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ENDOGENOUS GROWTH IN MULTISECTOR RAMSEY MODELS*

BY JIM DOLMAS[1]

In this paper, I give sufficient conditions for the existence of endogenously growing optimal paths in a general multisector Ramsey model of optimal capital accumulation. The key assumption involves the existence of a positive vector of capital stocks which admits strictly positive consumption and expansibility in inverse proportion to the utility discount factor. If the technology set contains the ray through such a point, in addition to standard convexity and interiority assumptions, then optimal paths grow without bound from any strictly positive initial stocks. The result unifies a number of existing models in the growth theory literature.

1. INTRODUCTION

Within both capital theory and macroeconomics there has been a resurgence of interest in models of capital accumulation which display endogenous growth—models without time-dependent technologies which nonetheless have the property that the optimal or equilibrium paths of capital and consumption which they generate grow without bound. It is thus surprising that little work has been done in establishing conditions which guarantee this property. A recent exception is Jones and Manuelli (1990), working in a variant of the standard one-sector Ramsey model of optimal growth. Earlier, Gale and Sutherland (1968) also proved a growth result for an undiscounted one-sector Ramsey model. By and large, though, this research program has been carried out in a series of particular examples with little suggestion of a general framework for achieving endogenous growth. This essay attempts to fill that gap, at least for models which may be cast in the convex Ramsey optimal growth framework.[2] The results below provide sufficient conditions for the existence of endogenously growing optimal paths in a convex multisector Ramsey model of optimal capital accumulation, thus unifying a number of particular examples in the growth literature, as well as providing simple conditions for guaranteeing growth in more complex models.

By way of motivation, consider the simplest of all endogenous growth models, the one-sector linear model, or A-k model, used by Rebelo (1991). There is a single, all-purpose consumption-investment good. An infinitely-lived representative con-

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[1] This paper is based on the second chapter of my Ph.D. thesis at the University of Rochester. The comments of John H. Boyd III, Lionel McKenzie, audiences at Rochester, Toronto and the 1991 Midwest Mathematical Economics meetings, and anonymous referees are gratefully acknowledged. Any remaining errors are my own.

[2] See, for example, Barro and Sala-i-Martin (1992), Bond, Wang and Yip (1993), King and Rebelo (1991) for models of this sort.

sumer chooses paths of consumption and capital so as to maximize lifetime utility

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$$

subject to the technological constraints $c_t + k_t \leq Ak_{t-1}$ for all $t = 1, 2\ldots$ given some $k_0 > 0$. Optimal paths grow without bound whenever $\delta A > 1$. Growth of optimal—and in this case equilibrium—paths depends only on the utility discount factor and properties of the production function. The specific value of k_0 and the parameters of u, aside from the general requirement of concavity, are not relevant.

Similar in structure, though somewhat more complicated, are Lucas's (1988) extension of the Uzawa (1965) model and King and Rebelo's (1990) two-sector model, the dynamics of which have recently been characterized by Caballé and Santos (1993) and Bond, Wang and Yip (1993). In both models, the consumer's preferences are the same as in the A-k model. On the production side, both have physical goods (consumption and physical capital) produced using physical capital and effective labor hours. Effective labor hours are simply raw labor hours multiplied by a measure of skills, the stock of human capital. Human capital in turn is produced using either effective labor alone or effective labor and physical capital. In the Lucas–Uzawa model, the constraints are $c_t + k_t \leq F(k_{t-1}, n_t h_{t-1})$ and $h_t = G(e_t h_{t-1})$, where h denotes human capital, n labor hours allocated to physical goods production, and e labor hours allocated to human capital production. At each date, n_t and e_t are constrained to sum up to the representative agent's endowment of time. In the King–Rebelo model, the human capital technology G also depends on k_{t-1}, which is then divided between the two production processes. In both models, as in the simple A-k model, assumptions relating to the utility discount factor δ and the properties of the production functions F and G suffice to guarantee growth.[3]

Finally, Jones and Manuelli (1990), described in more detail in a later section, consider a convex one-sector model with one consumption-investment good and multiple capital stocks. Their explicit aim is to extend the A-k model to incorporate labor income, which is lacking in the linear case. Their production function exhibits diminishing returns, and hence positive labor income, at low levels of output, followed by constant returns asymptotically. Here again, a condition relating δ and properties of the production function guarantees growth.

All these models can be shown to possess a common structure, and to derive endogenous growth in a common manner. That the conditions guaranteeing growth are independent of the specific value of initial capital and the parameters of u—abstracting from the spillovers in Lucas's model—suggests a connection with results on the existence of *steady states* in bounded convex models. In particular, even in the most general convex multisector models, steady states arise through

[3] Lucas (1988) includes an additional "spillover," or externality, term in the physical goods technology F, depending on the level of human capital h. This external effect is inessential to the derivation of endogenously growing paths, though it does divorce equilibria from optima in his model and yields a dependence of the equilibrium growth rate on parameters of u.

combining "expansibility" assumptions with a sufficient amount of "diminishing returns."[4]

The standard pictures of the long-run supply and demand for capital which one derives in the one-sector case can provide some intuition here. The same may be used to show why $\delta A > 1$ yields growth in the simple A-k model. The role of an expansibility assumption, $\delta f'(0) > 1$ in the one-sector model with production function f, is to guarantee that the technology is sufficiently productive at low levels of capital that the demand for capital lies initially above its long-run supply, which in the one-sector case is perfectly elastic at the rate of time preference. Expansibility assumptions thus involve only the utility discount factor and properties of the technology set. Diminishing returns, a property of the technology set alone, guarantees that eventually the demand curve for capital lies *below* the long-run supply curve. In the one-sector case, one typically assumes $\delta f'(k) < 1$ for large enough values of k. The Inada-type condition $f'(+\infty) = 0$ is an extreme version of this same assumption. If f' is continuous, and the usual Euler equations obtain, somewhere in between there must be a steady state. The condition $\delta A > 1$ guarantees growth in the A-k model precisely because the demand curve for capital—perfectly elastic at capital's net marginal product $A - 1$—lies everywhere above the long-run supply curve for capital—perfectly elastic at the rate of time preference $(1/\delta) - 1$. The result in this paper shows that for a large class of multisector models, as in the particular examples mentioned above, the one-sector, A-k intuition carries through. If we bring in only half the ingredients for a steady state, maintaining expansibility while dispensing with diminishing returns, we achieve unbounded growth of optimal paths. In this light, the result may seem trivial, and perhaps it is. But relying on one-sector intuition does not make a proof, in the same way that the one-sector steady-state conditions ($f'(0) > 1/\delta$, $f'(+\infty) < 1/\delta$ and f' continuous) do not prove that expansibility and diminishing returns guarantee the existence of steady states in more complicated models. Moreover, the one-sector conditions, whether with regard to steady states or growth, provide only a suggestion of what one must concretely assume in a model with multiple produced goods or costs of adjusting capital stocks or nondifferentiable technologies. A set of sufficient conditions for growth in a very general model of optimal capital accumulation may thus prove useful in applications, as in the construction of particular models.

