumption is required in the general non-stationary environment of Section 4, where we develop a quite different line of attack.*

Assumption 2 *: For all postures $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$, such that $u_1 > h_1(u_2)$,*

$$h_j(u_i) > d_j(m_j', m_i) \quad \forall m_j' \in M_j \quad i \neq j, \quad i, j = 1, 2.$$

Lemma 1 *Suppose Assumptions 1 and 2 and for any perfect Bayesian equilibrium σ, consider the continuation game following the choice of a pair of postures (γ_1, γ_2), such that $u_1 > h_1(u_2)$. Suppose that at time t player j reveals rationality and that player i still has not done so. Then the resultant equilibrium continuation payoff is $(h_j(u_i), u_i)$; player j, in effect, accepts i's offer immediately.*

Proof. See Appendix ■

Once each side has adopted a posture, players concede with constant hazard rates. At no time other than 0 does anyone concede with strictly positive *probability* (as opposed to conditional density). Again, the analogous results in Section 4 below take a much more complicated form, and require a novel approach.

Lemma 2 *Suppose Assumptions 1 and 2 and for any perfect Bayesian equilibrium σ, consider the continuation game following the choice of a pair of postures (γ_1, γ_2), such that $u_1 > h_1(u_2)$. This game has a unique perfect Bayesian equilibrium. In that equilibrium, at most one player concedes with positive probability at time zero. Thereafter, both players concede continuously with hazard rates $\lambda_i = \frac{r(h_j(u_i)-d_j)}{u_j-h_j(u_i)}$ $i \neq j$, $i, j = 1, 2$ until some common* *time $T^* < \infty$ at which the posterior probability that each player i is behavioral reaches 1. Furthermore the probability with which player j concedes to player i at the beginning of the continuation game is* $\max\left\{0, 1 - \frac{\eta_j}{(\eta_i)^{\lambda_j/\lambda_i}}\right\}$, *where η_i denotes the posterior probability that a player i who chooses γ_i is behavioral.*

The proof is omitted. It is similar to Proposition 1 of Abreu and Gul (2000) and follows as a special case of the discussion in the following section. We provide an intuitive treatment below starting with the analysis of the standard (without reputational elements) war of attrition.

Fixing an equilibrium σ and postures (γ_1, γ_2) denote by $F_i(t)$ the probability that player i (unconditional upon whether i is behavioral or normal) will reveal rationality (in the setting of this section, equivalent to conceding) by time t, conditional upon $j \neq i$ not revealing rationality prior to t.

Recall the analysis of the usual war of attrition in which the first player to drop out receives the "low" prize while his opponent wins the "high" prize. This game has two extreme asymmetric equilibria (in pure strategies) corresponding to 1 dropping out with probability 1 at $t = 0$, and conversely, to 2 dropping out immediately. There are in addition a continuum of mixed equilibria in which players randomize over concession times according to equilibrium distribution functions $F_i(\cdot)$.In any such (non-extreme) equilibrium $F_1(\cdot), F_2(\cdot)$ have no discontinuities except possibly at $t = 0$, identical supports, and no gaps. These basic equilibrium properties are consequences of the fact that the only reason for a player to delay conceding is the prospect that his opponent will concede in the interim. In addition, the equilibrium distribution functions are absolutely continuous.

Let $\lambda_1(t) = \frac{f_1(t)}{1-F_1(t)}$ denote 1's hazard rate of concession at t. This is calibrated to keep 2 indifferent between conceding at t or $t + \Delta$. In our context the cost to 2 of delaying concession is $(h_2(u_1) - d_2)\Delta$ while the benefit is $\frac{(u_2 - h_2(u_1))}{r}\lambda_1(t)\Delta$ (ignoring terms of order Δ^2 and higher). Equating costs and benefits yields

$$\lambda_1(t) = \frac{r(h_2(u_1) - d_2)}{u_2 - h_2(u_1)} \equiv \lambda_1, \quad \text{a constant independent of } t.$$

Hence,

$$1 - F_2(t) = c_2 e^{-\lambda_2 t}$$

where $c_2 \in (0, 1]$ is a constant of integration to be determined by equilibrium conditions. Observe that, $F_j(0) = 1 - c_j$, where $F_j(0)$ is the probability with which j concedes at $t = 0$. In the standard war of attrition, the constants c_1, c_2 define an equilibrium if and only if, $c_i \in [0, 1] \quad i = 1, 2 \quad$ and $\quad (1 - c_1)(1 - c_2) = 0$. In the event that neither player concedes with probability 1 at $t = 0$ (that is, $c_i > 0, i = 1, 2$) then neither player concedes with probability 1 by any time $t < \infty$.

In our *reputational* war of attrition, the above conditions are *necessary* but since behavioral types never concede we in addition require that,

$$1 - F_i(t) \geq \eta_i \quad \text{all } t \geq 0,$$

where

$$\eta_i = \frac{z_i \pi_i(\gamma_i)}{z_i \pi_i(\gamma_i) + (1 - z_i)\mu_i(\gamma_i)}$$

is the posterior probability that player i who chooses posture γ_i is behavioral.

This additional condition pins down the equilibrium uniquely. It follows from the latter condition that a normal player i must concede with probability 1 in finite time, indeed, at the latest, by T_i where

$$\begin{aligned} e^{-\lambda_i T_i} &= \eta_i \quad \text{and} \\ T_i &= \frac{-\log \eta_i}{\lambda_i} \end{aligned}$$

is the instant by which normal i would finish conceding if $c_i = 1$, or equivalently if player i did not concede with positive probability at $t = 0$. In equilibrium, normal types of the both players must finish conceding at the same instant, and at most one player can concede with positive probability at $t = 0$.

Let $T^* = \min\{T_1, T_2\}$. If $T_i = T^*$ then $c_i = 1$ and $c_j \in (0, 1]$ is determined by the requirement that

$$\begin{aligned} 1 - F_j(t) &= c_j e^{-\lambda_i T^*} = \eta_j \\ &\implies 1 - c_j = F_j(0) = 1 - \frac{\eta_j}{(\eta_i)^{\lambda_j/\lambda_i}} \end{aligned}$$

More generally,

$$F_j(0) = \max\{0, 1 - \frac{\eta_j}{(\eta_i)^{\lambda_j/\lambda_i}}\}$$

independently of whether $T_j < T_i$ or $T_j \geq T_i$.

Let $\eta_i(t)$ denote the posterior probability that player i is behavioral absent concession until time t. Then, $\eta_i(t) = \frac{\eta_i}{1 - F_i(t)} = \frac{1}{c_i} \eta_i e^{\lambda_i t}$. That is,

λ_i is the rate of growth of player i's reputation (for being behavioral). If $T_i > T^*$, then $c_i < 1$, and is chosen to boost i's reputation conditional upon non-concession at $t = 0$, by just enough for both players' reputations to reach 1 simultaneously at T^*.

It follows that, the player with the larger concession hazard rate is at an advantage in the war of attrition. Suppose for example that in equilibrium, after adopting some particular pair of profiles, players have the same initial reputations. If $\lambda_1 > \lambda_2$, player 1's reputation will reach 1 before 2's reputation does, in violation of Lemma 2. The only way to keep this from happening is for 2 to concede with enough probability at time zero so that in the event that she is observed *not* to have conceded, her reputation jumps just enough that the two players' reputations will reach 1 together after all. If initial reputations are tiny, even a small difference in hazard rates must be compensated for by concession at zero with probability close to 1. This follows from the formula for $F_j(0)$ given above.

Naive intuition might suggest that player i will tend to imitate the greediest possible type. But the formula in Lemma 2 indicates that by moderating the demand, i increases λ_i and decreases λ_j, which may serve i better in the war of attrition. The formula further shows that i should choose an action (while waiting) that hurts the opponent j without hurting i too much. Of course that is also what a player has in mind when choosing a threat in the Nash bargaining with threats (NBWT) game. The connection can be made precise as follows.

Lemma 3 *Suppose that player 1 adopts his NBWT posture. Then for* <u>all</u> *postures 2 could adopt, except ones that give 1 at least as much as he is asking for,* $\lambda_1 > \lambda_2$*.*

Proof. This is most easily seen graphically. Let 1 adopt the NBWT position $\gamma_i^* = (u_i^*, m_i^*)$ and 2 adopt any posture $\gamma_2 = (u_2, m_2)$ with $u_2 > u_2^*$. The NBWT threat point and allocation are denoted D and u^*, respectively. Let $D = d(m_1^*, m_2)$ and $A = (h_1(u_2), u_2)$. See Figure 1.

By Assumption 2, $D_1 < h_1(u_2)$. Since (m_1^*, m_2^*) is an equilibrium of the Nash threat game, D lies on or below the line through D^*u^* (if not, m_2 would be a strictly improving deviation for player 2 in the Nash threat game). By Nash's (1950) characterization of the Nash bargaining solution, the slope of the line D^*u^* equals the absolute value of some supporting hyperplane to the set F at u^*. Hence slope $DE >$ slope $Du^* \geq$ slope $D^*u^* \geq |\text{slope } Au^*|$.

But

$$\lambda_1 = \frac{r(u_2^* - D_2)}{h_2(a) - u_2^*} > \frac{r(a - D_1)}{u_1^* - a} = \lambda_2$$

if and only if

$$\text{slope } DE > |\text{slope } Au^*|.$$

■

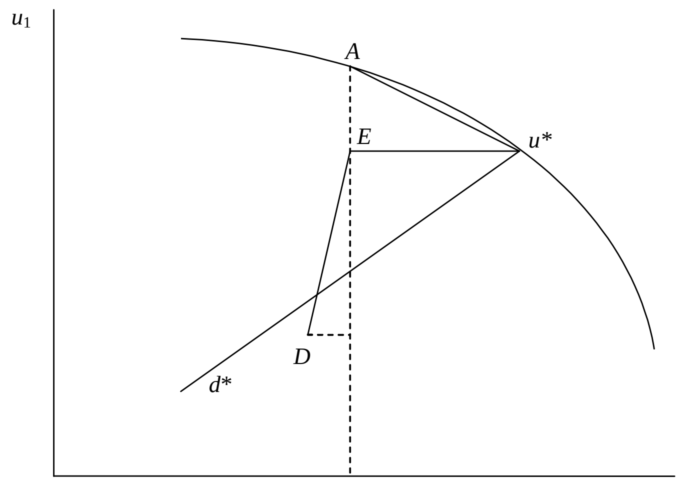

Figure 1

Lemma 4 *Suppose Assumptions 1 and 2. Then for any* $\xi > 0$, $R \in (0, \infty)$ *and* $\overline{\mu} > 0$ *there exists* $\delta > 0$ *such that if* $z_i \leq \delta$, $i = 1, 2$ *and* max $\left\{\frac{z_1}{z_2}, \frac{z_2}{z_1}\right\} \leq R$, *then for any perfect Bayesian equilibrium* σ *the payoff to a rational player 1 in the continuation game* (γ_1^*, γ_2) *is at least* $(u_1^* - \xi/2)$ *for any* $\gamma_2 \in \Gamma_2$ *which a rational player 2 adopts in equilibrium with probability* $\mu_2(\gamma_2) \geq \overline{\mu}$.

Proof. Consider the continuation game with (γ_1^*, γ_2). By Lemma 3, either γ_2 entails u_2 with $h_1(u_2) \geq u_1^*$, or $\lambda_1 > \lambda_2$. Suppose $u_1^* > h_1(u_2)$ and that rational 2 adopts γ_2 with at least probability $\underline{\mu} > 0$. Then

$$\begin{aligned}
\eta_1 &\geq \frac{z_1 \pi_1(\gamma_1)}{(1 - z_1) \cdot 1 + z_1 \pi_1(\gamma_1)} \\
\eta_2 &\leq \frac{z_2 \pi_2(\gamma_2)}{(1 - z_2) \cdot \overline{\mu} + z_2 \pi_2(\gamma_2)} \leq z_2 B \\
&\Rightarrow \frac{\eta_2}{\eta_1} \leq \frac{z_2}{z_1} \cdot \frac{\pi_2(\gamma_2)}{\pi_1(\gamma_1)} \cdot \frac{(1 - z_1) + z_1 \pi_1(\gamma_1)}{(1 - z_2) \cdot \overline{\mu} + z_2 \pi_2(\gamma_2)} \leq RC
\end{aligned}$$

for given R and some finite constants B, C independent of (z_1, z_2). Recall that the conditional probabilities $\pi_i(\gamma_i)$ are exogenous constants.

