

The Relationship Between Classical and Quantum Correlation in Games*

Adam Brandenburger[†]

First Version: June 2006
Current Version: August 2008

Abstract

Quantum games have been argued to extend classical games (Eisert-Wilkens-Lewenstein [8, 1999], Meyer [18, 1999], and subsequent papers). Levine [17, 2005] has questioned this assertion. We offer what we believe to be the simplest expression of the disagreement. In doing so, we also establish a general correspondence, at the probabilistic level, between the analyses of correlation in game theory and in quantum mechanics.

1 Introduction

Quantum games were introduced by Eisert-Wilkens-Lewenstein [8, 1999] and Meyer [18, 1999]. Additional contributions to the topic include Landsburg [16, 2005], La Mura [15, 2005], and Kargin [12, 2007].¹ In these papers, players make use of the phenomenon of quantum-mechanical entanglement—a kind of correlation that Einstein famously called “spooky action at a distance.” This is argued to allow outcomes that are impossible in a classical setting. Recently, Levine [17, 2005] has questioned the claim that quantum game theory really generalizes classical game theory. In this note, we try to identify the source of this disagreement in the simplest terms.

*This note owes a great deal to joint work with Amanda Friedenber and Noson Yanofsky. I am indebted to Samson Abramsky for asking a question which stimulated this investigation, and to David Levine, Eric Pacuit, and Gus Stuart for detailed comments. John Asker, Jean de Valpine, Ariel Rubinstein, Ted Temzelides, Johan van Benthem, and audiences at the 7th Conference on Logic and the Foundations of Game and Decision Theory (University of Liverpool, July 2006), the Colloquium on New Perspectives on Games and Interaction (Royal Netherlands Academy of Arts and Sciences, February 2007), the 3rd CSEF-IGIER Symposium on Economics and Institutions (Centro Internazionale per la Cultura Scientifica, Anacapri, June 2007), and Bocconi, Cornell, and Northwestern Universities provided important input. Financial support from the Stern School of Business is gratefully acknowledged. req-08-01-08

[†]Address: Stern School of Business, New York University, 44 West Fourth Street, New York, NY 10012, adam.brandenburger@stern.nyu.edu, www.stern.nyu.edu/~abranden

¹Lambert-Mogiliansky, Zamir, and Zwirn [14, 2003] and Temzelides [21, 2005] put quantum mechanics and decision theory together. They employ aspects of quantum mechanics—such as non-commutativity—to build non-classical decision theories.

In the next section, which is informal, we present an analysis of correlation in games and give a very simple example of correlation that cannot arise under the conditions of non-cooperative play. We then explain how the example is possible under entanglement, and what Levine’s objection is. Section 3 gives a formal treatment of what correlations are or are not possible under non-cooperative play. In Section 4, we show that our approach brings out a general correspondence, at the probabilistic level, between the analyses of correlation in game theory and in quantum mechanics. Section 5 contains a historical note.

2 Informal Treatment

In game theory, the idea of correlation goes back to von Neumann [22, 1928] and von Neumann-Morgenstern [24, 1944]. They defined a cooperative game by starting with a non-cooperative game and allowing correlated behavior among the players. (Formally, the characteristic function for a subset A of players is the maximin payoff to A in the associated zero-sum game between A and not- A .) Aumann [1, 1974], [2, 1987] introduced the idea of explaining correlation via the addition of “signals” to the underlying game.

		Bob	
		L	R
Ann	U	1/2	0
	D	0	1/2

Figure 1

The basic idea of correlation via signals is well known. Figure 1 is a typical example ([1, 1974], [2, 1987]). The observer has a correlated assessment that assigns probability $\frac{1}{2}$ to the event that Ann and Bob choose U and L , and probability $\frac{1}{2}$ to the event that they choose D and R .

