A Connection Between Correlation in Game Theory and Quantum Mechanics*

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Abstract

Correlation is a basic concept in both game theory and quantum mechanics. We show that there is a formal correspondence between the treatment of correlation in the two domains. We use this correspondence to comment on the notion of a quantum game.

1 Introduction

The idea of correlation in game theory (GT) goes back to von Neumann [26, 1928] and von Neumann-Morgenstern [28, 1944]. They defined a cooperative game by starting with a non-cooperative game and allowing correlated behavior among the players. (Formally, the characteristic function for a subset A of players is the maximin payoff to A in the associated zero-sum game between A and not-A.) Aumann [1, 1974] introduced the idea of explaining correlation via the addition of "signals" (formally, payoff-irrelevant moves by Nature) to the underlying game.

Correlation plays a fundamental role in quantum mechanics (QM). This was observed in a famous paper by Einstein-Podolsky-Rosen [10, 1935]. Here is the idea.[1] Two particles are prepared in a special way (called a "singlet" state) and then separated, and their spins in various directions are measured with detectors. The spin in any direction takes the value $+1$ or -1 (the spin is quantized). Also, the spin is equally likely to be found to be $+1$ or -1. But, if the same angle is chosen

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[1] To be precise, in the reformulation due to Bohm [5, 1951].

for both detectors, then the two spins will be perfectly correlated (actually, anti-correlated in the usual physical set-up). This is the phenomenon of quantum-mechanical entanglement, or, in Einstein's famous phrase, "spooky action at a distance." Knowledge of the outcome of a measurement performed on one particle appears to 'fix' the result of a measurement on another particle (which could be a long distance away).

In this note, we establish a formal correspondence between the treatment of correlation in these two domains. The correspondence is exact–the same mathematical conditions are used. Is the connection a coincidence? No. It reflects the fact that at some level there can only be one 'architecture' of correlation, regardless of the interpretation put on the mathematics. Of course, the interpretations are very different in the two domains, as we will note.

We use this connection to comment on the idea of a quantum game. These games were introduced by Eisert-Wilkens-Lewenstein [11, 1999] and Meyer [22, 1999]. (Recent papers on the topic include Landsburg [20, 2005], La Mura [19, 2005], and Kargin [16, 2007]. These papers contain additional references.) In a quantum game, players make use of entanglement to achieve outcomes that might not be possible in a classical setting. But there has been some question as to whether quantum game theory really generalizes classical game theory–see Levine [21, 2005]. We will comment on this issue.

2 Correlation in GT

Fix an underlying game and an observer's probability assessment of the players' strategy choices in the game. Figure 1 is a typical example (see Aumann [1, 1974], [2, 1987]). The observer has a correlated assessment that assigns probability $\frac{1}{2}$ to the event that Ann and Bob choose U and L, and probability $\frac{1}{2}$ to the event that they choose D and R.

		Bob	
		L	R
Ann	U	1/2	0
	D	0	1/2

Figure 1

The question is whether such a correlated assessment should be allowed under non-cooperative theory, which assumes that the players choose strategies independently. Aumann argued that the answer is yes. His idea is to modify the initial game by adding "signals" (formally, payoff-irrelevant moves by Nature). For example, Figure 2 is obtained from Figure 1 by adding a signal, which is l or r with equal probability. The players observe the signal and then choose independently. The observer thinks that: (i) if Nature chooses l, then Ann chooses U and Bob chooses L; and (ii) if Nature chooses r, then Ann chooses D and Bob chooses R. The two matrices give the observer's (degenerate) conditional probabilities.[2]