As will be seen below, a variety of particular models (including the ones cited above, as well as fixed-coefficient models, models with joint production, adjustment costs and, of course, differing numbers of consumption and capital goods) can fit within the framework of this paper. While the technology is described by a production correspondence and the necessary conditions are written in terms of supporting prices, they simply generalize the production functions and marginal conditions which characterize the differentiable models common in applications of growth theory. As noted above, the key conditions of the theorem have a simple interpretation in terms of the relationship between the long-run supply and demand curves for capital. I give several short examples of how the conditions may be

[4] See McKenzie (1986).

applied, as well as one extended example of a simple model which does not fit into the framework of previous results.

The structure of the paper is as follows. In Section 2, I describe the model and give an overview of the main results. Section 3 contains results guaranteeing the existence of optimal paths, while Section 4 characterizes those optimal paths in terms of supporting prices and profit-maximization conditions. These necessary conditions are fundamental to the growth result, which shows that under certain monotonicity assumptions and a productivity assumption, the vector of marginal utilities of consumption along an optimal path from positive initial stocks must go to zero in the limit. The main theorem is proven in Section 5.

In Section 6, I present an example of a model not encompassed by previous results, a one-sector Ramsey model with adjustment costs. The example shows how the results of the paper may be applied in practice. The Appendix contains proofs of the lemmata concerning existence of optimal paths and the necessary conditions for optimality.

2. RESULTS

The basic structure of the model is as follows. There are n consumption goods and m capital stocks at each date $t = 1, 2, \ldots$. The capital stocks at the beginning of each period determine, via a production correspondence, the feasible combinations of consumption for that period and capital stocks for the subsequent period. Utility in each period is derived from consumption in that period, and lifetime utility over the infinite horizon is the discounted sum of one-period utilities. An optimal path of consumption is one which maximizes lifetime utility over the set of feasible consumption paths.

Formally, the feasible set for the optimal growth problem is defined by a production correspondence Φ: $R^m_+ \to \{\text{subsets of } R^n_+ \times R^m_+\}$, where $(c, k') \in \Phi(k)$ has the interpretation that (c, k') is a feasible combination of current consumption and next-period's capital stocks given current-period capital stocks k. Call a path $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ feasible from initial stocks k if $(c_t, k_t) \in \Phi(k_{t-1})$ for all $t \geq 1$, and $k_0 = k$. Let $F(k)$ denote the set of paths of consumption $\{c_t\}_{t=1}^{\infty}$ such that $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ is a feasible path from k for some path of capital $\{k_{t-1}\}_{t=1}^{\infty}$.

Given a vector of initial capital stocks $k \in R^m_+$, the Ramsey problem is to choose a path of consumption which maximizes lifetime utility over $F(k)$. Lifetime utility is specified as

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t),$$

where u: $R^n_+ \to R \cup \{-\infty\}$ is the 'felicity' or 'momentary utility' function, and $\delta > 0$ is the discount factor.

Under standard continuity and compactness assumptions (A1–A3 below) there will exist a set K of initial capital stocks such that for any $k \in K$ an optimal path with $\sum_t \delta^{t-1} u(c_t) > -\infty$ exists. When momentary utility u is concave and the production correspondence Φ has a convex graph with nonempty interior (A4–A5), there will exist prices which support the optimal path in the sense that the optimal path is profit-maximizing at each date. These prices have the interpretation of marginal utilities of consumption and marginal values of capital. Given the profit-maximization conditions, we will show that if u and Φ satisfy monotonicity assumptions—u strictly increasing and Φ nondecreasing (A6)—then the marginal utility of current consumption goes to zero along any optimal path whenever Φ and δ satisfy the following productivity assumption:

(P). *There exist* $\bar{c} \gg 0$ *and* $\bar{k} > 0$ *such that* $\lambda(\bar{c}, \delta^{-1}\bar{k}) \in \Phi(\lambda \bar{k})$ *for every* $\lambda > 0$.

Assumption (P) is a natural generalization of Jones and Manuelli's Condition G, which for their model guarantees unbounded growth of consumption. In a one-sector model with a linear production function $f(k) = Ak$, (P) is equivalent to $\delta A > 1$. We see below that when the model here, which encompasses Jones and Manuelli's, is specialized to their framework, Assumption (P) is actually weaker than their key condition G.

Given the concavity and strict monotonicity of u, having the marginal utility of current consumption go to zero is tantamount to unbounded growth of consumption, that is, $\limsup \|c_t\| = +\infty$. Given that the technology is bounded at each date, given finite capital, the unbounded growth of consumption implies that capital stocks are growing without bound as well.

3. EXISTENCE OF OPTIMAL PATHS

The main results of the paper could be presented taking as given the existence of an optimal path. However, the conditions for existence of an optimal path and those for growth of an optimal path will often be in tension, in particular when u is unbounded above. Hence, it is worthwhile to present an existence result to clarify the nature of this tension. The result I present in this section is of the 'Weierstrass' variety, using the fact that an upper semicontinuous function on a compact set attains a maximum on that set. The method of proof adapts a partial summation technique exploited in Boyd (1990a).

We wish to make assumptions on the primitives u, δ and Φ such that lifetime utility is upper semicontinuous and the set of feasible consumption paths is compact in some common topology. That topology will be the product topology[5] on $R^n_+ \times R^n_+ \times \cdots$. The following three assumptions are sufficient for this purpose. The first two pertain to the production correspondence Φ and the felicity function u, respectively.

