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# CONDORCET JURY THEOREM: THE DEPENDENT CASE

by

**BEZALEL PELEG and SHMUEL ZAMIR**

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מרכז לחקר הרציונליות

**CENTER FOR THE STUDY OF RATIONALITY**

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

# Condorcet Jury Theorem: The dependent case

Bezalel Peleg and Shmuel Zamir
Center for the Study of Rationality, Hebrew University

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**Abstract**

We provide an extension of the Condorcet Theorem. Our model includes both the Nitzan-Paroush framework of "unequal competencies" and Ladha's model of "correlated voting by the jurors". We assume that the jurors behave "informatively", that is, they do not make a strategic use of their information in voting. Formally, we consider a sequence of binary random variables $X = (X_1, X_2, ..., X_n, ...)$ with range in $\{0, 1\}$ and a joint probability distribution $P$. The pair $(X, P)$ is said to satisfy the *Condorcet Jury Theorem* ($CJT$) if $\lim_{n\to\infty} P\left(\Sigma_{i=1}^n X_i > \frac{n}{2}\right) = 1$. For a general (dependent) distribution $P$ we provide necessary as well as sufficient conditions for the $CJT$. Let $p_i = E(X_i)$, $\overline{p}_n = (p_1 + p_2, ... + p_n)/n$ and $\overline{X}_n = (X_1 + X_2, ... + X_n)/n$. A consequence of our results is that the $CJT$ is satisfied if $\lim_{n\to\infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$ and $\Sigma_{i=1}^n \Sigma_{j\neq i} Cov(X_i, X_j) \leq 0$ for $n > N_0$. The importance of this result is that it establishes the validity of the $CJT$ for a domain which strictly (and naturally) includes the domain of independent jurors. Given $(X, P)$, let $\underline{p} = \liminf_{n\to\infty} \overline{p}_n$, and $\overline{p} = \limsup_{n\to\infty} \overline{p}_n$. Let $\underline{y} = \liminf_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2$, $\underline{y}^* = \liminf_{n\to\infty} E|\overline{X}_n - \overline{p}_n|$ and $\overline{y}^* = \limsup_{n\to\infty} E|\overline{X}_n - \overline{p}_n|$. Necessary conditions for the $CJT$ are that $\underline{p} \geq \frac{1}{2} + \frac{1}{2}\underline{y}^*$, $\underline{p} \geq \frac{1}{2} + \underline{y}$, and also $\overline{p} \geq \frac{1}{2} + \frac{1}{2}\overline{y}^*$. We exhibit a large family of distributions $P$ with $\liminf_{n\to\infty} \frac{1}{n(n-1)} \Sigma_{i=1}^n \Sigma_{j\neq i} Cov(X_i, X_j) > 0$ which satisfy the $CJT$. We do that by 'interlacing' carefully selected pairs $(X, P)$ and $(X', P')$. We then proceed to project the distributions $P$ on the planes $(\underline{p}, \underline{y}^*)$ and $(\underline{p}, \underline{y})$, and determine all feasible points in each of these planes. Quite surprisingly, many important results on the possibility of the $CJT$ are obtained by analyzing various regions of the feasible set in these planes.

# Introduction

The simplest way to present our problem is by quoting Condorcet's classic result (see Young(1997)):

**Theorem 1.** *(CJT–Condorcet 1785) Let n voters (n odd) choose between two alternatives that have equal likelihood of being correct a priori. Assume that voters make their judgements independently and that each has the same probability p of being correct* ($\frac{1}{2} < p < 1$)*. Then, the probability that the group makes the correct judgement using simple majority rule is*

$$\sum_{h=(n+1)/2}^{n} [n!/h!(n-h)!]p^h(1-p)^{n-h}$$

*which approaches* 1 *as n becomes large.*

We generalize Condorcet's model by presenting it as a *game with incomplete information* in the following way: Let $I = \{1, 2, \ldots, n\}$ be a set of jurors and let $D$ be the defendant. There are two *states of nature*: $g$ – in which $D$ is guilty and $z$ – in which $D$ is innocent. Thus the set of states of nature is $S = \{g, z\}$. Each juror has an action set $A$ with two actions: $A = \{c, a\}$. The action $c$ is to *convict* $D$. The action $a$ is to *acquit* $D$. Before the voting, each juror $i$ gets a private random signal $t^i \in T^i := \{t^i_g, t^i_z\}$. In the terminology of games with incomplete information, $T^i$ is the *type set* of juror $i$. The interpretation is that juror $i$ of type $t^i_g$ thinks that $D$ is guilty while juror $i$ of type $t^i_z$ thinks that $D$ is innocent. The signals of the jurors may be dependent and may also depend on the the state of nature. In our model the jurors act "*informatively*" (not "strategically") that is, the strategy of juror $i$ is $\sigma^i : T^i \to A$ given by $\sigma^i(t^i_g) = c$ and $\sigma^i(t^i_z) = a$. The definition of informative voting is due to Austen-Smith and Banks (1996) who question the validity of the CJT in a strategic framework. Informative voting was, and is still, assumed in the vast majority of the literature on the $CJT$, mainly because it is implied by the original Condorcet assumptions. More precisely assume, as Condorcet did, that $P(g) = P(z) = 1/2$ and that each juror is more likely to receive the 'correct' signal (that is, $P(t^i_g|g) = P(t^i_z|z) = p > 1/2$), then the strategy of voting informatively maximizes the probability of voting correctly, among all four pure voting strategies. Following Austen-Smith and Banks, strategic voting and Nash Equilibrium were studied by Wit (1998), Myerson (1998) and recently by Laslier and Weibull (2008) who discuss the assumption on preferences and beliefs under which sincere voting is a Nash equilibrium in a general deterministic majoritarian voting rule. As we said before, in this work we do assume informative voting and leave strategic consideration and equilibrium concepts to the next phase of our research. The action taken by a finite society of jurors $\{1, \ldots, n\}$ (i.e. the jury verdict) is determined by a simple majority (with some tie breaking rule e.g. by coin tossing). We are interested in the probability that the (finite) jury will reach the correct decision. Again in the style of games with incomplete information let $\Omega_n = S \times T^1 \times, \ldots, \times T^n$ be the set of *states of the world*. A state of the world consists of the state of nature and a list of the types of all jurors. Denote by $p^{(n)}$ the probability distribution on $\Omega_n$. This is a joint probability distribution on the state of nature and the signals of the jurors. For each juror $i$ let the random variable $X_i : S \times T^i \to \{0,1\}$ be the indicator of his correct voting i.e. $X_i(g, t^i_g) = X_i(z, t^i_z) = 1$ and $X_i(g, t^i_z) = X_i(z, t^i_g) = 0$. The probability distribution $p^{(n)}$ on $\Omega_n$ induces a joint probability distribution on the the vector $X = (X_1, \ldots, X_n)$ which we denote also by $p^{(n)}$. If $n$ is odd, then the probability that the jury reaches a correct decision is

$$p^{(n)}\left(\sum_{i=1}^{n} X_i > \frac{n}{2}\right)$$

.

Figure 1 illustrates our construction in the case $n = 2$. In this example, according to $p^{(2)}$ the state of nature is chosen with unequal probabilities for the two states: $P(g) = 1/4$ and $P(z) = 3/4$ and then the types of the two jurors are chosen according to a joint probability distribution which depends on the state of nature.

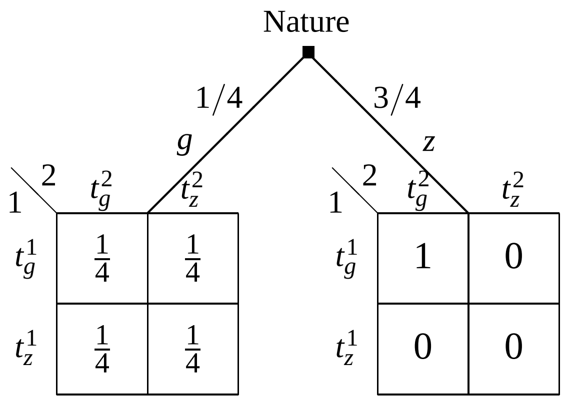


**Figure 1** The probability distribution $p^{(2)}$.

Guided by Condorcet, we are looking for limits theorems as the the size of the jury increases. Formally, as $n$ goes to infinity we obtain the sequence of increasing sequence of 'worlds', $(\Omega_n)_{n=1}^{\infty}$, such that for all $n$, the projection of $\Omega_{n+1}$ on $\Omega_n$ is the whole $\Omega_n$. The corresponding sequence of probability distributions is $(p^{(n)})_{n=1}^{\infty}$ and we assume that for every $n$, the marginal distribution of $p^{(n+1)}$ on $\Omega_n$ is $p^{(n)}$. It follows from the Kolmogorov extension theorem (see Loeve (1963), p. 93) that this defines a unique probability measure $P$ on the (projective, or *inverse*) limit

$$\Omega = \lim_{\infty \leftarrow n} \Omega_n = S \times T^1 \times \ldots \times T^n \ldots$$

such, for all $n$, that the marginal distribution of $P$ on $\Omega_n$ is $p^{(n)}$.