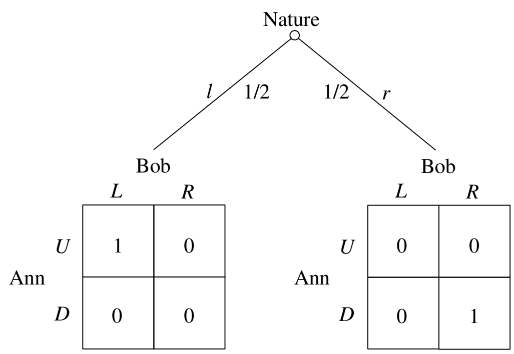

Figure 2

Here is a formalization of this idea, based on Brandenburger-Friedenberg [6, 2004]. Fix an underlying game (strictly, game form) and let S^a (resp. S^b) be Ann's (resp. Bob's) strategy set. In the extended game, Nature goes first and makes a payoff-irrelevant move that consists of choosing a point (λ^a, λ^b) from some finite product space $\Lambda^a \times \Lambda^b$. Ann (resp. Bob) observes the component λ^a (resp. λ^b).[3] These are the players' signals. The players then make choices as in the underlying game. Let p be the observer's probability measure (assessment) on $S^a \times S^b \times \Lambda^a \times \Lambda^b$. We stipulate two conditions (adapted from [6, 2004][4]):

Conditional Independence *The observer assesses Ann's and Bob's strategy choices as independent, conditional on the signals. Formally, whenever* $p(\lambda^a, \lambda^b) > 0$,

$$p(s^a, s^b|\lambda^a, \lambda^b) = p(s^a|\lambda^a, \lambda^b) \times p(s^b|\lambda^a, \lambda^b).$$

Sufficiency *If the observer knows Ann's signal, and comes to learn Bob's signal, this won't change the observer's assessment of Ann's strategy choice. Likewise with Ann and Bob interchanged. Formally, whenever* $p(\lambda^a, \lambda^b) > 0$,

$$\begin{aligned} p(s^a|\lambda^a, \lambda^b) &= p(s^a|\lambda^a), \\ p(s^b|\lambda^a, \lambda^b) &= p(s^b|\lambda^b). \end{aligned}$$

[2] Note the identification of moves in the two subtrees in Figure 2. This is the basis for saying that the assessments in Figures 1 and 2 agree.

[3] Finiteness of Λ^a and Λ^b is, of course, a restriction, but it does allow us to avoid any measure-theoretic issues. On the other hand, the product structure is without loss of generality. If, instead, Ann (resp. Bob) has a partition $\mathcal{H}^a$ (resp. $\mathcal{H}^b$) of a space Λ, just take Λ^a (resp. Λ^b) to be the quotient space $\{h^a : h^a \in \mathcal{H}^a\}$ (resp. $\{h^b : h^b \in \mathcal{H}^b\}$) and move the probabilities over in the obvious way.

[4] The focus of [6, 2004] is on developing a concept of "intrinsic" correlation in games. This is correlation that comes not from outside signals but from the players' own beliefs about the game. The conditions formulated in [6, 2004] are easily adapted to the current "extrinsic" setting (with signals), as we now do.

Here is an easy consequence of these conditions. (See Brandenburger-Friedenberg [6, 2004, Proposition 9.1] for a much more general result.)

Proposition 1 *Under Conditional Independence and Sufficiency, if* $p(\lambda^a, \lambda^b) = p(\lambda^a) \times p(\lambda^b)$ *for all* λ^a, λ^b*, then* $p(s^a, s^b) = p(s^a) \times p(s^b)$ *for all* s^a, s^b*.*

Proof. We have

$$\begin{aligned} p(s^a, s^b) &= \sum\nolimits_{\{(\lambda^a,\lambda^b)\ :\ p(\lambda^a,\lambda^b)>0\}} p(s^a, s^b|\lambda^a, \lambda^b)p(\lambda^a, \lambda^b) \\ &= \sum\nolimits_{\{(\lambda^a,\lambda^b)\ :\ p(\lambda^a,\lambda^b)>0\}} [p(s^a|\lambda^a) \times p(s^b|\lambda^b)]p(\lambda^a, \lambda^b) \\ &= \sum\nolimits_{\{\lambda^a\ :\ p(\lambda^a)>0\}} p(s^a|\lambda^a)p(\lambda^a) \times \sum\nolimits_{\{\lambda^b\ :\ p(\lambda^b)>0\}} p(s^b|\lambda^b)p(\lambda^b) \\ &= p(s^a) \times p(s^b), \end{aligned}$$

as required. ■

In words, under Conditional Independence and Sufficiency, if the observer assesses Ann's and Bob's signals as independent, then he assesses their strategy choices as independent. Taking the contrapositive, we see that Conditional Independence and Sufficiency guarantee that correlation in play implies ('comes from') correlation in signals. This is the non-cooperative view we wanted to formalize.