A1. Φ *is a continuous, compact-valued correspondence, satisfying "free disposal," that is, if* $(c, k') \in \Phi(k)$, *then* $(\tilde{c}, \tilde{k}') \in \Phi(\tilde{k})$ *for all* $0 \le \tilde{c} \le c$, $0 \le \tilde{k}' \le k'$ *and* $\tilde{k} \ge k$. *There exist constants* $\alpha, \eta, \theta \ge 0$ *and* $\beta \ge 1$ *such that* $(c, k') \in \Phi(k)$ *implies* $\|c\| \le \eta + \theta\|k\|$ *and* $\|k'\| \le \alpha + \beta\|k\|$.

[5] The product topology is the topology of pointwise convergence on the space of consumption paths.

A2. *u: $R_{+}^{n} \to R \cup \{-\infty\}$ is upper semicontinuous with $u(c) > -\infty$ for $c \gg 0$. There are constants ν, μ and γ, with $\mu \geq 0$, such that $u(c) \leq \nu + \mu \|c\|^{\gamma}/\gamma$ for every $c \in R_{+}^{n}$.*

As shown in the Appendix, the last part of A1 implies that if $\{c_t\}_{t=1}^{\infty}$ is a feasible path of consumption from k, each c_t resides in a compact subset of R_{+}^{n}. By Tychonoff's theorem, then, the feasible set $F(k)$ is contained in a set which is compact in the product topology. Closure of $F(k)$, which would then imply its compactness, follows easily from the first part of A1, which assumes that Φ is continuous and compact-valued.

The assumptions on u contained in A2, together with the following joint restriction on preferences and technology, will guarantee that lifetime utility $\sum_t \delta^{t-1} u(c_t)$ is upper semicontinuous in the product topology on $F(k)$. The Weierstrass theorem then yields the existence of an optimal path.

A3. *The constants β and γ from A1 and A2 and the discount factor δ satisfy $0 < \delta < 1$ and $\beta^{\gamma}\delta < 1$.*

The first part of Assumption A3 is akin to the familiar condition of Brock and Gale (1969) relating the maximal growth rate of capital, the discount factor and the asymptotic curvature of the felicity function.[6] The requirement that $\delta < 1$ is inessential at this point, but would eventually be required if consumption is to grow without bound, and if we consider momentary utilities which are unbounded above.[7] In the Appendix I prove:

LEMMA 3.1. *Let Φ, u and δ satisfy A1–A3. Then, there exists from any $k \in R_{+}^{m}$ a path $\{c_t\}_{t=1}^{\infty} \in F(k)$ which attains*

$$\sup\left\{ \sum_{t=1}^{\infty} \delta^{t-1} u(c_t) : \{c_t\}_{t=1}^{\infty} \in F(k) \right\}.$$

Call this supremum $V(k)$. The one problem which remains is that since u has been assumed to be an upper semicontinuous function taking values in $R \cup \{-\infty\}$, we may have $V(k) = -\infty$ for some values of k. Let $K \subset R_{+}^{m}$ denote the set of k satisfying $V(k) > -\infty$. Given free disposal, K will be nonempty when we make the productivity Assumption (P), which we employ below. To see this, let (P) hold, and suppose k is such that $k \geq \rho \bar{k}$ for some $\rho > 0$, where $\bar{k}$ is as defined in (P). Then, by free disposal, consumption every period of $\lambda \bar{c}$ is feasible, where $\bar{c} \gg 0$ is as defined

[6] McFadden (1973) gives a thorough analysis of existence conditions of the 'Brock–Gale' sort for one-sector Ramsey models and for multisector models with 'input–output' technology sets of the form considered by von Neumann (1945), Malinvaud (1953), Gale (1967) and others. The sort of technologies common in the recent growth theory literature, which fit very neatly into the capital accumulation framework presented above, are often less easily put within the framework which McFadden analyzes. The adjustment-cost model considered in Section 6 below is one such example.

[7] Here, $\delta > 1$, $\beta \geq 1$ and $\delta\beta^{\gamma} < 1$ imply $\gamma < 0$. With a productive technology, 'upcounting' (setting $\delta > 1$) is potentially permissible if u is bounded above, if $\sup\{u(c)\} = 0$, and if consumption grows sufficiently fast. I discuss this possibility in Section 5.

in (P) and λ is some positive real number. Thus, for all such values of k, (P) and the free disposal assumption imply

$$V(k) \geq \frac{u(\lambda \bar{c})}{1-\delta} > -\infty.$$

In particular, $\operatorname{int}(R^m_+) \subset K$.[8]

4. SUPPORTING PRICES

We now proceed to characterize the optimal path in terms of necessary conditions. In standard fashion, the derivation of the necessary conditions here relies upon convexity and interiority assumptions which allow the use of certain results in convex optimization theory.

Recall that the value function V: $R^m_+ \to R \cup \{-\infty\}$ has been defined as

$$V(k) = \sup\left\{ \sum_{t=1}^{\infty} \delta^{t-1} u(c_t) \colon \{c_t\}_{t=1}^{\infty} \in F(k) \right\},$$

and $K \subset R^m_+$ has been defined as the set of capital stocks for which $V(k) > -\infty$. When u is concave and $F(k)$ satisfies $F(\alpha k + (1-\alpha)\tilde{k}) \supset \alpha F(k) + (1-\alpha)F(\tilde{k})$ for all $k, \tilde{k} \in R^m_+$ and $\alpha \in [0,1]$, the value function must be concave as well. F in turn will have the desired property whenever the graph of the production correspondence,

$$\operatorname{Gr}(\Phi) \equiv \{(k, c, k') \in R^m_+ \times R^n_+ \times R^m_+ \colon (c, k') \in \Phi(k)\},$$

is convex. When $\operatorname{Gr}(\Phi)$ is convex, the convex combination of two feasible consumption paths is feasible by employing the convex combination of the associated capital paths. Hence I assume:

A4. *The function u is concave.* $\operatorname{Gr}(\Phi)$ *is convex.*

Note that when V is concave, the set $K = \{k \colon V(k) > -\infty\}$ is convex. It is also straightforward to show that V satisfies Bellman's equation:

$$V(k) = \sup\{u(c) + \delta V(k') \colon (c, k') \in \Phi(k)\}.$$

The derivation of the necessary conditions will rely heavily on the fact that $V(k_{t-1}) = u(c_t) + \delta V(k_t)$ for all t along an optimal path.