From Lemma 2,

$$F_2(0) = 1 - \frac{\eta_2}{\eta_1}(\eta_1)^{1-\frac{\lambda_2}{\lambda_1}}$$

if the latter term is non-negative. By the preceding inequalities,

$$\begin{aligned} F_2(0) &\geq 1 - RC(z_2 B)^{1-\frac{\lambda_2}{\lambda_1}} \\ &\geq 1 - \overline{R}\delta^{1-\frac{\lambda_2}{\lambda_1}} \end{aligned}$$

where $\overline{R} = RCB^{1-\frac{\lambda_2}{\lambda_1}} < \infty$. Hence for δ small enough, $F_2(0)$ is close to 1.

Player 1's payoff is:

$$\begin{aligned} & F_2(0)u_1^* + (1 - F_2(0))h_1(u_2) \\ \geq\ & u_1^* - \frac{\xi}{2} \end{aligned}$$

for δ small enough and (consequently) $F_2(0)$ close enough to 1. ∎

Proposition 1 *Suppose Assumptions 1 and 2. Then for any* $\varepsilon > 0$ *and* $R \in (0, \infty)$ *there exists* $\delta > 0$ *such that if* $z_i \leq \delta$, $i = 1, 2$ *and* $\max\left\{\frac{z_1}{z_2}, \frac{z_2}{z_1}\right\} \leq R$ *then for any perfect Bayesian equilibrium* σ *of* $\mathcal{G}(z)$, $|U(\sigma) - u^*| < \varepsilon$.

Proof. For any given perfect Bayesian equilibrium σ, and $\overline{\mu} > 0$, let $\widehat{\Gamma}_2 = \{\gamma_2 \in \Gamma_2 \mid \mu_2(\gamma_2) \leq \overline{\mu}\}$. Then $\sum_{\gamma_2 \in \widehat{\Gamma}_2} \mu_2(\gamma_2) \leq |\widehat{\Gamma}_2|\overline{\mu} \leq |\Gamma_2|\overline{\mu}$. Hence $\sum_{\gamma_2 \in \Gamma_2/\widehat{\Gamma}_2} \mu_2(\gamma_2) = 1 - \sum_{\gamma_2 \in \widehat{\Gamma}_2} \mu_2(\gamma_2) \geq 1 - |\Gamma_2|\overline{\mu}$. Under the conditions of Lemma 4, for any $\gamma_2 \in \Gamma_2/\widehat{\Gamma}_2$, the payoff to a rational player 1 in the continuation game (γ_1^*, γ_2) is at least $(u_1^* - \xi/2)$, and consequently the payoff to adopting γ_1^* is at least

$$(1 - |\Gamma_2|\overline{\mu})(u_2^* - \frac{\xi}{2}) + |\Gamma_2|\,\overline{\mu}\underline{w}_i$$

where $\underline{w}_i$ is the lowest payoff to i in the (finite) stage game G.

Clearly we can choose $\overline{\mu} > 0$ such that $|\Gamma_2|\overline{\mu} \leq 1$ and

$$(1 - |\Gamma_2|\overline{\mu})(u_2^* - \frac{\xi}{2}) + |\Gamma_2|\,\overline{\mu}\underline{w}_i \geq u_2^* - \xi$$

For such a $\overline{\mu} > 0$ Lemma 4 immediately implies that under the stated conditions, the payoff to adopting γ_1^* is at least $u_1^* - \xi$, in *any* PBE σ. If follows that $U_1(\sigma) \geq u_i^* - \xi$. Since this is true for both players and u^* is a (strongly) efficient feasible payoff[6] of the stage game, the Proposition follows directly. ∎

[6] That is, there does not exist feasible u' s.t. $u_i' > u_i^*$ and $u_j' \geq u_j^*$.

4 Nonstationary Postures

Following Fernandez and Glazer (1991) and Haller and Holden (1992), Busch and Wen (1995) have provided a general analysis for repeated games with complete information where a long-run enforceable contract can be signed. In conformity with the earlier result, in many games there is a significant multiplicity of equilibrium outcomes.[7] Our goal is to be able to say that any rich perturbation of such a game leads to an essentially unique outcome, and that the outcome is not sensitive to the small ex ante probabilities of the respective behavioral types. That is true if perturbations are restricted to stationary strategies, as Section 3 has shown. Which of the results there survive the introduction of nonstationary strategies?

We revert now to the general model specified in Section 2. Behavioral types are finite automata that announce and follow repeated game strategies that may have complicated intertemporal features and can respond to the opponent's play. Suppose one asks how well player 1's stationary Nash bargaining with threats (NBWT) strategy would do against any nonstationary posture 2 might adopt. How different from Section 3 would the analysis look, and does 1 do himself harm by not taking advantage of the opportunity to use a dynamic closed-loop strategy himself?

We formulate a hybrid discrete/continuous model of time that simplifies the war of attrition calculations without introducing any of the logical difficulties associated with games played in continuous time. Each round of bargaining lasts for a period of length one. A player can accept at any moment within a round but can only change his own offer/demand and action between rounds. The analysis is facilitated by assuming that there is no discounting *between* rounds: one can interpret this as the limit of bargaining protocols in which the time *between* rounds goes to zero. We operationalize this assumption by the following device. Our primitive notion of time is a 'date'. The set of dates is $\mathcal{T}$. A date $\tau \in \mathcal{T}$ has two dimensions; $\tau = (t, k)$ where $t \in R_+$ and $k \in \{-1, 0, 1\}$. For $\tau \in \mathcal{T}$ let $t(\tau)$ denote the first dimension and $k(\tau)$ the second. For $n \in \mathcal{N} \equiv \{0, 1, 2, ...\}$, $\{(n, -1), (n, 0), (n, +1)\} \subseteq \mathcal{T}$. For τ such that $t(\tau) \notin \mathcal{N}$, $k(\tau) = 0$. At date $(n, 0)$, $n \in \mathcal{N}$ players can make new (offer, action) choices. The new offer can be accepted

[7] Busch and Wen work with an alternating offer structure in the spirit of the Rubinstein model. In our formulation offers are simultaneous and it is easy to establish a folk theorem type result. See Section 5 for a discussion of this point.

at dates $(n,+1)$, $((n+1),-1)$ and all dates in between. Thus the beginning of a round and the end of a round are well-defined. There is no discounting between the dates $(n,-1)$, $(n,0)$ and $(n,+1)$: discounting only depends on the pure time component of a date. The ordering on $\mathcal{T}$ is lexicographic: for any $\tau,\tau' \in \mathcal{T}$, $\tau' \equiv (t',k') \succ (t,k) \equiv \tau$ if $t' > t$ or if $t' = t$ and $k' > k$. A player's choices at date $(n,+1)$, say, can be conditioned on observed choices at dates $(n,0)$, $(n,-1)$ and, of course, all preceding dates.

For later reference, we define the infimum of a set of dates $\Theta \subseteq \mathcal{T}$. Denote $\underline{\theta} \equiv \inf \Theta$. Let $\Theta_t = \{t(\theta) \mid \theta \in \Theta\}$ and $\underline{t} \equiv \inf \Theta_t$. If $\underline{t} \notin \mathcal{N}$, then $\underline{\theta} \equiv (\underline{t},0)$. If $\underline{t} \in \mathcal{N}$, define $\Theta(\underline{t}) = \{\theta \in \Theta \mid t(\theta) = \underline{t}\}$. If $\Theta(\underline{t}) \neq \emptyset$ then $\underline{\theta} \equiv \{\theta \in \Theta(\underline{t}) \mid k(\theta) \leq k(\theta') \text{ all } \theta' \in \Theta(\underline{t})\}$. If $\Theta(\underline{t}) = \emptyset$ then $\underline{\theta} \equiv (\underline{t},+1)$. The supremum is defined analogously. These are the natural extensions of the usual definitions.

With 1's strategy fixed at the stationary NBWT action and demand, player 2's situation is similar in some ways to what she faced in Section 3. Whenever 2 reveals rationality, one can show that she does so by, in effect, accepting 1's offer. This is the one-sided analog of Lemma 1 in Section 3. But the same is *not* true for player 1. Suppose that 1 is offered 5 until some date τ, and 10 thereafter. Rather than wait to get 10 at τ, at an earlier time τ' he might offer a Pareto-superior contract: give me 9 right now. Player 2 might accept this (as long as she doesn't expect to do better in the subgame in which she instead reveals rationality *without* accepting 1's offer). Thus, the offer from 2's machine γ_2, at τ' is just a lower bound on 1's equilibrium expectation of the payoffs he would receive if τ' arrives without either player having revealed rationality. This simple example is illustrated in Figure 2. The fact that 1 expects a payoff of 9, conditional on arriving at period τ', is indicated by the dot at point $(\tau',9)$ on the graph.

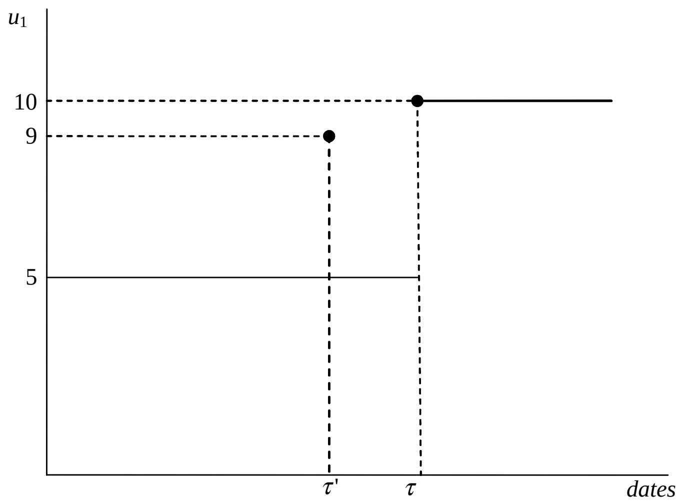

Figure 2

The reader may wonder why 1 would wait until τ' to make this suggestion, and for that matter, why 1 doesn't ask for an even greater amount. The answer lies in the full-information subgames after 1 and 2 have both revealed rationality. These typically have a continuum of subgame perfect equilibria, and in the construction of a solution of the full game, the selection from this set can depend on arbitrary details of the history of play. For example, if 1 demands 9 at τ'' prior to τ' instead of at τ', or 9.3 at τ, say, 2 could believe that she would fare extremely well, and 1 badly, if she revealed rationality instead of accepting 1's offer. The problems this multiplicity of expectations causes in behaviorally perturbed repeated games with two long-run players are explored in Schmidt (1993), and are the reason the ensuing literature[8] has been unable to provide precise payoff predictions for repeated games with players of comparable patience.

The example of Figure 2 might leave the impression that 1's expected payoff at τ' could exceed 2's offer there only because γ_2 later makes a more generous offer in response to 1's constant play of his NBWT position. This is not true. At τ, for example, if 1 reveals rationality without accepting the offer of 10, he may be able to manipulate γ_2 into offering him 15. His expected payoff at τ could therefore easily exceed 10.

To summarize, when 1's static NBWT strategy faces more complex

[8](to be completed)

strategies of 2, 1's expected payoff in a particular continuation game is no longer given by what 2 offers him, and may vary greatly across different equilibria of that continuation game. A rational player 1 may want to reveal rationality (by abandoning the NBWT posture at some point) but *not* accept 2's offer. Further we shall see that nonstationarity in 2's posture induces *discontinuities* in the war of attrition, with one or more players conceding *away from time zero* with strictly positive probability.

All of the above makes it impossible to replicate the line of attack of Section 3. Perhaps surprisingly, the main result concerning players' payoffs is essentially unchanged, along with the power of the static NBWT posture. The proofs, however, are quite different, and much more elaborate. This Section states and proves the main Proposition.

To avoid complicating an already difficult argument, we make a genericity assumption about behavioral types.

Assumption 3 (Generic Types) *For all $\gamma_2 \in \Gamma_2/\{\gamma_2^*\}$ in the continuation game defined by (γ_1^*, γ_2), and for all n, $h_1(u_2(n)) \neq d_1(n)$. A corresponding assumption applies to types $\gamma_1 \in \Gamma_1/\{\gamma_1^*\}$.*

Proposition 2 *Invoke Assumptions 1 and 3. Then for any $\varepsilon > 0$ and $R \in (0, \infty)$ there exists $\delta > 0$ such that if $z_i \leq \delta$, $i = 1, 2$ and $\max\left\{\frac{z_1}{z_2}, \frac{z_2}{z_1}\right\} \leq R$ then for any perfect Bayesian equilibrium σ of $\mathcal{G}(z)$, $|U(\sigma) - u^*| < \varepsilon$.*

Proposition 2 says that no matter how high you allow the bound on the relative probabilities that the respective players are behavioral to be, and no matter how close to u_i^* you want 1's expected utility to be, you can achieve this uniformly across all perfect Bayesian equilibria, by ensuring that behavioral players have sufficiently low prior probabilities.