Note that this treatment of correlation is somewhat different from the usual one. Aumann [1, 1974] proposed the concept of correlated equilibrium as the embodiment of non-cooperative play in the presence of signals. We don't want to impose this or any other solution concept. We want to capture just the idea that the players are acting non-cooperatively. We make no assumptions (e.g., maximizing behavior) beyond this. The players simply do whatever they do–rather like particles, in fact. Indeed, our formalization will yield the connection to QM (to come in Section 4).

3 A Positive and a Negative Result

We explained the correlation in Figure 1 by transforming it into Figure 2. Can this always be done? The first answer is yes. Fix an observer's assessment q on $S^a \times S^b$. We want spaces Λ^a and Λ^b and a measure p on $S^a \times S^b \times \Lambda^a \times \Lambda^b$, such that p agrees with q on $S^a \times S^b$ and satisfies Conditional Independence and Sufficiency. To get this, set $\Lambda^a = S^a$ and $\Lambda^b = S^b$, and build p on the diagonal by setting

$$p(s^a, s^b, \lambda^a, \lambda^b) = \begin{cases} q(s^a, s^b) & \text{if } s^a = \lambda^a \text{ and } s^b = \lambda^b, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that Conditional Independence and Sufficiency both hold. (The basic idea of the construction–though not couched in terms of these conditions–is well known.)

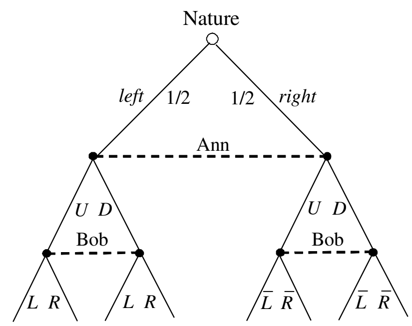

Figure 3

Next, though, is a more complicated game where, under certain conditions, the answer is no. Figure 3 is essentially Figure 1 in Forges [13, 1986] (but without payoffs). Notice that the game already contains a move by Nature (which could be payoff-relevant). This is not to be confused with any (payoff-irrelevant) moves by Nature we decide to add to the game.

Let the observer have an assessment q satisfying $q(U|left) = 1$ and $q(D|right) = 1$. As before, we ask: Can we find spaces Λ^a and Λ^b and a build a measure p on the extended game that agrees with q and that captures non-cooperative play?

To address this, we first have to extend our definitions to situations where the underlying game already contains moves by Nature. We won't consider a general tree, but will suppose that there is a basic game (form) $S^a \times S^b$. We then build a larger game in which Nature goes first and chooses a point (μ^a, μ^b) from some finite product space $M^a \times M^b$. Ann (resp. Bob) observes the component μ^a (resp. μ^b). Ann (resp. Bob) then makes a choice $s^a \in S^a$ (resp. $s^a \in S^a$). Thus, in Figure 3, $M^a = \{\emptyset\}$, $M^b = \{left, right\}$, and we identify Bob's moves in the two subtrees.) Note: We use μ rather than λ to distinguish (payoff-relevant) moves by Nature in the underlying game from (payoff-irrelevant) moves by Nature in an extended game.

Here are the extensions of Conditional Independence and Sufficiency from earlier (we won't give them new names):

Conditional Independence *Whenever* $p(\mu^a, \mu^b, \lambda^a, \lambda^b) > 0$,

$$p(s^a, s^b|\mu^a, \mu^b, \lambda^a, \lambda^b) = p(s^a|\mu^a, \mu^b, \lambda^a, \lambda^b) \times p(s^b|\mu^a, \mu^b, \lambda^a, \lambda^b). \tag{1}$$

Sufficiency *Whenever* $p(\mu^a, \mu^b, \lambda^a, \lambda^b) > 0$,

$$\begin{aligned} p(s^a|\mu^a, \mu^b, \lambda^a, \lambda^b) &= p(s^a|\mu^a, \lambda^a), \\ p(s^b|\mu^a, \mu^b, \lambda^a, \lambda^b) &= p(s^b|\mu^b, \lambda^b). \end{aligned} \tag{2}$$