The supergradients of u and V will play the role of prices in our subsequent analysis. Formally, for a function f: $R^l \to R$, w is a *supergradient* of f at a point x if

[8] Clearly, the existence of a constant, strictly positive path of consumption, which is implied by (P) and free disposal, is more than sufficient to give $V(k) > -\infty$ when u is unbounded below. For $c \in R_+$, consider $u(c) = c^\gamma/\gamma$ for $\gamma < 0$. A path $\{c_t\}$ with $c_t = \theta^{t-1} c_1$ and $0 < \theta < 1$ has $\sum_t \delta^{t-1} u(c_t) > -\infty$ provided $\theta^\gamma \delta < 1$, even though $c_t \to 0$ and $u(c_t) \to -\infty$. If our concern were with less productive technologies, we would want to take this fact into account.

$w \in R^l$ and $f(x)+w(y-x) \geq f(y)$ $\forall y \in R^l$. Proper concave functions which are bounded below always have supergradients, which may be thought of as generalized derivatives. The set of supergradients of f at x is denoted $\partial f(x)$. If $f'(x)$ exists, then $\partial f(x) = \{f'(x)\}$. If A is a set in R^l, and $x \in A$, the notation supp$\{A, x\}$ denotes the *support* of A at x—i.e., the collection of all w with $wx \geq wy$ for all $y \in A$. We will use below (in Lemma 4.1) the following result from convex optimization theory.

FACT (ABSTRACT KUHN-TUCKER THEOREM). *Suppose f: $R^l \to R$ is concave and bounded below on a convex set D with nonempty interior. Then x^* solves* $\max\{f(x)$: $x \in D\}$ *if and only if* $\partial f(x^*) \cap \text{supp}\{D, x^*\} \neq \varnothing$.[9]

Let G denote the subset of $R^m_+ \times R^n_+ \times R^m_+$ obtained by intersecting Gr(Φ) with $K \times R^n_+ \times K$. Note that, given our assumptions, Gr(Φ) and K are both convex, so that G is convex as well. Looking ahead to applying the Kuhn-Tucker theorem, we also assume:

A5. *G has a nonempty interior.*

The main result for the theorems given in the next section is the following lemma, which establishes necessary conditions for optimality. The conditions should appear familiar; they can be interpreted either as a generalization of the duality-based necessary conditions from the reduced form models of the turnpike literature or as an analogy to the profit maximization conditions in Malinvaud-type models. The lemma shows that optimal paths from initial stocks which are interior to K are necessarily price supported in the sense that marginal utilities of consumption and marginal values of capital support the optimal choices of current capital, consumption and next-period's capital out of the set G at each date:

LEMMA 4.1. *Assume A1–A5, and let $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ denote an optimal path from $k_0 \in \text{int}(K)$. Then, there are prices $\{q_t, p_{t-1}\}_{t=1}^{\infty}$ with $q_t \in \partial u(c_t)$, $p_{t-1} \in \partial V(k_{t-1})$ and such that $(-p_{t-1}, q_t, \delta p_t)$ supports G at (k_{t-1}, c_t, k_t) at each t.*

The condition '$(-p_{t-1}, q_t, \delta p_t)$ supports G at (k_{t-1}, c_t, k_t)' can be restated as

$$q_t c_t + \delta p_t k_t - p_{t-1} k_{t-1} \geq q_t c + \delta p_t k' - p_{t-1} k$$

for all $(k, c, k') \in G$. In other words, a firm with technology set given by G—producing consumption and capital, with capital as an input—would find the optimal path to be profit-maximizing if it faced the sequence of prices derived in the lemma.

[9] If x^* maximizes f over all of X, then $0 \in \partial f(x^*)$, by definition of ∂f. The abstract Kuhn-Tucker theorem follows from noting that maximizing f over some constraint set D is the same thing as maximizing $f + \Gamma_D$ over all of X, where $\Gamma_D(x) = 0$ if $x \in D$ and $\Gamma_D(x) = -\infty$ otherwise. Then, under the given assumptions, zero must be in the supergradient of $(f + \Gamma_D)(x^*)$ at an optimal x^*, which supergradient is in turn simply $\partial f(x^*) + \partial \Gamma_D(x^*)$. But $\partial \Gamma_D(x^*)$ is simply $-\text{supp}\{D, x^*\}$. For results concerning supergradients, see Clarke (1983).

The proof of the lemma, given in the Appendix, proceeds inductively by showing that if ∂V is ever nonempty along an optimal path with $k_t \in K$ for all t, then ∂V is nonempty thereafter, as is ∂u. Further, the prices contained in the supergradients, appropriately discounted, support the optimal path in the sense described above. The condition $k_t \in K$ for $t = 1, 2, \ldots,$ is a ready consequence of the assumption that $k_0 \in K$, given that V satisfies Bellman's equation. To begin the induction, an appeal to standard results shows that if $k_0 \in \text{int}(K)$, we will have $\partial V(k_0) \neq \varnothing$.

Note that since $\text{int}(R_+^m) \subset K$ when Assumption (P) is made, we will ultimately have supporting prices from any $k_0 \gg 0$.

5. ENDOGENOUS GROWTH

We now combine the necessary conditions derived in the last section with monotonicity assumptions on u and Φ and the productivity Assumption (P) regarding Φ and δ. The monotonicity assumptions imply, and we will show, that the prices at each date are such that q_t, the vector of consumption prices, is strictly positive, and p_t, the vector of capital values, is nonzero and weakly positive. The productivity assumption yields an even sharper restriction: along any optimal path which is price-supported, the q_t's converge to zero. Combining this with the concavity and monotonicity of u yields the conclusion that the optimal path of consumption must grow without bound. The monotonicity assumption is:

A6. *Φ is non-decreasing ($\tilde{k} \geq k$ implies $\Phi(k) \subset \Phi(\tilde{k})$), and u is strictly increasing ($\tilde{c} > c$ implies $u(\tilde{c}) > u(c)$).*

Recall that the productivity Assumption (P) is:

(P). *There are $\bar{c} \gg 0$ and $\bar{k} > 0$ with $\lambda(\bar{c}, \delta^{-1}\bar{k}) \in \Phi(\lambda\bar{k})$ for all $\lambda \geq 0$.*

Another way of stating (P) is that $\text{Gr}(\Phi)$ contains the ray through $(\bar{k}, \bar{c}, \delta^{-1}\bar{k})$. Note that G contains this ray less the origin, since any positive scalar multiple of $\bar{k}$ is in the set K.