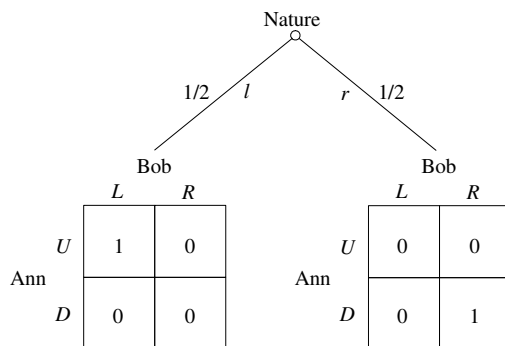


Figure 2

The question is whether such a correlated assessment should be allowed under non-cooperative theory, which assumes that the players choose strategies independently. Aumann argued that the answer is yes. His idea is to modify the initial game by adding signals (formally, payoff-irrelevant moves by Nature). Thus, Figure 2 is obtained from Figure 1 by adding a signal, which is l or r with equal probability. The players observe the signal and then choose independently. The observer thinks that: (i) if Nature chooses l , then Ann chooses U and Bob chooses L ; and (ii) if Nature chooses r , then Ann chooses D and Bob chooses R . The two matrices give the observer's (degenerate) conditional probabilities.²

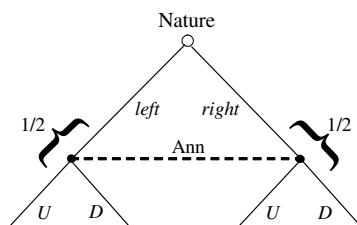


Figure 3

Now a more complicated example, which includes moves by Nature in the underlying game. See Figure 3. (This is essentially Figure 1 in Forges [9, 1986], but without payoffs. We come back to this connection later.) Note that the moves *left* and *right* by Nature could be payoff-relevant, unlike the moves l and r in Figure 2 (which were added signals). The observer has an assessment that assigns probability $\frac{1}{2}$ to the event that Nature chooses *left* and Ann chooses U , and probability $\frac{1}{2}$ to the event that Nature chooses *right* and Ann chooses D .

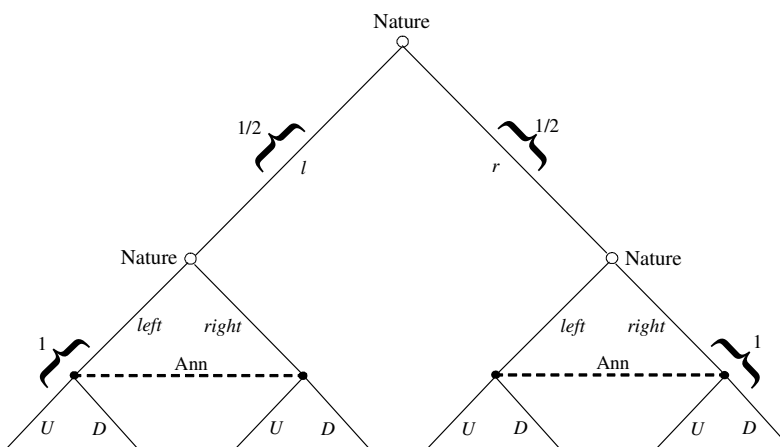


Figure 4

This time, the correlation is between a player, Ann, and Nature. Still, as before, we can explain

²Note the identification of moves in the two subtrees in Figure 2. This is the basis for saying that the assessments in Figures 1 and 2 agree.

the correlation in the observer’s assessment by adding signals; see Figure 4.

But, suppose the move *left* or *right* by Nature in the underlying game of Figure 3 corresponds to a private coin toss by another player (Bob). In this case, it seems reasonable to require that signals such as *l* or *r* that are added to the game are independent of the underlying move by Nature. If so, the assessment in Figure 4 is disallowed and, indeed, it looks like there is now no way to add signals to explain the correlation in the original assessment in Figure 3. In the next section, we will give a formal treatment of what it means to explain correlation and why this can’t necessarily be done under an independence assumption.