The rationale for these conditions is as before, with the difference that now we take account of the fact that Ann observes both μ^a (from the underlying game) and λ^a (from the extended game), and likewise for Bob.[5]

Under conditions (1) and (2), there is no difficulty in deriving the assessment $q(U|\mathit{left}) = 1$ and $q(D|\mathit{right}) = 1$ by adding signals to Figure 3. The construction (which works for any game of the type described above) is essentially the same as before: Set $\Lambda^a = S^a \times M^a$ and $\Lambda^b = S^b \times M^b$ and build p via a diagonal construction.

But, suppose the move *left* or *right* by Nature in Figure 3 corresponds to a private coin toss by Bob and is therefore independent of any signals that we add to the game. (This is the scenario that Forges [13, 1986, pp.1378-9] considers.) In this case, we should impose an additional condition:

> **λ-Independence** *The observer assesses moves by Nature in the underlying game and moves by Nature in the extended game as independent. Formally,*
>
> $$p(\mu^a, \mu^b, \lambda^a, \lambda^b) = p(\mu^a, \mu^b) \times p(\lambda^a, \lambda^b). \tag{3}$$

Now, we can't derive the assessment $q(U|\mathit{left}) = 1$ and $q(D|\mathit{right}) = 1$ via signals. This is intuitively clear. Ann's choice can depend on her signal, which may be correlated with Bob's signal. But it cannot depend on Bob's coin toss if this is independent of the signals. An assessment with $q(U|\mathit{left}) \neq q(U|\mathit{right})$ fails this requirement. Here is the formal statement:

Proposition 2 *Consider the game in Figure 3. There is a probability measure q on $S^a \times S^b \times M^a \times M^b$ for which there are no (finite) spaces Λ^a and Λ^b, and probability measure p on $S^a \times S^b \times M^a \times M^b \times \Lambda^a \times \Lambda^b$, such that p agrees with q on $S^a \times S^b \times M^a \times M^b$ and satisfies Sufficiency and λ-Independence.*

Proof. Suppose $q(\mathit{left}) > 0$, $q(\mathit{right}) > 0$, and $q(U|\mathit{left}) \neq q(U|\mathit{right})$. Under the contrary hypothesis, we can write

$$\begin{aligned} q(U|\mathit{left}) &= p(U|\mathit{left}) \\ &= \sum\nolimits_{\{(\lambda^a,\lambda^b)\,:\,p(\mathit{left},\lambda^a,\lambda^b)>0\}} p(U|\mathit{left},\lambda^a,\lambda^b)p(\lambda^a,\lambda^b|\mathit{left}) \\ &= \sum\nolimits_{\{(\lambda^a,\lambda^b)\,:\,p(\mathit{left},\lambda^a,\lambda^b)>0\}} p(U|\lambda^a)p(\lambda^a,\lambda^b|\mathit{left}) \\ &= \sum\nolimits_{\{(\lambda^a,\lambda^b)\,:\,p(\lambda^a,\lambda^b)>0\}} p(U|\lambda^a)p(\lambda^a,\lambda^b), \end{aligned}$$

where the first line uses agreement, the third line uses Sufficiency, and the fourth line uses λ-Independence. But we can repeat the argument for $q(U|\mathit{right})$, to get $q(U|\mathit{left}) = q(U|\mathit{right})$, a contradiction. ∎

[5] In a general tree, with more conditioning events than just Nature's initial move, we would need to extend (1) and (2) appropriately.

Forges [13, 1986] uses this example to conclude that communication in games can sometimes yield outcomes not possible under correlation. The idea is to let Bob inform Ann of Nature's choice (*left* or *right*). Now, an observer's assessment with $q(U|left) = 1$ and $q(D|right) = 1$ becomes reasonable. Formally, following each of Nature's moves (*left* or *right*) in Figure 3, we would add payoff-irrelevant moves for Bob (say "*left*" or "*right*") which are observed by Ann. We come back to the issue of communication in Section 5.