In this paper we address the the following problem: Which probability measures $P$ derived in this manner satisfy the *Condorcet Jury Theorem* ($CJT$) that is, Which probability measures $P$ satisfy

$$\lim_{n\to\infty} P\left(\Sigma_{i=1}^{n} X_i > \frac{n}{2}\right) = 1.$$

As far as we know, the only existing result on this general problem is that of Berend and Paroush (1998) which deals only with independent jurors.

Rather than working with the space $\Omega$ and its probability measure $P$, it will be more convenient to work with the infinite sequence of binary random variables $X = (X_1, X_2, ..., X_n, ...)$ (the indicators of 'correct voting') and the induced probability measure on it, which we will denote also by $P$. Since the pair $(X,P)$ is uniquely determined by $(\Omega,P)$ , in considering all pairs $(X,P)$ we cover all pairs $(\Omega,P)$. A secondary advantage of working with $(X,P)$ is that our results can be interpreted also as forms of laws of large numbers for *general* infinite sequences of binary random variables.

# 1 Sufficient conditions

Let $X = (X_1, X_2, ..., X_n, ...)$ be a sequence of binary random variables with range in $\{0,1\}$ and with joint probability distribution $P$. The sequence $X$ is said to satisfy the *Condorcet Jury Theorem* ($CJT$) if

$$\lim_{n\to\infty} P\left(\Sigma_{i=1}^{n} X_i > \frac{n}{2}\right) = 1 \tag{1}$$

We shall investigate necessary as well as sufficient conditions for $CJT$.

Given a sequence of random binary variables $X = (X_1, X_2, ..., X_n, ...)$ with joint distribution $P$ denote $p_i = E(X_i)$, $Var(X_i) = E(X_i - p_i)^2$ and $Cov(X_i, X_j) = E[(X_i - p_i)(X_j - p_j)]$, for $i \neq j$, where $E$ denotes, as usual, the expectation operator. Also let $\overline{p}_n = (p_1 + p_2, ... + p_n)/n$ and $\overline{X}_n = (X_1 + X_2, ... + X_n)/n$.

Our first result provides a sufficient condition for $CJT$:

**Theorem 2.** *Assume that:* $\Sigma_{i=1}^{n} p_i > \frac{n}{2}$ *for all* $n > N_0$ *and*

$$\lim_{n\to\infty} \frac{E(\overline{X}_n - \overline{p}_n)^2}{(\overline{p}_n - \frac{1}{2})^2} = 0, \tag{2}$$

*or equivalently assume that:*

$$\lim_{n\to\infty} \frac{\overline{p}_n - \frac{1}{2}}{\sqrt{E(\overline{X}_n - \overline{p}_n)^2}} = \infty, \tag{3}$$

*then the* $CJT$ *is satisfied.*

*Proof.*

$$\begin{aligned} P\left(\Sigma_{i=1}^n X_i \le \frac{n}{2}\right) &= P\left(-\Sigma_{i=1}^n X_i \ge -\frac{n}{2}\right) \\ &= P\left(\Sigma_{i=1}^n p_i - \Sigma_{i=1}^n X_i \ge \Sigma_{i=1}^n p_i - \frac{n}{2}\right) \\ &\le P\left(|\Sigma_{i=1}^n p_i - \Sigma_{i=1}^n X_i| \ge \Sigma_{i=1}^n p_i - \frac{n}{2}\right) \end{aligned}$$

By Chebyshev's inequality (assuming $\Sigma_{i=1}^n p_i > \frac{n}{2}$) we have

$$P\left(|\Sigma_{i=1}^n p_i - \Sigma_{i=1}^n X_i| \ge \Sigma_{i=1}^n p_i - \frac{n}{2}\right) \le \frac{E\left(\Sigma_{i=1}^n X_i - \Sigma_{i=1}^n p_i\right)^2}{\left(\Sigma_{i=1}^n p_i - \frac{n}{2}\right)^2} = \frac{E(\overline{X}_n - \overline{p}_n)^2}{(\overline{p}_n - \frac{1}{2})^2}$$

As this last term tends to zero by (2), the $CJT$ (1) then follows. □

**Corollary 3.** *If $\Sigma_{i=1}^n \Sigma_{j\neq i} Cov(X_i, X_j) \le 0$ for $n > N_0$ (in particular if $Cov(X_i, X_j) \le 0$ for all $i \neq j$) and $\lim_{n\to\infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$ then the $CJT$ is satisfied.*

*Proof.* Since the variance of a binary random variable $X$ with mean $p$ is $p(1-p) \le 1/4$ we have for $n > N_0$,

$$\begin{aligned} 0 \le E(\overline{X}_n - \overline{p}_n)^2 &= \frac{1}{n^2} E\left(\Sigma_{i=1}^n (X_i - p_i)\right)^2 \\ &= \frac{1}{n^2}\left(\Sigma_{i=1}^n Var(X_i) + \Sigma_{i=1}^n \Sigma_{j\neq i} Cov(X_i, X_j)\right) \le \frac{1}{4n} \end{aligned}$$

Therefore if $\lim_{n\to\infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$, then

$$0 \le \lim_{n\to\infty} \frac{E(\overline{X}_n - \overline{p}_n)^2}{(\overline{p}_n - \frac{1}{2})^2} \le \lim_{n\to\infty} \frac{1}{4n(\overline{p}_n - \frac{1}{2})^2} = 0$$

□

**Remark 4.** *When $X_1, X_2, \ldots, X_n, \ldots$ are independent then, under mild conditions, $\lim_{n\to\infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty$ is a necessary and sufficient condition for $CJT$ (see D.Berend and J. Paroush (1998)).*

Given a sequence $X = (X_1, X_2, \ldots, X_n, \ldots)$ of binary random variables with a joint probability distribution $P$, we define the following parameters of $(X, P)$:

$$\underline{p} := \liminf_{n\to\infty} \overline{p}_n \tag{4}$$

$$\overline{p} := \limsup_{n\to\infty} \overline{p}_n \tag{5}$$

$$\underline{y} := \liminf_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2 \tag{6}$$

$$\overline{y} := \limsup_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2 \tag{7}$$

$$\underline{y}^* := \liminf_{n\to\infty} E|\overline{X}_n - \overline{p}_n| \tag{8}$$

$$\overline{y}^* := \limsup_{n\to\infty} E|\overline{X}_n - \overline{p}_n| \tag{9}$$

We first observe the following:

**Remark 5.** *If $\underline{p} > 1/2$ and $\overline{y} = 0$ then the $CJT$ is satisfied.*

*Proof.* As $E(\overline{X}_n - \overline{p}_n)^2 \geq 0$, if $\overline{y} = 0$ then $\lim_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2 = 0$. Since $\underline{p} > 1/2$, there exists $n_0$ such that $\overline{p}_n > (1/2 + \underline{p})/2$ for all $n > n_0$. The result then follows by Theorem 2. □

## 2 Necessary conditions using the $L_1$-norm

Given a sequence $X = (X_1, X_2, ..., X_n, ...)$ of binary random variables with a joint probability distribution $P$, if $\overline{y} > 0$, then we cannot use Theorem 2 to conclude $CJT$.

To derive necessary conditions for the $CJT$, we first have:

**Proposition 6.** *If the $CJT$ holds then $\underline{p} \geq \frac{1}{2}$.*

*Proof.* Define a sequence of events $(B_n)_{n=1}^{\infty}$ by $B_n = \{\omega | \overline{X}_n(\omega) - 1/2 \geq 0\}$. Since the $CJT$ holds, $\lim_{n\to\infty} P\left(\Sigma_{i=1}^n X_i > \frac{n}{2}\right) = 1$ and hence $\lim_{n\to\infty} P(B_n) = 1$. Since

$$\overline{p}_n - \frac{1}{2} = E(\overline{X}_n - \frac{1}{2})) \geq -\frac{1}{2} P(\Omega \backslash B_n),$$

taking the lim inf, the right hand side tends to zero and we obtain:
$\liminf_{n\to\infty} p_n = \underline{p} \geq \frac{1}{2}$. □

We shall first consider a stronger violation of Theorem 2 than $\overline{y} > 0$ namely assume that $\underline{y} > 0$. We shall prove that in this case, there is a range of distributions $P$ for which the $CJT$ is false.

