Willpower and the Optimal Control of Visceral Urges*

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First Version: October, 2004

This Version: April, 2006

Abstract

Common intuition and experimental psychology suggest that the ability to self-regulate, willpower, is a depletable resource. We investigate the behavior of an agent who optimally consumes a cake (or paycheck or workload) over time and who recognizes that restraining his consumption too much would exhaust his willpower and leave him unable to manage his consumption. Unlike prior models of self-control, a model with willpower depletion can explain the increasing consumption sequences observable in high frequency data (and corresponding laboratory findings), the apparent links between unrelated self-control behaviors, and the altered economic behavior following imposition of cognitive loads. At the same time, willpower depletion provides an alternative explanation for a taste for commitment, intertemporal preference reversals, and procrastination. Accounting for willpower depletion thus provides a more unified theory of time preference.

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*We thank Roy Baumeister, Gérard Gaudet and Miles Kimball for their important contributions to this paper. We also thank Roland Benabou, Yoram Halevy, Robert Mendelsohn, Joel Sobel, Kathleen Vohs, Itzhak Zilcha, and participants in seminars at IAS/Princeton University, the University of Michigan, University of Milano-Bicocca, and the University of Montreal for their helpful comments on a previous draft. Silverman gratefully acknowledges the support and hospitality of the Institute for Advanced Study.

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1 Introduction

Patterns of intertemporal choice that are common but inconsistent with standard models have inspired a growing literature on the economics of self-control. The human tendencies to seek self-commitment, to seize small immediate rewards despite their important future costs, and to procrastinate have motivated a number of studies of quasi-hyperbolic time discounting (e.g., Laibson 1997, O'Donoghue and Rabin, 1999), temptation costs (e.g., Gul and Pesendorfer 2001, 2004), and conflicts between "dual selves" (e.g., Thaler and Shefrin, 1988; Fudenberg and Levine, 2006). Such models of self control are consistent with a great deal of experimental evidence, and have been fruitfully applied to a number of economic problems ranging from portfolio choice to labor supply to health investment.

While capturing important aspects of both experimental and field data, these economic models of self control cannot explain three other "anomalous" features of intertemporal choices: (1) people behave differently if they first engage in cognitively demanding tasks (e.g. pilots making a second demanding landing of the day are known to be more prone to accident; see Wald, 2006); (2) agents sometimes exhibit *negative* time preference (e.g. they consume at an increasing rate, "save the best for last" when consuming and "get the hard part out of the way " when working); (3) people sometimes behave as if completely unrelated activities requiring self-control are linked (e.g. they do not resist smoking or overeating when preparing for an exam or important presentation).

In this paper, we present a tractable model of self control that explains each of these three anomalies. At the same time, our model provides an alternative explanation for prominent anomalies addressed by prior models of self-control. We accomplish this by taking into account a single cognitive constraint that is both intuitively appealing and consistent with a large literature in experimental psychology: exerting self control depletes a fungible but scarce cognitive resource, "willpower."

Anecdotes consistent with the notion of depletable willpower are common. Many who resist unhealthy foods and fruitless websurfing all day and who might prefer to go to bed early after a light dinner find themselves at the end of the day gorging on junk food and unable to stop watching television. More generally, dieters can often maintain their discipline for short periods but find such self-restraint unsustainable over the long term. Profligate spending or drinking to excess is frequently the "reward" for a hard week at work.

Experimental psychology (Baumeister *et al.,* 1994; Baumeister and Vohs, 2003) has gone beyond these anecdotes and has demonstrated in a variety of settings that individuals depleted by prior acts of self-restraint later behave as though they have less capacity for self-control. The experiments in this literature typically have two phases. Every subject in the experiment participates in the second phase but only a subset, randomly selected, participates in the first; the remainder is used as the control group. In the first phase, subjects are asked to perform a task that is meant to deplete their willpower; in the second phase, their endurance in an entirely unrelated activity also requiring self control is measured.[1] Subjects who participated in the first phase display substantially less endurance in the second phase. This apparent link between the exercise of self-control in one activity and later self-discipline in another activity has been observed repeatedly, with many different manipulations and measures of self-regulation (Baumeister and Vohs, 2003, Vohs and Faber, 2004). Two recent experiments (Vohs and Faber, 2004 and Dewitte et al., 2005) show that willpower depletion and prior cognitive loads affect subsequent *economic* behavior.[2] While individual experiments have weaknesses and leave open questions about the nature of willpower depletion and its empirical relevance, we regard the collection of these experimental findings as reinforcing the intuitive notion of willpower as a depletable cognitive resource.

In this paper, we study the effects of incorporating willpower depletion in the simplest model of intertemporal decision-making: the canonical cake-eating problem. In our formulation, moderating consumption requires willpower; the greater restraint the consumer exercises, the faster his willpower erodes. In addition, consistent both with introspection and experimenter observation (Baumeister *et al.,* 1994), we assume that a given level of consumption depletes willpower at a (weakly) faster rate when a person's reserves of willpower are lower. Finally, we assume that consuming nothing requires no self-restraint when no cake remains.

Ours is the first formal model in the economics literature to take explicit account of willpower depletion. Because we do so in a simple and familiar economic environment, we are able to clarify what phenomena willpower depletion can explain.

Willpower depletion provides an alternative explanation for a preference for commitment, apparent time-inconsistencies, and profound procrastination—the hallmarks of self-control models in-

[1] For example, in the first phase subjects have been asked not to eat tempting foods, not to drink liquids when thirsty, or to take a so-called "Stroop test." In each round of such a test, the name of a color (e.g. "green") is written in ink of a different color and the respondent is asked to report the color of the ink. A respondent's accuracy and speed at the test is thought to depend on his ability to "inhibit" or control his "automatic" response which is to read the word rather than identify the color of the ink in which it is written.

[2] In Vohs and Faber (2004), subjects who were willpower-depleted purchased a wider assortment of merchandise and spent a larger portion of their experimental earnings than the control group. The experiments of Dewitte et al. should be distinguished from prior cognitive-load experiments that showed the effects of such loads on *contemporaneous* intertemporal choices. In these earlier studies, respondents were more likely to choose cake over fruit (Shiv and Fedorikin, 1999) or a smaller earlier reward over a larger later one (Hinson, et al. 2003) when *simultaneously* asked to perform a memory task. In contrast, Dewitte et al. (2005) found that even when such memory tasks were performed prior to the consumption decision, they affected the choice of how much candy to eat.

cluding temptation cost, dual-selves, and hyperbolic discounting. As in prior models of self-control, an agent in our model would strictly prefer to have his "cake" or paycheck doled out to him by a savings club. For if the entire amount were available, resisting spending it would deplete his scarce willpower. Optimal behavior when willpower is depletable also provides an alternative explanation for apparent intertemporal preference reversals.[3] When asked to choose between a smaller (consumption) reward that will arrive immediately and a larger one that would arrive after some delay, an agent with limited willpower may choose the former. The willpower cost (measured in utility terms) of resisting the immediate, smaller reward may outweigh the utility gain from receiving the later, larger reward. However, if asked to choose now one of these same two options set off in the temporal distance, the consumer may choose the later, larger prize. The choice now of a prize to be delivered in the future enables the consumer to commit irreversibly to an option. He then prefers the later, larger prize since willpower ceases to be required to resist the earlier, smaller prize. A willpower-constrained agent may therefore appear to behave in a time-inconsistent manner. Optimal behavior in our model also provides an alternative explanation for procrastination. Suppose the agent must expend a fixed amount of time to complete an assignment before a certain deadline, but can allocate the remaining leisure time optimally.[4] If the agent is willpower constrained, he may enjoy leisure early in the program, and then work non-stop until the deadline. While doing marginally more work early on and enjoying marginally more leisure later would strictly increase his utility, he lacks the willpower to implement this alternative program. Working harder in the first phase would require more willpower but so would restraining himself from squandering the positive stock of leisure he would carry into the second phase.

Our model also explains anomalies inconsistent with other models of self-control. Prior cognitive loads affect the subsequent path of consumption by reducing the initial stock of willpower. Moreover, an agent in our model may increase his consumption over time because exercising self control later, when his stock of willpower is reduced, may require more willpower than exercising the same self control earlier. Our model also explains linkages between seemingly unrelated activities, since the same cognitive resource is used to exercise self control in different activities.

This linkage has important implications for consumption behavior. One might anticipate that

[3]While experimental evidence consistent with declining rates of time discount is extensive, it should be noted that the interpretation of these experiments as supporting (quasi-) hyperbolic discounting is not uncontroversial. See, e.g., Rubinstein (2003), Read (2003), and Benhabib, et al.(2006), for alternative interpretations.

[4]This is exactly the setup Fischer (2001) used to investigate procrastination but we have introduced a willpower depletion constraint. Without it, Fischer finds that the discount factor required to explain procrastination in a model where geometrically-discounted, additively-separable utility is maximized is unrealistic; our model generates procrastination (zero consumption of leisure for the final phase of the planning horizon) for any discount rate.

perfect smoothing of consumption would always occur if the agent had sufficient willpower. While this result always holds in our model if restraining consumption is the only use of willpower, it almost never obtains if willpower has alternative uses. Suppose that willpower can be used to implement intertemporal saving (cake-eating) as well as some other activities (e.g. cramming for exams, training for musical performances, maintaining a diet, or preparing an important presentation) that require self-discipline. We show that the optimal allocation almost never results in consumption smoothing over the entire horizon, even when it is feasible. Intuitively, consumption smoothing over the finite horizon is optimal only if there is so much willpower that allocating any more toward regulating this intertemporal activity would not increase the utility which can be achieved from it. But allocating so much willpower to the intertemporal consumption can never be optimal provided that redirecting more toward some other activity will strictly increase the utility obtained in that activity.