To see how the quantum games approach explains correlation, continue to think of *left* vs. *right* in Figure 3 as corresponding to a private coin toss by Bob. Depending on the outcome of the toss, Bob makes one or other of two measurements on a certain kind of particle. Without seeing either the coin toss or what measurement Bob makes, Ann makes her own measurement on the particle. Depending on the outcome of her measurement, she moves *U* or *D*. Quantum mechanics allows the (highly counter-intuitive!) situation where the outcome of Ann’s measurement can depend on what measurement Bob makes. This follows from the Kochen-Specker Theorem ([13, 1967]) of quantum mechanics.³ (It is related to the famous Bell’s Theorem [3, 1964]. More on this in Section 4.) Quantum game theorists conclude that the observer’s assessment in Figure 3 is, in fact, consistent with non-cooperative play.

Levine [17, 2005] is a critique of quantum games, taking the position that they can be fully encompassed within classical games. Levine argues that since the quantum approach allows a kind of communication between the players, a fair comparison should also allow communication in the classical case. Then, Bob could simply inform Ann of the outcome of his coin toss. This would immediately justify the observer’s assessment in Figure 3. (Forges [9, 1986] used this game in exactly this way—to argue that (classical) communication is different from correlation.) Quantum game theorists, presumably, take the view that the kind of “communication without communication” allowed by Kochen-Specker is nevertheless of interest. Indeed, Levine [17, 2005, p.6] does add: “. . . quantum pseudo-communication may have advantages of security; or may be available when other ‘true’ communication devices are not.”

Our small contribution to this issue is to try to state it in the very simplest terms, which is what we believe Figure 3 does.

³More detail for the interested reader: In the actual Kochen-Specker set-up, the spin of a particle is measured in various directions. The spin in any direction can take values +1, 0, or −1. The arrangement is such that if the spin is measured in each of three orthogonal directions, and the squares of the three spins are calculated, we will always get two 1’s and one 0. But, it is impossible to assign to each point on a sphere a 1 or a 0 so that: (i) every set of three orthogonal points has two 1’s and one 0; and (ii) antipodal points are both 0’s or both 1’s. The conclusion is that the outcome of a measurement of (the square of) spin in one direction depends on which other directions of measurement are also chosen. Experimental demonstrations of Kochen-Specker now exist; see Hasegawa et al. [10, 2006].

Kochen-Specker actually gave an argument involving only finitely many different directions of measurement. The number of directions needed has been reduced over time. An important reference on this is Peres [19, 1991].

3 Classical Correlation

We now give the promised formal statement of what it means to “explain”—in classical terms—the correlation in an observer’s assessment. We want to do so without imposing any particular solution concept (such as Aumann’s [1, 1974] correlated equilibrium). We want to capture just the idea that the players are acting non-cooperatively. We make no assumptions (such as maximizing behavior) beyond this.

Fix an underlying game (strictly, game form) and let S^a (resp. S^b) be Ann’s (resp. Bob’s) strategy set. The observer has a probability assessment q on $S^a \times S^b$. We want to make such an assessment consistent with the assumption of non-cooperative play by building an extended game with signals. In the extended game, Nature goes first and makes a payoff-irrelevant move that consists of choosing a point (λ^a, λ^b) from some finite product space $\Lambda^a \times \Lambda^b$. Ann (resp. Bob) observes the component λ^a (resp. λ^b).⁴ The players then make choices as in the underlying game. Let p be the observer’s probability measure on the extended game $S^a \times S^b \times \Lambda^a \times \Lambda^b$. We want p to be consistent with q , that is, we want $\text{marg}_{S^a \times S^b} p = q$. But we also want p to be consistent with non-cooperative play. To ensure this, we adapt some conditions from Brandenburger-Friedenberg [5, 2008]:⁵

Conditional Independence *The observer assesses Ann’s and Bob’s strategy choices as independent, conditional on the signals. Formally, whenever $p(\lambda^a, \lambda^b) > 0$,*

$$p(s^a, s^b | \lambda^a, \lambda^b) = p(s^a | \lambda^a, \lambda^b) \times p(s^b | \lambda^a, \lambda^b).$$