First we notice that for $-1 \leq x \leq 1$, $|x| \geq x^2$. Hence $E|\overline{X}_n - \overline{p}_n| \geq E(\overline{X}_n - \overline{p}_n)^2$ for all $n$ and thus $\underline{y} > 0$ implies $\underline{y}^* > 0$

We are now ready to state our first impossibility theorem which can readily translated into a necessary condition.

**Theorem 7.** *Given a sequence $X = (X_1, X_2, ..., X_n, ...)$ of binary random variables with joint probability distribution P. If $\underline{p} < \frac{1}{2} + \frac{\underline{y}^*}{2}$, then the $(X, P)$ violates the $CJT$.*

*Proof.* If $\underline{y}^* = 0$, then the $CJT$ is violated by Proposition 6. Assume then that $\underline{y}^* > 0$ and choose $\tilde{y}$ such that $0 < \tilde{y} < \underline{y}^*$ and $2t := \frac{\tilde{y}}{2} + \frac{1}{2} - \underline{p} > 0$. First we notice that, since $E(\overline{X}_n - \overline{p}_n) = 0$, we have $E \max(0, \overline{p}_n - \overline{X}_n) = E \max(0, \overline{X}_n - \overline{p}_n)$, thus since $\underline{y}^* > 0$, we have

$$E \max(0, \overline{p}_n - \overline{X}_n) > \frac{\tilde{y}}{2} \;\text{ for }\; n > \overline{n}. \tag{10}$$

If $(\Omega, P)$ is the probability space on which the sequence $X$ is defined, for $n > \overline{n}$ define the events

$$B_n = \{\omega | \overline{p}_n - \overline{X}_n(\omega) \geq \max(0, \frac{\tilde{y}}{2} - t)\} \tag{11}$$

By (10) and (11), $P(B_n) > q > 0$ for some $q$ and

$$\overline{p}_n - \overline{X}_n(\omega) \geq \frac{\tilde{y}}{2} - t \text{ for } \omega \in B_n,\ n > \overline{n}. \tag{12}$$

Choose now a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$\overline{p}_{n_k} < \frac{\tilde{y}}{2} + \frac{1}{2} - t = \underline{p} + t,\ \ k = 1, 2, \ldots \tag{13}$$

By (12) and (13), for all $\omega \in B_{n_k}$ we have,

$$\overline{X}_{n_k}(\omega) \leq \overline{p}_{n_k} - \frac{\tilde{y}}{2} + t < \frac{1}{2},$$

and thus $P(\overline{X}_{n_k} > \frac{1}{2}) \leq 1 - q < 1$ which implies that $P$ violates the $CJT$. $\square$

**Corollary 8.** *If* $\liminf_{n\to\infty} \overline{p}_n \leq \frac{1}{2}$ *and* $\liminf_{n\to\infty} E|\overline{X}_n - \overline{p}_n| > 0$, *then* $P$ *violates the* $CJT$.

# 3 Necessary conditions using the $L_2$-norm

Let $X = (X_1, X_2, \ldots, X_n, \ldots)$ be a sequence of binary random variables with a joint probability distribution $P$. In this section we take a closer look at the relationship between the parameters $\underline{y}$ and $\underline{y}^*$(see (7) and (9)). We first notice that $\underline{y} > 0$ if and only if $\underline{y}^* > 0$. Next we notice that $\overline{p}_n \geq \frac{1}{2}$ for $n > \overline{n}$ implies that $\overline{X}_n - \overline{p}_n \leq \frac{1}{2}$ for $n > \overline{n}$. Thus, by corollary 8, if $\underline{y} > 0$ and the $CJT$ is satisfied then $\max(0, \overline{X}_n - \overline{p}_n) \leq \frac{1}{2}$ for $n > \overline{n}$. Finally we observe the following Lemma, the proof of which is straightforward:

**Lemma 9.** *If* $\liminf_{n\to\infty} P\{\omega | \overline{p}_n - \overline{X}_n(\omega) \geq \overline{p}_n/2\} > 0$ *then the* $CJT$ *is violated.*

We now use the previous discussion to prove the following theorem:

**Theorem 10.** *If (i)* $\liminf_{n\to\infty} \overline{p}_n > \frac{1}{2}$ *and*
*(ii)* $\liminf_{n\to\infty} P(\overline{X}_n > \overline{p}_n/2) = 1$, *then* $\underline{y}^* \geq 2\underline{y}$.

*Proof.* As we have observed, $(i)$ implies that $\max(0, \overline{X}_n - \overline{p}_n) \leq \frac{1}{2}$. Also $(ii)$ implies that $\lim_{n\to\infty} P(\overline{p}_n - \overline{X}_n \leq \frac{1}{2}) = 1$, thus

$$\lim_{n\to\infty} P(-\frac{1}{2} \leq \overline{X}_n - \overline{p}_n \leq \frac{1}{2}) = 1 \tag{14}$$

Define the events $B_n = \{\omega | -\frac{1}{2} \leq \overline{X}_n(\omega) - \overline{p}_n \leq \frac{1}{2}\}$ then by (14)

$$\liminf_{n\to\infty} \int_{B_n} (\overline{X}_n - \overline{p}_n)^2 dP = \underline{y} \tag{15}$$

and

$$\liminf_{n\to\infty} \int_{B_n} |\overline{X}_n - \overline{p}_n| dP = \underline{y}^*. \tag{16}$$

Since any $u \in [-\frac{1}{2}, \frac{1}{2}]$ satisfies $|u| \geq 2u^2$, it follows from (15) and (16) that $\underline{y}^* \geq 2\underline{y}$. $\square$

**Corollary 11.** *Let $\underline{p} = \liminf_{n\to\infty} \overline{p}_n$ and $\underline{y} = \liminf_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2$. Then if $\underline{p} < \frac{1}{2} + \underline{y}$ then $P$ does not satisfy the $CJT$.*

*Proof.* Assume $\underline{p} < \frac{1}{2} + y$. If $\underline{y} = 0$, then $CJT$ is not satisfied by Proposition 6 Hence we may thus assume that $\underline{y} > 0$ which also implies that $\underline{y}^* > 0$. Thus, if $\liminf_{n\to\infty} \overline{p}_n \leq \frac{1}{2}$ then $CJT$ fails by Corollary 8. Assume then that $\liminf_{n\to\infty} \overline{p}_n > \frac{1}{2}$. By Lemma 9 we may also assume that $\liminf_{n\to\infty} P\left(\overline{X}_n > \overline{p}_n/2\right) = 1$ and thus by Theorem 10 we have $\underline{y}^* \geq 2\underline{y}$ and hence $\underline{p} < \frac{1}{2} + \underline{y} \leq \frac{1}{2} + \frac{\underline{y}^*}{2}$ and the $CJT$ fails by Theorem 7. $\square$

# 4 Dual Conditions

A careful reading of sections (2) and (3) reveals that it is possible to obtain "dual" results to Theorems 7 and 10 and Corollary 11 by replacing "$\liminf$" by "$\limsup$". More precisely for a sequence $X = (X_1, X_2, \ldots, X_n, \ldots)$ of binary random variables with joint probability distribution $P$, we let $\overline{p} = \limsup_{n\to\infty} \overline{p}_n$ and $\overline{y}^* = \limsup_{n\to\infty} E|\overline{X}_n - \overline{p}_n|$, and we have:

**Theorem 12.** *If $\overline{p} < \frac{1}{2} + \frac{\overline{y}^*}{2}$, then the $(X,P)$ violates the $CJT$.*

*Proof.* As we saw in the proof of Corollary 11, we may assume that $\liminf_{n\to\infty} \overline{p}_n \geq \frac{1}{2}$ and hence also

$$\overline{p} = \limsup_{n\to\infty} \overline{p}_n \geq \liminf_{n\to\infty} \overline{p}_n \geq \frac{1}{2},$$

and hence $\overline{y}^* > 0$. Choose $\tilde{y}$ such that $0 < \tilde{y} < \overline{y}^*$ and $2t = \frac{\tilde{y}}{2} + \frac{1}{2} - \overline{p} > 0$. Let $(X_{n_k})_{k=1}^{\infty}$ be a subsequence of $X$ such that

$$\lim_{k\to\infty} E|\overline{X}_{n_k} - \overline{p}_{n_k}| = \overline{y}^*.$$