In reality, there are always alternative uses of willpower, and our formal analysis of this case yields some surprising implications. Behavioral differences between rich and poor people sometimes attributed to differences in self-control skills may reflect wealth differences and nothing more. Consider how two agents, who differ only in the size of their initial cakes (interpreted as stocks of either wealth or leisure), would choose to consume in the first phase if they recognize that utility from the alternative activity depends on the amount of willpower that is left over for that activity. Assume they have the same initial willpower, the same self-control technology, and the same preferences. We provide an example where the rich agent chooses to smooth his consumption more than his poorer counterpart even though the richer agent invests more of his willpower in alternative activities such as market work or exercise regimes. We thus illustrate how the poor may appear to exert less self-discipline (have lower rates of saving, higher rates of obesity), not because they have different preferences, willpower endowments or self-discipline technology but merely because they have fewer material resources.[5]

We proceed as follows. In the next section we present our model and interpret its first-order conditions. In section 3 we describe qualitative properties of the optimal consumption program with and without an alternative use for willpower. In section 4 we discuss extensions to the case where the exercise of self control depletes willpower over the short term but builds it over the

[5] Allowing willpower to have alternative uses has other implications. For example, using the typical two-stage protocol, Muraven (1998) finds that a given first-stage depletion activity has less of an effect on second stage self-control when subjects are paid more for exerting that latter control. This finding is consistent with a model where willpower has alternative uses. If we fix the marginal value of willpower remaining after the experiment, and increase the incentive for exerting self-control in the second phase activity, then the agent will optimally reallocate willpower to that second phase self-control activity and leave less in reserve for self-regulation after the experiment.

longer term and to the case where an individual is uncertain about the size of his willpower stock. In section 5 we present a detailed discussion of the advantages of our formulation over existing self-control models. That discussion includes a summary of evidence of increasing consumption paths from high frequency field data. In section 6 we conclude.

2 Consumption with Limited Willpower

In the canonical cake-eating problem in continuous time, a consumer maximizes his discounted utility by choosing his consumption path $c(t)$ over a fixed horizon ($t \in [0, T]$). We denote the size of the cake at time t as $R(t)$ and assume that $R(0)$ is given. The rate of decline in the cake at t (denoted $-\dot{R}(t)$) is, therefore, $c(t)$.

We depart from the canonical model by assuming that the agent is endowed with a given stock of willpower $W(0)$ and depletes it when he restrains his consumption. We denote the rate of willpower depletion as $f(W(t), c(t))$. Allowing the rate of depletion to depend not merely on the level of self restraint but on the remaining willpower reserves captures the experimental observation that the same restraint depletes willpower at a faster rate when one's willpower reserves are lower. Because neither experiments nor introspection suggests the sign of the relationship, we assume that $f(\cdot)$ is not affected by the stock of cake remaining ($R(t)$), provided some remains.[6]

An important feature of willpower depletion is that as long as even a morsel of cake remains, one has to use willpower to consume nothing; but consuming nothing requires no willpower when there is nothing left to eat. We refer to this feature as the "fundamental discontinuity of willpower depletion" and take account of it in our formulation.[7]

In anticipation of the analysis in Section 3, we assume that any willpower remaining after the conclusion of intertemporal consumption is used in an alternative activity and generates additional utility $m(\cdot)$.

Since the agent is not permitted to choose a consumption path which results in negative willpower, the willpower constraint may preclude the path which exhausts the cake while equalizing discounted marginal utility up through T. We refer to that path, which is the hallmark of the canonical model, as "perfect smoothing." Even when perfect smoothing is feasible, the agent may choose to forgo it. Our goal throughout is to investigate how the presence of the willpower constraint alters the agent's chosen consumption plan relative to the predictions of the canonical

[6] Assuming $f(\cdot)$ is not a function of $R(t)$ simplifies our analysis. If, however, f were increasing (decreasing) in $R(t)$, we conjecture that this would mitigate (reinforce) the agent's incentive to increase consumption over time.

[7] This discontinuity would persist even if $f(\cdot)$ were a function of $R(t)$ so long as $\lim_{R(t)\to 0} f(\cdot) > 0$. While it would simplify our analysis if the discontinuity conveniently disappeared, there is no reason to expect that it does.

model.

2.1 Formulation of the Model

The agent chooses $c(t) \geq 0$ to maximize

$$
\begin{aligned}
& V(0) = \int_0^T e^{-\rho t} U[c(t)]\,dt + e^{-\rho t'} m\left(W\left(t'\right)\right) \\
\text{subject to} \quad & \dot{R}(t) = -c(t) \qquad\qquad (P1) \\
& \dot{W}(t) = \begin{cases} -f(W(t), c(t)) & \text{if } R(t) > 0 \\ 0, & \text{otherwise} \end{cases} \\
& R(T) \geq 0,\ W(T) \geq 0 \\
& R(0) = \overline{R} \geq 0 \\
& W(0) = \overline{W} \geq 0,
\end{aligned}
$$

where ρ is the subjective rate of time discount and $t' = \sup\{t \in [0,T] : R(t) > 0\}$. Note that the law of motion for willpower is discontinuous reflecting the fundamental discontinuity of willpower depletion discussed above.

We make the following assumptions on the willpower technology f and the utility function U. We assume that, for all W, $f(W,c) > 0$ for $c \in [0,\bar{c})$ and $f(W,c) = 0$ for $c \in [\bar{c},\infty)$ for some $\bar{c} > 0$. While following a given consumption path, if the willpower stock becomes zero before exogenous time T, consumption must weakly exceed $\bar{c}$ thereafter until the cake is depleted. Therefore, this assumption guarantees that the set of feasible consumption paths is nonempty. We assume that f is twice differentiable everywhere except at $\bar{c}$ and continuous at $\bar{c}$. We assume that f is strictly decreasing ($f_c < 0$) and weakly convex ($f_{cc} \geq 0$) in c for $c \in [0,\bar{c})$ and weakly decreasing ($f_W \leq 0$) in W. As for cross effects, we assume that $f_{cW} \geq 0$. Thus, we assume that the more the agent restrains his consumption, the faster he depletes his willpower reserves; moreover, the same level of restraint may result in faster depletion of willpower if the agent's reserves of willpower are lower. We assume $U(0) = 0$[8], $U(c)$ is differentiable, strictly increasing and strictly concave. We are interested in modeling non-addictive behaviors like those considered in the experiments by Baumeister and colleagues; so $U(c)$ is a function only of contemporaneous consumption, and not past consumption.[9]

[8] As long as $U(0) > -\infty$, we can always renormalize the utility function U so that $U(0) = 0$.

[9] There is a related literature (Bernheim and Rangel, 2004; Laibson 2001) concerning self-regulation when the good consumed is addictive, or preferences are state-dependent. In these models self-control is perfect in a cool state but absent in a hot state. By changing current consumption, the agent may alter the probability that he enters the hot state, thereby indirectly exerting future self control. While these models explain important issues regarding consumption of addictive goods, they do not adress essential features of willpower depletion. For example they do not

This problem, though simple in its formulation, is non-standard since the law of motion for one state variable has a discontinuity at the point where the other state variable equals zero. We solve this nonstandard problem by examining the solution to a related problem (P2 below) which is established in Appendix A to have the same solution. In P2, the agent chooses both an optimal consumption path $c(t)$ and the date $s \leq T$ after which consumption ceases, where $\dot{W} = c(t) = 0$ for all $t \in (s, T]$.

$$
\begin{aligned}
& V(0) = \int_0^s e^{-\rho t} U[c(t)]\, dt + e^{-\rho s} m(W(s)) \\
\text{subject to} \quad & \dot{R}(t) = -c(t) \qquad\qquad (P2) \\
& \dot{W}(t) = -f(W(t), c(t)) \\
& R(s) \geq 0,\ W(s) \geq 0 \\
& R(0) = \overline{R} \geq 0 \\
& W(0) = \overline{W} \geq 0.
\end{aligned}
$$

The Hamiltonian for problem $(P2)$ is given by

$$H(c(t), R(t), W(t), t, \alpha(t), \lambda(t)) = e^{-\rho t} U(c(t)) - \alpha(t) c(t) - \lambda(t) f(W(t), c(t)).$$

To reduce notation, we shall refer to this Hamiltonian as $H(t)$ when no confusion arises. The first-order conditions are:

$$c(t) \geq 0,\ e^{-\rho t} U'(c(t)) - \alpha(t) - \lambda(t) f_c \leq 0 \text{ and c.s.} \tag{1}$$

$$\dot{W}(t) = -f \tag{2}$$

$$\dot{\alpha}(t) = 0 \tag{3}$$

$$\dot{\lambda}(t) = \lambda(t) f_W \tag{4}$$

$$T - s \geq 0,\ H(s) - \rho e^{-\rho s} m(W(s)) \geq 0 \text{ and c.s.} \tag{5}$$

$$R(s) \geq 0,\ \alpha(s) \geq 0 \text{ and c.s.} \tag{6}$$

$$W(s) \geq 0,\ \lambda(s) - m'(W(s)) \geq 0 \text{ and c.s.} \tag{7}$$

It should be noted for future use that, whenever willpower is strictly positive, consumption varies continuously with time. This result follows because (1) the Hamiltonian is strictly concave in c; (2) α and λ vary continuously with time when $W(t) > 0$; (3) U' is continuous in c; (4) f_c is continuous in both c and W; and (5) W varies continuously with time.

explain consumption behavior for non-addictive goods, and in particular the effect of cognitive loads on consumption, and do not link behavior across seemingly unrelated activities.

The first-order conditions can be interpreted intuitively. When $c(t) > 0$, we can rewrite (1) as:

$$\underbrace{e^{-\rho t}U'(c(t))}_{\text{direct marginal benefit}} + \underbrace{\lambda(t)(-f_c)}_{\text{indirect marginal benefit}} = \underbrace{\alpha(t)}_{\text{marginal cost}} .$$

Grouped in this way, condition (1) implies that consuming at a slightly faster rate at time t generates two marginal benefits and one marginal cost. The *direct* marginal benefit $(e^{-\rho t}U'(c(t)))$ is the increase in utility at time t (expressed in utils at $t = 0$) that results from consuming more then. Increasing consumption also has an *indirect* marginal benefit $(-f_c\lambda(t))$ since willpower is depleted at a slower rate $(-f_c)$ and each unit of willpower saved at time t is worth $\lambda(t)$. The marginal cost of consuming at a faster rate at t $(\alpha(t))$ is the utility lost because the additional cake consumed at t can no longer be consumed at another time. At an interior optimum $(c(t) > 0)$, the sum of the two marginal benefits equals the marginal cost (all expressed in utils at a common time, $t = 0$).