Sufficiency *If the observer knows Ann’s signal, and comes to learn Bob’s signal, this won’t change the observer’s assessment of Ann’s strategy choice. Likewise with Ann and Bob interchanged. Formally, whenever $p(\lambda^a, \lambda^b) > 0$,*

$$\begin{aligned} p(s^a | \lambda^a, \lambda^b) &= p(s^a | \lambda^a), \\ p(s^b | \lambda^a, \lambda^b) &= p(s^b | \lambda^b). \end{aligned}$$

Here is an easy consequence of these conditions. (See [5, 2008, Proposition 9.1] for a much more general result.)

Proposition 1 *Under Conditional Independence and Sufficiency, if $p(\lambda^a, \lambda^b) = p(\lambda^a) \times p(\lambda^b)$ for all λ^a, λ^b , then $p(s^a, s^b) = p(s^a) \times p(s^b)$ for all s^a, s^b .*

⁴Finiteness of Λ^a and Λ^b is, of course, a restriction, but allows us to avoid any measure-theoretic issues. On the other hand, the product structure is without loss of generality. If, instead, Ann (resp. Bob) has a partition \mathcal{H}^a (resp. \mathcal{H}^b) of a space Λ , just take Λ^a (resp. Λ^b) to be the quotient space $\{h^a : h^a \in \mathcal{H}^a\}$ (resp. $\{h^b : h^b \in \mathcal{H}^b\}$) and move the probabilities over in the obvious way.

⁵The focus of [5, 2008] is on developing a concept of “intrinsic” correlation in games. This is correlation that comes not from outside signals but from the players’ own beliefs about the game. The conditions formulated in [5, 2008] are easily adapted to the current “extrinsic” setting (with signals), as we now do.

Proof. We have

$$\begin{aligned}
p(s^a, s^b) &= \sum_{\{(\lambda^a, \lambda^b) : p(\lambda^a, \lambda^b) > 0\}} p(s^a, s^b | \lambda^a, \lambda^b) p(\lambda^a, \lambda^b) \\
&= \sum_{\{(\lambda^a, \lambda^b) : p(\lambda^a, \lambda^b) > 0\}} [p(s^a | \lambda^a) \times p(s^b | \lambda^b)] p(\lambda^a, \lambda^b) \\
&= \sum_{\{\lambda^a : p(\lambda^a) > 0\}} p(s^a | \lambda^a) p(\lambda^a) \times \sum_{\{\lambda^b : p(\lambda^b) > 0\}} p(s^b | \lambda^b) p(\lambda^b) \\
&= p(s^a) \times p(s^b),
\end{aligned}$$

as required. ■

In words, under Conditional Independence and Sufficiency, if the observer assesses Ann’s and Bob’s signals as independent, then he assesses their strategy choices as independent. Taking the contrapositive, we see that Conditional Independence and Sufficiency guarantee that correlation in play implies (“comes from”) correlation in signals. This is the non-cooperative view we want to formalize.

It is easy to check that the assessment in Figure 2 satisfies Conditional Independence and Sufficiency—so, this was indeed an example of explaining correlation as we have now formalized the idea. But can such a maneuver always be carried out? If the underlying game doesn’t contain any moves by Nature, the answer is yes. The idea of the construction is well known. Set $\Lambda^a = S^a$ and $\Lambda^b = S^b$, and build p on the diagonal by setting

$$p(s^a, s^b, \lambda^a, \lambda^b) = \begin{cases} q(s^a, s^b) & \text{if } s^a = \lambda^a \text{ and } s^b = \lambda^b, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\text{marg}_{S^a \times S^b} p = q$, and Conditional Independence and Sufficiency both hold.

But, what if the underlying game does already contain moves by Nature? The example of Figure 3 suggests that now the answer might be no: Not all correlations can be explained (at least classically).