Before providing the proof, we give a quick account of the main ideas. Given the unavoidable fact that a typical continuation game (following the choice of postures) suffers from a vast multiplicity of perfect Bayesian equilibria, our strategy is as follows. Any *particular equilibrium* of the full game offers player 1 expected payoffs at each date in each continuation game, following the realization of 2's choice of posture γ_2. Just as one can graph the offers γ_2 makes to 1 over time, one can graph the payoffs 1 would get by revealing rationality at any L, *without* accepting 2's offer. It is the maximum of these two values that drives the war of attrition. In analyzing that war of attrition, one can treat the stream of these maxima as exogenous variation, just as one accepts the possibility of arbitrary strategies γ_2. Once the characterization result is established for all possible streams, it holds a fortiori for all graphs that could actually arise in equilibrium.

Recall from Section 3 that the more player i demands, the slower i's rate of concession must be, and the slower i's reputation will grow. If i's demand is sufficiently greedy, this will require i to concede at time 0 with high probability. The same basic force is at work here. If 2 is asking for more than her NBWT payoff, she has to concede slower than 1 (if he chooses his NBWT posture). The rate changes as her demands change, and one has to integrate these rates and add them to discrete probability concessions. It is necessary to make cross-player comparisons of payoff discontinuities of different sizes and with qualitatively different effects. This is the most delicate part of the argument. But the same picture ultimately emerges: over all, 2's reputation grows more slowly than 1's and this becomes decisive when prior behavioral probabilities are low.

Non-stationarities in player 2's posture typically induce discrete concession episodes by both players. The simplest case, which we call a "downward jump", involves a decrease in the value of 2's offer to 1. Suppose that between rounds n and $(n+1)$ (that is, at date $(n,0)$) and before τ^*, player 2's offer falls from a to $b < a$. If 1 ever accepts the offer of b in equilibrium after $(n,0)$, he must be compensated at $(n,+1)$ for letting the offer fall from a to b, by a probabilistic concession from 2. The probability P_2 of 2's conceding at $(n,+1)$ that makes 1 indifferent between accepting the offer of a or waiting satisfies:[9]

$$a = P_2 u_1^* + (1 - P_2) b$$

"Upward jumps" have more interesting repercussions. Assume for simplicity that 2's action choice is constant and that at some date $\tau \in (0, \tau^*]$,

[9] More generally, $b < a$ could be player 1's expected equilibrium payoff at $(n,+1)$. This could exceed 2's starting offer at $(n,+1)$ because of future (offer, action) choices of 2 which are more attractive to 1 or future equilibrium payoffs from revealing rationality. Furthermore a could be player 1's equilibrium payoff from revealing rationality at $(n,0)$ and exceed 2's offer at $(n,-1)$ and $(n,+1)$.

2's offer jumps up from b to $a > b$ (or alternatively, that at τ, the equilibrium implicitly offers 1 the payoff a for revealing rationality at τ without accepting 2's offer). For some time interval of length Δ before τ, 1 would rather wait until τ to get a, than to concede immediately and get b (see Figure 3). Since 1 experiences flow payoffs while waiting, Δ solves:

$$\frac{b}{r} = \int_{t_1-\Delta}^{t_1} d_1(s)e^{-r(s-(t_1-\Delta))}ds + e^{-r\Delta}\frac{a}{r}$$

where $t_1 \equiv t(\tau)$ and $d_1(s)$ is player 1's flow payoff (given (γ_1^*, γ_2)) at time s.

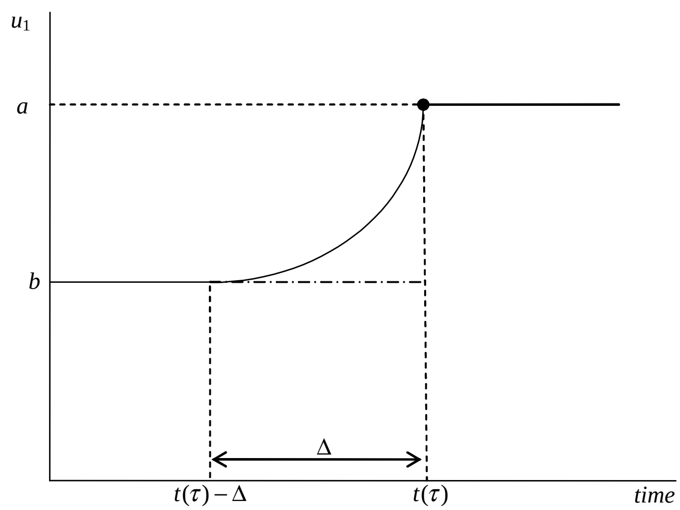

Figure 3

Notice that 2 will not concede in the Δ interval before $t(\tau)$ either: since 1 never concedes in that interval, 2 should either concede before the interval is reached, or wait until it is over. For 2 to be just compensated for waiting through the barren interval Δ, 1 must concede at $t(\tau)$ with probability P_1 solving:

$$\frac{u_2^*}{r} = \int_{t_0}^{t_1} d_2(s)e^{-r(s-t_0)}ds + e^{-r(t_1-t_0)}\left[P_1\frac{v_2^2(\tau)}{r} + (1-P_1)\frac{u_2^*}{r}\right] \quad (1)$$

where $t_0 \equiv t_1 - \Delta$ and $v_2^2(\tau)$ is player 2's payoff when player 1 reveals rationality at τ.

We say that the jump at τ "casts a shadow" of length Δ over the time period preceding τ. What if no $P_1 \leq 1$ solves the equation? Then 2 cannot be induced to wait, and rational 2 should concede with probability 1 weakly before the shadow begins (contradicting our assumption that $\tau \in (0, \tau^*]$). This expression makes it clear that *changes* in flow payoffs $d_1(s)$ can also contribute to or even cause "shadows". For instance even if $b = a$, if there are changes in 2's action choices so that initially player 1's flow payoffs $d_1(s)$ are less than a and later $d_1(s)$ exceeds a, so that

$$\int_{t_1-\Delta}^{t_1} d_1(s)e^{-r(s-(t_1-\Delta))}ds = a\int_{t_1-\Delta}^{t_1} e^{-r(s-(t_1-\Delta))}ds$$

then we have a shadow of length Δ generated exclusively by changes in flow payoffs.

Interestingly, there can be an upward jump at τ, followed by a downward jump "at the same instant". Suppose that 2's posture γ_2 is as illustrated in step 3 above, but that the equilibrium offers $c > a$ at $\tau \equiv (n, 0)$ (and nowhere else, for simplicity). Clearly 1's option of getting c at $(n, 0)$ casts a shadow (a longer one than that cast by a) over an interval in which neither 1 nor 2 will concede. Player 1 reveals rationality probabilistically at $(n, 0)$ (without accepting 2's offer). In the event that he *does not* concede, he faces an immediate drop in expected payoff to a. To make 1 indifferent between revealing rationality and waiting, 2 must concede with probability

$$P_2 = \frac{c - a}{u_1^* - a}$$

at $(n, +1)$, conditional on 1's *not* revealing rationality at $(n, 0)$.

Proof of Proposition 2. Proposition 2, the analogue of Proposition 1, follows from Lemma 5 below in the same way as Proposition 1 follows from Lemma 4 (see the proof of Proposition 1). Lemma 5 establishes the effectiveness of player 1's NBWT posture γ_1^* against any relevant posture of player 2. The following notation will be used in the proof.

Fix $z = (z_1, z_2)$ and an equilibrium of the overall game, and consider the continuation game following the choice of (arbitrary) postures (γ_1, γ_2). The dependence of various functions and terms introduced below on z, the equilibrium in question, and on (γ_1, γ_2) is not made explicit in the notation but should be understood in what follows.

Let $u_i(t(\tau))$ denote player *i's* demand at time $t(\tau)$ if both players conform with their postures until and including date τ. Let $d_i(t(\tau))$ denote the flow payoff to i at time $t(\tau)$ when neither player has revealed rationality prior to or at τ.

Associated with the continuation game are 'distribution functions' $F_i(\cdot)$, $i = 1, 2$ where $F_i(\tau)$ is the probability that player i reveals rationality by

τ conditional upon player j not revealing rationality prior to τ. Note that the distribution functions and the terms defined below are specific to the equilibrium in question.

Define $v_i^j(\tau)$ as the supremum over possible (given player j's strategy) payoffs to i, conditional upon revealing rationality at τ (given that i and j have not revealed rationality prior to τ). If $t(\tau) \notin \mathcal{N}$, and player j does not reveal rationality at τ with strictly positive probability, $v_i^j(\tau) = h_i(u_j(\tau))$; the only way for i to reveal rationality within a round is to accept the opponent's standing offer at τ. Let $v_i^i(\tau)$ denote player i's expected equilibrium payoff conditional upon player j revealing rationality at τ.

The proof proceeds by demonstrating the effectiveness of player 1's NBWT posture γ_1^* against *any* relevant posture of player 2. This is formalized in Lemma 5 below. ■

Lemma 5 *Invoke Assumption 1. Then for any* $\xi > 0$, $R \in (0, \infty)$ *and* $\overline{\mu} > 0$ *there exists* $\delta > 0$ *such that if* $z_i \leq \delta$, $i = 1, 2$ *and* $\max\left\{\frac{z_1}{z_2}, \frac{z_2}{z_1}\right\} \leq R$, *then for any perfect Bayesian equilibrium* σ *the payoff to a rational player 1 in the continuation game* (γ_1^*, γ_2) *is at least* $(u_1^* - \xi/2)$ *for any* $\gamma_2 \in \Gamma_2$ *which a rational player 2 adopts in equilibrium with probability* $\mu_2(\gamma_2) \geq \overline{\mu}$.

The proof of Lemma 5 is presented in eleven steps.

1.*Implications of Stationarity of* γ_1^*

Fix a PBE σ and a posture γ_2 for 2, and consider the continuation game starting from date (0,0) after 1 has adopted his NBWT posture and 2 has adopted γ_2. The profile σ induces an equilibrium on that continuation game. Denote by η_i, $i = 1, 2$ the initial probability that player i is behavioral, conditional on the observed posture. Recall that $\eta_i = \frac{z_i \pi_i(\gamma_i)}{z_i \pi_i(\gamma_i) + (1 - z_i)\mu_i(\gamma_i)}$ is the posterior probability that player i who chooses γ_i, is behavioral.

Because of the *stationarity* of 1's offer and the nature of the Nash threat a rational type of player 2 must concede with probability 1 in finite time (see Lemma 6 in the Appendix). Moreover a rational player 2 reveals rationality by, in effect, accepting 1's offer (see Lemma 7 in the Appendix). These "Coasean" results are closely related to Lemma 1 of Section 3, and do not hold for arbitrary non-stationary γ_1.

Definition 1 $\tau^* = \inf\{\tau \mid u_1^* \leq h_1(\,u_2(\tau))$ *or* $1 - F_1(\tau) = \eta_1$ *or* $1 - F_2(\tau) = \eta_2\}$

Thus τ^* is the first date by which (1) a rational type of either players 1 or 2 reveals rationality (i.e. does not follow γ_i), or (2) the demands generated by the pair of postures (γ_1^*, γ_2) are mutually compatible.

2. *Concession Distribution Functions*

Concession behavior strictly within rounds is driven by the familiar logic of the war of attrition, with parameters given by the constant offers and flow payoffs corresponding to the round in question. Specifically, suppose that τ', τ'' are dates within round $n \in \mathcal{N}$ with $\tau^* \succ \tau'' \succ \tau'$, and $(n+1) > t(\tau'') > t(\tau') > n$, and that $F_i(\tau'') > F_i(\tau')$ for some $i = 1, 2$.