The conditions (3) and (4) highlight an important conceptual distinction between the two depletable resources in the model. Additional cake is equally valuable whenever it arrives since its availability before it is depleted provides no services and its future arrival can be fully anticipated by drawing down reserves faster in advance. That explains why the imputed value of additional cake is constant over time ($\dot{\alpha}(t) = 0$ from (3)). In contrast, when $f_W < 0$ and the willpower constraint is binding, additional willpower is more valuable the earlier it arrives because its mere presence provides a service: the more willpower one has available at t, the less must be depleted over a short interval to restrain consumption by a given amount. This is not a service upon which one can draw merely by recognizing that one will have more willpower in the future. For this reason, the imputed value of willpower declines over time ($\dot{\lambda}(t) < 0$ from (4) whenever $f_W < 0$). Of course, this distinction between the two depletable resources disappears in the special case where $f_W = 0$. In that case, additional willpower—like additional cake—is equally valuable whenever it arrives.

We are now in a position to explain intuitively how consumption changes over time. To isolate the new effects arising in our model, we assume that $\rho = 0$. Since the marginal cost of increased consumption never varies over time ($\dot{\alpha}(t) = 0$), the sum of the direct and indirect marginal benefit of increased consumption can never vary over time: whenever the direct marginal benefit path is increasing the indirect marginal benefit path must be decreasing at the same rate (and vice versa). Since utility is stationary and subject to diminishing returns, an increasing consumption path must always result in a decreasing direct marginal benefit path. Hence, an increasing consumption path must be accompanied by an increasing indirect marginal benefit path. By a similar argument, a decreasing consumption path must be accompanied by a decreasing indirect marginal benefit path.

In short, $\text{sign}\, \dot{c}(t) = \text{sign}\, \frac{d}{dt}(\lambda(t))(-f_c)$.

Consumption can be constant over time if and only if the indirect marginal benefit path is constant over time. This can occur for any of three distinct reasons: (1) willpower has no value ($\lambda(t) = 0$); (2) willpower, although valuable, provides no services ($f_W = 0$), resulting in a constant imputed value of willpower and the same reduction in the speed of willpower depletion when consumption expands regardless of the size of the stock of willpower remaining (neither f_c nor $\lambda(t)$ changes with time); and (3) the willpower released by increased consumption just happens to increase over time by exactly what is required to offset the concurrent decrease in the utility value of each unit of released willpower ($-\lambda(t)f_c$ is constant).

More formally, suppose $\rho = 0$ and $c(t) > 0$ for $t \leq s$. Condition (1) implies

$$U''(c)\,\dot{c} - [\dot{\lambda} f_c + \lambda f_{cc}\dot{c} + \lambda f_{cW}\dot{W}] = \dot{\alpha}$$

$$U''(c)\,\dot{c} - [\lambda f_W f_c + \lambda f_{cc}\dot{c} - \lambda f_{cW} f] = 0$$

$\iff$

$$\dot{c} = \lambda(t)\,\frac{f_W f_c - f_{cW} f}{U''(c) - \lambda(t) f_{cc}}. \tag{8}$$

Given our curvature assumptions (U strictly concave and f weakly convex in c), the denominator of the right-hand side of equation (8) is strictly negative. So if the willpower constraint does not bind ($\lambda(t) = 0$), equation (8) implies that consumption is constant. On the other hand, if $\lambda(t) > 0$, then when $c(t) > 0$, $\dot{c} \gtreqless 0$ as $(f_{cW} f - f_W f_c) \gtreqless 0$.

Increasing consumption paths must emerge in our model whenever the indirect marginal benefit path increases over time. To illustrate how this surprising phenomenon can arise, consider the following simple example of a willpower depletion function f:

$$f = A + K(W)\,g(c) \text{ with } K' \leq 0 \text{ and } g' < 0.$$

In this simple example, a given act of will is more depleting when the agent is already depleted, and willpower is either being depleted ($A > 0$) or replenished ($A < 0$) at a constant rate. Note that,

$$f_W = K'g$$

$$f_c = Kg'$$

$$f_{cW} = K'g'.$$

Thus,

$$\text{sign}(\dot{c}) = \text{sign}\left(-K'gKg' + K'g'(A + Kg)\right)$$

$$\text{sign}(\dot{c}) = \text{sign}\left(K'g'A\right).$$

Since g is strictly decreasing and K is weakly decreasing, there are two possibilities. At any t where $c(t) > 0$, $\dot{c} = 0$ if $K' = 0$. But if $K' < 0$ then $\dot{c} \gtreqless 0$ as $A \gtreqless 0$. So whenever the willpower constraint is binding, $K' < 0$, and $A > 0$, the path of optimal consumption must increase over time. The increasing path emerges for two reasons. First, because A is positive, willpower would be disappearing as time elapses; this is a "use it or lose it" situation. The basic incentive that this depletion gives for using willpower earlier is magnified by the fact that K' is negative; a given act of will is more depleting when the willpower stock is low. So the agent optimally takes the opportunity to exert acts of will when they have a relatively low opportunity cost (when W is large) and before time erodes the willpower stock. Thus consumption grows in that case because the indirect marginal benefit of increased consumption grows over time.

3 Optimal Consumption When Willpower Has Alternative Uses

Intuitively, willpower has many uses besides the regulation of intertemporal consumption and if the agent anticipates these uses their existence will affect his consumption profile. Muraven (1998) demonstrated in the laboratory that subjects do alter their behavior in anticipation of future uses of willpower. When subjects appear to have nearly exhausted their willpower, they may in fact be holding willpower in reserve for future activities. In one experiment (Muraven, 1998), some subjects were given two tasks to be performed consecutively and some were told in advance that a third task would follow the first two. When performing the second task, those who anticipated the third task gave up sooner.

If an agent lacks any alternative use of willpower, a necessary and sufficient condition for him to consume at a constant rate over the entire horizon ("perfect smoothing" for short) is that he have enough willpower initially for smoothing to be feasible. If willpower has alternative uses, however, he may refrain from perfect smoothing no matter how large his initial stock of willpower. In this section we derive necessary and sufficient conditions for perfect smoothing to occur with and without alternative uses of willpower.

3.1 The Benchmark Case

As a benchmark, we consider first the case where willpower has no alternative uses ($m(W) \equiv 0$). When willpower can only be used to regulate intertemporal consumption, perfect smoothing is optimal whenever feasible. Intuitively, since there is no shortage of willpower, there is no marginal value to having more of it ($\lambda(t) = 0$ for $t \geq 0$) and characteristics of the willpower technology do not induce time-varying consumption.

In the absence of discounting, perfect smoothing implies that consumption is constant over time for $t \in [0, T]$. More generally, Proposition 1 describes the qualitative properties of the optimal consumption path when $\rho = m(W) = 0$.

Proposition 1 *Let W_H be the minimum level of initial willpower such that setting $c(t) = \frac{\overline{R}}{T}$ for $t \in [0, T]$ is feasible. Denote the optimal consumption path as $c^*(t)$. If $\overline{W} \geq W_H$ then the cake is exhausted, and $c^*(t) = \frac{\overline{R}}{T}$ for $t \in [0, T]$. If $\overline{W} < W_H$ then both the cake and willpower are exhausted $(R(s) = W(s) = 0)$, and when consumption is strictly positive it is strictly increasing (resp. constant, strictly decreasing) if and only if $f_{cW} f - f_W f_c$ is strictly positive (resp. zero, strictly negative).*

Proof. From (3) $\alpha(t)$ is a constant function, and with a slight abuse of notation we denote this constant as $\alpha \geq 0$.

First assume $\overline{W} \geq W_H$. Consider the case where $\lambda(t) = 0$ for all $t \in [0, s]$. Since $U'(\cdot) > 0$, (1) requires $\alpha > 0$ and $c(t)$ constant. With a slight abuse of notation, we denote this constant as $c \geq 0$. Since $\alpha > 0$, (6) requires $R(s) = 0$. Since $R(0) = \overline{R} > 0$, then $c > 0$. From (5), $T - s \geq 0$. If $T - s > 0$, (5) would require $U(c) - U'(c)c = 0$. But since $U'(\cdot) > 0, U''(\cdot) < 0$, and $c > 0$, $U(c) - U'(c)c > 0$; hence, $T - s > 0$ must be ruled out. It follows that when $\lambda(t) = 0$ for all $t \in [0, s]$, then $s = T$ and $c = \frac{\overline{R}}{T}$. By definition of W_H, $W(T) \geq 0$. Thus (7) is satisfied. This proves the first statement in the proposition.

Now assume $\overline{W} < W_H$. Then $\lambda(t) > 0$ for all $t \in [0, s]$, for suppose to the contrary that $\lambda(t) = 0$ for some $t \in [0, s]$. Equation (4) implies that $\lambda(t)$ is weakly decreasing and can be written as:

$$\lambda(t) = \lambda(0) e^{\int_{n=0}^{t} f_W(W(n), c(n)) dn}.$$

Since $e^{\int_{n=0}^{t} f_W(W(n), c(n)) dn} > 0$ for all t, $\lambda(t) = 0$ for some $t \in [0, s]$, implies that $\lambda(t) = 0$ for all $t \in [0, s]$. But as seen above the conditions above then imply that $c = \frac{\overline{R}}{T}$ which is infeasible when $\overline{W} < W_H$. So if $\overline{W} < W_H$ then $\lambda(t) > 0$ for all $t \in [0, s]$, and by (7) $W(s) = 0$. To satisfy (1) with $U'(\cdot) > 0$ and $-\lambda f_c > 0$ requires $\alpha > 0$; and, again, since $\alpha > 0$, (6) requires that the cake is entirely consumed $(R(s) = 0)$. In this case, by our earlier observations, $(f_{cW} f - f_W f_c) \gtreqless 0$ for all $t \in [0, s]$ if and only if $\dot{c} \gtreqless 0$ for all t where $c(t) > 0$. This concludes the proof of the proposition. ■

Note that, when $\overline{W} < W_H$, Proposition 1 describes the rate of change in optimal consumption when consumption is strictly positive. This description does not, however, preclude intervals of zero consumption. Consider the example presented at the end of section 2.1 where $f = A + K(W) g(c)$ with $K' \leq 0$ and $g' < 0$. Recall that, when $\overline{W} < W_H$ and consumption is strictly positive, the rate of change in consumption takes the sign of A. Indeed, consumption is strictly positive for $t \in [0, s]$, but the last moment of positive consumption (s) may occur before T. In that case, the optimal path has $c(t)=0$ for $t\in(s,T]$. Thus, in this example, the agent may (if the good is leisure and A is positive) work relatively hard on a project at the beginning, slack off as time goes by, but then cram for some interval just before the deadline. This optimal cramming just before the deadline reflects a basic tension between the willpower and cake (leisure) budgets. The extra leisure needed to stretch positive consumption a bit longer at the end implies harder work earlier in the program, and thus less willpower left over at the end. The increased consumption at the end does not reduce willpower depletion then since before, when no cake remained, no willpower was being depleted. Once the worker frees up time to enjoy leisure in the final phase, exercising restraint increases willpower depletion in that phase as well. At the optimum, therefore, smoothing the time spent working is either infeasible or implies such a distortion of the previous path (perhaps from complete slacking as willpower is depleted) as to lower utility.