As in (10) we get

$$E\max(0, \overline{p}_{n_k} - \overline{X}_{n_k}) > \frac{\tilde{y}}{2} \;\text{ for }\; k > \overline{k}. \tag{17}$$

Define the events $(B_{n_k})_{k=1}^{\infty}$ by

$$B_{n_k} = \{\omega | \overline{p}_{n_k} - \overline{X}_{n_k}(\omega) \geq \frac{\tilde{y}}{2} - t\}. \tag{18}$$

By (17) and (18), $P(B_{n_k}) > q$ for some $q > 0$ and

$$\overline{p}_{n_k} - \overline{X}_{n_k}(\omega) \geq \frac{\tilde{y}}{2} - t \;\text{ for }\; \omega \in B_{n_k} \;\text{ and }\; k > \overline{\overline{k}}. \tag{19}$$

Now

$$\limsup_{n\to\infty} \overline{p}_n = \overline{p} < \frac{\tilde{y}}{2} + \frac{1}{2} - t. \tag{20}$$

Thus, for $n$ sufficiently large $\overline{p}_n < \frac{\tilde{y}}{2} + \frac{1}{2} - t$. Hence, for $k$ sufficiently large and all $\omega \in B_{n_k}$,

$$\overline{X}_{n_k}(\omega) \leq \overline{p}_{n_k} - \frac{\tilde{y}}{2} + t < \frac{1}{2}. \tag{21}$$

Therefore $P(\overline{X}_{n_k} > \frac{1}{2}) \leq 1 - q < 1$ for sufficiently large $k$ in violation of the $CJT$. $\square$

Similarly we have the ”dual” results to those of Theorem 10 and Corollary 11:

**Theorem 13.** *If (i)* $\liminf_{n\to\infty} \overline{p}_n > \frac{1}{2}$ *and*
*(ii)* $\liminf_{n\to\infty} P(\overline{X}_n > \overline{p}_n/2) = 1$, *then* $\overline{y}^* \geq 2\overline{y}$.

**Corollary 14.** *If* $\overline{p} < \frac{1}{2} + \overline{y}$ *then P does not satisfy the CJT.*

The proofs which are similar respectively to the proofs of Theorem 10 and Corollary 11 are omitted.

# 5 Existence of distributions satisfying the *CJT*

In this section we address the issue of the existence of distributions that satisfy the *CJT*. In particular we shall exhibit a rather large family of distributions $P$ with $\underline{y} > 0$ (and $\underline{p} > 1/2$) for which the *CJT* holds. Our main result is the following:

**Theorem 15.** *Let* $t \in [0, \frac{1}{2}]$. *If F is a distribution with parameters* $(\underline{p}, \underline{y}^*)$, *then there exists a distribution H with parameters* $\underline{\tilde{p}} = 1 - t + t\underline{p}$ *and* $\underline{\tilde{y}}^* = t\underline{y}^*$ *that satisfy the CJT.*

*Proof.* To illustrate the idea of the proof we first prove (somewhat informally) the case $t = 1/2$. Let $X = (X_1, X_2, ..., X_n, ...)$ be a sequence of binary random variables with a joint probability distribution $F$. Let $G$ be the distribution of the sequence $Y = (Y_1, Y_2, ..., Y_n, ...)$, where $EY_n = 1$ for all $n$ (that is, $Y_1 = Y_2 = ...Y_n = ...$ and $P(Y_i = 1) = 1\ \forall i$). Consider now the following “interlacing” of the two sequences $X$ and $Y$:

$$Z = (Y_1, Y_2, X_1, Y_3, X_2, Y_4, X_3, ..., Y_n, X_{n-1}, Y_{n+1}, X_n ...),$$

and let the probability distribution $H$ of $Z$ be the product distribution $H = F \times G$. It is verified by straightforward computation that the parameters of the distribution $H$ are in accordance with the theorem for $t = \frac{1}{2}$ namely, $\underline{\tilde{p}} = \frac{1}{2} + \frac{1}{2}\underline{p}$ and $\underline{\tilde{y}}^* = \frac{1}{2}\underline{y}^*$. Finally, as each initial segment of voters in $Z$ contains a majority of $Y_i$’s (thus with all values 1), the distribution $H$ satisfies the *CJT*, completing the proof for $t = \frac{1}{2}$.

The proof for a general $t \in [0, 1/2)$ follows the same lines: We construct the sequence $Z$ so that any finite initial segment of $n$ variables, includes “about, but not more” than the initial $tn$ segment of the $X$ sequence, and the rest is filled with the constant $Y_i$ variables. This will imply that the *CJT* is satisfied.

Formally, for any real $x \geq 0$ let $\lfloor x \rfloor$ be the largest integer smaller or equal to $x$ and let $\lceil x \rceil$ be smallest integer larger or equal to $x$. Note that for any $n$ and any $0 \leq t \leq 1$ we have $\lfloor tn \rfloor + \lceil (1-t)n \rceil = n$ thus, one and only one of the following holds:

**(i)** $\lfloor tn \rfloor < \lfloor t(n+1) \rfloor$ or

**(ii)** $\lceil (1-t)n \rceil < \lceil (1-t)(n+1) \rceil$

From the given sequence $X$ and the above defined sequence $Y$ (of constant 1 variables) we define now the sequence $Z = (Z_1, Z_2, \ldots, Z_n, \ldots)$ as follows: $Z_1 = Y_1$ and for any $n \geq 2$, let $Z_n = X_{\lfloor t(n+1) \rfloor}$ if (i) holds and $Z_n = Y_{\lceil (1-t)(n+1) \rceil}$ if (ii) holds. This inductive construction guarantees that for all $n$, the sequence contains $\lfloor tn \rfloor$ $X_i$ coordinates and $\lceil (1-t)n \rceil$ $Y_i$ coordinates. The probability distribution $H$ is the product distribution $F \times G$. The fact that $(Z, H)$ satisfies the $CJT$ follows from:

$$\lceil (1-t)n \rceil \geq (1-t)n > tn \geq \lfloor tn \rfloor,$$

and finally $\underline{\tilde{p}} = 1 - t + t\underline{p}$ and $\underline{\tilde{y}}^* = t\underline{y}^*$ is verified by straightforward computation. $\square$

**Remark 16.** *The "interlacing" of the two sequences $X$ and $Y$ described in the proof of Theorem 15 may be defined for any $t \in [0,1]$. We were specifically interested in $t \in [0,1/2]$ since this guarantees the $CJT$.*

# 6 Feasibility considerations

The conditions developed so far for a sequence $X = (X_1, X_2, \ldots, X_n, \ldots)$ with joint probability distribution $P$ to satisfy the $CJT$ involved only the parameters $\underline{p}, \overline{p}, \underline{y}, \overline{y}, \underline{y}^*$ and $\overline{y}^*$. In this section we pursue our characterization in the space of these parameters. We shall look at the distributions in two different spaces: The space of points $(\underline{p}, \underline{y}^*)$, which we call the $L_1$ space, and the space $(\underline{p}, \underline{y})$, which we call the $L_2$ space.

## 6.1 Feasibility and characterization in $L_1$

With the pair $(X, P)$ we associate the point $(\underline{p}, \underline{y}^*)$ in the Euclidian plane $\mathbb{R}^2$. It follows immediately that $0 \leq \underline{p} \leq 1$. We claim that $\underline{y}^* \leq 2\underline{p}(1-\underline{p})$ holds for all distributions $P$. To see that we first observe that $E|X_i - p_i| = 2p_i(1-p_i)$ hence

$$E|\overline{X}_n - \overline{p}_n| = \frac{1}{n} E|\sum_{i=1}^{n} (X_i - p_i)| \leq \frac{1}{n} E\left(\sum_{i=1}^{n} |X_i - p_i|\right) \leq \frac{2}{n} \sum_{i=1}^{n} p_i(1-p_i).$$

The function $\sum_{i=1}^{n} p_i(1-p_i)$ is (strictly) concave hence:

$$E|\overline{X}_n - \overline{p}_n| \leq 2 \sum_{i=1}^{n} \frac{1}{n} p_i(1-p_i) \leq 2\overline{p}_n(1-\overline{p}_n). \tag{22}$$

Finally let $\underline{p} = \lim_{k\to\infty} \overline{p}_{n_k}$, then

$$\underline{y}^* = \liminf_{n\to\infty} E|\overline{X}_n - \overline{p}_n| \leq \liminf_{k\to\infty} E|\overline{X}_{n_k} - \overline{p}_{n_k}| \leq 2 \lim_{k\to\infty} \overline{p}_{n_k}(1-\overline{p}_{n_k}) = 2\underline{p}(1-\underline{p}).$$

The second inequality is due to (22).