The monotonicity assumption on Φ implies that V is non-decreasing; hence, if $p \in \partial V(k)$, then $p \geq 0$. Since u is strictly increasing, $q \in \partial u(c)$ implies $q \gg 0$. This implies that the sequence of prices $\{q_t\}_{t=1}^{\infty}$ from the previous lemma satisfies $q_t \gg 0$ for all t. Combining this with the fact that G contains the ray through $(\bar{k}, \bar{c}, \delta^{-1}\bar{k})$, the $\{p_t\}_{t=1}^{\infty}$ of Lemma 4.1 must in fact satisfy $p_t \neq 0$ for all t. To see this, suppose that $p_{t-1} = 0$ for some t. If the profit-maximization condition is to be satisfied at t, we must have $0 \geq q_t\bar{c} + \delta p_t(\delta^{-1}\bar{k}) - p_{t-1}\bar{k} = q_t\bar{c} + p_t\bar{k}$. But, $q_t \gg 0$, $p_t \geq 0$, $\bar{c} \gg 0$ and $\bar{k} > 0$ together imply that $q_t\bar{c} + p_t\bar{k}$ is strictly greater than zero, so the inequality cannot hold at t, in violation of the previous lemma. Hence, we must have $p_t > 0$ for all t.

Some discussion of (P) is in order. Clearly, (P) is an assumption of some measure of constant returns to scale. Constant returns to scale implies that there are no essential fixed factors of production. In a model with primary resources such as labor and land, one would have to view those resources as being measured not in terms of physical stocks, but rather in terms of the services which they provide. This is the standard view in human capital-based growth models in which hours of labor are in fixed supply, but the services of labor may be augmented by skill accumulation.

(P) also implies that the β of Assumption A1 can be no less than δ^{-1}, or $\beta\delta \geq 1$; in most particular examples, we will in fact have $\beta\delta > 1$. Here, the tension between existence and growth of optimal paths becomes clear. If u is unbounded above, so that the γ from A2 is positive, the dual requirements of $\beta^{\gamma}\delta < 1$ and $\beta\delta > 1$ can place tight restrictions on the primitives of the model, if one is to have both existence and growth. The simple one-sector model with $f(k) = Ak$, for $A \geq 1$, and $u(c) = c^{\gamma}/\gamma$, for $\gamma \neq 0$, provides a good illustration of this tension. The conditions for existence and growth in this case are $A^{\gamma}\delta < 1$ and $\delta A > 1$. Optimal paths, when they exist, have a simple form; because of the homogeneity of utility and the linearity of the technology we must have $k_t = \theta A k_{t-1}$ and $c_t = (1-\theta)Ak_{t-1}$ for some $\theta \in (0,1)$.[10] In fact, from the Euler equations for the problem,[11] one can show that $\theta = (A^{\gamma}\delta)^{(1/1-\gamma)}$. The common growth factor shared by consumption and capital is then $(\delta A)^{(1/1-\gamma)}$, which is greater than one whenever $\delta A > 1$. Momentary utility at date t along such a path will be proportional to $[(\delta A)^{(\gamma/1-\gamma)}]^{t-1}$; discounting by δ^{t-1} gives $[(\delta A^{\gamma})^{(1/1-\gamma)}]^{t-1}$, so the utility sum converges whenever $A^{\gamma}\delta < 1$. For $\gamma \in (0,1)$, $\delta < 1$ is necessary for there to exist an A which meets both conditions; given $\delta < 1$ and $\gamma \in (0,1)$, an interval of feasible A's exists, the size of which shrinks as either γ or δ approach one. It is in this case, with utility unbounded above, that the tension between existence and growth is most pronounced. For $\gamma < 0$, so utility is bounded above but unbounded below, any $A > 1/\delta$ will meet both requirements if $\delta < 1$. This is not surprising since, when utility is unbounded below, a more productive technology enhances, rather than harms, the possibility for existence.

As noted in Section 3, upcounting—having $\delta > 1$—is in fact possible when $\gamma < 0$. If $\gamma < 0$ and $\delta > 1$, the requirements for both existence and growth are met by any A with $A^{\gamma} < 1/\delta$, since $A^{\gamma} < 1/\delta < 1$ implies $A > 1 > 1/\delta$. In this case, existence actually presupposes growth. If we think of (P) as a constraint on the primitives of the model, that constraint is slack in this case.

Note, too, that (P) renders inadmissible for optimal growth considerations certain types of momentary utility functions. In particular, if u is homogenous of degree one, an optimum will fail to exist from strictly positive initial stocks. This follows from the fact that if $k \gg 0$, then the path of consumption given by $c_t = \lambda(1/\delta)^{t-1}\bar{c}$ is feasible from k for some $\lambda > 0$, because of free disposal. If u is homogeneous of degree one, $u(c_t) = (1/\delta)^{t-1}u(\lambda\bar{c})$, and $\sum_{t=1}^{T} \delta^{t-1}u(c_t)$ diverges as T goes to infinity. In particular, linear or Cobb-Douglas felicities are ruled out.

A simple multisector model illustrating (P) is the fixed coefficients model with

$$\Phi(k) = \{(c, k') \in R_+^n \times R_+^m : Qc + Rk' \leq k\},$$

[10] See Boyd (1990b).
[11] See Section 6, below.

where Q is an $m \times n$ nonnegative matrix and R is an $m \times m$ nonnegative matrix. The i, jth element of Q, q_{ij}, is the amount of capital good i needed at the outset of the period per unit of consumption good j produced within the period, while r_{ij} is the amount of the ith capital good required per unit of capital good j taken out of the period. A sufficient condition for a model with this Φ to satisfy (P) is that the matrix $(I - \delta^{-1}R)$ have a strictly positive inverse. When $n = m = 1$ and $Q = R = A^{-1}$, Φ reduces to the one-sector linear technology, and the condition that $(I - \delta^{-1}R)$ have a positive inverse becomes the condition $\delta A > 1$.