3.2 Necessary and Sufficient Conditions for Perfect Smoothing When Willpower Has Alternative Uses

Having characterized how the consumer would behave if willpower had no other uses, we now turn to the more relevant case where the agent anticipates needing the remaining willpower to regulate other urges. We denote the maximized expected utility derived from these alternative uses of willpower by the bequest function, $m(\cdot)>0$, which we assume is strictly increasing and weakly concave. Denote perfectly smooth consumption as $c_H=\frac{\bar{R}}{T}$. When willpower has alternative uses, perfect smoothing may be eschewed even when feasible. We assume throughout the remainder of this section that perfect smoothing is feasible: $\overline{W}>W_H$. To isolate the influence of willpower concerns we will continue to assume $\rho=0$.

Denote by $\hat{W}$ the willpower available at T if perfect smoothing (c_H) has been implemented; clearly $\hat{W}$ depends on the initial levels of willpower and cake ($\bar{W},\bar{R}$) but we suppress this dependence for simplicity. If perfect smoothing has been implemented, first-order condition (1) implies that $U'(c_H)-\lambda(T)f_c\left(\hat{W},c_H\right)=\alpha$. Multiplying both sides by c_H we get,

$$\left[U'(c_H)-\lambda(T)f_c\left(\hat{W},c_H\right)\right]c_H=\alpha c_H. \tag{9}$$

Since $s=T$, (5) requires $U(c_H)-\alpha c_H-\lambda(T)f\left(\hat{W},c_H\right)\geq 0$. Substituting (9) into (5), we obtain:

$$U(c_H)-\left[U'(c_H)-\lambda(T)f_c\left(\hat{W},c_H\right)\right]c_H-\lambda(T)f\left(\hat{W},c_H\right)\geq 0.$$

We therefore have the following implication of $s=T$:

$$\lambda(T)\leq m_H(\hat{W},c_H), \tag{10}$$

where we define $m_H(W,c) = \frac{U(c)-U'(c)c}{f(W,c)-f_c(W,c)c}$.

On the other hand, if $s < T$, the first-order conditions imply that

$$\lambda(s) = m_H(W(s), c(s)). \tag{11}$$

As the following proposition makes clear, when willpower has alternative uses perfect smoothing need not occur no matter how large the stock of initial willpower:

Proposition 2 *If perfect smoothing occurs, the following conditions must hold:*
(1) $f_W f_c - f_{cW} f = 0$, for $c = \frac{\bar{R}}{T}$, $W \in \left[\hat{W}, \bar{W}\right]$, and
(2) $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$.

Proof. If smoothing is perfect then $c(t) = \frac{\bar{R}}{T}$, implying $\dot{c} = 0$ and $c(t) > 0$ for $t \in (0,T)$. Since $\bar{W} > W_H$ and therefore $\hat{W} > 0$, (7) implies that $\lambda(T) = m'\left(\hat{W}\right) > 0$, which in turn implies that $\lambda(t) > 0$ for all $t \leq T$. It then follows from (8) that perfect smoothing requires $f_W f_c - f_{cW} f = 0$ where $f(\cdot,\cdot)$ and its partial derivatives are evaluated at $c = \frac{\bar{R}}{T}$ and any $W \in \left[\hat{W}, \bar{W}\right]$. This confirms condition (1) of the proposition. Even then, since $\lambda(T) = m'\left(\hat{W}\right) > 0$, (10) requires $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$, confirming condition (2) of the proposition. ■

By widening the domain of condition (1) and retaining condition (2), we can also obtain a sufficient condition for perfect smoothing to occur:

Proposition 3 *The following conditions are sufficient for perfect smoothing:*
(1′) $f_W f_c - f_{cW} f = 0$, for all $c(t) > 0$, $W \in \left[\hat{W}, \bar{W}\right]$, and (2) from Proposition (2),

Proof. To show that conditions (1′) and (2) imply perfect smoothing, assume they do not. Perfect smoothing could fail to occur for two reasons: varying consumption or constant consumption terminating at $s < T$. Since (1′) is assumed to hold, consumption must be constant. If it terminates at $s < T$, then $c = \frac{\bar{R}}{s} > c_H$. Since less restraint will be exercised over a shorter interval $W(s) > \hat{W} > 0$. This in turn has two implications. Since more willpower will be bequeathed to the alternative activity, $m'(\hat{W}) \geq m'(W(s))$. In addition, since $W(s) > 0$, (7) implies that $\lambda(s) = m'(W(s)) > 0$. By hypothesis, $s < T$. Then, as shown above, the first-order conditions imply $\lambda(s) = m_H(W(s), c)$. Since $m_H(\cdot,\cdot)$ is weakly increasing in the first argument and strictly increasing in the second argument, $m_H(W(s), c) > m_H(\hat{W}, c_H)$. Given the hypothesis that $s < T$ we therefore conclude that $m'(\hat{W}) \geq m'(W(s)) = \lambda(s) = m_H(W(s), c) > m_H(\hat{W}, c_H)$. But $m'(W(s)) > m_H(\hat{W}, c_H)$ violates condition (2). Hence, (1′) and (2) are sufficient for perfect smoothing. ■

In the benchmark case where willpower has no alternative uses, perfect smoothing occurs whenever feasible. Characteristics of the willpower technology ($f(W(t), c(t))$) play no role. In contrast, whenever willpower has any alternative use, the consumer abandons perfect smoothing unless the willpower technology satisfies condition (1). Recall the example of Section 2.1. If $K'(W) < 0$ and $A \neq 0$, perfect smoothing *always* occurs when feasible if willpower has no alternative use but *never* occurs—no matter how large the initial stock of willpower—when there is *any* other use of willpower. In cases where condition (1) in Proposition (2) fails, consumption increases or decreases over time as summarized by equation (8).

3.3 An Example

Unless condition (1) of Proposition 2 holds, there can be no perfect smoothing when there are alternative uses of willpower. To see some of surprising implications of this result and to clarify the importance of condition (2), we conclude with an example where condition ($1'$) and *a fortiori* condition (1) hold. In that case, perfect smoothing must occur when condition (2) holds and must not occur when condition (2) fails. We will give two interpretations of our example. In the first, the cake is interpreted as wealth; in the second it is interpreted as a stock of leisure. Under the first interpretation, we see from this example that behavioral differences between rich and poor people sometimes attributed to differences in self-control skills may reflect wealth differences and nothing more.

Suppose two agents, one poor the other rich, have the same willpower technology ($f = K(\bar{c} - c)$). Notice that this technology satisfies condition ($1'$) and hence condition (1). Proposition (1) indicates that optimal consumption paths are constant for both rich and poor as long as consumption is strictly positive. Perfect smoothing occurs if and only if $s = T$.

Assume the rich agent has more cake than the poor but the two are identical in every other respect: same preferences over consumption profiles, same initial endowment of willpower, same willpower technology, and same bequest function. To simplify further, assume their bequest functions are linear with slope m

To distinguish whether a variable pertains to the poor or the rich agent, we append a subscript "p" or "r", respectively. Thus, for consumption we write c_i (for $i = p, r$). Because, in this example, each agent has the same utility function and self-control technology, the *function* $m_H(\cdot, \cdot)$ is identical for each and there is no need to add the subscript i.

Assume that, for $i = p, r,\ m > m_H(\hat{W}_i, c_{H,i})$ in violation of condition (2). Then, by Proposition (2) , the optimal consumption path can not involve perfect smoothing. Since consumption is constant and exhausts the cake, $s_i < T$ and $c_i > c_{H,i}$, for $i = p, r$.

Since, by assumption, both the rich and the poor carry some willpower into the second activity, additional willpower must have the same marginal utility in the two activities and, from (7), $\lambda_i(s_i) = m$. However, since $s_i < T$, (11) requires that $\lambda_i(s_i) = m_H(W_i, c_i)$. Hence,

$$m_H(W_i, c_i) = m \text{ for } i = p, r. \tag{12}$$

Since the left-hand side of (12) is independent of its first argument and strictly increasing in its second argument, this equation defines the same consumption for the two types of agent: $c_p = c_r$ as long as both consumptions are positive. Since the poor agent has a strictly smaller cake ($\bar{R}_r > \bar{R}_p$), he must run out of cake sooner ($s_p < s_r$), after which his consumption drops to zero. Having started with the same willpower and having exercised the same restraint over a shorter time interval, the poor agent will have more willpower remaining to invest in alternative activities: $W_p(s_p) > W_r(s_r)$.

Consider what happens if the initial cake size of the rich agent is expanded. The larger the initial cake size, the longer the interval of constant consumption and the smaller the bequest. When the cake size of the rich reaches $\bar{R}_r = Tc_p,\ m = m_H(\hat{W}, c_p)$ and condition 2 is no longer violated. Further increases in the initial cake size, require $c_r = \frac{\bar{R}_r}{T} > c_p$ and since m_H is strictly increasing in its second argument, $m < m_H(\hat{W}, c_r)$. Since condition 2 holds in this region, perfect smoothing must occur. Further increases in the initial cake size cause faster but constant consumption over the entire horizon. Hence in this region willpower remaining at T becomes an increasing, instead of a decreasing, function of the initial cake size.

Interestingly, if we compare the behavior of the poor and the rich, in this region where remaining willpower is increasing in cake size, it appears as though the rich agent is more disciplined. He not only does a better job smoothing consumption, he also has more willpower left to expend on, say, exercise or diet regimes. In fact, the rich agent is endowed the same intertemporal preferences, the same self-control technology, and the same willpower stock as the poor agent. In this example, the poor agent seems less disciplined simply because he is poor.