To examine this formally, we first have to extend our definitions to situations where the underlying game includes moves by Nature. We won’t consider a general tree, but will suppose that there is a basic game $S^a \times S^b$. We then build a larger game in which Nature goes first and chooses a point (μ^a, μ^b) from some finite product space $M^a \times M^b$. Ann (resp. Bob) observes the component μ^a (resp. μ^b). Ann (resp. Bob) then makes a choice $s^a \in S^a$ (resp. $s^b \in S^b$). Thus, in Figure 3, $S^a = \{U, D\}$, $S^b = \{\emptyset\}$, $M^a = \{\emptyset\}$, and $M^b = \{left, right\}$. Note: We use μ rather than λ to distinguish (possibly payoff-relevant) moves by Nature in the given game from (payoff-irrelevant) moves by Nature in an extended game. Here are the extensions of Conditional Independence and Sufficiency (we won’t give them new names):

Conditional Independence Whenever $p(\mu^a, \mu^b, \lambda^a, \lambda^b) > 0$,

$$p(s^a, s^b | \mu^a, \mu^b, \lambda^a, \lambda^b) = p(s^a | \mu^a, \mu^b, \lambda^a, \lambda^b) \times p(s^b | \mu^a, \mu^b, \lambda^a, \lambda^b). \quad (1)$$

Sufficiency Whenever $p(\mu^a, \mu^b, \lambda^a, \lambda^b) > 0$,

$$\begin{aligned} p(s^a | \mu^a, \mu^b, \lambda^a, \lambda^b) &= p(s^a | \mu^a, \lambda^a), \\ p(s^b | \mu^a, \mu^b, \lambda^a, \lambda^b) &= p(s^b | \mu^b, \lambda^b). \end{aligned} \quad (2)$$

The rationale for these conditions is as before, with the difference that now we take account of the fact that Ann observes both μ^a (from the given game) and λ^a (from the extended game), and likewise for Bob.

There is no difficulty in building an extended game with signals, to satisfy these new Conditional Independence and Sufficiency conditions. The construction is an analog to the previous one: Set $\Lambda^a = S^a \times M^a$ and $\Lambda^b = S^b \times M^b$ and build p via a diagonal construction as before.

But, what if we ask for an independence requirement between the two kinds of moves by Nature? (We said that in Figure 3, this would be appropriate if, for example, the set M^b describes a private coin toss by Bob.) That is, we might require:

λ -Independence *The observer assesses moves by Nature in the underlying game and moves by Nature in the extended game as independent. Formally,*

$$p(\mu^a, \mu^b, \lambda^a, \lambda^b) = p(\mu^a, \mu^b) \times p(\lambda^a, \lambda^b). \quad (3)$$

Proposition 2 *Consider the game and assessment q in Figure 3. There do not exist (finite) spaces Λ^a and Λ^b , and probability measure p on $S^a \times S^b \times M^a \times M^b \times \Lambda^a \times \Lambda^b$, such that p agrees with q on $S^a \times S^b \times M^a \times M^b$ and satisfies Sufficiency and λ -Independence.*

Proof. We can write

$$\begin{aligned} q(U|left) &= p(U|left) \\ &= \sum_{\{(\lambda^a, \lambda^b) : p(left, \lambda^a, \lambda^b) > 0\}} p(U|left, \lambda^a, \lambda^b) p(\lambda^a, \lambda^b | left) \\ &= \sum_{\{(\lambda^a, \lambda^b) : p(left, \lambda^a, \lambda^b) > 0\}} p(U|\lambda^a) p(\lambda^a, \lambda^b | left) \\ &= \sum_{\{(\lambda^a, \lambda^b) : p(\lambda^a, \lambda^b) > 0\}} p(U|\lambda^a) p(\lambda^a, \lambda^b), \end{aligned}$$

where the first line uses agreement, the third line uses Sufficiency, and the fourth line uses λ -Independence. But we can repeat the argument for $q(U|right)$, to get $q(U|left) = q(U|right)$, contradicting $q(D|right) = 1$. ■

This is the formal treatment of our claim in the previous section that Figure 3 involves an assessment that is inconsistent with non-cooperative play and the independence assumption. Note that Proposition 2 uses only Sufficiency and λ -Independence. What about Conditional Independence, which we said is also conceptually appropriate for non-cooperative analysis? In fact, there is always an extended game in which Conditional Independence and λ -Independence hold. (This follows from Brandenburger-Yanofsky [6, 2007, Theorem 3.2].) We touch on Conditional Independence again below.