Note that Proposition 2 uses only Sufficiency and λ-Independence. What about Conditional Independence, which we said is also conceptually appropriate for non-cooperative analysis? In fact, there is always an extended game in which Conditional Independence and λ-Independence hold. (This follows from Brandenburger-Yanofsky [7, 2007, Theorem 3.2].) We touch on Conditional Independence again below.

4 QM and Hidden Variables

Now the connection to QM. Einstein-Podolsky-Rosen [10, 1935, p.777] drew the lesson from their thought-experiment that QM needed to be made "complete." That is, additional variables, usually called hidden variables, needed to be added to the theory to explain the correlations. This is like the move in GT from Figure 1 to Figure 2, where an extra variable (a move by Nature) is added. In the QM setting, a hidden variable would be introduced that determines–when the two particles are prepared–whether the outcomes of the measurements on them will be $+1$ and -1, or -1 and $+1$.

More generally, the goal of the hidden-variable program in QM is to build extended models that satisfy certain desiderata and that reproduce the predictions of QM. What are these desiderata? They are precisely conditions (1)-(3) above.

To see this, reinterpret the variables. We now think of μ^a and μ^b as the measurements Ann and Bob make, s^a and s^b as the outcomes of the measurements, and (as before) λ^a and λ^b as extra (hidden) variables. (The scenario might involve two different particles, or two measurements on one particle.)

Condition (1) (reinterpreted this way) is called **Outcome Independence** in QM (Jarrett [15, 1984], Shimony [24, 1986]). Note that this can be rewritten as the pair of conditions

$$\begin{aligned} p(s^a|s^b, \mu^a, \mu^b, \lambda^a, \lambda^b) &= p(s^a|\mu^a, \mu^b, \lambda^a, \lambda^b), \\ p(s^b|s^a, \mu^a, \mu^b, \lambda^a, \lambda^b) &= p(s^b|\mu^a, \mu^b, \lambda^a, \lambda^b), \end{aligned}$$

whenever $p(s^b, \mu^a, \mu^b, \lambda^a, \lambda^b) > 0$ and $p(s^a, \mu^a, \mu^b, \lambda^a, \lambda^b) > 0$. This says that conditional on the values of the hidden variables and the measurements undertaken, the outcome of a measurement is (probabilistically) unaffected by the outcome of another measurement. Condition (2) is called **Parameter Independence** ([15, 1984], [24, 1986]). It says that, conditional on the values of the hidden variables, the outcome of a measurement depends (probabilistically) only on that measurement and not on another measurement. Condition (3) says that the process determining the hidden variables is independent of what measurements are conducted. The term **λ-Independence** for this condition is found in Dickson [9, 2005, p.140].

There are also two derived conditions in QM. One is **Locality** (Bell [3, 1964]), which requires the factorization

$$p(s^a, s^b|\mu^a, \mu^b, \lambda^a, \lambda^b) = p(s^a|\mu^a, \lambda^a) \times p(s^b|\mu^b, \lambda^b),$$

whenever $p(\mu^a, \mu^b, \lambda^a, \lambda^b) > 0$. It is easily seen to be equivalent to the conjunction of (1) and (2) (Jarrett [15, 1984]). The **Non-Contextuality** condition (Kochen-Specker [17, 1967]) is that

$$\begin{aligned} q(s^a|\mu^a, \mu^b) &= q(s^a|\mu^a, \tilde{\mu}^b), \\ q(s^b|\mu^a, \mu^b) &= q(s^b|\tilde{\mu}^a, \mu^b), \end{aligned}$$

whenever $q(\mu^a, \mu^b) > 0$, $q(\mu^a, \tilde{\mu}^b) > 0$, and $q(\tilde{\mu}^a, \mu^b) > 0$. In words, this says the probability (without introducing hidden variables) of obtaining a particular outcome of a measurement does not depend on what other measurement is performed. It is not hard to see that Non-Contextuality is implied by the conjunction of (2) and (3). (We showed this for a particular case in the course of proving Proposition 2. But the argument is clearly general. See also Brandenburger-Yanofsky [7, 2007, Proposition 2.2].)