We can reinterpret this example for the case where the cake represents leisure. Consider agents obligated to complete tasks of different sizes on or before T. Those agents with lighter workloads are, therefore, "richer" in leisure. If an agent's workload is sufficiently light, he may work at a light but steady pace throughout the horizon, consuming leisure at a constant pace as well. A larger workload will require him to work harder and consume leisure at a slower but constant rate over the entire time horizon. Once the workload exceeds some threshold, however, the optimal program has two phases. Until time $s < T$, the agent works at a steady pace and consumes leisure at a constant rate as well; after s, the agent works relentlessly without any leisure breaks. Further increases in the workload do not alter the rate at which the agent works in the first phase. Instead, he shortens the phase during which he combines leisure and work and lengthens the phase when he works relentlessly.

4 Extensions

In the previous sections we have assumed that willpower is simply depleted by its use and that the agent always knows his stock of willpower with certainty. In this section, we discuss the consequences of relaxing each assumption.

4.1 Self-Control Builds Willpower Like a Muscle

Common intuition and some experimental psychology indicate that willpower may be like a muscle: controlling visceral urges depletes willpower over the short term, but the regular exercise of self-restraint may eventually build willpower. Indeed, there is some evidence that willpower can be built up in one domain and then used to advantage in other arenas (Muraven *et al.,* 1999 and Muraven and Baumeister, 2000).[10] In this section, we consider how allowing willpower to be built through its exercise may affect the optimal consumption paths described in the previous sections. Of particular interest is an understanding of when the ability to build willpower reinforces and when it counteracts the incentives for increasing paths of consumption we observed in previous sections. Our analysis indicates that the ability to build willpower through its exercise may introduce variation in consumption rates over time even when willpower concerns alone do not. Because the analysis of this extension is more technical, we relegate details to Appendix B.

To evaluate the effects of buildable willpower, we introduce a third state variable, muscle, the level of which is denoted by $M(t)$. We augment our earlier model in two ways. First the rate of change of willpower is now given by $\dot{W}(t) = \gamma M(t) - f(W(t), c(t))$. As before, willpower is depleted by restraining consumption, $f(W(t), c(t))$, but this depletion is moderated by the service flow γ from the stock of muscle, $M(t)$. Since the stocks of willpower and muscle cannot jump, the only way to alter the rate of willpower depletion immediately is by altering contemporaneous consumption. In the future, however, muscle may be developed by the previous exercise of willpower and this muscle provides additional willpower at rate γ. We assume, for simplicity, that the rate at which muscle develops (or deteriorates) is given by $\dot{M} = f(W(t), c(t)) - \sigma M(t)$. The idea is that exercising willpower today contributes to one's future muscle but the contributions decay.

Assuming that unused willpower has no terminal value, the agent's problem is to choose $c(t)$

[10] In one experiment (Muraven *et al.,* 1999), subjects who participated in two-week self-control drills (regulating moods, improving posture, etc.) later showed significant increases in the length of time the would squeeze a handgrip relative to those who did not participate in the drills.

to maximize

$$V(0) = \int_0^T e^{-\rho t} U[c(t)]\, dt \tag{$P3$}$$

subject to

$$\dot{R}(t) = -c(t)$$

$$\dot{W}(t) = \begin{cases} \gamma M(t) - f(W(t), c(t)) & \text{if } R(t) > 0 \\ \gamma M(t), & \text{otherwise} \end{cases} \tag{13}$$

$$\dot{M}(t) = \begin{cases} f(W(t), c(t)) - \sigma M(t) & \text{if } R(t) > 0 \\ -\sigma M(t), & \text{otherwise} \end{cases} \tag{14}$$

$$R(t) \geq 0,\ W(t) \geq 0,\ M(t) \geq 0 \text{ for } t \in [0, T] \tag{15}$$

$$R(0) = \overline{R} > 0,\ W(0) = \overline{W} > 0,\ M(0) = \overline{M} > 0.$$

The optimal consumption path in this muscle model shares some qualitative features with that in the model without muscle. First, the cake is entirely consumed. Second, for every initial level of muscle there is a willpower level $W_{\tilde{H}}$ above which the optimal path entails perfect smoothing. This is true because, for every initial stock of muscle there is an initial stock of willpower sufficiently large such that the Hotelling path is feasible and therefore optimal.[11] If we start with an initial level of willpower sufficient for perfect smoothing, decreases in that stock will eventually lead to a willpower level $\left(W_{\tilde{H}}\right)$ where any further reduction in the initial stock of willpower will make the perfectly smooth path infeasible.

Next, we ask how allowing buildable willpower alters our previous conclusions about the time path of optimal consumption in the absence of discounting $(\rho = 0)$. As in the model with out muscle, on the optimal path the *sum* of the direct and indirect marginal benefits of increased consumption must remain equal to the marginal cost of that consumption (having a bit less cake for the future). Hence, if consumption is strictly increasing over time, which would depress the direct marginal benefits of consumption, then the indirect marginal benefits must also be strictly increasing. Formally, we find

$$\dot{c} = \frac{\overbrace{(\lambda - \pi)(f_W f_c - f_{cW} f)}^{\text{direct willpower effect}}}{\Delta} + \frac{\overbrace{(\lambda - \pi) f_{cW} \gamma M(t)}^{\text{muscle service flow}}}{\Delta} + \frac{\overbrace{\left(\lambda\left(\frac{\gamma}{\sigma}\right) - \pi\right) \sigma f_c}^{\text{muscle building}}}{\Delta} \tag{16}$$

where $\Delta = [U''(c) - (\lambda - \pi) f_{cc}] < 0$, λ is the costate associated with the law of motion for willpower and π is the costate associated with the law of motion for muscle. That is, $\lambda(t)$ is the shadow value of willpower at time t and $\pi(t)$ is the shadow value of muscle at t. Note that if $\lambda(t) = \pi(t) = 0$ for $t \geq 0$, equation (16) yields the classical result that consumption is constant as long as it is positive and indeed the first order conditions will require that it be positive until T.

[11] Indeed, if the initial muscle level is large enough, the agent will be able to achieve perfect smoothing without any initial willpower.

To clarify how muscle building influences the time path of optimal consumption when perfect smoothing is infeasible, we consider two special cases where, in the absence of muscle, the optimal consumption path is particularly simple. Specifically, we consider first the case where the rate of willpower depletion is determined only by rate of consumption and not by the willpower remaining ($f_W = 0$), and second the case where $(f_W f_c - f_{cW} f) = 0$. In each of these cases, when muscle is absent, optimal consumption is constant until some time s when consumption drops to zero. In other words, willpower concerns alone would induce no time preference except at the point when the cake (and willpower) is depleted entirely.

Referring to equation (16), case 1 ($f_W = 0$) implies that both the direct willpower effect and the muscle service flow are absent. The time path of consumption is therefore determined only by muscle building. First consider the case where muscle decays at a faster rate than it contributes to willpower $[(\gamma/\sigma) \leq 1]$. In this case, the muscle building term of equation (16) is always positive. Thus when $(\gamma/\sigma) \leq 1$ and $f_W = 0$, consumption is always increasing. Relative to the optimal path in the absence of muscle, the ability to build willpower through its exercise leads the agent to bear down at the beginning of the program in order to enjoy a greater willpower later.

When muscle decays more slowly $[(\gamma/\sigma) > 1]$, more complex consumption paths may emerge. Depending on initial conditions, optimal behavior may have the same qualitative features as when $(\gamma/\sigma) \leq 1$. Consumption is always increasing. In other circumstances detailed in Appendix B, however, consumption will decrease with time and later increase. That is, the consumption profile is $\cup$ - shaped.

Referring again to equation (16), case 2 $[(f_W f_c - f_{cW} f) = 0]$ implies that only the direct willpower effect is inactive. Relative to the optimal path in the absence of muscle, the ability to build of willpower with exercise again induces time preference. Consider, for example, a situation where the initial muscle stock is zero $(M(0) = 0)$, and thus, at the beginning of the program, the muscle service flow term is inactive. In the beginning of the program, optimal behavior in this case is like that in the case where $f_W = 0$. For example, when $(\gamma/\sigma) \leq 1$, consumption will increase in these early stages of the optimal path. Thus we see in each of these cases that the ability to build willpower through its exercise induces a time preference for consumption even when willpower concerns themselves do not.

4.2 Depleting a Stock of Willpower of Unknown Size

Loewenstein (2000) points out that an agent may not be a good judge of the willpower he has in reserve. He may in retrospect find that he overestimated his self-control resources or he may be pleasantly surprised to have more willpower than he anticipated and discover that he has "a second wind." Gilbert (1979), among others, has addressed a similar issue when there is a single depletable resource. His analysis can easily be adapted to the case where the agent knows his cake size but not his initial reserves of willpower.

A two-state example will clarify the basic ideas. Suppose an agent has one of two initial stocks of willpower and assigns positive probability to each state. Assume that the agent has no alternative use of willpower ($m(W) = 0$) and that even the larger stock of willpower is insufficient to implement perfect smoothing. Assume that that $f_W = 0$ so that consumption is constant when strictly positive.

If the agent knew that his willpower stock was low, he would conserve it by consuming at a rapid, constant rate and would deplete his cake quickly, after which he would have nothing left to consume until the end of the horizon. Denote as s_l the date when he would exhaust his cake if he was sure his initial willpower stock was low. Similarly, denote as $s_h(> s_l)$ the date when he would exhaust his cake if he was sure his initial willpower stock was high. Suppose, however, that the consumer could not observe his willpower reserves directly. Then, whatever consumption path he chooses, he would obtain no information about his initial willpower stock until the date when his willpower would have run out if his initial stock had been small. At that point, he could infer the size of his initial reserves: if he loses control because he has no more willpower, then his initial reserves were small; if he retains control because he has more willpower, then his initial reserves were large.

Under uncertainty of this kind, it is optimal to consume at a constant rate intermediate between the alternative consumption rates he would choose under certainty. Since this would involve restraining his consumption more than is optimal if he was certain his willpower stock was low, he would run out of willpower sooner in the event that his is initial stock was in fact low. If we denote as s_{ul} the date when he would run out under uncertainty if his initial stock of willpower was in fact low, then $s_{ul} < s_l$. In that event, lacking willpower reserves to control his consumption, he would exhaust the remaining cake at the rate $\bar{c}$ and then would have nothing left to consume until the end of the horizon.