4 Quantum Correlation

In Section 2, we said that the quantum games approach explains the correlated assessment in Figure 3 by having Bob make one or other of two measurements on a particle, depending on the realization of his coin toss (in the set M^b). We can illuminate the connection to quantum mechanics (QM) if, instead, we take the set M^b to describe directly the measurement that Bob makes. Likewise, we will now think of M^a as a set of possible measurements that Ann can make. Again, we said before that Ann will choose a strategy from S^a according to the outcome of her measurement. Now, we will take S^a simply to be the set of possible outcomes. Likewise, S^b will now be the set of possible outcomes of Bob's measurement. Ann and Bob might make their measurements on a common particle (as in Kochen-Specker) or on different particles.

An important agenda within QM, called the hidden-variable (h.v.) program, is building extended models that include additional variables to explain the counter-intuitive correlations that arise (so-called entanglement). If we denote these variables by λ^a and λ^b (λ is the customary notation for h.v.s in QM), then the conditions usually imposed on h.v. models are precisely conditions (1), (2), and (3) from the previous section. In QM, condition (1) is called Outcome Independence (Jarrett [11, 1984], Shimony [20, 1986]). It says that conditional on the values of the measurements undertaken and the hidden variables, the outcomes of the measurements are independent. Condition (2) is called Parameter Independence ([11, 1984], [20, 1986]) and says that, given Ann's choice of measurement and the value of the a -h.v., knowledge of Bob's choice of measurement and the value of the b -h.v. doesn't change (probabilistically) the outcome of Ann's measurement. Condition (3) says that the process determining the values of the h.v.s is independent of what measurements are conducted. The term λ -Independence for this condition is found in Dickson [7, 2005, p.140]. (We deliberately borrowed the term in the previous section.)

The implication is that, at the probabilistic level, the following two exercises are exactly the same: (i) explain correlations in QM via h.v.s; and (ii) explain correlations in game theory via signals.

Two famous impossibility results in QM show that (i) is not always possible. Bell [3, 1964] produced a physical model obeying the rules of QM that cannot be derived from an h.v. model satisfying (1), (2), and (3). The stronger Kochen-Specker [13, 1967] Theorem exhibits a physical

model obeying the rules of QM that cannot be derived from an h.v. model satisfying just (2) and (3).

The equivalence between (i) and (ii) is what we exploited less formally in Section 2: Since (i) is not always possible, (ii) is not always possible. Moreover, since the physical scenarios in Bell or Kochen-Specker can actually arise, the conclusion is that there are physically possible game scenarios involving correlations that cannot be explained classically. At least, this is so if “classical” includes signals but not direct communication; remember our discussion of the Levine [17, 2005] critique.

5 A Historical Note

There is, of course, a historical connection between game theory and QM, via the towering figure of von Neumann.

Von Neumann initiated the hidden-variable program in QM and gave an impossibility argument. In fact, his argument is now known to use too-strong assumptions (Bell [4, 1966]). It is the modern impossibility results of Bell [3, 1964] and Kochen-Specker [13, 1967] that are now considered decisive. Still, von Neumann’s position was clear, as in the often-quoted: “[T]he present system of quantum mechanics would have to be objectively false, in order that another description of the elementary processes than the statistical one [i.e., in order that a hidden-variable description] be possible” ([23, 1955, p.325]).

We noted that von Neumann initiated the study of correlation ([22, 1928], [24, 1944]) in game theory. However, there does not appear to be any evidence that he saw any formal connection between these two enterprises.⁶

⁶Jean de Valpine brought to my attention that there are several references—but only of a rather general kind—to QM in von Neumann-Morgenstern [24, 1944, pp.3, 33, 148, 401].