A simple consequence of our assumptions thus far is the nonexistence of a nonzero optimal steady state.[12]

THEOREM 5.1. (*Nonexistence of an optimal steady state.*) *Make, in addition to the assumptions of Lemma* 4.1, *Assumptions A*6 *and* (*P*). *Then, the optimal growth model cannot have a nonzero optimal steady state.*

PROOF. *Suppose that* (k^*, c^*) *is an optimal steady state. By Lemma 4.1, there is* $q^* \in \partial u(c^*)$ and $p^* \in \partial V(k^*)$ such that $(-p^*, q^*, \delta p^*)$ supports G at (k^*, c^*, k^*). Let $(\bar{k}, \bar{c})$ be as in Assumption (P)—i.e., $\bar{c} \gg 0$, $\bar{k} > 0$ and $\lambda(\bar{k}, \bar{c}, \delta^{-1}\bar{k}) \in \mathrm{Gr}(\Phi)$ $\forall \lambda > 0$. Thus, we must have

$$q^*c^* + \delta p^*k^* - p^*k^* \geq \lambda\big(q^*\bar{c} + \delta p^*(\delta^{-1}\bar{k}) - p^*\bar{k}\big)$$

$$\geq \lambda q^*\bar{c}$$

for all $\lambda > 0$. But $q^* \gg 0$ since u is strictly increasing, so $\bar{c} \gg 0$ implies $q^*\bar{c} > 0$—implying the above inequality cannot be maintained for all $\lambda > 0$. □

It's interesting that the existence of a capital stock expansible by δ^{-1}, when taken in conjunction with the assumption of *bounded* feasible paths, is instrumental in proving the *existence* of an optimal steady state. Here, with boundedness relaxed, the expansible stock assumption is instrumental in proving the *nonexistence* of an optimal steady state.[13] Theorem 5.1 also shows the sense in which the determinants of growth in this model are related to the determinants of a steady state in the standard neoclassical model with an essential fixed factor of production. Basically, the list of ingredients is the same except for the constant returns to scale with respect to the expansible stock. The intuitive picture is that of a demand curve for capital which lies everywhere above capital's long run supply curve, which is flat at the rate of time preference. The "expansibility" part of (P) puts the demand curve initially above the supply curve, just as in the basic neoclassical model, while the "constant returns to scale" part keeps it there. The lack of an intersection between the demand for capital and its long-run supply vitiates the possibility of an optimal

[12] An optimal steady state in this context is a pair (k^*, c^*) such that the path $\{c_t, k_{t-1}\}_{t=1}^{\infty}$, where $c_t = c^*$ and $k_t = k^*$ $\forall t$, is optimal from $k_0 = k^*$.

[13] In the standard reduced-form model from the turnpike literature, where consumption is not explicitly introduced, the boundedness assumption typically takes the form: There are constants $\kappa > 0$ and $\theta < 1$ such that if (k_{t-1}, k_t) is a feasible combination of current and next-period capital, then $\|k_t\| \leq \theta \|k_{t-1}\|$ whenever $\|k_{t-1}\| \geq \kappa$.

steady state and, as Theorem 5.2 shows, guarantees the endogenous growth of optimal paths.

The next result shows that the marginal utilities of consumption along the optimal path—the prices q_t—must go to zero as t goes to infinity. Given the concavity and monotonicity of utility, this is tantamount to the level of consumption going to infinity for some subset of the n consumption goods. Whether consumption of all n goods goes to infinity or not will depend on the specific assumptions made in a given model as regards the function u. It is conceivable that, given substitutabilities between goods, consumption of some goods may go to infinity while consumption of other goods remains bounded, perhaps even going to zero. A model that predicted eventual unbounded consumption of *all* goods would hardly be realistic if goods are distinguished with even moderate precision.[14] In more aggregative models it is perhaps reasonable to view all goods within a period as complements, in which case $q_t \to 0$ would imply $c_{it} \to \infty$ for all $i = 1, 2 \dots n$. Obviously, if u takes the form $u(c_t) = v_1(c_{1t}) + v_2(c_{2t}) + \cdots + v_n(c_{nt})$, with each v_i strictly increasing and concave, then $q_t \to 0$ implies $c_{it} \to \infty$ for all $i = 1, 2 \dots n$.

THEOREM 5.2. *Let* $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ *denote an optimal path from initial stocks* $k \gg 0$, *and let* $\{q_t, p_{t-1}\}_{t=1}^{\infty}$ *be as derived in Lemma* 4.1. *Then* $\lim q_t = 0$.

PROOF. Since G contains the ray through $(\bar{k}, \bar{c}, \delta^{-1}\bar{k})$ we must have

$$0 \geq q_t \bar{c} + \delta p_t(\delta^{-1}\bar{k}) - p_{t-1}\bar{k}$$

or

$$0 \geq q_t \bar{c} + p_t \bar{k} - p_{t-1}\bar{k} \;\; \forall t.$$

We've already noted that q_t and $\bar{c}$ are both strictly positive, so $q_t \bar{c} > 0$. Thus

$$0 > p_t \bar{k} - p_{t-1}\bar{k},$$

or $p_{t-1}\bar{k} > p_t \bar{k}$ for all t. Since p_t and $\bar{k}$ are both positive, $p_t \bar{k} \geq 0$. Thus, $\{p_{t-1}\bar{k}\}_{t=1}^{\infty}$ is a decreasing sequence of real numbers, bounded below by zero—hence convergent, hence Cauchy. So, for any $\epsilon > 0$, there is a T with $|p_t \bar{k} - p_{t-1}\bar{k}| < \epsilon$ whenever $t \geq T$. Hence,

$$\begin{aligned} 0 &\geq q_t \bar{c} + p_t \bar{k} - p_{t-1}\bar{k} \\ &\geq q_t \bar{c} - |p_t \bar{k} - p_{t-1}\bar{k}| \\ &> q_t \bar{c} - \epsilon \end{aligned}$$

for all $t \geq T$. In other words, for any $\epsilon > 0$, there is a T with $\epsilon > q_t \bar{c} > 0$ for all $t \geq T$. Since $\bar{c} \gg 0$, the result in the statement of the theorem is immediate. □

[14] To borrow an example from Stokey (1988), one would not want consumption of both gruel and steak to grow without bound in a reasonable model.