We first argue that $h_i(\ u_j(\tau)) > d_i(\tau)$ for all $\tau \in ((n, +1), \tau'')$ and $i = 1, 2$. For $i = 2$ this follows from the definition of the NBWT posture γ_1^* (and our regularity assumption which excludes the exceptional case $u_2^* = d_2(m_1^*, m_2^*)$). Recall also our second regularity assumption that for all γ_2, the pair (γ_1^*, γ_2) generate offers and flow payoffs such that $h_1(\ u_2(\tau)) \neq d_1(\tau)$. Finally suppose that $h_1(\ u_2(\tau) < d_1(\tau)$. We show that this contradicts our initial assumption that $F_i(\tau'') > F_i(\tau')$ for some $i = 1, 2$. The inequality $h_1(\ u_2(\tau) < d_1(\tau)$ implies that player 1 is strictly better-off conceding at the end of the round than at any date within the round, independently of 2's concession behavior. Hence $F_1(\tau'') = F_1((n, 0))$. Since $u_2^* > d_2(\tau)$ it follows that 2 is strictly better-off conceding at $(n, +1)$ than at τ'' or at any date in between. Hence $F_2(\tau'') = F_2(\tau')$ also, a contradiction.

For all $n \in \mathcal{N}$ let

$$\bar{\tau}(n) = \inf\{\tau \mid F_i(\tau) = F_i((n+1, -1))\}.$$

By the preceding argument, $h_i(u_j(t(\tau))) > d_i(t(\tau))$ for all $\tau \in ((n, +1), \bar{\tau})$ and $i = 1, 2$. The analysis within the time interval $(n, t(\bar{\tau}))$ is as in the usual war of attrition, with equilibrium behavior governed by the basic principle that a normal player *delays* conceding only in the expectation that the other player might concede in the interim. Thus we have the familiar result that the players concede with constant hazard rates $\lambda_i(s)$ for $s \in (n, t(\bar{\tau}\,(n)))$ where

$$\lambda_i(s) = r \cdot \frac{h_i\,(u_j\,((n, +1))) - d_i\,(n)}{u_i\,((n, +1)) - h_i\,(u_j\,((n, +1)))}$$

Integrating this expression yields:

$$(1 - F_i(\tau)) = e^{-\lambda_i(s)(t(\tau)-n)}(1 - F_i((n, +1)))$$

This discussion is summarized in Lemma 8 below where $\bar{\tau}$ is defined as above.

Lemma 8 *For all τ', τ'' and $n \in \mathcal{N}$ with $\tau^* \succ \tau'' \succ \tau'$, and $(n+1) > t(\tau'') > t(\tau') > n$, if $F_i(\tau'') > F_i(\tau')$ for some $i = 1, 2$ then for $k = 1, 2$*

$$(1 - F_k(\tau)) = e^{-\lambda_k(n)(t(\tau)-n)}(1 - F_k((n, +1))) \textit{ for all } \tau \in ((n, +1), \bar{\tau})$$

Let

$$\lambda_i(s) = \begin{cases} r \cdot \frac{h_i(u_j((n,+1))) - d_i(n)}{u_i((n,+1)) - h_i(u_j((n,+1)))} & \text{for } s \in (n, t(\bar{\tau}\,(n))) \\ 0 & \text{otherwise} \end{cases}$$

Note for later use that $\lambda_1\,(s) > \lambda_2\,(s)$ for $s \in (n, t(\bar{\tau}\,(n)))$. The argument is exactly the same as in Lemma 3 of Section 3.

Let $Q_i(\tau)$ denote the probability with which i reveals rationality at τ conditional upon not having revealed rationality prior to τ.

Define $P_i(n)$ by

$$(1 - P_i(n)) = \prod_{k \in \{-1,0,1\}} (1 - Q_i((n,k)))$$

An implication of Lemma 8 is that positive probability concessions can only occur at the end, between or at the beginning of rounds, but not strictly within rounds. Thus the only dates at which player i might reveal rationality with strictly positive probability are those τ for which $t(\tau) \in \mathcal{N}$. (In fact, in addition, it is necessary that $k(\tau) \neq -1$.)

Hence,

$$1 - F_i(\tau) = e^{-\int_0^{t(\tau)} \lambda_i(s)ds} \prod_{\nu \leq \tau} (1 - Q_i(\nu))$$

and for $\tau \notin \mathcal{N}^0$,

$$(1 - F_i(\tau)) = e^{-\int_0^{t(\tau)} \lambda_i(s)ds} \prod_{\substack{n \in \mathcal{N} \\ n \leq t(\tau)}} (1 - P_i(n))$$

3. *Discrete Concessions by Player 2*

We seek to show that after time 0, player 1 reveals rationality faster than 2. This is the case in regions of continuous concession, for the same reasons as in Section 3. It will also be necessary to compare discrete concession probabilities by 1 and 2.

Each discrete concession by 2 is tightly linked to a contemporaneous reduction in what 1 can extract from 2, that is, to a "down jump" (see the preamble to the proof of Proposition 2). Lemma 9 provides an upper bound on the concession probability by 2 that can be provoked by a down jump from value a to $b < a$.

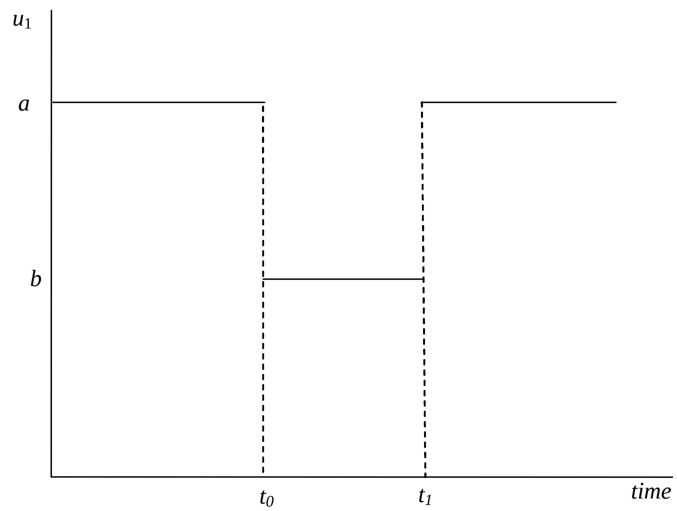

Figure 4

Let $w_1(\tau)$ be the expected equilibrium payoff to 1 (discounted to τ) conditional upon neither player revealing rationality prior to and including τ.

The total size of the down jump at round n is denoted $J_d(n)$ and :

$$J_d(n) = \max\left\{0, \max\{v_1^2((n,-1), v_1^2((n,0)\} - w_1((n,+1)\right\}$$

Lemma 9

$$P_2(n) \leq \frac{J_d(n)}{u_1^* - w_1((n,+1))}$$

Proof. Suppose by way of contradiction that $P_2(n) > \frac{J_d(n)}{u_1^* - w_1((n,+1))}$. Then

$$\begin{aligned} P_2(n)u_1^* + (1 - P_2(n))w_1((n,+1)) &= P_2(n)(u_1^* - w_1((n,+1)) + w_1((n,+1)) \\ &> \max\{v_1^2((n,-1), v_1^2((n,0), w_1(n,+1)\} \end{aligned}$$

That is, 1's payoff from conceding after $(n,+1)$ strictly exceeds 1's payoff from conceding at $(n,+k)$ or just prior to $(n,+k)$ for $k = -1, 0, +1$. Hence $F_1((n,+1)) = F_1(\tau)$ for some $\tau \prec (n,+1)$ with $t(\tau) < n$. It follows that 2's payoffs from conceding at $\tau' \in (\tau, (n,+1)]$ strictly exceeds 2's payoffs from conceding at $(n,+k)$ for $k = -1, 0, +1$. However our initial hypothesis implies that $P_2(n) > 0$. Hence $Q_2((n,k)) > 0$ for some $k = -1, 0, +1$, a contradiction. ■

4. *Neutral Subdivision of Downward Jumps*

The nonstationarity of some postures γ_2 may induce frequent fluctuations in 1's continuation values. The discrete concessions by 2 associated with the numerous down jumps could give 2 an insurmountable advantage in the war of attrition, unless the fluctuations induce concessions by 1 of similar or greater magnitude. Comparing the effects of up and down jumps of different sizes is difficult. Fortunately, if down jumps are subdivided arbitrarily into smaller jumps, this is neutral with respect to the overall probability of concession by 2. Neutrality is demonstrated in the following paragraph, and used in Step 6.

Let $P_2 > 0$ be associated with a down jump from $u_1 \equiv \max\{v_1^2((n,-1), v_1^2((n,0)\}$ to $w_1 \equiv w_1((n,+1)$. By Lemma 9, $1 - P_2 \geq \frac{u_1^* - u_1}{u_1^* - w_1}$. Consider the strictly decreasing sequence u_1^l, $l = 0, 1, \ldots, L$ such that $u_1^0 = u_1$ and $u_1^L = w_1$, and define $P_2^l = \frac{u_1^{l-1} - u_1^l}{u^* - u_1^l}$. Then $1 - P_2^l = \frac{u_1^* - u_1^{l-1}}{u_1^* - u_1^l}$. Consequently

$$(1 - P_2^1)(1 - P_2^2) \cdots (1 - P_2^L) = \frac{u_1^* - u_1}{u_1^* - w_1} \leq 1 - P_2$$

Thus a down jump may be broken up into an 'equivalent' sequence of smaller down jumps which span the same range.

5. *Paired Up and Down Jumps*

In general, it is possible to have multiple up and down jumps in 1's continuation value, all in a single interval of non-concession by player 2. Comparison of the respective concession probabilities of players 1 and 2 can be extremely involved, and this is relegated to the Appendix. To provide a more accessible treatment, we limit attention here to a simple case involving two perfectly complementary jumps. Readers interested in the Appendix may find it useful to get a motivating overview by looking at Steps 5 and 6 here before turning to the material in the Appendix concerning Section 4.

Figure 4 illustrates a scenario in which 1's continuation value is initially $a < u_1^*$, then falls to $b < a$, and later returns to a. We assume for simplicity that these continuation values coincide with what 2's posture γ_2 offers 1 (there are no endogenous rewards to 1 that augment what γ_2 offers). One can solve for the concession probability P_1 induced by the increase in value, and the concession probability P_2 induced by the earlier fall in value. As noted earlier:

$$P_2 = \frac{a-b}{u_1^* - b} \tag{2}$$

and

$$\frac{b}{r} = \int_{t_0}^{t_1} d_1(s) e^{-r(s-t_0)} ds + e^{-r(t_1-t_0)} \frac{a}{r}$$

or,

$$(b - d_1) = e^{-r(t_1-t_0)}(a - d_1) \tag{3}$$

where $\Delta \equiv (t_1 - t_0)$, $d_i \frac{(1-e^{-r\Delta})}{r} \equiv \int_{t_0}^{t_1} d_i(s) e^{-r(s-t_0)} ds$.
(d_i is the 'average' 'disagreement' payoff over the interval (t_0, t_1)).
Equation (1) may be rewritten as:

$$(u_2^* - d_2)(1 - e^{-r\Delta}) = e^{-r\Delta} P_1 (h_2(a) - u_2^*) \tag{4}$$

where we have replaced $v_2^2(\tau)$ with $h_2(a)$.

Combining (3) and (4) yields

$$P_1 = \frac{u_2^* - d_2}{h_2(a) - u_2^*} \cdot \frac{a-b}{b-d_1}$$

Hence $P_1 > P_2$ if and only if

$$\frac{u_2^* - d_2}{b - d_1} > \frac{h_2(a) - u_2^*}{u_1^* - b}$$

To see that this inequality does hold, note that:

1. (d_1, d_2) must be on or below the line joining $d(m_1^*, m_2^*)$ and u^*
2. the slope of the latter line equals the (absolute value) of the slope of some supporting hyperplane to the set of feasible payoffs, at u^*
3. a lies to the left of u^* on the (concave) frontier of the feasible set and b lies to the left of a.

See Figure 5.

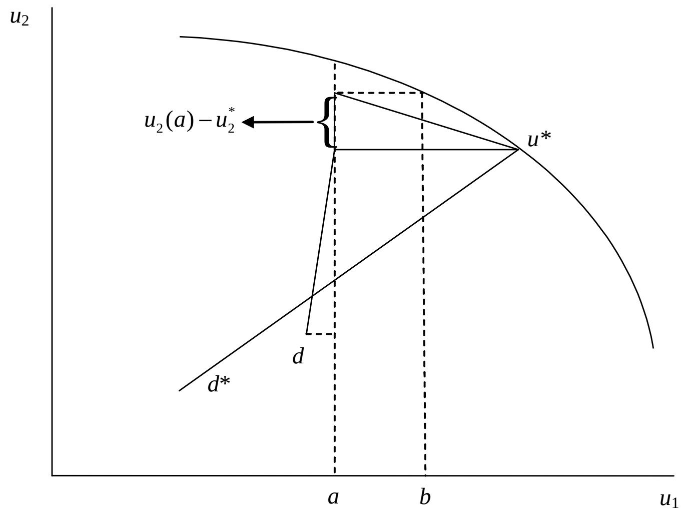

Figure 5

Thus, although the decline in 1's value from a to b appears to give 2 an advantage in the war of attrition (by inducing a discrete concession by 2), this advantage is outweighed by the larger discrete concession by 1 induced by the return from b to a. Player 1's overall advantage is even greater if there are many of these paired discrete concessions, rather than the single pair illustrated here.