Thus, if $(u,w)$ denote a point in $\mathbb{R}^2$ then any feasible pair $(\underline{p},\underline{y}^*)$ is in the region

$$FE_1 = \{(u,w) | 0 \le u \le 1, \;\; 0 \le w \le 2u(1-u)\} \tag{23}$$

We shall now prove that all points in this region are feasible that is, any point in $FE_1$ is attainable as a pair $(\underline{p},\underline{y}^*)$ of some distribution $P$. Then we shall indicate the sub-region of $FE_1$ where the $CJT$ may hold. We first observe that any point $(u_0,w_0) \in FE_1$ on the parabola $w = 2u(1-u)$, for $0 \le u \le 1$, is feasible. In fact such $(u_0,w_0)$ is attainable by the sequence $X = (X_1,X_2,\ldots,X_n,\ldots)$ with identical variables $X_i$, $X_1 = X_2 = \ldots = X_n \ldots$ and $EX_1 = u_0$ (clearly $\underline{p} = u_0$, and $\underline{y}^* = 2u_0(1-u_0)$ follows from the dependence and from $E|X_i - p_i| = 2p_i(1-p_i) = 2u_0(1-u_0)$).

Let again $(u_0,w_0)$ be a point on the parabola which is thus attainable. Assume that they are the parameters $(\underline{p},\underline{y}^*)$ of the pair $(X,F)$. Let $(Y,G)$ be the pair (of constant variables) described in the proof of Theorem 15 and let $t \in [0,1]$. By Remark 16 the $t$-interlacing of $(X,F)$ and $(Y,G)$ can be constructed to yield a distribution with parameters $\underline{\tilde{p}} = t\underline{p} + (1-t)$ and $\underline{\tilde{y}}^* = t\underline{y}^*$ (see the proof of Theorem 15). Thus, the line segment defined by $\tilde{u} = tu_0 + (1-t)$ and $\tilde{w} = tw_0$ for $0 \le t \le 1$, connecting $(u_0,w_0)$ to $(1,0)$ consists of attainable pairs contained in $FE$. Since any point $(u,w)$ in $FE$ lies on such a line segment, we conclude that *every point in $FE$ is attainable*. We shall refer to $FE$ as *the feasible set* which is shown in Figure 2.

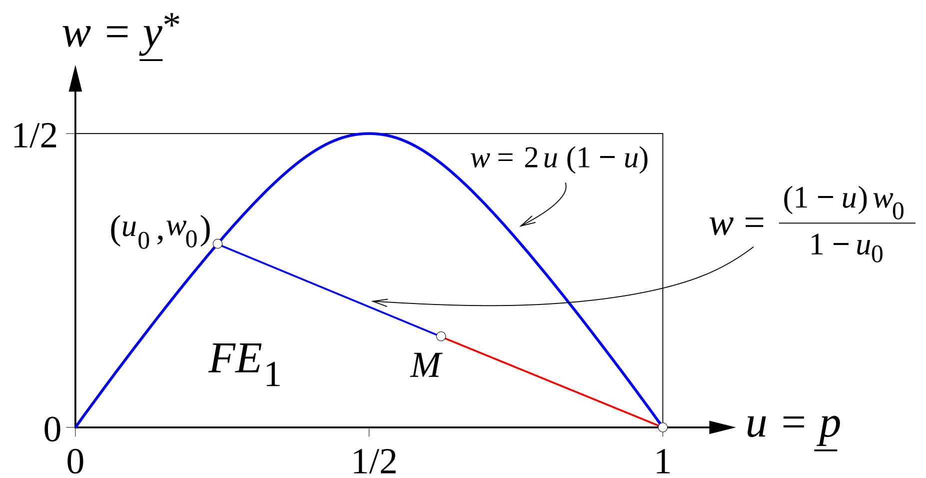


**Figure 2** The feasible set $FE_1$.

We now attempt to characterize the points of the feasible set according to whether the $CJT$ is satisfied or not. For that we first define:

**Definition 17.**

- *The* strong $CJT$ *set, denoted by sCJT is the set of all points* $(u,w) \in FE_1$ *such that any pair* $(X,P)$ *with parameters* $\underline{p} = u$ *and* $\underline{y}^* = w$ *satisfies the CJT.*

- *The* weak $CJT$ *set, denoted by* $wCJT$ *is the set of all points* $(u,w) \in FE_1$ *for which there exists a pair* $(X,P)$ *with parameters* $\underline{p} = u$ *and* $\underline{y}^* = w$ *which satisfies the* $CJT$.

*We denote* $-sCJT = FE_1 \backslash sCJT$ *and* $-wCJT = FE_1 \backslash wCJT$.

For example $(1,0) \in sCJT$ and $(1/2,0) \in wCJT$.

By Theorem 7, if $u < 1/2 + 1/2w$, then $(u,w) \in -wCJT$. Next we observe that if $(u_0,w_0)$ is on the parabola $w = 2u(1-u)$ and $M$ is the midpoint of the segment $[(u_0,w_0),(1,0)]$ then, by the proof of Theorem 15 (adapted for $L_1$, replacing $\underline{y}$ by $\underline{y}^*$), the segment $[M,(1,0)] \subseteq wCJT$ (see Figure 2). To find the upper boundary of the union of all these segments that is, the locus of the mid points $M$ in Figure 2, we eliminate $(u_0,w_0)$ from the equations $w_0 = 2u_0(1-u_0)$, and $(u,w) = 1/2(u_0,w_0) + 1/2(1,0)$ and obtain

$$w = 2(2u-1)(1-u) \tag{24}$$

This is a parabola with maximum $1/4$ at $u = 3/4$. The slope of the tangent at $u = 1/2$ is 2 that is, the tangent of the parabola at that point is the line $w = 2u-1$ defining the region $-wCJT$. Finally, a careful examination of the proof of Theorem 15, reveals that for every $(u_0,w_0)$ on the parabola $w = 2u(1-u)$, the line-segment $[(u_0,w_0),M]$ is in $-sCJT$ (see Figure 2).

Our analysis so far leads to the conclusions summarized in Figure 3 describing the feasibility and and regions of $CJT$ possibility for all pairs $(X,P)$.

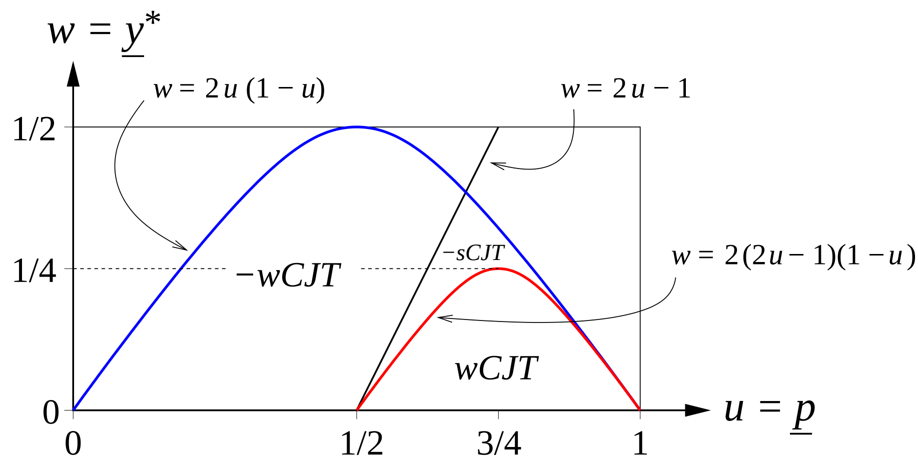


**Figure 3** Regions of possibility of $CJT$ in $L_1$.

Figure 3 is not complete in the sense that the regions $wCJT$ and $-sCJT$ are not disjoint as it may mistakenly appear in the figure. More precisely we complete Definition 17 by defining:

**Definition 18.** *The* mixed $CJT$ *set, denoted by* $mCJT$ *is the set of all points* $(u,w) \in FE_1$ *for which there exists a pair* $(X,P)$ *with parameters* $\underline{p} = u$ *and* $\underline{y}^* = w$ *which satisfies the* $CJT$*, and a pair* $(\hat{X},\hat{P})$ *with parameters* $\underline{\hat{p}} = u$ *and* $\underline{\hat{y}}^* = w$ *for which the* $CJT$ *is violated.*

Then the regions $sCJT$, $-wCJT$ and $mCJT$ are disjoint and form a partition of the feasible set of all distribution $FE$;

$$FE_1 = -wCJT \cup sCJT \cup mCJT \tag{25}$$

To complete the characterization we have to find the regions of this partition, and for that it remains to identify the region $mCJT$ since by definition, $wCJT \backslash mCJT \subset sCJT$ and $-sCJT \backslash mCJT \subset -wCJT$.