In the case of a single consumption good, given the concavity and monotonicity of u, $\lim q_t = 0$ is equivalent to $\limsup c_t = +\infty$. With more than one consumption good, the relationship between the asymptotic behavior of q_t and that of c_t will, as noted above, depend on aspects of utility such as the presence of complementarities or substitutabilities between goods within a given period. Nonetheless, consumption of some subset of the n consumption goods must grow without bound:

COROLLARY 5.3. *Let c_t and q_t be as in Theorem* 5.2. *The condition* $\lim q_t = 0$ *implies* $\limsup \|c_t\| = +\infty$.

PROOF. Suppose that c_t is bounded. Let $c^* = \sup c_t$, which is then finite. By definition of q_t

$$u(c_t) + q_t(c - c_t) \geq u(c) \ (\forall c \in R^n_+) \ (\forall t).$$

In particular, $u(c_t) + q_t(c^* + e - c_t) \geq u(c^* + e)$ where $e = (1, 1, \dots, 1)$. Rearranging, we obtain $q_t(c^* + e) \geq u(c^* + e) - u(c_t) + q_t c_t$. The right-hand side of this last inequality is bounded away from zero by a strictly positive number, since u is strictly increasing and $c^* \geq c_t \ \forall t$. But the condition $\lim q_t = 0$ implies there is eventually a t with $q_t(c^* + e)$ less than any fixed positive number, an obvious contradiction. Thus c_t is *not* bounded, and $\limsup \|c_t\| = +\infty$. □

As an example to illustrate the possibilities here, consider the felicity function

$$u(c_1, c_2) = \frac{-1}{1 + c_1 + c_2}.$$

This u is differentiable, with $Du(c) = (1/(1 + c_1 + c_2)^2, 1/(1 + c_1 + c_2)^2)$. Thus, if $q_t = Du(c_t)$ goes to zero, we may conclude that either c_{1t} has gone to infinity or c_{2t} has gone to infinity, but not necessarily both c_{1t} and c_{2t}. On the other hand, if, for example,

$$u(c_1, c_2) = c_1^a c_2^b$$

where $a + b < 1$, then $Du(c_t)$ going to zero is equivalent to both c_{1t} and c_{2t} going to infinity.

What does Theorem 5.2 imply for the behavior of capital stocks along the optimal path? Clearly, since Φ is compact-valued, unbounded growth of any subset of consumption goods can only occur if some subset of the capital stocks grows without bound as well. As with the consumption goods, more specific assumptions on the primitives u and Φ would yield more precise implications for the behavior of capital along the optimal path. For example, in the fixed coefficients model described above, if u is separable across consumption goods, so $c_{it} \to \infty$ for all $i = 1, 2 \dots n$, and if each capital good is an input in the production of some consumption good, which means for each $i \in \{1, 2 \dots m\}$ there is a $j \in \{1, 2 \dots n\}$ with $q_{ij} > 0$, then $k_{it} \to \infty$ for all $i = 1, 2 \dots m$.

Note that all that is essential to the proof of Theorem 5.2 is that the input–output combination $(\bar{k}, \bar{c}, \delta^{-1}\bar{k})$ earn a non-positive profit at the supporting prices. A simple technology (simple in an æsthetic sense) which accommodates this requirement is that Gr(Φ) contains a convex cone which contains $(\bar{k}, \bar{c}, \delta^{-1}\bar{k})$. This is substantially the assumption made by Jones and Manuelli (1990) in their variant of the one-sector model.

A comparison with Jones and Manuelli's result is perhaps in order here. The model which Jones and Manuelli work with is a Ramsey model with multiple capital stocks, but a single produced consumption-investment good. Formally, if $k \in R^m_+$ is current capital, then current output is $f(k)$ where f is assumed to satisfy the usual conditions of concavity, continuity and differentiability. The all-purpose produced good is divided between consumption, c, and next-period's capital, $\Sigma_{i=1}^m k_i'$. For convenience, I've subsumed the depreciation of capital, which Jones and Manuelli keep explicit, into the definition of f. As the manner of investment makes clear, while there are many capital goods, all capital goods are perfect substitutes on the supply side.

To guarantee growth of the optimal path, Jones and Manuelli assume first that there is a degree-one homogeneous, concave function h with $f(k) \geq h(k)$ for all k. Further, they assume that there is a positive vector of capital stocks $\bar{k}$ such that if $\bar{k}_i > 0$, then $\delta h_i(\bar{k}) > 1$, where δ is the discount factor, and h_i denotes the ith partial derivative of h. Under this assumption and standard convexity and continuity assumptions, they show that any optimal path must satisfy $\limsup c_t = +\infty$.

We may show that our Assumption (P) is an implication of Jones and Manuelli's assumption. Suppose that h and $\bar{k}$ are as in Jones and Manuelli's assumption, that is, h is homogeneous of degree one, with $h \leq f$ and $\delta h_i(\bar{k}) > 1$ whenever $\bar{k}_i > 0$. Since h is degree-one homogeneous, Euler's theorem implies $\delta h(\bar{k}) = \delta \Sigma_{i=1}^n h_i(\bar{k})\bar{k}_i > \Sigma_{i=1}^n \bar{k}_i$. Since $f \geq h$, we have $\delta f(\bar{k}) > \Sigma_{i=1}^n \bar{k}_i$, or $f(\bar{k}) > \Sigma_{i=1}^n \delta^{-1}\bar{k}_i$. In other words, given initial capital $\bar{k}$, it is feasible to produce next-period's capital in the amount $\delta^{-1}\bar{k}$, and still have strictly positive consumption of $\bar{c} \equiv f(\bar{k}) - \Sigma_{i=1}^n \delta^{-1}\bar{k}_i$ left over. Furthermore, since h is degree-one homogeneous, any scalar multiple of this plan is also feasible.

6. A SIMPLE EXAMPLE

In this section, I consider a simple one-sector Ramsey model with adjustment costs. I show how the results on the existence of optimal paths and the existence of endogenous growth can be applied in practice. Despite the model's simplicity, it is not encompassed by previous growth results such as Jones and Manuelli (1990).