What if there are more (or bigger) decreases in value than increases? For example, if value decreases from a to b, and then stays there forever, 2 has a discrete concession not matched by one from 1. This turns out to have an effect similar to 2's having a moderate reputational advantage over 1. It is swamped by other effects as the $z_i's$ approach 0. The argument in the Appendix shows that as long as all *repeated* down jumps are matched by (or "covered by" – see the Appendix) up jumps, 1's asymptotic advantage will be decisive. But repeated down jumps are indeed matched by up jumps: if value falls from 6 to 4, say, it can't fall through that range again until it has first risen through that range. Among the difficulties dealt with in the Appendix is the fact that where 1 has multiple concession episodes in the same interval of non-concession by 2, the respective concession probabilities often are not uniquely defined.

We are implicitly assuming that 1 obtains the payoff a by accepting an improved offer from b; it might also be that 1 obtains a by revealing rationality but not accepting 2's offer. In this case the resultant equilibrium payoff to 2 may be less than $h_2(a)$. It may be seen that this only strengthens the 'gap' between P_1 and P_2. This subtlety and related issues are dealt with in the Appendix.

6. *Modified Distribution Functions.*

By Lemma 10 in the Appendix, when $t(\tau^*) = 0$, the conclusion of Lemma 5 follows straightforwardly. Now suppose $t(\tau^*) > 0$. Recall that $\eta_2(\tau)$ is the posterior probability that 2 is behavioral conditional upon 2 not revealing rationality up until and including date τ. By Lemma 12 in the Appendix, there exists $\tau \preceq \tau^*$ such that $\eta_2(\tau) \geq \widetilde{\eta}$. Let $\widetilde{\tau} = \inf\{\tau \mid \eta_2(\tau) \geq \widetilde{\eta}\}$ and η_i denote the posterior probability (at the start of the continuation game) that a player who adopts the posture γ_i is behavioral. Then

$$\eta_2(\widetilde{\tau}) = \frac{\eta_2}{1 - F_2(\widetilde{\tau})} \geq \widetilde{\eta}.$$

Furthermore,

$$\eta_1(\widetilde{\tau}) = \frac{\eta_1}{1 - F_1(\widetilde{\tau})} \leq 1.$$

The goal is to establish that for small $z_i's$ the only way for the above inequalities to be satisfied is for $P_2(0)$ to be close to 1. However the true distribution functions are difficult to work with. Instead we define modified functions $\widehat{F}_i(\tau)$ for which $\widehat{P}_i(0)$=$P_i(0)$ but which otherwise (weakly) underestimate 1's probability of concession and overestimate 2's. That is,

$$\begin{aligned} \eta_2 &\geq \widetilde{\eta}\,(1 - F_2(\widetilde{\tau})) \geq \widetilde{\eta}(1 - \widehat{F}_2(\widetilde{\tau})) \\ \text{and} \quad (1 - \widehat{F}_1(\widetilde{\tau}) &\geq 1 - F_1(\widetilde{\tau}) \geq \eta_1 \end{aligned}$$

We show below that for small $z_i's$ the above inequalities imply that $\widehat{P}_2(0)$ is close to 1. That is, 2 concedes "too slowly" relative to 1 even when we overestimate 2's rate of concession and underestimate 1's.

Recall from Step 2 that

$$1 - F_i(\widetilde{\tau}) = e^{-\int_0^{t(\widetilde{\tau})} \lambda_i(s)ds}(1 - P_i^0)(1 - P_i^1)...(1 - P_i^{L_i})$$

where $l = 0, 1, ..., L_i$ indexes positive probability concessions by player i until date $\widetilde{\tau}$.

For Player 2, any positive probability concession must be associated with a down jump (Lemma 9). Let the l^{th} down jump occur at date $\tau(l)$ (assumed to be increasing in l) and entail a drop in payoff to player 1 from $a(l)$ to $b(l)$. For any l such that $a(l+1) > b(l)$. Lemma 13 in the Appendix establishes the intuitively plausible result that between dates $\tau(l)$ and $\tau(l+1)$ there must exist a consecutive sequence of "shadows" corresponding to up jumps in 1's payoffs from a payoff $b \leq b(l)$ to $a \geq a(l+1)$. Down jumps over payoff drops which have also occurred at an earlier date are offset by up-jumps covering (at least) the same range. By Step 4, we can match (by subdividing and agglomerating as required) such 'repeated' down jumps with up jumps which span the same range. Thus one can construct new functions $\widehat{F}_i$, $i = 1, 2$ for which up jumps and down jumps are matched as follows. Let

$$\begin{aligned} 1 - \widehat{F}_1(\widetilde{\tau}) &= (1-\widehat{P}_1^0)(1-\widehat{P}_1^1)\dots(1-\widehat{P}_1^K)e^{-\int_0^{t(\widetilde{\tau})}\lambda_1(s)ds} \\ 1 - \widehat{F}_2(\widetilde{\tau}) &= (1-\widehat{P}_2^0)(1-\widehat{P}_2^1)\dots(1-\widehat{P}_2^K)(1-\widehat{P}_2^{K+1})e^{-\int_0^{t(\widetilde{\tau})}\lambda_2(s)ds} \end{aligned}$$

where for $k = 1, \dots, K$, $\widehat{P}_2^k$ corresponds to a 'down-jump' from some u_1^k to $u_1^k - \Delta$ and $\widehat{P}_1^k$ corresponds to a matched up-jump from $u_1^k - \Delta$ to u_1^k between time times $\underline{t}^k$ and $\bar{t}^k$ respectively. We set $\widehat{P}_i^0 = P_i^0$. The latter is the probability of revealing rationality at the very start of the game and is the same for the original and modified distributions. The unmatched term $\widehat{P}_2^{K+1}$ accounts for the possibility of 'non-repeating' down-jumps . The modified distribution function $\widehat{F}_1$ neglects some concession episodes, since

1. it is possible that some $P_1^l > 0$ are not associated with up jumps (see the remarks preceding the proof of Proposition 2), and
2. some up jumps might not simply be 'offsetting' repeated down jumps.

   Of course this is consistent with underestimating 1's distribution function, and as desired we have:

$$(1 - \widehat{F}_1(\widetilde{\tau})) \geq (1 - F_1(\widetilde{\tau})).$$

By setting $\widehat{P}_2^{K+1}$ "generously" we can furthermore guarantee that $(1 - \widehat{F}_2(\widetilde{\tau})) \leq (1 - F_2(\widetilde{\tau}))$. By Lemma 11, $u_1^* - \varepsilon$ is an upper bound on player 1's expected equilibrium payoff at any $\tau \preceq \widetilde{\tau}$. The highest possible $\widehat{P}_2^{K+1}$ is associated with an offer that drops from $u_1^* - \varepsilon$ to $\underline{u}_1$, the smallest payoff to

1 in the efficiency frontier of the stage game. Thus a generous specification of $\widehat{P}_2^{K+1}$ is

$$(1-\widehat{P}_2^{K+1}) = \frac{\varepsilon}{u_1^* - \underline{u}_1} \equiv a_2 > 0.$$

By the analysis of Step 4, all non-repeating down jumps are covered by the term $\widehat{P}_2^{K+1}$ as defined above.

7. *Comparing Matched Concession Episodes.*

As noted following Lemma 8,

$$\begin{array}{ll} \lambda_1(s) > \lambda_2(s) & \text{for } s \in (n, t(\bar{\tau}(n))) \\ \lambda_1(s) = \lambda_2(s) = 0 & \text{otherwise.} \end{array}$$

Can we compare $\hat{P}_1^k$ corresponding to an up jump from $\underline{w}_1^k$ to $\bar{w}_1^k$ between times $\underline{t}_k$ and $\bar{t}^k$ to $\hat{P}_2^k$ corresponding to a down jump from $\bar{w}_1^k$ to $\underline{w}_1^k$? As defined in the Appendix (see proof of Lemma 14), $\hat{P}_1^k$ solves

$$\frac{u_2^*}{r} = \int_{\underline{t}_k}^{\bar{t}_k} e^{-r(s-\underline{t}_k)} d_2(s)\, ds + e^{-r(\bar{t}_k - \underline{t}_k)} \left[\frac{h_2(\bar{w}_1^k)}{r} \hat{P}_1^k + \frac{u_2^*}{r}\left(1-\hat{P}_1^k\right) \right]$$

where $w_1^k \in (\underline{w}_1^k, \bar{w}_1^k]$, and $\hat{P}_2^k = \frac{\bar{w}_1^k - \underline{w}_1^k}{u_1^* - \underline{w}_1^k}$ as usual.

Now it follows that $\hat{P}_1^k > \hat{P}_2^k$ exactly as in Step 5. The only difference is that instead of $h_2(\bar{w}_1^y)$ ($\bar{w}_1^y$ plays the same role as a in Step 5) we have $h_2(w_1^y)$. This does not impact the argument.

By Lemma 11 there exists $\varepsilon > 0$ such that $\tau \preceq \bar{\tau}$, $w_1^+(\tau) \le u_1^* - \varepsilon$ uniformly across $(z_1, z_2) \in (0,1)^2$ and possible equilibria. Hence $u_1((n,+1)) \le u_1^* - \varepsilon$ and $\bar{w}_1^k \le u_1^* - \varepsilon$. It follows that there exists $\alpha > 1$ such that

$$\begin{array}{ll} \lambda_1(s) \ge \alpha \lambda_2(s) & \text{for all } s \text{ and} \\ \left(1-\hat{P}_1^l\right) \le \left(1-\hat{P}_2^l\right)^{\alpha} & \text{for all } l = 1, ..., L \end{array}$$

uniformly across z_1, z_2 and possible equilibria.

Recall that

$$\begin{array}{lll} \eta_1 & \le & 1-\widehat{F}_1(\tau^*) = (1-\widehat{P}_1^0)A_1 \\ \eta_2 & \ge & 1-F_2(\widehat{\tau}^*) = (1-\widehat{P}_2^0)A_2(1-\widehat{P}_2^{L+1}) \end{array}$$

where $A_i = e^{-\int_0^{t(\tau^*)} \lambda_i(s)ds}(1-\widehat{P}_i(1))...(1-\widehat{P}_i(L))$, and $\widehat{P}_i(0)$ is the initial probability with which player i concedes.

It follows that

$$
\begin{aligned}
\eta_1 &\leq (1 - P_1(0))A_2^\alpha \\
\frac{\eta_2}{(1 - P_2(0))(1 - \widehat{P_2}(L+1))} &\geq [\frac{\eta_1}{(1 - P_1(0))}]^{1/\alpha} \\
\frac{\eta_2}{\eta_1} \cdot \eta_2^{\alpha-1} &\geq \frac{(1 - P_2(0))^\alpha}{(1 - P_1(0))} \cdot a_2^\alpha
\end{aligned} \tag{5}
$$

where $a_2 \equiv \left(1 - \hat{P}_2(L+1)\right) > 0$.

This analysis applies to any z_1, z_2. Suppose $\frac{z_2}{z_1} \leq R$ and $\mu_2(\gamma_2; z_2) \geq \underline{\mu}$. Then

$$
\begin{aligned}
\eta_1 &\geq \frac{z_1 \pi_1(\gamma_1)}{(1 - z_1) \cdot 1 + z_1 \pi_1(\gamma_1)} \\
\eta_2 &\leq \frac{z_2 \pi_2(\gamma_2)}{(1 - z_2) \cdot \overline{\mu} + z_2 \pi_2(\gamma_2)} \leq z_2 B \\
&\Rightarrow \frac{\eta_2}{\eta_1} \leq \frac{z_2}{z_1} \cdot \frac{\pi_2(\gamma_2)}{\pi_1(\gamma_1)} \cdot \frac{(1 - z_1) + z_1 \pi_1(\gamma_1)}{(1 - z_2) \cdot \overline{\mu} + z_2 \pi_2(\gamma_2)} \leq RC
\end{aligned}
$$

for a given R and some finite constants B, C independent of (z_1, z_2). Returning to (5) we obtain

$$
RC(z_2 B)^{\alpha-1} \geq \frac{(1 - P_2^0)^\alpha}{(1 - P_1^0)} \cdot a_2^\alpha
$$

Hence $P_2^0 \geq 1 - (RC)^{\frac{1}{\alpha}} \left(1 - P_1^0\right)^{\frac{1}{\alpha}} \frac{1}{\alpha} (z_2 B)^{\frac{\alpha-1}{\alpha}}$, which is close to 1 for $\delta > 0$ small enough and $z_2 \leq \delta$. Normal player 1's payoff is at least $P_2(0) u_1^* + (1 - P_2(0)) \underline{d}_1$ which in turn is at least $u_1^* - \frac{\xi}{2}$ for δ small enough and (consequently) $P_2(0)$ close enough to 1. (Recall that $\underline{d}_1$ is the lowest possible payoff to 1 in the stage game G.)