**Proposition 19.** *All three regions $sCJT$, $-wCJT$ and $mCJT$ are not empty.*

*Proof.* As can be seen from Figure 3, $-wCJT$ is clearly not empty; It contains for example the points $(0,0)$ and $(1/2,1/2)$. The region $sCJT$ contains the point $(1,0)$ since this point corresponds to a unique pair $(X,P)$ in which $X_i = 1$ for all $i$ with probability 1. This trivially satisfies the $CJT$. Finally we observe that the point $(1/2,0)$ is in the region $mCJT$. To see that we use the Berend and Paroush necessary and sufficient condition for $CJT$ in the independent case (see Remark 4) namely:

$$\lim_{n\to\infty} \sqrt{n}(\overline{p}_n - \frac{1}{2}) = \infty \tag{26}$$

First consider the pair $(\tilde{X},\tilde{P})$ in which $(\tilde{X}_i)_{i=1}^{\infty}$ are $i.i.d$ with $P(\tilde{X}_i = 1) = 1/2$ and $P(\tilde{X}_i = 0) = 1/2$. Clearly $\sqrt{n}(\overline{p}_n - \frac{1}{2}) = 0$ for all $n$ and hence condition (26) is not satisfied implying that $CJT$ is not satisfied.

Now consider $(X,P)$ in which $X = (1,1,0,1,0,1\cdots)$ with probability 1. This pair corresponds to the point $(1/2,0)$ since

$$\overline{X}_n = \overline{p}_n = \begin{cases} \frac{1}{2} + \frac{1}{n} & \text{if } n \text{ is even} \\ \frac{1}{2} + \frac{1}{2n} & \text{if } n \text{ is odd} \end{cases},$$

and hence $\underline{p} = 1/2$ and $\underline{y}^* = 0$. Finally this sequence satisfies the $CJT$ as $\overline{X}_n > \frac{1}{2}$ with probability one for all $n$. $\square$

## 6.2 Feasibility and characterization in $L_2$

Replacing $\underline{y}^* = \liminf_{n\to\infty} E|\overline{X}_n - \overline{p}_n|$ by the parameter $\underline{y} = \liminf_{n\to\infty} E(\overline{X}_n - \overline{p}_n)^2$, we obtain results in the space of points $(\underline{p},\underline{y})$ similar to those obtained in the previous section in the space $(\underline{p},\underline{y}^*)$.

Given a sequence of binary random variable $X$ with its joint distribution $P$, we first observe that for any $i \neq j$,

$$Cov(X_i,X_j) = E(X_iX_j) - p_ip_j \leq \min(p_i,p_j) - p_ip_j.$$

Therefore,

$$E(\overline{X}_n-\overline{p}_n)^2 = \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j\neq i} Cov(X_i,X_j)+\sum_{i=1}^{n} p_i(1-p_i)\right\} \tag{27}$$

$$\leq \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j\neq i}[\min(p_i,p_j)-p_ip_j]+\sum_{i=1}^{n} p_i(1-p_i)\right\}. \tag{28}$$

We claim that the maximum of the last expression (28), under the condition $\sum_{i=1}^{n} p_i=\overline{p}_n$ is $\overline{p}_n(1-\overline{p}_n)$. This is attained when $p_1=\cdots=p_n=\overline{p}_n$. To see that this is indeed the maximum, assume to the contrary that the maximum is attained at $\tilde{p}=(\tilde{p}_1,\cdots,\tilde{p}_n)$ with $\tilde{p}_i\neq\tilde{p}_j$ for some $i$ and $j$. Without loss of generality assume that: $\tilde{p}_1\leq\tilde{p}_2\leq\cdots\leq\tilde{p}_n$ with $\tilde{p}_1<\tilde{p}_j$ and $\tilde{p}_1=\tilde{p}_\ell$ for $\ell<j$. Let $0<\varepsilon<(\tilde{p}_j-\tilde{p}_1)/2$ and define $p^*=(p_1^*,\cdots,p_n^*)$ by $p_1^*=\tilde{p}_1+\varepsilon,\ \ p_j^*=\tilde{p}_j-\varepsilon$ and $p_\ell^*=\tilde{p}_\ell$ for $\ell\notin\{1,j\}$. A tedious, but straightforward computation shows that the expression (28) is higher for $p^*$ than for $\tilde{p}$ in contradiction to the assumption that it is maximized at $\tilde{p}$. We conclude that

$$E(\overline{X}_n-\overline{p}_n)^2\leq\overline{p}_n(1-\overline{p}_n).$$

Let now $(\overline{p}_{n_k})_{k=1}^{\infty}$ be a subsequence converging to $\underline{p}$ then

$$\begin{aligned}\underline{y} &= \liminf_{n\to\infty} E(\overline{X}_n-\overline{p}_n)^2\leq\liminf_{k\to\infty} E(\overline{X}_{n_k}-\overline{p}_{n_k})^2\\ &\leq \liminf_{k\to\infty}\overline{p}_{n_k}(1-\overline{p}_{n_k})=\underline{p}(1-\underline{p}).\end{aligned}$$

We state this as a theorem:

**Theorem 20.** *For every pair $(X,P)$, The corresponding parameters $(\underline{p},\underline{y})$ satisfy $\underline{y}\leq\underline{p}(1-\underline{p})$.*

Next we have the analogue of Theorem 15, proved in the same way.

**Theorem 21.** *Let $t\in[0,\frac{1}{2}]$. If $F$ is a distribution with parameters $(\underline{p},\underline{y})$, then there exists a distribution $H$ with parameters $\underline{\tilde{p}}=1-t+t\underline{p}$ and $\underline{\tilde{y}}=t^2\underline{y}$ that satisfy the CJT.*

We can now construct Figure 4 which is the analogue of Figure 2 in the $L_2$ space $(\underline{p},\underline{y})$. The feasible set in this space is

$$FE_2=\{(u,w)|0\leq u\leq 1,\ \ 0\leq w\leq u(1-u)\} \tag{29}$$

The geometric locus of the midpoints $M$ in Figure 4 is derived from:
(1) $u=\frac{1}{2}u_0+\frac{1}{2}$; (2) $w=\frac{1}{4}w_0$ and (3) $w_0=u_0(1-u_0)$ and is given by
$w=\frac{1}{2}(2u-1)(1-u)$. This yields Figure 5 which is the analogue of Figure 3. Note, however, that unlike in Figure 3, the straight line $w=u-\frac{1}{2}$ *is not tangent* to the small parabola $w=(u-\frac{1}{2})(1-u)$ at $(\frac{1}{2},0)$.