On the other hand, if at s_{ul} the agent discovered that he still had the ability to restrain consumption, then he would rationally conclude that his initial willpower stock had been high. Since he had been consuming at a faster rate than if he had known this state from the outset, he would find himself at s_{ul}^+ with more willpower and less cake. In this circumstance, the consumer would take advantage of this "second wind" by restricting his consumption below the constant rate he would have chosen if he had known from the outset of his high reserves of willpower.

5 Discussion and Comparison with Costly Self-Control Models

The standard discounted utility model has proved very useful in economics but, as noted in the introduction, violations of the model seem common.[12] One set of violations consists of present-biased time preferences, intertemporal preference reversals, and tastes for commitment. Another violation of the standard model is negative time discounting; sometimes people prefer increasing sequences of consumption. Present-biased preferences, preference reversals and a taste for commitment have previously been explained using models of self-control. Negative time preference is usually regarded as a separate phenomenon requiring a separate model. As we have shown, however, willpower depletion can *simultaneously* explain present-biased preferences, intertemporal preference reversals, a taste for commitment *and* preferences for increasing consumption sequences. At the same time, our model accommodates previously unexplained links between (seemingly unrelated) self-control behaviors. Willpower depletion thus provides a more unified theory of time preference.

We begin this section with a discussion of the empirical evidence that people often consume from a fixed stock at an increasing rate for intervals of time. We then demonstrate that this behavior is inconsistent with prior models of self-control. Indeed models of costly self-control rule out perfect consumption smoothing as well. We conclude this section by distinguishing our treatment of self control as an endogenous shadow cost from prior treatments as an exogenous direct cost.

While consumption paths which decrease over time seem unremarkable, there is evidence that agents sometimes prefer increasing sequences of consumption. See, among many examples, Loewenstein and Prelec (1993) on food and entertainment consumption, Frank and Hutchens (1993) on wages, and Chapman (2000) on health outcomes. This evidence, drawn largely from responses to hypothetical choice scenarios, is consistent with the seemingly common desire to "save the best for last" when consumption is desirable or "get the hard part out of the way," when the intertemporal choice involves labor versus leisure.

These previous studies did not link preferences for increasing sequences of consumption to the problems of self control that, as we have seen, can generate preferences for commitment, induce profound procrastination, and create links between behavior in seemingly unrelated self-regulation activities. Instead, these models of negative time preference focus on anticipation (time non-separable) utility and on reference dependent preferences. Because they are not focussed on issues of self control, the choice scenarios used in these studies do not emphasize whether the decision maker is committed to the consumption path he has chosen.[13] This issue of commitment is important

[12] See Frederick, et al. (2002), for a recent review.

[13] The subjects in these studies are simply asked which sequence of consumption or income they prefer.

because, like all other models of self-control with time-separable utility, our willpower model predicts perfect smoothing of consumption if the consumer can commit to that path at any time prior to the moment consumption begins. So, for example, with commitment and no subjective rate of time discount, even a willpower constrained consumer would choose a perfectly flat consumption path from time 0 to time T.

There is, however, field evidence of unambiguously uncommitted consumers choosing increasing paths of consumption expenditure. Studies of high frequency data on individual expenditures in Britain indicate that consumers sometimes choose intervals of increasing consumption between the arrival of paychecks.[14] Both Kelly and Lanot (2002) and Huffman and Barenstein (2004) find that the non-durable expenditures of British consumers exhibit a U-shaped pattern over the course of the four weeks between paychecks. (See, for example, their Figures 1 and 4, respectively). Such behavior can never occur in the conventional model under certainty, with time-separable utility and exponential discounting; nor can it result from allowing the conventional model to incorporate hyperbolic discounting. Models of anticipatory utility or reference dependent preferences can accommodate increasing consumption paths, but cannot explain by themselves a taste for commitment, or profound procrastination, or links between seemingly unrelated acts of self-regulation. Kelly and Lanot (2002) attribute the U-shaped pattern of consumption expenditure to uncertainty and precautionary saving within the pay period. Huffman and Barenstein (2004) attribute the interval of apparently increasing average consumption to measurement error.[15] As we have seen, our model can explain these increasing sequences of consumption.

Although present in both experimental and field data, increasing consumption sequences are inconsistent with prior models of self control. The prior literature can be grouped into three strands. The first focuses on the present-biased choices which result from (quasi-) hyperbolic time discounting; the second models self-control problems which arise from temptation costs; the third

[14] There is also some, less statistically certain evidence of intervals of increasing consumption in high frequency data on daily expendiure by US consumers. See Stephens (2003), Tables 2 and 3 for total expenditure and Figures 4c-4g for non-durable expenditures.

[15] The Family Expenditure Survey used in both studies asks respondents to record their expenditures in a diary for two weeks. The day upon which the diary begins is randomly assigned. Public use data aggregate each household's expenditures by week. Researchers can identify whether a respondent is paid monthly and the date his last paycheck arrived but not, precisely, the date of his next check. Therefore, diary weeks that begin 25 days or more after the last check arrived will typically include expenditures from at least one day after the next check has arrived. If consumption is higher on days just after the check arrives, this measurement error may explain the higher levels of consumption in diary weeks that began 25-30 days after the last check arrived (see Huffman and Barenstein, 2004, Figure 4). This error would not seem to explain, however, the increase in average expenditure between weeks starting 21 (perhaps 17) and 24 days after the last check. Neither can this measurement error explain why average expenditure is often higher in the third week after the arrival of the last check than it is in the second (Kelly and Lanot, 2002, Figure 1).

models self-control problems which derive from internal conflicts between multiple "selves," one a long-run self which may, at a direct utility cost, exert control over the choices of a sequence of short-term selves. Models with hyperbolic discounting (Laibson, 1997; O'Donoghue and Rabin, 1999) do not explain strictly increasing consumption since the impatience posited would always lead to larger immediate consumption and smaller consumption later. Demonstrating that dual self-models (Thaler and Sheffrin, 1988; Fudenberg and Levine, 2006) and models with temptation costs (Gul and Pesendorfer, 2001, 2004a,b; Benhabib and Bisin, 2004; Dekel, Lipman and Rustichini, 2005) do not explain the preference for increasing consumption sequences requires a more detailed investigation.

Consider a finite-horizon, undiscounted version of Fudenberg-Levine's saving model and assume no growth. Although they restrict attention to a specific functional form of the utility function and the cost function, we consider the more general case where utility is any strictly increasing and strictly concave function and cost is any strictly increasing and weakly convex function, each twice differentiable. Then in the Fudenberg-Levine model the single-agent chooses $(c_1, ..., c_t)$ to maximize

$$\sum_{t=1}^{T} \{u(c_t) - g(u(R_t) - u(c_t))\} \text{ subject to } R_{t+1} = R_t - c_t \text{ and } R_1 = \bar{R}, \tag{17}$$

where $c_t \geq 0$, $R_t \geq 0$ and $\bar{R} > 0$ and we assume that the utility function $u(\cdot)$ is strictly increasing and strictly concave and the cost function $g(\cdot)$ is strictly increasing and weakly convex. As before $\bar{R}$ is the size of the cake, and R_t is the remaining cake in period t. To show that $c_{t+1} \geq c_t$ can never occur in their model, we verify that it would always violate the first-order condition for this problem. Assuming differentiability, the following condition must hold at an optimum:

$$u'(c_t)\cdot[1+g'(u(R_t)-u(c_t))]-u'(c_{t+1})\cdot[1+g'(u(R_{t+1})-u(c_{t+1}))]+u'(R_{t+1})[g'(u(R_{t+1})-u(c_{t+1}))] = 0 \tag{18}$$

The final term of (18) is strictly positive since it is the product of two strictly positive factors. As for the other two terms, they would combine to something weakly positive, violating condition (18) whenever $c_{t+1} \geq c_t$.[16] Weakly increasing consumption sequences are, therefore, inconsistent with Fudenberg-Levine's model.

For similar reasons, weakly increasing consumption sequences are inconsistent with Gul-Pesendorfer's

[16] To see this, note that each term is the product of two factors. If consumption is weakly increasing the first factor of the first term ($u'(c_t)$) is weakly larger than its counterpart ($u'(c_{t+1})$). Moreover, the argument of g' in the first term is also strictly larger than its counterpart in the second term: $u(R_t) - u(c_t) > u(R_{t+1}) - u(c_{t+1})$ since the cake remaining will be strictly smaller and consumption at $t+1$ will be weakly larger. Since g is assumed weakly convex, the first product of two factors must weakly exceed the second product of two factors and hence these terms make the left-hand side of the first-order condition even more positive.

model. In the finite horizon, undiscounted version of their problem the agent maximizes:

$$\sum_{t=1}^{T} \{[u(c_t) + v\,(c_t)] - v(R_t)\} \text{ subject to } R_{t+1} = R_t - c_t \text{ and } R_1 = \bar{R}, \tag{19}$$

where $c_t \geq 0$ and $R_t \geq 0$. Both the u and v functions are assumed to be strictly increasing. To insure concavity of the objective function, the u is assumed to be strictly concave, v is convex and $u + v$ is strictly concave. It is straightforward to verify that this formulation insures strictly decreasing consumption.[17] Intuitively, an agent smoothing consumption between adjacent periods can always do better by marginally redistributing consumption to the earlier of the two periods. This perturbation in consumption has no first-order effects on the sum of utilities of consumption in the two periods while strictly reducing the uneaten cake and hence the temptation cost. Thus, their model also predicts strictly decreasing consumption. In contrast, increasing consumption sequences can arise in a model with willpower depletion because the same amount of self-control is less costly in terms of willpower expended if it is exercised when the willpower stock is larger. Hence, there is an incentive to exercise more self control in the earlier of two periods ("use it or lose it").