Appendix

Section 3: Stationary Postures

Proof of Lemma 1

Proof. Suppose w.l.o.g. that $i = 1$ and $j = 2$.

Step 1 Exists $\widetilde{T} < \infty$ such that 2 accepts 1's demand with probability 1 by $t + \widetilde{T}$ if γ_1 continues to be played until $t + \widetilde{T}$.

Since (1) $h_2(u_1) > \max_{m_2'} d_2(m_2', m_1)$ (2) (Player) 2 is impatient (3) 2's payoffs in G are bounded above (G is finite), it follows that there exist $\beta > 0$ and $T < \infty$ such that player 2 will accept 1's offer right away unless 2 believes that 1 will reveal rationality with probability at least β, between t and $t + T$.

Let β satisfy

$$\beta\overline{u_2} + (1 - \beta)\max_{m_2'} d_2(m_2', m_1) < h_2(u_1)$$

and $T < \infty$ satisfy

$$\beta\overline{u_1} + (1 - \beta)[(1 - e^{-rT})\max_{m_2'} d_2(m_2', m_1) + e^{-rT}\overline{u_1}] < h_2(u_1)$$

Conditional upon player 2 not accepting 1's offer and upon 1 continuing to conform with γ_1 until $t + T$, a similar conclusion follows between $t + T$ and $t + 2T$, and so on. Since $\pi_1(\gamma_1) > 0$ the posterior probability η_1 that 1 is behavioral at t is strictly positive, and conditional upon conformity by 1 and non-acceptance by 2 the posterior probability that 1 is behavioral at $t + nT$ is $\frac{\eta_1}{(1-\beta)^n}$. Since it is also necessary that $\frac{\eta_1}{(1-\beta)^n} \leq 1$, this leads to contradiction for large n. It follows that there exists $\overline{T} < \infty$ such that player 2 accepts 1's demand u_1 by $\overline{T}$ with probability 1, conditional upon 1 continuing to conform with γ_1 between t and $t + \overline{T}$. Suppose $\overline{T}$ is chosen such that the preceding statement is false for any $\widetilde{T} < \overline{T}$.

Step 2 $\overline{T} = 0$

Suppose not. Then $\widetilde{u_2}$, 2's demand immediately prior to $\overline{T} + t$, exceeds $h_2(u_1)$, and there exists $\varepsilon > 0$ such that 1 strictly prefers $u_1 e^{-\varepsilon}$ to $h_1(\widetilde{u_2})$. It follows that conditional upon sticking with γ_1 until $\overline{T} + t - \varepsilon$, 1 will continue to stick with γ_1 with probability 1 until $\overline{T} + t$. It follows that 2 should accept 1's demand u_1 with probability 1 strictly prior to $\overline{T} + t$, contradicting the definition of $\overline{T}$. ■

Section 4: Nonstationary Postures

Lemma 6 *There exists τ with $t(\tau) < \infty$ such that $1 - F_2(\tau) = \eta_2$.*

Proof. By our regularity assumption , $\max_{m_2'} d_2(m_1^*, m_2') < u_2^*$. The rest of the argument is virtually identical to Step 1 of the proof of Lemma 1. ■

Lemma 7 *Consider equilibrium in the continuation game following the choice of postures (γ_1^*, γ_2). Suppose that player 2 is the first to reveal rationality and does so at date τ. Then the resultant equilibrium continuation payoff is (u_1^*, u_2^*); player 2, in effect, always reveals rationality by accepting 2's stationary offer.*

Proof. The lemma follows directly from the proof of Lemma 1. ■

Lemma 10 *If $t(\tau^*) = 0$ then a rational player 1's payoff is at least* $(1 - \eta_2)u_1^* + \eta_2\underline{u}_1$

Proof. We argue that if $t(\tau^*) = 0$ the strategy "always conform with γ_1^*" yields a rational player 1 a payoff which is at least $(1 - \eta_2)u_1^* + \eta_2\underline{u}_1$. If

$u_1^* \leq h_1(u_2(\tau^*))$ then the conclusion follows directly. If $(1 - F_2(\tau^*)) = \eta_2$ then the result follows since by Lemma 4, player 2 reveals rationality by, in effect, accepting 1's stationary demand. Finally, if $(1 - F_1(\tau^*)) = \eta_1$ then, in the event of player 1 not revealing rationality at τ^*, a rational player 2 should reveal rationality immediately thereafter. The conclusion again follows directly. ■

The faster rate of concession by 1 (both continuous and lumpy) is driven by the GAP between what 1 can extract from 2 by revealing rationality and 1's 'reasonable' demand u_1^*. If the gap goes to zero then the difference in the rates goes to zero also and 1 no longer 'wins' the 'race' by an overwhelming margin and the argument that 2 needs to give in at the start with probability close to 1 breaks down. The next lemma establishes that this gap (which is date dependent) is uniformly bounded above zero, until 2's posterior probability of being behavioral reaches a threshold $\tilde{\eta}$.

Let $w_1^+(\tau)$ be the expected equilibrium payoff to 1 at τ conditional upon neither player revealing rationality strictly prior to τ.

Recall that $\underline{d}_i$ is the lowest possible payoff to Player i in G, and that $\underline{u}_i$ and $\overline{u}_i$ are respectively the minimum and maximum payoffs to i on the (strictly) Pareto efficient frontier of G.

Lemma 11 *For all $\gamma_2 \in \Gamma$, there exists $\tilde{\eta} \in (0,1)$ and $\varepsilon > 0$ such that $\tilde{\eta}\underline{u}_1 + (1 - \tilde{\eta})\, u_1^* > u_1^* - \varepsilon$ and such that for all $(z_1, z_2) \in (0,1)^2$ and for* *all perfect Bayesian equilibria in the continuation game given* (z_1, z_2)*, when* (γ_1^*, γ_2) *are chosen, either* (1) $t(\tau^*) = 0$ *or* (2) *for all* $\tau \preceq \tau^*$ *if* $\eta_2(\tau) \leq \tilde{\eta}$ *then* $w_1^+(\tau) \leq u_1^* - \varepsilon$.

Proof. Suppose $t(\tau^*) > 0$. Since γ_1^* and γ_2 have a finite number of states, then exists $\varepsilon_1 > 0$ such that $h_1(u_2(\tau)) < u_1^* - \varepsilon_1$ for all τ such that $u_2(\tau) > u_2^*$ (equivalently $h_1(u_2(\tau)) < u_1^*$). Hence there exists $0 < \varepsilon_2 < \varepsilon_1$ and $\Delta > 0$ such that $u_1^* - \varepsilon_1 < e^{-r\Delta}(u_1^* - \varepsilon_2) + \left(1 - e^{-r\Delta}\right)\underline{d}_1$ where $\underline{d}_1$ is the lowest possible payoff to 1 in G. (The rhs of the preceding inequality is the payoff to 1 if 1 waits for time Δ to receive $(u_1^* - \varepsilon_2)$ while receiving the lowest possible flow payoff in the interim.) It follows that if at some τ, $w_1^+(\tau) \geq u_1^* - \varepsilon_2$ for the first time, then player 1 will reveal rationality with probability zero for an interval of time $\Delta > 0$, prior to τ. (Note that we can choose $\Delta > 0$ such that $\Delta < t(\tau)$.)

For normal 2 not to concede with probability 1 prior to τ, it must be the case that $u_2^* \leq e^{-r\Delta}u_2 + \left(1 - e^{-r\Delta}\right)d_2$, where u_2 is normal 2's expected equilibrium payoff at τ and d_2 is 2's (discounted average) flow payoff in the interim. By the definition of γ_1^* and m_1^*, 2's payoff in any round must be less than or equal to u_2^*, and by our regularity assumption must, in fact, be *strictly* less. Since each posture has a finite number of states there exists $a > 0$ such that $d_2 < u_2^* - a$.Hence $u_2 \geq u_2^* + b$ for some $b > 0$. It follows that, conditional upon 2 being normal, 1's expected payoff at τ is at most $u_1^* - \varepsilon_3$ for some $\varepsilon_3 > 0$. Consequently $w_1^+(\tau) \leq \eta_2(\tau)\overline{u}_1 + (1 - \eta_2(\tau))(u_1^* - \varepsilon_3)$. Let $\varepsilon = \frac{\varepsilon_3}{2}$. Clearly there exists $\tilde{\eta} \in (0,1)$ such that $\tilde{\eta}\underline{u}_1 + (1 - \tilde{\eta})u_1^* > u_1^* - \varepsilon$ and $\eta_2(\tau)\overline{u}_1 + (1 - \eta_2(\tau))(u_1^* - \varepsilon_3) < u_1^* - \varepsilon$ for all $\eta_2(\tau) \leq \tilde{\eta}$. (Set $\tilde{\eta} = \min\left\{\frac{\varepsilon}{u_1^* - \underline{u}_1}, \frac{\varepsilon}{\overline{u}_1 - u_1^* + 2\varepsilon}\right\}$). ■

Lemma 12 *For any equilibrium consider the continuation game following the choice of* (γ_1^*, γ_2)*. Let* $\tilde{\eta}$ *be defined as in Lemma 11. Then either* $t(\tau^*) = 0$ *or* $\eta_2(\tau^*) \geq \tilde{\eta}$*.*

Proof. Suppose $t(\tau^*) > 0$ and $\eta_2(\tau^*) < \tilde{\eta}$. Then $1 - F_2(\tau^*) > \eta_2$. (If $1 - F_2(\tau^*) = \eta_2$ then $\eta_2(\tau^*) = 1 > \tilde{\eta}$). Also, by Lemma 11, $w_1^+(\tau) < u_1^* - \varepsilon$ for all $\tau \preceq \tau^*$ (since $\eta_2(\tau) \leq \eta_2(\tau^*) \leq \tilde{\eta}$ for all $\tau \preceq \tau^*$). From the definition of τ^* it therefore follows that $1 - F_1(\tau^*) = \eta_1$. Consequently normal player 2 must reveal rationality/concede immediately after τ^*. Hence, $w_1^+(\tau^*) \geq \tilde{\eta}\underline{u}_1 + (1 - \tilde{\eta})u_1^* > u_1^* - \varepsilon$ (see Lemma 11), a contradiction. ■

The discussion below elaborates elements of Step 10 in the text, in particular the discussion of *repeated* down jumps. As in the text, consider the l^{th} down jump and suppose that player 1's payoff $b(l)$ after the l^{th} down jump is strictly less than 1's payoff $a(l+1)$ at the 'start' of the $(l+1)^{\text{th}}$ down jump. Between these down-jumps we wish to argue that there are offsetting up jumps.

Recall from step 5 of the proof of Proposition 2 that there is a formula for the conditional concession probability by player 1 that is needed to compensate player 2 for waiting while player 1 waits for an upward jump of a given size in 1's value. Call this the canonical formula. There are complicated cases in which this formula does not apply directly. For example, suppose that an increase in value from b to a at some time τ_2 casts a shadow over the interval $[\tau_0, \tau_2]$. There might be some date $\tau_3 \in (\tau_0, \tau_2)$ at which the continuation equilibrium rewards 1 for revealing rationality (but *not* conceding) just enough so that he is indifferent between doing so or waiting until τ_2. His indifference means that there are many combinations of concession probabilities at τ_1 and τ_2 by 1 that are compatible with maximizing his utility, and which exactly compensate 2 for her wait from τ_0 to τ_2. In such cases one cannot use the canonical formula to associate with the jumps at τ_2, a particular concession probability by 1.

Because of the indeterminacy just described, it is important to look at the interval $[\tau_0, \tau_2]$ as a whole, rather than at the concession episodes at τ_1 and τ_2 separately (and hence the introduction of Definition 2).