Can the hidden-variable program actually be carried out? Two famous impossibility results showed that there are limits to what is possible. Bell [3, 1964] produced a physical model obeying the rules of QM that could not be derived from a hidden-variable model satisfying Locality and λ-Independence–i.e., satisfying (1), (2), and (3). The stronger Kochen-Specker [17, 1967] result is that there is a physical model in QM that fails Non-Contextuality. (A fortiori, the model cannot be derived from a hidden-variable model satisfying (2) and (3).)

We see the continuing parallel. The game of Figure 3 plays the same role in GT as Kochen-Specker does in QM. (The Kochen-Specker set-up is more complicated than a direct translation of Figure 3 would indicate. The reason is the constraints needed to get Non-Contextuality in an actual physical model.[6])

[6] More detail for the interested reader: In Kochen-Specker, the spin of one particle is measured in various directions. The spin in any direction can take values $+1$, 0, or -1. The arrangement is such that if the spin is measured in each of three orthogonal directions, and the squares of the three spins are calculated, we will always get two 1's and one 0. But, it is impossible to assign to each point on a sphere a 1 or a 0 so that: (i) every set of three orthogonal points has two 1's and one 0; and (ii) antipodal points are both 0's or both 1's. The conclusion is that the outcome of a measurement of (the square of) spin in one direction must depend on which other directions of measurement are also chosen. Non-Contextuality fails.

Kochen-Specker actually give an argument involving only finitely many different directions of measurement. The number of directions needed has been reduced over time. An important reference on this is Peres [23, 1991].

5 Quantum Games

Now put GT and QM together. This is what the quantum games literature does. (References were given earlier.[7]) We can get the essential idea from the game of Figure 3. Ann and Bob decide to peg their choices on a quantum device, as follows. If Bob sees Nature move *left*, he makes a certain measurement on a particle. If he sees Nature move *right*, he makes another measurement. Without seeing any of this, Ann makes her own measurement on the particle. Depending on the outcome of her measurement, Ann moves U or D. But the set-up is of the Kochen-Specker kind, so that the outcome of Ann's measurement depends on what measurement Bob makes.[8] (Non-Contextuality fails.) In this case, an observer might well hold an assessment with $q(U|\mathit{left}) \neq q(U|\mathit{right})$. Proposition 2 says that such an assessment is impossible under classical correlation alone.

Levine [21, 2005] is a critique of quantum games, taking the position that they can be fully encompassed within classical games. We can see his argument with the help, once again, of Figure 3. Levine would point out that $q(U|\mathit{left}) \neq q(U|\mathit{right})$ can be obtained classically, if the players are allowed to communicate–i.e., if "cheap talk" à la Crawford-Sobel [8, 1982] and Farrell [12, 1987] is allowed. (We said that for Forges [13, 1986], the point of Figure 3 was to show that communication–of the classical kind–is different from correlation.) Quantum game theorists, presumably, take the view that the kind of "communication without communication" allowed by Kochen-Specker is nevertheless of interest. Levine [21, 2005, p.6] does add: "... quantum pseudo-communication may have advantages of security; or may be available when other 'true' communication devices are not."

Our small contribution (if any) to this issue is to try to state it in the very simplest terms, which is what we believe Figure 3 does.

6 A Historical Note

There is, of course, another connection between GT and QM, via the towering figure of von Neumann.

In QM, von Neumann initiated the hidden-variable program and gave an impossibility argument. In fact, his argument is now known to use too-strong assumptions (Bell [4, 1966]). It is the modern impossibility results of Bell [3, 1964] and Kochen-Specker [17, 1967] that are now considered decisive. Still, von Neumann's position was clear, as in the often-quoted: "[T]he present system of quantum mechanics would have to be objectively false, in order that another description of the elementary processes than the statistical one [i.e., in order that a hidden-variable description] be possible" ([27, 1955, p.325]).

In GT, we noted that von Neumann initiated the study of correlation ([26, 1928], [28, 1944]). However, there does not appear to be any evidence that he saw any formal connection between these

[7] Lambert-Mogiliansky, Zamir, and Zwirn [18, 2003] and Temzelides [25, 2005] put QM and GT together in a different way. They employ aspects of QM–such as non-commutativity–in building non-classical decision theories.

[8] Experimental demonstrations of Kochen-Specker now exist; see Hasegawa et al. [14, 2006].

two enterprises.[9]

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