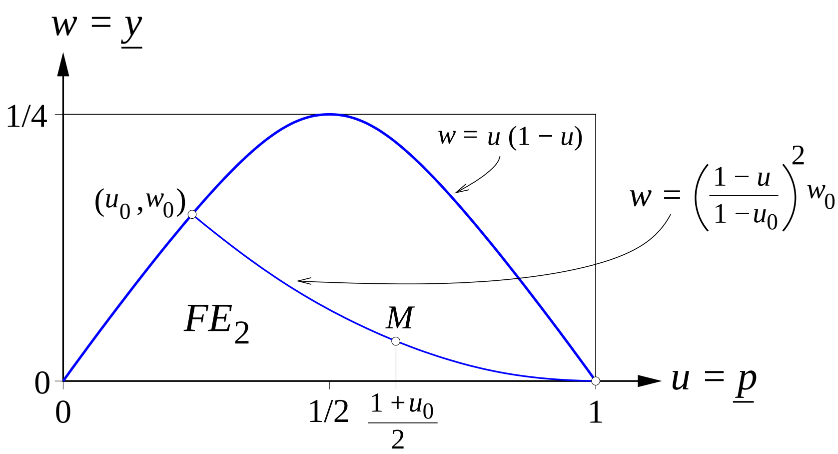


**Figure 4** The feasible set $FE_2$.

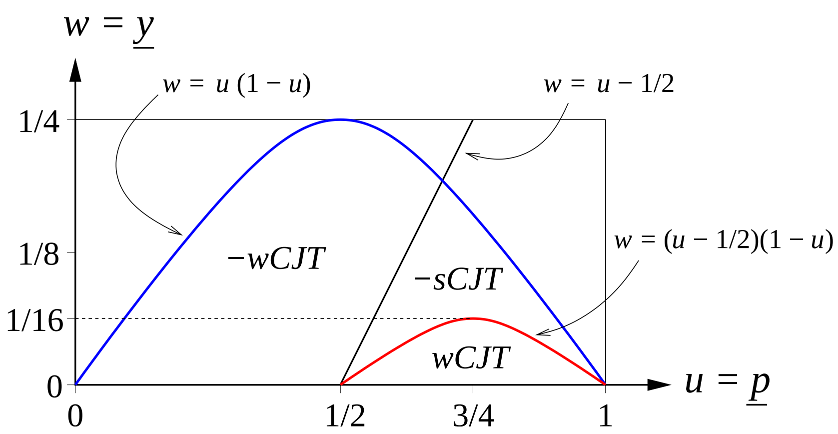


**Figure 5** Regions of possibility of $CJT$ in $L_2$.

The next step toward determining the region $mCJT$ in the $L_2$ space (Figure 5) is the following:

**Proposition 22.** *For any* $(u,w) \in \{(u,w) | \frac{1}{2} < u < 1\ ;\ \ 0 \le w \le u(1-u)\}$, *there is a pair* $(Z,H)$ *such that:*

**(i)** $E(Z_i) = u, \forall i.$

**(ii)** $\liminf_{n\to\infty} E(\overline{Z}_n - u)^2 = w.$

**(iii)** *The distribution* $H$ *does not satisfy the* $CJT$.

*Proof.* Let $(X,F)$ be given by $X_1 = X_2 = \ldots = X_n = \ldots$ and $E(X_i) = u$. That is, $(X,F)$ corresponds to the point $(u,u(1-u))$ on the large parabola in Figure 5. Further, let $Y,G)$ be a sequence of of $i.i.d.$ random variables $(Y_i)_{i=1}^{\infty}$ with expectation $u$. We first observe that

$$\lim_{n\to\infty} E(\overline{Y}_n - u)^2 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \lim_{n\to\infty} \frac{1}{n^2}\, nu(1-u) = 0.$$

Thus $(Y,G)$ corresponds to the point $(u,0)$ in Figure 5.

Let $S_p = \{0,1\}^{\infty}$ be the space of infinite binary sequences (or equivalently, the space of all the realizations of infinite sequences of binary variables), and consider the product probability space $(S_p \times S_p, F \times G)$ and denote $H = F \times G$. The idea of the desired construction is along the following lines: In a sequence $(Z_i)_{i=1}^{\infty}$ consisting of blocks of $X_i$ and $Y_i$, the average $\overline{p}_n$ is constantly $u$ and

- The sequence $(X,F)$ satisfies $F(\overline{X}_n > \frac{1}{2}) = u < 1$.
- As soon as there is a majority of $X$'s in the sequence $Z = (Z_1,\cdots,Z_n)$, the probability that the majority votes 0 is at least $1-u$. Hence $H(\overline{Z}_n > \frac{1}{2}) \le u$.
- Adding more variables $X_i$ increases $E(\overline{Z}_n - u)^2$ (in steps that can be made arbitrarily small with $n$).
- Adding more variables $Y_i$ decreases $E(\overline{Z}_n - u)^2$ (in steps that can be made arbitrarily small with $n$).
- By starting with a block of $X_i$ and appropriately choosing the sizes of the blocks we get:
  - $E(\overline{Z}_n - u)^2 \ge w$ for all $n$.
  - For a subsequence $(n_k)_{k=1}^{\infty}$ (namely the ends of the $Y_i$ blocks) $E(\overline{Z}_{n_k} - u)^2$ approaches $w$. Combined with the previous point this implies $\underline{y} = \liminf_{n\to\infty} E(\overline{Z}_n - u)^2 = w$.
  - For a subsequence $(m_k)_{k=1}^{\infty}$ (namely the ends of the $X_i$ blocks) $E(\overline{Z}_{m_k} - u)^2$ approaches $u(1-u)$, that is, the sequence has a majority of $X$'s and hence the probability the majority votes 0 is at least $(1-u)$ implying $H(\overline{Z}_{m_k} > \frac{1}{2}) \le u$ on the subsequence $(m_k)_{k=1}^{\infty}$. Consequently, $(Z,H)$ does not satisfy the $CJT$.

Formally, we define a sequence of random variables $(Z_i)_{i=1}^{\infty}$ in the following way: Let $Z_1 = X_1,\ Z_2 = X_2$ and set $k_1 = 2$ (the length of the first block) and $B_1 = \{1,2\}$ (the set of indices of the first block). Then

$$H(\overline{Z}_2 > \frac{1}{2}) = F(X_1 = 1) = u < 1.$$

Next, we choose the second block $B_2 = \{k_1+1,\cdots,k_1+k_2\}$ of $k_2$ variables $Y_i$ so that if $Z_i = Y_{i-k_1},\quad i \in B_2$, then $E(\overline{Z}_j - u)^2 \geq w$ for all $j \leq k_1+k_2$ and $E(\overline{Z}_{k_1+k_2} - u)^2 \in [w, w+\frac{K}{k_1+k_2}]$, where $K$ is some constant, fixed throughout our construction. We show below that such choice of $k_2$ is possible. We now continue to choose alternated blocks of $X$'s and $Y$'s. The third block, which is a block of $X$'s, is $B_3 = \{k_1+k_2+1,\cdots,k_1+k_2+k_3\}$ is chosen such that $k_1+k_3 > k_2$ and $Z_i = X_{i-k_1-k_2},\quad i \in B_3$. So, in the first three blocks there is a majority of $X$'s which imply $H(\overline{Z}_{k_1+k_2+k_3} > \frac{1}{2}) \leq u < 1$. Next, $B_4 = \{\sum_{i=1}^{3} k_i + 1,\cdots,\sum_{i=1}^{4} k_i\}$ is chosen and $Z_i = X_{i-\sum_{i=1}^{3} k_i}$ for $i \in B_4$, so that $E(\overline{Z}_{\sum_{i=1}^{4} k_i} - u)^2 \in [w, w+\frac{K}{\sum_{i=1}^{4} k_i}]$ and $E(\overline{Z}_j - u)^2 \geq w$ for all $j \leq \sum_{i=1}^{4} k_i$. We continue to construct the sequence $Z$ in this manner: At the end of each odd block (of $X$'s) there is a majority of $X$'s which guarantees that at the end of each odd bock $B_{2r+1}$ we have $H(\overline{Z}_{\sum_{i=1}^{2r+1} k_i} > \frac{1}{2}) \leq u < 1$. At the end of an even block $B_{2r}$ we have $E(\overline{Z}_{\sum_{i=1}^{2r} k_i} - u)^2 \in [w, w+\frac{K}{\sum_{i=1}^{2r} k_i}]$ and $E(\overline{Z}_j - u)^2 \geq w$ for all $j \leq \sum_{i=1}^{2r} k_i$. The result of the construction is the desired pair $(Z,H)$. Note that by our construction, an even block $B_{2r}$ may be empty because of the constraint: $E(\overline{Z}_j - u)^2 \geq w$ for all $j \leq \sum_{i=1}^{2r} k_i$, (however, $\sum_{r=1}^{\infty} k_{2r} = \infty$ if $w < u(1-u)$). On the other hand odd blocks can be made all non empty since adding more $X$'s to an odd block with the desired properties maintains those properties.

It remains to show that such a construction is possible. Let $Z = (Z_1,\cdots,Z_n)$ be a finite of binary random variables consisting of $x$ variables $X_i$ and $y$ variables $Y_i$, whose joint distribution is the marginal of the product distribution $H = F \times G$. Then, assuming that both $x$ and $y$ are at least 2,

$$\begin{aligned} E(\overline{Z}_n - u)^2 &= \frac{x^2 u(1-u) + yu(1-u)}{(x+y)^2} \\ &= u(1-u)\frac{x^2+y}{(x+y)^2}, \text{ for } x,y \geq 2. \end{aligned}$$

The function $f(x,y) := \frac{x^2+y}{(x+y)^2}$ has the following properties:

**(i)** It is (strictly) increasing in $x$ for fixed $y$ $\left(\frac{\partial f}{\partial x} = \frac{2y(x-1)}{(x+y)^3}\right)$.