Definition 2 *The interval $\mathcal{I}$ is an interval of zero concession by player 2 if for all $\tau', \tau'' \in \mathcal{I}$, $F_2(\tau') = F_2(\tau'')$. Such an interval is a maximal interval of zero concession by player 2, if for all $\mathcal{I}^+ \supseteq \mathcal{I}, \mathcal{I}^+ \neq \mathcal{I}$ there exist $\tau', \tau'' \in \mathcal{I}^+$ such that $F_2(\tau'') > F_2(\tau')$.*

Lemma 13 asserts that between any two episodes in which 1's value falls over a certain range, say from 20 to 14, there must be a sequence (called a "spanning sequence" - see Definition 5 following Lemma 13) of (weakly overlapping) up-jumps whose union covers the interval $[14, 20]$. For example, if the value falls from 22 to 13, it might later fall from 20 to 14, but before doing so it would have to somehow rise to at least 20.

Lemma 13 *Suppose for some $n', n'' \in N$, $n' < n''$, $P_2(n) = 0$ for all $n \in N, n' < n < n''$, $P_2(n'), P_2(n'') > 0$ and $w_1((n', +1)) < \max\{v_1^2((n'', -1)), v_1^2((n'', 0))\}$. Then there exists a sequence of maximal intervals of zero concession by 2, $\mathcal{I}(q)$, $q = 1, \ldots, Q$, with associated left and right end-points $\underline{\tau}(q) \equiv \inf \mathcal{I}(q)$*

and $\overline{\tau}(q) \equiv \sup \mathcal{I}(q)$ *respectively, such that*

$$\begin{aligned}
w_1(\underline{\tau}(1)) &\leq w_1(\tau'),\, w_1(\overline{\tau}(Q)) \geq w_1(\tau'') \\
w_1(\underline{\tau}(q+1)) &\leq w_1(\overline{\tau}(q)) \quad q = 1,\ldots,Q-1 \text{ and} && (6)\\
w_1(\underline{\tau}(q)) &< w_1(\overline{\tau}(q)) \quad q = 1,\ldots,Q && (7)\\
\tau' &\preccurlyeq \underline{\tau}(1) \prec \overline{\tau}(Q) \preccurlyeq \tau'',\, \overline{\tau}(q) \prec \underline{\tau}(q+1), \quad q = 1,\ldots,Q && (8)
\end{aligned}$$

Proof. Since $w_1((n',+1)) < \max\{v_1^2((n'',-1)), v_1^2((n'',0))\}$, there must exist a first date $\tilde{\tau} \succ (n',+1)$ at which $v_1^2(\tilde{\tau}) > w_1((n',+1))$. It follows that Player 1 does not concede immediately prior to $\tilde{\tau}$. Hence neither does 2 in an interval prior to $\tilde{\tau}$. It follows that there exists a maximal interval $\mathcal{I}$ with associated left and right end-points $\underline{\tau}$ and $\overline{\tau}$ respectively, containing $\tilde{\tau}$, such that $w_1(\underline{\tau}) \leq w_1((n',+1))$. It is however possible that $w_1(\overline{\tau}) < w_1((n',+1))$. In this case, $t(\overline{\tau}) < n''$ and we can repeat the preceding argument replacing $(n',+1)$ with $\overline{\tau}$. Proceeding in this manner we obtain a first maximal interval $(\underline{\tau}(1), \overline{\tau}(1))$ for which $w_1(\underline{\tau}(1)) \leq w_1((n',+1))$ and $w_1(\underline{\tau}(1)) < w_1(\overline{\tau}(1))$. If $w_1(\overline{\tau}(1)) < \max\{v_1^2((n'',-1)), v_1^2((n'',0))\}$, $\overline{\tau}(1)$ now plays the role of $(n',+1)$ in the initial argument. And so on, until the required sequence is obtained. Since $P_2(n'') > 0$, $t(\overline{\tau}(q)) \leq n''$ for all q. ■

Definition 3 *A sequence as specified in Lemma 13 is said to span* $[b,a]$, *where* $b = w_1((n',+1))$ *and* $a = \max\{v_1^2((n'',-1)), v_1^2((n'',0))\}$.

By our regularity assumption regarding generic type sets (Assumption 2), $h_1(u_2(\tau)) \neq d_1(\tau)$ for all $\tau \preccurlyeq \tau^*$. It follows that if within a round n there is zero concession by player 2, conceding at $(n,+1)$ strictly dominates conceding at any subsequent date within round n, for player 1 if $h_1(u_2(\tau)) > d_1(\tau)$, while conceding at $(n+1,-1)$ strictly dominates conceding at a prior date within the round, if the opposite inequality is satisfied. Hence within an interval such as $\mathcal{I}(q)$, player 1 reveals rationality or concedes only at the beginning, in between, or at the end of rounds contained within $\mathcal{I}(q)$.

For a sequence of maximal intervals as in Lemma 13 and Definition 5, let $x = 1,\ldots,X$ index the finite set of instances at which 1 concedes at a date in $\mathcal{I}(q)$ for some $q \in \{1,\ldots,Q\}$. Let P_1^x denote the corresponding conditional probability of concession by 1.

Lemma 14 translates the probabilities P_1^x just defined, into modified probabilities $\hat{P}_1^1,\ldots,\hat{P}_1^Y$ such that (i) the overall probability of concession by 1 is weakly lower according to the modified probabilities than the true probabilities, and (ii) the modified probabilities are less than or equal to the numbers one would obtain by applying the canonical formula (see Step 5 in the text) to the respective up jumps in 1's value that occur in the maximal interval in question. Property (ii) is useful because if a probability P_1 is obtained by applying the canonical formula to an up jump, it can be compared easily (see Step 5) to the concession probability by 2 associated with a down jump over the same interval. Both (i) and (ii) are consistent with our need to underestimate 1's concession probabilities (see Step 6).

Lemma 14 *Consider a sequence of maximal intervals of zero concession by 2 which span $[a, b]$ and suppose that $\eta_2(\overline{\tau}(Q)) \leq \widetilde{\eta}$ where $\overline{\tau}(Q)$ is as defined in Lemma 13. Let $P_1^1, ..., P_1^X$ be a sequence of (conditional) probabilities as specified above. Then there exists a sequence of probabilities $\hat{P}_1^1, \ldots, \hat{P}_1^Y$ and a corresponding sequence of values and dates $\underline{w}_1^y, \bar{w}_1^y, w_1^y \in (\underline{w}_1^y, \bar{w}_1^y], \underline{t}_y$ and $\bar{t}_y$, $y = 1, \ldots, Y$, such that:*

$$\begin{aligned} \underline{w}_1^y &< \bar{w}_1^y \\ \underline{w}_1^y &< \underline{w}_1^{y+1} \leq \bar{w}_1^y \\ \bar{w}_1^y &< \bar{w}_1^{y+1} \\ \underline{t}_y &\leq \bar{t}_y \\ \underline{w}_1^1 &\leq a, \qquad \bar{w}_1^Y \geq b \end{aligned}$$

and such that the $\hat{P}_1^y$'s solve the 'canonical' equation

$$\frac{u_2^*}{r} = \int_{\underline{t}_y}^{\bar{t}_y} e^{-r\left(s-\underline{t}_y\right)} d_2\left(s\right) ds + e^{-r\left(\bar{t}_y-\underline{t}_y\right)} \left[\frac{h_2\left(w_1^y\right)}{r} \hat{P}_1^y + \frac{u_2^*}{r}\left(1-\hat{P}_1^y\right)\right]$$

and

$$\left(1-P_1^1\right) \ldots \left(1-P_1^X\right) \leq \left(1-\hat{P}_1^1\right) \ldots \left(1-\hat{P}_1^Y\right)$$

We prove the lemma by establishing the result for any single maximal interval (of zero concession by player 2) $\mathcal{I}$ and the finite set of instances $l = 1, \ldots, L$ (with corresponding conditional probabilities $P_1^l, l = 1, \ldots, L$) at which 1 concedes within such an interval.

Proof. Let $\mathcal{I}$ be a maximal interval as defined above and let $\underline{\tau}$ and $\overline{\tau}$, respectively be the left and right end points of the interval. Let $\{\tau_1, \ldots, \tau_L\} \subseteq \mathcal{N}^+ \cap \mathcal{I}$ be the finite set of dates at which 1 reveals rationality within $\mathcal{I}$.

Define

$$\begin{aligned} \tau_0 &\equiv \begin{cases} (t(\underline{\tau}), +1) & \text{if } t(\underline{\tau}) \in \mathcal{N} \\ \underline{\tau} & \text{otherwise} \end{cases} \\ t_l &\equiv t(\tau_l). \end{aligned}$$

Let w_2^l be player 2's expected (discounted average) payoff at date τ_l, conditional upon player 1 not having revealed rationality until τ_l (inclusive). Then

$$\frac{w_2^l}{r} = \left[\int_{t_l}^{t_{l+1}} d_2(s)e^{-r(s-t_l)}ds\right] + e^{-r(t_{l+1}-t_l)}\left[\frac{v_2^2(\tau_l)}{r}P_1^{l+1} + \frac{w_2^{l+1}}{r}(1-P_1^{l+1})\right]$$

where $v_2^2(\tau)$ denotes the expected equilibrium payoff to 2, conditional upon 1 revealing rationality at τ.

Let $\tau_0, \ldots, \tau_L$ be as defined above. We will define a new sequence $\hat{\tau}_0, \ldots, \hat{\tau}_K$ and corresponding sequence of probabilities $\hat{P}_1^1, \ldots, \hat{P}_1^K$ such that

$$(1-P_1^1)\cdots(1-P_1^L) \leq (1-\hat{P}_1^1)\cdots(1-\hat{P}_1^K)$$

where $\hat{P}_1^k$ corresponds to an up-jump from $\underline{w}_1^k$ to $\overline{w}_1^k$ which can be matched with a down-jump from $\overline{w}_1^k$ to $\underline{w}_1^k$. Furthermore, $\underline{w}_1^1 \leq w_1^1$, $\underline{w}_1^k < \underline{w}_1^{k+1} \leq \overline{w}_1^k$, and $\overline{w}_1^K = w_1^L$.

The argument proceeds by modifying the original P_1^l's in successive steps such that the modified P_1^l's (call them $P_1^l(h)$'s in step h) yield a product $(1-P_1^1(h))\cdots(1-P_1^L(h))$ which is weakly higher than the corresponding product from the preceding step.

Within any maximal interval of concession (by 2), we wish to assign concession probabilities by 1 in the most conservative way, so that our approximate concession distribution function $\hat{F}_1$, will underestimate 1's probability of conceding by any date, as desired. The following procedure achieves this, while respecting the incentive constraints of both players. Begin by defining a sequence

$$\begin{aligned} q(0) &= 0, \text{ and for } k = 1, \ldots, K \\ q(k) &= \max\left\{ l \;\middle|\; \begin{array}{c} v_1^2(\tau_l) \leq v_1^2(\tau), \\ \tau = \tau_{q(k-1)+1}, \ldots, \tau_l. \end{array} \right\} \end{aligned}$$

The new sequence terminates at K such that $q(K) = L$. Observe that $v_1^2(\tau_{q(k)})$ is strictly increasing in k.

Define

$$k^*(l) = \min\{k \mid q(k) \geq l\}$$

Let

$$\begin{aligned} P_1^l(0) &= P_1^l \quad l = 1, \dots, L \\ \text{and } w_2^l(0) &= w_2^l \quad l = 0, 1, \dots, L \end{aligned}$$

We seek to inductively define $P_2^l(1)$ and $w_2^l(1)$ starting with $l = L$ and moving backwards to $l = 0$ (or 1 as the case may be). Recall that h in $P_2^l(h)$ refers to the h^{th} step in modifying the initial P_1^l's. Each step *itself* involves an inductive definition starting with $l = L$ and moving backwards to $l = 1$.

We set $w_2^L(1) = w_2^L(0)$.