We conclude by distinguishing our treatment of self-control as an endogenous shadow cost from the treatment in the previous literature as an exogenous direct cost.[18] Presumably there are circumstances in which the classical model predicts well. After all, it has served as the canonical model for decades. It is, therefore, an attractive feature of any new model supplanting a classical one that it include the classical model as a special case and that the circumstances in which the classical results hold are predicted endogenously. Other models of self-control do not have this feature; there are no circumstances under which they collapse to the classical model and predict consumption smoothing over time.[19] In contrast, if willpower depletion is appended as a constraint,

[17] The first-order condition is $[u'(c_t) + v'\,(c_t)] - [u'(c_{t+1}) + v'\,(c_{t+1}) - v'(R_{t+1})] = 0$. Given that $u + v$ is strictly concave and v is strictly increasing, the first-order condition can only be solved by a strictly decreasing consumption sequence.

[18] In discussing an appropriate formulation to capture Baumeister's experiments, Loewenstein and O'Donoghue (2004) regard the exertion of willpower as generating disutility although Loewenstein (2000) regards the matter as an open question. Whether or not willpower depletion has a utility cost seems difficult to resolve empirically. What we have resolved analytically, however, is that including the stock of willpower (or its rate of change) in the utility function is not necessary to capture either the behaviors psychologists have documented in their laboratories or to explain a number of prominent "anomalies" behavioral economists have reported in the field. For these purposes, it is sufficient to append to the conventional formulation of intertemporal utility maximization the additional constraint that the consumption path chosen must not overexhaust the agent's willpower.

[19] Of course $(\beta - \delta)$ models of quasi-hyperbolic discounting nest the standard model (when $\beta = 1$). However, this nesting derives, exogenously, from unobserved preferences rather than endogenously from changes in observable features of the environment.

there will be circumstances (e.g. large stocks of initial cake or willpower or a short horizon) in which perfect smoothing does not violate the constraint and will be optimal.

6 Conclusion

This paper has explored the consequences of including in a conventional model of intertemporal choice a cognitive constraint well-documented by experimental psychologists. Specifically, we assumed that if an agent restricts his consumption, then exercising self-restraint depletes his finite stock of willpower. This willpower constraint captures the common notion, consistent with laboratory experiments, that an individual has a limited, though positive, capacity to regulate his own impulsive behaviors.

To study how a binding willpower constraint affects choices over time, we introduced it in the simplest intertemporal model, where an agent decides how to consume a cake (or paycheck or stock of leisure time) over a fixed time period. With sufficient willpower, an agent perfectly smooths, consuming cake at a constant rate (in the absence of discounting) until the end of the planning horizon.

If the agent lacks the willpower to implement this program, however, his willpower acquires scarcity value and his optimal response is to behave in ways that might seem anomalous and which the prior self-control literature has sought to explain. Specifically, a consumer subject to a binding willpower constraint exhibits a taste for commitment, an apparent present bias, and a tendency for profound procrastination. Moreover, he displays other anomalous behaviors that, although well-documented, are inconsistent with existing models of self control. For example, prior cognitive loads affect his subsequent conduct because they reduce the stock of willpower he can use to regulate his behavior. Moreover, an agent in our model may *increase* his consumption over time because exercising self control later, when his stock of willpower is reduced, requires more willpower than exercising the same self control earlier. Finally, a willpower constrained consumer regards seemingly unrelated activities as linked since he uses the same cognitive resource to exercise self control in different activities. This linkage has important implications. If willpower has alternative uses besides restraining consumption (e.g. cramming for exams), the agent may never smooth consumption regardless of his initial willpower.

We considered two extensions of the model. First, we investigated what would happen if current exercise of self control, while immediately depleting willpower, also builds willpower reserves in the future. We found that renewable willpower induces a time preference even when consumption would have been constant if willpower had been nonrenewable. Second, we considered the possibility that the decision maker is uncertain about his stock of willpower. We found that the agent would optimally consume at a higher rate as a hedge against the risk of running out of willpower and, if surprised to infer that he has larger willpower reserves, experiences a "second wind" that permits further self-discipline later on the optimal path.

That willpower is scarce and depletable accords with both introspection and experiment. To explore the behavioral consequences of taking this constraint into account, we appended it to the canonical model of intertemporal choice. To our surprise, we found that the augmented model accounts *both* for prominent anomalies of intertemporal choice that have been the focus of the self-control literature *and* for other anomalies that were unaddressed by that literature and which have instead been treated as altogether separate phenomena requiring separate models. Given that taking account of willpower depletion is so tractable and illuminating, future research should clarify experimentally the form of the willpower depletion function and should embed that function as a constraint in other economic models.

Appendix A: The Equivalence of Solving Problems $P1$ and $P2$

Here we show that by solving the problem in which the date when consumption ceases is a choice variable ($P2$), we in fact solve our original problem ($P1$). In problem ($P1$), the optimal consumption path either finishes the cake at time $t' < T$, or it does not. If the cake is exhausted at time $t' < T$, then the law of motion governing the depletion of willpower jumps to zero, and for any time $t \in (t', T]$, $\dot{W}(t) = c(t) = 0$. These same paths of consumption and willpower could be achieved in problem ($P2$) by choosing $s = t'$, and would generate the same payoff. Similarly, if in the original problem ($P1$) the cake is not exhausted before time T $(R(t) > 0$ for $t < T)$, these paths of consumption and willpower could also be achieved in the related problem by choosing $s = T$, and would generate the same payoff. Since the two problems share objective functions and laws of motion up to time $s = t'$, any program that is feasible in the original problem is also feasible in the related problem and will generate the same payoff.

Given that any consumption path that is feasible in problem ($P1$) is also feasible in problem ($P2$), if the optimal consumption path in problem ($P2$) is feasible in problem ($P1$) then it is also optimal in problem ($P1$).[20] A sufficient condition for any consumption path in problem ($P2$) to be feasible in problem ($P1$) is that the consumption path depletes the cake by time s $(R(s) = 0)$. After all, that consumption path generates the same willpower path up to time s in both problems and therefore $W(t) \geq 0$ up to time s. After time s in problem ($P1$), willpower remains constant (at $W(s)$) since the cake is depleted. Hence any such consumption path is feasible in problem ($P1$) as well.

To see that the optimal consumption path in ($P2$) exhausts the cake at time s, suppose the contrary—that the "optimal" program leaves $R(s) > 0$ cake uneaten at s. To dominate this program, consider a different program that duplicates the "optimal" consumption path up to time $s - \Delta$, for any $\Delta > 0$, and consumes $\frac{R(s)}{\Delta}$ more during the remaining interval of length Δ. This exhausts the cake, draws willpower down to the same level at $s - \Delta$ and depletes less willpower during $(s - \Delta, s)$. Indeed, one can always choose Δ small enough that $\frac{R(s)}{\Delta} \geq \bar{c}$. In this case, depletion of willpower during $(s - \Delta, s)$ ceases altogether. Not only would utility from the alternative activity be weakly larger than on the "optimal" path but utility from intertemporal consumption would be strictly larger. This follows since the alternative consumption path is uniformly higher throughout and strictly higher from time $s - \Delta$ to s. This contradicts the claim that any feasible path with

[20] To see this, suppose that the optimal consumption path in problem ($P2$) is feasible but not optimal in problem ($P1$). Then there is a strictly preferred consumption path that is feasible in problem ($P1$). But by our earlier argument this path is also feasible in problem ($P2$) and would dominate the program we claimed was optimal in problem ($P2$). Hence we have a contradiction.

$R(s) > 0$ can be optimal. Hence, without loss of generality, we can confine our attention to problem $(P2)$.

Appendix B: When Willpower May be Built Through Its Exercise: Details

Again we consider a related but more tractable problem and argue that, by solving it, we solve problem $(P3)$. In the related problem, the agent chooses both an optimal consumption path $c(t)$ and the date $s \leq T$ after which consumption ceases, where $\dot{W}(t) = \dot{M}(t) = c(t) = 0$ for all $t \in (s, T]$, to maximize:

$$V(0) = \int_0^s e^{-\rho t} U[c(t)]\, dt \qquad (P4)$$

subject to the constraints of problem $(P3)$ except constraints (13)-(15) which are replaced by:

$$\dot{W}(t) = \gamma M(t) - f(W(t), c(t))$$
$$\dot{M}(t) = f(W(t), c(t)) - \sigma M(t)$$
$$R(t) \geq 0,\ W(t) \geq 0,\ M(t) \geq 0 \text{ for } t \in [0, s].$$

As in the problem without muscle, to show that this tractable problem has the same consumption path for $t \in [0, s]$ as the solution to the actual problem $(P3)$, it suffices to show that in problem $(P4)$ the entire cake is consumed by $t = s$ in the optimal solution. To see this note that if $R(s) > 0$ in the optimal program then $M(s) > 0$ and $W(s) \geq 0$. But then we could choose Δ small enough that $\frac{R(s)}{\Delta} \geq \bar{c}$. We could then duplicate the proposed optimal path until $s - \Delta$ and augment it by $\frac{R(s)}{\Delta}$ in this final interval. The payoff would be strictly higher and the program would be feasible since the willpower left at $s - \Delta$ is the same in the two programs ($W(s - \Delta) \geq 0$) and no willpower is depleted in the final interval.

Having established that the solution of our related problem $(P4)$ solves the actual problem $(P3)$, we make two observations which simplify the analysis. First, since the muscle is initially positive and decays exponentially even if it is never augmented, muscle will be strictly positive and will at no time violate the nonnegativity constraint. Second, since the stock of cake can only decline, requiring that it is nonnegative at s insures that it will be nonnegative previously.