**(ii)** It is (strictly) decreasing in $y$ for fixed $x$ $\left(\frac{\partial f}{\partial y} = \frac{x(1-2x)-y}{(x+y)^3}\right)$.

**(iii)** There exist a constant $K > 0$ such that $\max\left\{|\frac{\partial f}{\partial x}(x,y)|, |\frac{\partial f}{\partial y}(x,y)|\right\} \leq \frac{K}{x+y},\ \ x,y \geq 2.$

These properties make our previous construction possible. Indeed $E(\overline{Z}_j - u)^2$ is strictly decreasing with $j$ in a block of $Y$'s, say $j \in B_2$ and goes to zero if the size of the block is infinite, so there is a maximal $j$ for which $E(\overline{Z}_j - u)^2 \geq w$. As for the size of the $X$ blocks, these have to be large enough so as to have a majority of $X$'s at the end of the block. For example, the size of $B_3$ is $k_3$ which satisfies $k_1 + k_3 > k_2$ (which implies $H(\overline{Z}_{k_1+k_2+k_3} > \frac{1}{2}) \leq u < 1$). The same argument guarantees, inductively, the possibility of the construction of all steps. □

Combining Proposition 22 and Theorem 21 yields the following conclusions which are also presented in Figure 6

**Corollary 23.** *1. The region below the small parabola in Figure 5, with the exception of the point* $(1,0)$*, is in mCJT that is:*

$$\left\{ (\underline{p}, \underline{y}) | \frac{1}{2} \leq \underline{p} < 1; \text{ and } \underline{y} \leq \frac{1}{2}(2\underline{p} - 1)(1 - \underline{p}) \right\} \subseteq mCJT.$$

2. *The point* $(\underline{p}, \underline{y}) = (1,0)$ *is the* only *point in sCJT. It corresponds to a single sequence with* $X_1 = \cdots = X_n = \cdots$ *with* $F(X_i = 1) = 1$.

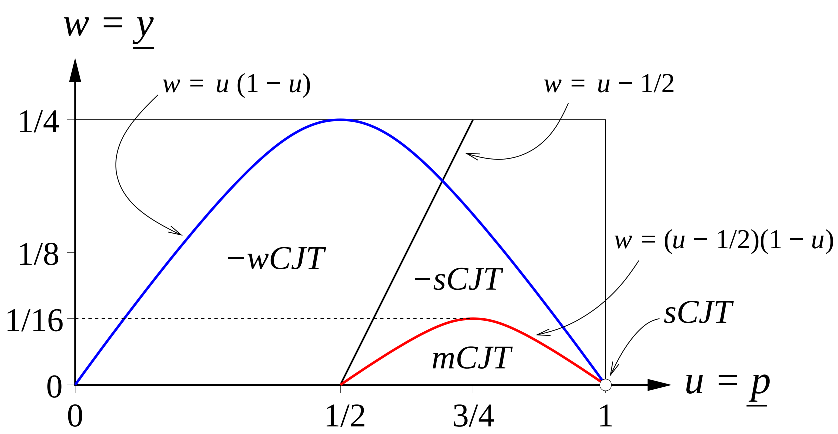


**Figure 6** *mCJT* and *sCJT* in the $L_2$ space.

# 7 General interlacing

We now generalize the main construction of the proof of Theorem 15. This may be useful in advancing our investigations.

**Definition 24.** *Let* $X = (X_1, X_2, \ldots, X_n, \ldots)$ *be a sequence of binary random variables with joint probability distribution* $F$ *and let* $Y = (Y_1, Y_2, \ldots, Y_n, \ldots)$ *be another sequence of binary* *random variables with joint distribution $G$. For $t \in [0,1]$, the $t$*-interlacing *of $(X,F)$ and $(Y,G)$ is the pair $(Z,H) := (X,F) *_t (Y,G)$ where for $n = 1,2,\ldots$,*

$$Z_n = \begin{cases} X_{\lfloor tn \rfloor} & \text{if} \quad \lfloor tn \rfloor > \lfloor t(n-1) \rfloor \\ Y_{\lceil (1-t)n \rceil} & \text{if} \quad \lceil (1-t)n \rceil > \lceil (1-t)(n-1) \rceil \end{cases}, \tag{30}$$

*and $H = F \times G$ is the product probability distribution of $F$ and $G$.*

The following lemma is a direct consequence of Definition 24.

**Lemma 25.** *If $(X,F)$ and $(Y,G)$ satisfy the $CJT$ then for any $t \in [0,1]$ the pair $(Z,H) = (X,F) *_t (Y,G)$ also satisfies the $CJT$.*

*Proof.* We may assume that $t \in (0,1)$. Note that

$$\left\{\omega | \overline{Z}_n(\omega) > \frac{1}{2}\right\} \supseteq \left\{\omega | \overline{X}_{\lfloor tn \rfloor}(\omega) > \frac{1}{2}\} \cap \{\omega | \overline{Y}_{\lceil (1-t)n \rceil}(\omega) > \frac{1}{2}\right\}$$

By our construction and the fact that both $(X,F)$ and $(Y,G)$ satisfy the $CJT$,

$$\lim_{n\to\infty} F\left(\overline{X}_{\lfloor tn \rfloor} > \frac{1}{2}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty} G\left(\overline{Y}_{\lceil (1-t)n \rceil} > \frac{1}{2}\right) = 1.$$

As

$$H\left(\overline{Z}_n > \frac{1}{2}\right) \geq F\left(\overline{X}_{\lfloor tn \rfloor} > \frac{1}{2}\right) \cdot G\left(\overline{Y}_{\lceil (1-t)n \rceil} > \frac{1}{2}\right),$$

the proof follows. □

**Corollary 26.** *The region $wCJT$ is* star-convex *in the $L_1$ space . Hence, in particular, it is path connected in this space.*

*Proof.* Let $(u,w)$ be a point in $wCJT$ in the $L_1$ space. Then, there exists a pair $(X,F)$ which satisfies $CJT$, where $X$ is the sequence of binary random variables with joint probability distribution $F$ satisfying $\underline{p} = u$ and $\underline{y}^* = w$. By Remark 16, Lemma 25 and the proof of Theorem 15, the line segment $[(u,w),(1,0)]$ is contained in $wCJT$ proving that $wCJT$ is star-convex. □

**Corollary 27.** *The region $wCJT$ is* path connected *in the $L_2$ space.*

*Proof.* In the $L_2$ space a point $(u,w)$ corresponds to $\underline{p} = u$ and $\underline{y} = w$. By the same arguments as before, the arc of the parabola $w = ((1-u)/(1-u_0))^2 w_0$ connecting $(u,w)$ to $(1,0)$ (see Figure 4) is contained in $wCJT$, and thus $wCJT$ is path connected. □

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# Appendix
# $CJT$ in the space of all probability distributions.

We look at the space $S_p = \{0,1\}^\infty$ already introduced in the proof of Proposition 22 on page 16. We consider $S_p$ both as a measurable product space and as a topological product space. Let $\mathscr{P}$ be the space of all probability distributions on $S_p$. $\mathscr{P}$ is a compact metric space in the weak topology.

**Lemma 28.** *If $P_1$ and $P_2$ are two distributions in $\mathscr{P}$, and if $P_2$ does not satisfy the CJT then for any $0 < t < 1$, the distribution $P_3 = tP_1 + (1-t)P_2$ does not satisfy the $CJT$.*

*Proof.* For $n = 1, 2, \cdots$ let

$$B_n = \left\{ x = (x_1, x_2, \cdots) \in S_p \middle| \frac{1}{n} \sum_{i=1}^{n} x_i > \frac{1}{2} \right\}$$

There exists a subsequence $(B_{n_k})_{k=1}^{\infty}$ and $\varepsilon > 0$ such that $P_2(B_{n_k}) \leq 1-\varepsilon$ for $k = 1, 2, \cdots$. then

$$P_3(B_{n_k}) = tP_1(B_{n_k}) + (1-t)P_2(B_{n_k}) \leq t + (1-t)(1-\varepsilon) = 1 - \varepsilon(1-t),$$

implying that $P_3$ does not satisfy the $CJT$. $\square$

**Corollary 29.** *The set of probability distributions that do not satisfy the* $CJT$ *is dense in* $\mathscr{P}$ *(in the weak topology), and is convex.*