At each stage, $P_1^{l+1}(1) = \max\{0, x\}$ where x solves:

$$w_2^l(0) = r\int_{t_i}^{t_{i+1}} d_2(s)e^{-r(s-t_i)}ds + e^{-r(t_{i+1}-t_i)}\left[w_2^{l+1}(1) + x(h_2(v_1^2(k^*(l))) - w_2^{l+1}(1)\right] \tag{9}$$

The definition of P_1^{l+1} also leads to the definition of $w_1^l(1)$ as follows:

$$w_1^l(1) = r\int_{t_i}^{t_{i+1}} d_2(s)e^{-r(s-t_i)}ds + e^{-r(t_{i+1}-t_i)}\left[w_2^{l+1}(1) + P_1^{l+1}(h_2(v_1^2(k^*(l))) - w_2^{l+1}(1)\right]$$

Because $v_2^2(\tau_L) \leq h_2(v_1^2(k^*(L)))$, these definitions imply:

$$\begin{aligned} P_1^L(1) &\leq P_1^L(0) \text{ and} \\ \text{and } w_2^{L-1}(1) &\geq w_2^{L-1}(0) \end{aligned}$$

Since

$$v_2^2(\tau_{l+1}) \leq h_2(v_1^2(k^*(l+1))),\ w_2^{l+1}(1) \geq w_2^{l+1}(0)$$

at stage $(l+1)$ of the inductive definition,

$$P_1^{l+1}(1) \leq P_1^{l+1}(0) \text{ and } w_2^l(1) \geq w_2^l(0)$$

The next step in the argument relies on the result that $w_2^l(1) \geq u_2^*$ $l = 0, 1, ..., L-1$. To demonstrate these inequalities we first establish the following useful fact for $l = 1, ...L-1$: *If* $w_2^l(0) < u_2^*$ *then* $k(\tau_l) = -1$, $\tau_{l+1} = (t(\tau_l), 0)$ *and* $v_2^2(\tau_{l+1}) < u_2^*$. To see this note that, strictly within a round player 1 can only reveal rationality by conceding to player 2's current demand, which prior to τ^* must exceed u_2^* when $\eta_2(\tau) \leq \tilde{\eta}$ (See Lemmas 11 and 12). Furthermore, strictly within a round player 2 can always obtain u_2^* by conceding to player 1. The only possibility of payoffs below u_2^* arises due to 1 revealing rationality between rounds in a manner which yields 1 less than u_2^*. Player 2 accepts this eventuality precisely because of the possibility of positive probability concession by player 1 a moment earlier at the end of the preceding round $[\tau_l = (t(\tau_l), -1)]$ which yields player 2 strictly more than u_2^* (that is, her demand in the preceding round).

These considerations, and the definition of τ_0 directly imply that $w_2^0 \geq u_2^*$ also.

Finally, we argue that $w_2^{L-1}(1) \geq u_2^*$. Recall that by definition, $w_2^L(1) = w_2^L(0) \equiv w_2^L$. Suppose that $w_2^{L-1}(1) < u_2^*$. By Lemma 14, this is only possible if $k(\tau_{L-1}) = -1$, $\tau_L = (t(\tau_{L-1}), 0)$ and $v_2^2(\tau_L) < u_2^*$. Now $k(\tau_L) = 0$ implies (by the definition of $w_2(\cdot)$) that $w_2(\tau_L) \equiv w_2^L \geq u_2^*$. By lemma ?? $v_1^2(\tau_L) < u_1^*$, hence $h_2(v_1^2(\tau_L)) > u_2^*$.Then the definition of P_1^L yields:

$$\begin{aligned} P_1^L(1) &= 0 \\ \text{and } w_2^{L-1}(1) &= 0 + e^{-r0} w_2^L(1) \\ &= w_2^L \geq u_2^*, \end{aligned}$$

a contradiction.

Continue to suppose that the lemma is false and let

$$l = \max\{m \leq L-1 \mid w_2^m(1) < u_2^*\}$$

Now we can repeat the preceding argument with l replacing $L-1$ to obtain the same contradiction as before.

This demonstrates that $w_2^l(1) \geq u_2^* \quad l = 0, 1, ..., L-1,$ as required.

Let

$$\begin{aligned} w_2^L(2) &\equiv w_2^L(1) \\ w_2^l(2) &= u_2^* \quad l = 1, \ldots, L-1 \\ w_2^0(2) &= w_2^0(1) \end{aligned}$$

The $P_1^l(2)$'s are uniquely defined by the equations:

$$\begin{aligned} w_2^{l-1}(2) &= d_2^{l-1}\left(1 - e^{-r(t_l - t_{l-1})}\right) \\ &\quad + e^{-r(t_L - t_{L-1})}\left[w_2^l(2) + P_1^l(2)\left(h_2(k^*(l)) - w_2^l(2)\right)\right] \end{aligned}$$

where d_2^{l-1} is the average discounted flow payoff to 2 between t_{l-1} and t_l.

Since $d_2^{l-1} < u_2^*$ (see proof of Lemma 11), $P_1^l(2)$ so defined exist, are strictly positive and unique. Furthermore, we show that

$$\begin{aligned} & (1 - P_1^1(1))(1 - P_1^2(1)) \cdots (1 - P_1^L(1)) \\ \leq\; & (1 - P_1^1(2))(1 - P_1^2(2)) \cdots (1 - P_1^L(2)) \end{aligned}$$

as required.

To see why this is the case, consider

$$\begin{aligned}
w_2 &= \hat{d}^1\left(1-e^{-r\Delta_1}\right)+e^{-r\Delta_1}\left[a+x(h^1-a)\right]\\
b &= \hat{d}^2\left(1-e^{-r\Delta_2}\right)+e^{-r\Delta_2}\left[w_2+y(h^2-w_2)\right]
\end{aligned}$$

where $a,\ b,\ h^1,\ h^2,\ \hat{d}^1$ and $\hat{d}^2$ are fixed, and we think of the probabilities x and y as functions of w_2.

Totally differentiating these equations with respect to w_2 yields

$$\begin{aligned}
1 &= e^{-r\Delta_1}(h_1-a)\frac{dx}{dw_2}\\
0 &= \left(h^2-w_2\right)\frac{dy}{dw_2}+(1-y)
\end{aligned}$$

It follows that

$$\frac{d\left(1-x(w_2)\right)\left(1-y(w_2)\right)}{dw_2}=-(1-y)\frac{dx}{dw_2}-(1-x)\frac{dy}{dw_2}<0$$

$$\begin{aligned}
&\Leftrightarrow e^{-r\Delta_1}\left(h^2-w_2\right)>(1-x)(h^1-a)\\
&\Leftrightarrow h^2-w^2>e^{-r\Delta_1}h^1-e^{-r\Delta_1}a-e^{-r\Delta_1}x\left(h^1-a\right)\\
&\Leftrightarrow h^2-e^{-r\Delta_1}h^1>\hat{d}^1\left(1-e^{-r\Delta_1}\right)
\end{aligned}$$

It follows that if, $\hat{d}^1,\hat{d}^2<u_2^*$, $h^2\geq h^1>u_2^*$ and $b\leq u_2^*$, then indeed

$$\frac{d\left(1-x(w_2)\right)\left(1-y(w_2)\right)}{dw_2}<0 \tag{10}$$

Now, if we set

$$\begin{aligned}
a &= w_2^L(2),\ \Delta_1=t_L-t_{L-1}\\
\hat{d}^1\left(1-e^{-r\Delta_1}\right) &= r\int_{t_{L-1}}^{t_L}d_2(s)e^{-r(s-t_{L-1})}ds\\
h^1 &= h_2\left(v_1^2\left(k^*\left(L\right)\right)\right)\\
b &= w_2^{L-2}(1),\ \Delta_2=t_{L-1}-t_{L-2}\\
\hat{d}^2\left(1-e^{-r\Delta_2}\right) &= r\int_{t_{L-2}}^{t_{L-1}}d_2(s)e^{-r(s-t_{L-2})}ds
\end{aligned}$$

then the latter inequalities indeed hold. Hence (10) implies

$$\begin{aligned} & \left(1-x\left(u_2^*;L\right)\right)\left(1-y\left(u_2^*;L\right)\right) \\ \geq \; & \left(1-P_1^L(1)\right)\left(1-P_1^{L-1}(1)\right) \end{aligned}$$

since

$$\begin{aligned} P_1^L(1) &= x\left(w_2^{L-1}(1)\right) \\ P_1^{L-1}(1) &= y\left(w_2^{L-1}(1)\right) \end{aligned}$$

and $w_2^{L-1}(1) \geq u_2^*$. (The argument L in $x\left(u_2^*;L\right)$, indexes the values chosen for $a,\ b,\ h^1,\ h^2,\ \hat{d}^1,\ \hat{d}^2$ and the time arguments in the integral.)

Notice that $P_1^L(2) = x\left(u_2^*;L\right)$. Proceeding inductively in this manner we see that $P_1^{L-1}(2) = x\left(u_2^*;L-1\right),\ P_1^{L-2}(2) = x\left(u_2^*;L-2\right)$, and so on. For instance, the next step would entail $a = w_2^{L-1}(2)$, $\Delta_1 = t_{L-1} - t_{L-2}$, $b = w_2^{L-3}(2), h^1 = h_2\left(v_1^2\left(k^*\left(L-1\right)\right)\right)$ and so on. It follows that,

$$\begin{aligned} & (1-P_1^1(1))(1-P_1^2(1))\cdots(1-P_1^L(1)) \\ \leq \; & (1-P_1^1(2))(1-P_1^2(2))\cdots(1-P_1^L(2)) \end{aligned}$$

Finally, we set

$$\begin{aligned} \hat{P}^k &= P_1^{q(k)+1}(2) \\ \underline{w}_1^k &= v_1^2(\tau_{q(k)}) \\ \overline{w}_1^k &= v_1^2(\tau_{q(k)+1}) \\ \underline{t}_k &= t\left(\tau_{q(k)}\right) \\ \bar{t}_k &= t\left(\tau_{q(k)+1}\right) \end{aligned}$$

to obtain the desired result for a single interval $\mathcal{I}(q)$. The extension to the collection of intervals is immediate. ■

Lemma 15 uses the collection of up-jump intervals constructed in Lemma14 to define modified conditional concession probabilities for 2, to be used in the modified distribution functions of Step 6 in the text. It applies the formula for P_2 from Step 5 to those constructed intervals to get the modified probabilities for 2; this overestimates (as desired) 2's probability of concession (away from 0) because, as Lemma 15 shows, there is a partition of the actual down-jump range whose elements are subsets of the constructed intervals in question (and by the neutrality result of Step 4, every partition of that range has the same aggregate implication for concession probability). Lemma 14 guarantees that the modified concession probabilities it assigns to player 1 yield lower overall concession probability than the true value for 1 (as desired). The first paragraph of Step 7 adapts the analysis for perfectly paired jumps in Step 5 to ensure that the modified up-jump probabilities (uniformly) outweigh the modified down-jump probabilities.

Lemma 15 *Consider the sequence of values $\underline{w}^y$, $\bar{w}^y$ $y = 1, ..., Y$ from the previous lemma. Define*

$$\hat{P}_2^y = \frac{\bar{w}_1^y - \underline{w}_1^y}{u_1^* - \underline{w}_1^y} \qquad\qquad y = 1, ..., Y$$

and

$$P_2 = \frac{a - b}{u_1^* - b}$$

then,

$$(1 - P_2) \geq \left(1 - \hat{P}_2^1\right) ... \left(1 - \hat{P}_2^Y\right)$$

Proof. Consider the sequence of values as defined in Lemma 14, and construct the following new sequences $\underline{v}_1^y, \bar{v}_1^y$ $y = 1, \ldots, Y$ where $\underline{v}_1^y = \min\left\{b, \bar{w}^{y-1}\right\}$ and $\bar{v}_1^y = \min\left\{a, \bar{w}^y\right\}$.

The sequence of intervals $[\underline{v}_1^y, \bar{v}_1^y]$ are mutually exclusive and partition $[b, a]$. Down jumps over the range $[b, a]$ may be neutrally (see Step 4) subdivided into Y down jumps from $\underline{v}_1^y$ to $\bar{v}_1^y$, $y = 1, \ldots, Y$ respectively. Let $\tilde{P}_2^y$ denote the positive probability of concession by 2 associated with a down jump from $\underline{v}_1^y$ to $\bar{v}_1^y$. Then $\left(1 - \tilde{P}_2^y\right) \leq \frac{u_1^* - \bar{v}_1^y}{u_1^* - \underline{v}_1^y}$ by Lemma 9. Let $\hat{P}_2^y$ be defined by $\left(1 - \hat{P}_2^y\right) = \frac{u_1^* - \bar{w}_1^y}{u_1^* - \underline{w}_1^y}$, that is correspond to a down jump from $\bar{w}_1^y$ to $\underline{w}_1^y$. Then clearly $\left(1 - \tilde{P}_2^y\right) \leq \left(1 - \hat{P}_2^y\right)$. Consequently, $(1 - P_2) = \left(1 - \tilde{P}_2^1\right) ... \left(1 - \tilde{P}_2^Y\right) \leq \left(1 - \hat{P}_2^1\right) ... \left(1 - \hat{P}_2^Y\right)$ ■

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