References

- [1] Aumann, R., "Subjectivity and Correlation in Randomized Strategies," *Journal of Mathematical Economics*, 1, 1974, 76-96.
- [2] Aumann, R., "Correlated Equilibrium as an Expression of Bayesian Rationality," *Econometrica*, 55, 1987, 1-18.
- [3] Bell, J., "On the Einstein-Podolsky-Rosen Paradox," *Physics*, 1, 1964, 195-200.
- [4] Bell, J., "On the Problem of Hidden Variables in Quantum Theory," *Reviews of Modern Physics*, 38, 1966, 447-452.
- [5] Brandenburger, A., and A. Friedenberg, "Intrinsic Correlation in Games," *Journal of Economic Theory*, 141, 2008, 28-67.
- [6] Brandenburger, A., and N. Yanofsky, "A Classification of Hidden-Variable Properties," 2007. Available at <http://arxiv.org/abs/0711.4650>.
- [7] Dickson, W.M., *Quantum Chance and Non-Locality: Probability and Non-Locality in the Interpretations of Quantum Mechanics*, Cambridge University Press, 2005.
- [8] Eisert, J., M. Wilkens, and M. Lewenstein, "Quantum Games and Quantum Strategies," *Physical Review Letters*, 83, 1999, 3077-3088.
- [9] Forges, F., "An Approach to Communication Equilibria," *Econometrica*, 54, 1986, 1375-1385.
- [10] Hasegawa, Y., Loidl, R., Badurek, G., Baron, M., Rauch, H., "Quantum Contextuality in a Single-Neutron Optical Experiment," *Physical Review Letters*, 97, 2006, 230401, 1-4.
- [11] Jarrett, J., "On the Physical Significance of the Locality Conditions in the Bell Arguments," *Noûs*, 18, 1984, 569-589.
- [12] Kargin, V., "On Coordination Games with Quantum Correlations," forthcoming in *International Journal of Game Theory*, 2007. Available via Online First at www.springerlink.com/content/101791.
- [13] Kochen, S., and E. Specker, "The Problem of Hidden Variables in Quantum Mechanics," *Journal of Mathematics and Mechanics*, 17, 1967, 59-87.
- [14] Lambert-Mogiliansky, A., S. Zamir, and H. Zwirn, "Type Indeterminacy: A Model for the KT (Kahneman-Tversky)-Man," 2003. Available at www.ma.huji.ac.il/~zamir.
- [15] La Mura, P., "Correlated Equilibria of Classical Strategic Games with Quantum Signals," *International Journal of Quantum Information*, 3, 2005, 183-188.

- [16] Landsburg, S., “Nash Equilibria in Quantum Games,” 2005. Available at www.landsburg.com.
- [17] Levine, D., “Quantum Games Have No News for Economists,” 2005. Available at www.dklevine.com.
- [18] Meyer, D., “Quantum Strategies,” *Physical Review Letters*, 82, 1999, 1052-1055.
- [19] Peres, A., “Two Simple Proofs of the Kochen-Specker Theorem,” *Journal of Physics A*, 24, 1991, L175-L178.
- [20] Shimony, A., “Events and Processes in the Quantum World,” in Penrose, R., and C. Isham (eds.), *Quantum Concepts in Space and Time*, Oxford University Press, 1986, 182-203.
- [21] Temzelides, T., “An Uncertainty Principle for Social Science Experiments,” 2005. Available at www.pitt.edu/~tedt.
- [22] Von Neumann, J., “Zur Theorie der Gesellschaftspiele,” *Mathematische Annalen*, 100, 1928, 295-320.
- [23] Von Neumann, J., *Mathematische Grundlagen der Quantenmechanik*, Springer-Verlag, 1932. (Translated as *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.)
- [24] Von Neumann, J., and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944 (60th Anniversary Edition, 2004).