Given that for $t \in [0, s)$ these two state variables must be nonnegative, we can simplify our formulation by replacing $R(t) \geq 0$ and $W(t) \geq 0$ by $R(s) \geq 0$ and $W(s) \geq 0$. However, since $W(s) \geq 0$ no longer implies that $W(t) \geq 0$ for $t < s$, the conditions which must necessarily hold at an optimum whenever $W(t) > 0$ will differ from those that hold while $W(t) = 0$. Given our focus, we consider only the former situation in detail.[21] The Hamiltonian for this problem is:

$$
\begin{aligned}
&H\left(c\left(t\right),R\left(t\right),W\left(t\right),t,\alpha\left(t\right),\lambda\left(t\right),\pi\left(t\right)\right)\\
&=e^{-\rho t}U\left(c\left(t\right)\right)-\alpha\left(t\right)c\left(t\right)+\lambda\left(t\right)\left(\gamma M\left(t\right)-f\left(W\left(t\right),c\left(t\right)\right)\right)+\\
&\pi\left(t\right)\left(f\left(W\left(t\right),c\left(t\right)\right)-\sigma M\left(t\right)\right).
\end{aligned}
$$

The first order conditions are given by,

$$c\left(t\right)\geq 0,\ e^{-\rho t}U'\left(c\left(t\right)\right)-\alpha\left(t\right)-\left(\lambda\left(t\right)-\pi\left(t\right)\right)f_c\leq 0 \text{ and c.s.} \tag{20}$$

$$\dot{W}\left(t\right)=\gamma M\left(t\right)-f \tag{21}$$

$$\dot{M}\left(t\right)=f-\sigma M\left(t\right) \tag{22}$$

$$\dot{\alpha}\left(t\right)=0 \tag{23}$$

$$\dot{\lambda}\left(t\right)=f_W\left(\lambda\left(t\right)-\pi\left(t\right)\right) \tag{24}$$

$$\dot{\pi}\left(t\right)=\pi\left(t\right)\sigma-\lambda\left(t\right)\gamma=-\gamma\left(\lambda\left(t\right)-\frac{\sigma}{\gamma}\pi\left(t\right)\right) \tag{25}$$

$$T-s\geq 0,\ \ H(c\left(s\right),R\left(s\right),W\left(s\right),s,\alpha\left(s\right),\lambda\left(s\right),\pi\left(s\right))\geq 0 \text{ and c.s.} \tag{26}$$

$$W\left(t\right)>0 \text{ and c.s.} \tag{27}$$

$$R\left(s\right)\geq 0,\ \ \alpha\left(s\right)\geq 0 \text{ and c.s.} \tag{28}$$

$$W\left(s\right)\geq 0,\ \ \lambda\left(s\right)\geq 0 \text{ and c.s.} \tag{29}$$

$$M\left(s\right)\geq 0,\ \ \pi\left(s\right)\geq 0 \text{ and c.s.} \tag{30}$$

To analyze the dynamics of consumption, it will be useful to sign $\lambda\left(t\right)-\pi\left(t\right)$ and $\lambda\left(t\right)\left(\frac{\gamma}{\sigma}\right)-\pi\left(t\right)$. First assume that $\gamma/\sigma\leq 1$. In this case, we show that each of the preceding terms is weakly positive while $\dot{\lambda}$ and $\dot{\pi}$ are weakly negative. These results can be most readily understood using the phase diagram depicted in Figure 1. Provisionally assume that $f_W<0$ and $\gamma>0$. Then we can plot the locus of (λ,π) pairs such that $\dot{\lambda}=0$. By the $\dot{\lambda}$ equation (24), these points lie on the 45^0 line $\pi=\lambda$. Horizontal motion above this locus is to the right and below it is to the left. Similarly, we can plot the locus of (λ,π) pairs such that $\dot{\pi}=0$. By the $\dot{\pi}$ equation (25), these points lie on a flatter ray provided $\frac{\gamma}{\sigma}<1$. In the extreme case where $\frac{\gamma}{\sigma}=1$, the two rays coincide. Vertical motion above the $\dot{\pi}=0$ locus is upward and below this locus it is downward. As long as muscle exists at any time in

[21] To derive conditions which must hold across both cases, Seierstad and Sydsœter (1987), and also Léonard and Long (1992)) begin by forming the Lagrangean $H+\Theta(t)W(t)$, where $\Theta(t)$ is a Lagrange multiplier. In such problems the multiplier (λ) on the state variable (willpower) may jump discontinuously as the nonnegativity constraint is just reached or as it becomes slack. If that multiplier does jump, then consumption would jump as well at such dates.

the program, some will remain at the end ($M(T) > 0$), because it at most decays exponentially and therefore never reaches zero. It follows from condition (30) that the endpoint condition $\pi(T) = 0$ is satisfied. As long as willpower considerations matter ($\lambda(0), \pi(0) \neq 0$), the endpoint condition and dynamics preclude initial multipliers set at or above the lower of the two rays since then $\dot{\pi} \geq 0$ implying $\pi(T) > 0$. Thus $\pi(T) = 0$ requires that the initial multipliers be set below the lower of the two rays. But this in turn implies that $\lambda - \pi > 0, \frac{\gamma}{\sigma}\lambda - \pi > 0, \dot{\lambda} < 0$, and $\dot{\pi} < 0$.

Now consider the case where $\gamma/\sigma > 1$, depicted in Figure 2. The endpoint condition $\pi(T) = 0$, together with these dynamics imply that there will be a final phase in which the multipliers will lie strictly below the $\dot{\lambda} = 0$ locus, and thus, again, $\lambda - \pi > 0, \frac{\gamma}{\sigma}\lambda - \pi > 0, \dot{\lambda} < 0$, and $\dot{\pi} < 0$.

To see that the optimal consumption path in the muscle model involves consuming the entire cake, note that having assumed $U'(\cdot) > 0$ and $f_c < 0$, and shown $(\lambda - \pi) > 0$, we can satisfy (20) only if $\alpha > 0$; and then (28) requires that the cake be exhausted by time s. To see that, if we start with an initial level of willpower sufficient for perfect smoothing, decreases in that stock eventually lead to a willpower level $\left(W_{\tilde{H}}\right)$ where any further reduction in the initial stock of willpower will make the perfectly smooth path infeasible, note that since $\lambda(0) \geq 0$, we know that utility is increasing in the initial stock of willpower. Because perfect smoothing is the optimal path in the absence of willpower concerns path, it follows that such a path is infeasible for any $W(0) < W_{\tilde{H}}$.

Finally, we turn to the time path of optimal consumption in the absence of discounting $(\rho = 0)$. Differentiating condition (20), we obtain:

$$\begin{aligned} U''(c)\dot{c} &= \dot{\alpha} + \left(\dot{\lambda} - \dot{\pi}\right) f_c + (\lambda - \pi)\left(f_{cc}\dot{c} + f_{cW}\dot{W}\right) \\ &= \left(f_W(\lambda - \pi) - (\pi\sigma - \lambda\gamma)\right) f_c + (\lambda - \pi)\left(f_{cc}\dot{c} + f_{cW}\dot{W}\right) \\ &= (\lambda - \pi)\left(f_W f_c + f_{cc}\dot{c} + f_{cW}\dot{W}\right) - (\pi\sigma - \lambda\gamma) f_c \end{aligned}$$

which implies

$$\dot{c} = \overbrace{\frac{(\lambda - \pi)(f_W f_c - f_{cW} f)}{\Delta}}^{\text{direct willpower effect}} + \overbrace{\frac{(\lambda - \pi) f_{cW}\gamma M(t)}{\Delta}}^{\text{muscle service flow}} + \overbrace{\frac{\left(\lambda\left(\frac{\gamma}{\sigma}\right) - \pi\right)\sigma f_c}{\Delta}}^{\text{muscle building}} \tag{31}$$

where $\Delta = [U''(c) - (\lambda - \pi) f_{cc}] < 0$. The inequality follows because c maximizes the Hamiltonian. If $\lambda(t) = \pi(t) = 0$ for $t \geq 0$, equation (31) yields the classical result that consumption is constant as long as it is positive and equation (26) requires that it be positive until T.

Referring to equation (31), if $f_W = 0$ then both the direct willpower effect and the muscle service flow are absent. First consider the case where muscle decays at a faster rate than it contributes to willpower $[(\gamma/\sigma) \leq 1]$. In this case, the muscle building term of equation (31) is always positive (see Figure 1) and the associated discussion). Thus when $(\gamma/\sigma) \leq 1$ and $f_W = 0$, consumption is always increasing.

When muscle decays more slowly $[(\gamma/\sigma) > 1]$, more complex consumption paths may emerge. If the multipliers λ and π are initial located in regions II or III of Figure 2, optimal behavior has the same qualitative features as when $(\gamma/\sigma) \leq 1$. Consumption is always increasing. If, however, the multipliers are initially located in region I of Figure 2, consumption will decrease with time until the multipliers pass into region II. That is, the consumption profile is $\cup$ - shaped.

Referring again to equation (16), if $(f_W f_c - f_{cW} f) = 0$ then only the direct willpower effect is inactive. Relative to the optimal path in the absence of muscle, the ability to build of willpower with exercise again induces time preference. Consider, for example, a situation where the initial muscle stock is zero $(M(0) = 0)$,and thus, at the beginning of the program, the muscle service flow term is inactive. In the beginning of the program, optimal behavior in this case is like that in the case where $f_W = 0$. For example, when $(\gamma/\sigma) \leq 1$, consumption will increase in these early stages of the optimal path.

Figures

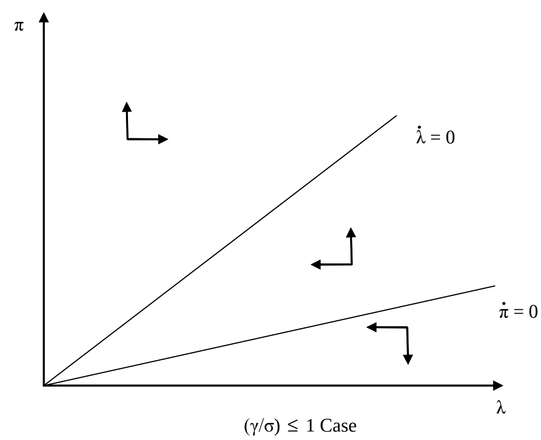

Figure 1: Phase Diagram of Costate Variables in Muscle Model Where $\frac{\gamma}{\sigma} \leq 1$.

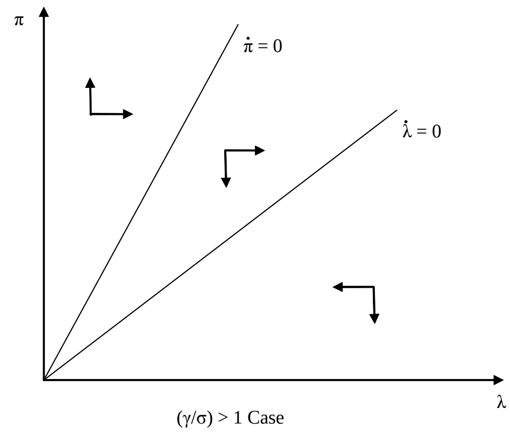

Figure 2: Phase Diagram of Costate Variables in Muscle Model Where $\frac{\gamma}{\sigma} > 1$.

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