In this model, output is produced from capital according to a linear production function $f(k) = Ak$. Output is divided between consumption, c, and investment, i. Next period's stock of capital depends on current capital and the rate of investment, i/k. In particular, assume that $k' = kg(i/k)$, where g is continuous, strictly increasing, concave and satisfies $\lim_{k \to 0} kg(i/k) = 0$ for each $i \geq 0$. The production correspondence Φ is then given by

$$\Phi(k) = \{(c, k') \in R^2_+ : c + i \leq Ak, k' = kg(i/k) \text{ for some } i \geq 0.\}.$$

On the preference side, assume for simplicity that $u(c) = c^\gamma/\gamma$, for $\gamma \neq 0$, and $\delta \in (0,1)$.

In order to check for the existence of optimal paths, it is enough to verify the last part of A1, that $(c, k') \in \Phi(k)$ implies $c \leq \eta + \theta k$ and $k' \leq \alpha + \beta k$ for some $\alpha, \eta, \theta \geq 0$ and $\beta \geq 1$, and the Brock–Gale condition, A3. Clearly Φ satisfies the first part of A1—compactness, continuity and free disposal—and u obviously satisfies A2—upper semicontinuity on R_+, boundedness below on $\text{int}(R_+)$ and $u(c) \leq \nu + \mu c^\gamma/\gamma$ for constants ν, μ and γ.

From the definition of Φ, for any $k \in R_+$ we must have $0 \leq c \leq Ak$ and $0 \leq k' \leq kg(Ak/k) = kg(A)$. Thus, $\theta = A$, $\beta = \max\{g(A), 1\}$ and any $\alpha, \eta \geq 0$ will meet the conditions of A1. If we also have $\delta\beta^\gamma < 1$ (A3), we may conclude that an optimal path exists from any $k \geq 0$, though when u is unbounded below, we may have $V(k) = -\infty$. However, just as with the more general analysis of Section 3, when (P) is assumed to hold we will have $V(k) > -\infty$ from any $k > 0$.

We now turn to the question of growth. Under what parameter restrictions will the optimal paths in this model display endogenous growth? Obviously, u and Φ satisfy all the basic continuity and convexity assumptions. Also, Φ is nondecreasing and u is strictly increasing, as required by A6. We need only verify the key Assumption (P). For (P), first note that Φ displays constant returns to scale. To see this, note that multiplying k by $\lambda > 0$ multiplies feasible choices of consumption and investment by λ as well. The feasible rates of investment i/k are unchanged. Since next-period's capital is linear in k given the rate of investment, feasible choices of next-period's capital scale by λ as well.

To check the rest of (P), note that what we want are a $\bar{k} > 0$ and a $\bar{c} > 0$ such that $A\bar{k} \geq \bar{c} + i$ and $\delta^{-1}\bar{k} = \bar{k}g(i/\bar{k})$ for some $i \geq 0$. This condition may be restated as: there exists a $\bar{k} > 0$ such that $\delta g(i/\bar{k}) = 1$ and $i/\bar{k} < A$ for some i. Since g is continuous and strictly increasing, a sufficient condition is $\delta g(A) > 1$, since we can then take $\bar{k}$ to be any positive number and set $i = \alpha(A\bar{k})$ for $\alpha \in (0,1)$. With $\delta g(A) > 1$, there will be an $\alpha < 1$ such that $\delta g(\alpha A) = 1$ and $\bar{c} = (1-\alpha)A\bar{k} > 0$. Since $\delta < 1$, we must have $g(A) > 1$. Thinking back to the discussion of existence, we then have $g(A) = \max\{g(A), 1\}$, and the Brock–Gale condition becomes $\delta g(A)^\gamma < 1$.

We have an analogy to the simple one-sector linear model. There, growth was guaranteed by the restriction $\delta A > 1$; assuming $A > 1$, the existence condition for that model would be $\delta A^\gamma < 1$. Both conditions can be recovered here by letting $g(i/k) = i/k$. We also can see again that the dual requirements of existence and growth can put fairly sharp restrictions on the primitives of the model. Here, feasible choices of δ, A, γ and $g(\cdot)$ are circumscribed by the conditions $\delta g(A)^\gamma < 1$ for existence, and $\delta g(A) > 1$ for growth.

One can see how growth is implied by the condition $\delta g(A) > 1$ by considering the Euler equations which characterize the optimum for this model. For simplicity, let z_t denote the rate of investment at time t, so $k_t = k_{t-1}g(z_t)$. The Euler equations are:

$$\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} = \delta g'(z_t)\left\{A + \frac{g(z_{t+1})}{g'(z_{t+1})} - z_{t+1}\right\} \ (\forall t).$$

Despite the adjustment costs, the technology is still constant returns to scale. Couple this with homogeneous utility, and the optimal choices for consumption and next-period's capital must be linear in current capital, implying that investment is also linear in current capital.[15] Thus, $i_t = \theta k_{t-1}$ for some θ, and $z_t = \theta$ for all t. The Euler equations then reduce to:

$$\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} = \delta\{g(\theta) + g'(\theta)(A - \theta)\}.$$

Also, $c_{t+1}/c_t = k_t/k_{t-1} = g(\theta)$. It's quite simple to see, given this expression for the Euler equation, that our condition $\delta g(A) > 1$ generates growth. To see this, note that since g is concave, $g(z) + g'(z)(\tilde{z} - z) \geq g(\tilde{z})$ for all z and $\tilde{z}$. In particular,

$$g(\theta) + g'(\theta)(A - \theta) \geq g(A)$$

which, from the Euler equation, implies

$$\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} \geq \delta g(A) > 1.$$

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7. APPENDIX: PROOFS OF LEMMAS 3.1 AND 4.1

7.1. *Lemma* 3.1. Let A1–A3 hold, and let $b > \beta \geq 1$ and such that $b^{\gamma}\delta < 1$. Since $\beta^{\gamma}\delta < 1$ by A3, such a b exists. Given A1, if $\{c_t\}_{t=1}^{\infty}$ is a feasible path of consumption, we must have

$$\begin{aligned} \|c_t\| &\leq \eta + \theta\|k_{t-1}\| \\ &\leq \eta + \theta\left[\alpha\left(\frac{b^{t-1} - 1}{b - 1}\right) + b^{t-1}\|k\|\right] \end{aligned}$$

or

$$\|c_t\| \leq \eta + \theta b^{t-1}(\alpha + \|k\|),$$

since $b > 1$. This verifies the claim made in the text, that each c_t along a feasible path resides in a compact subset of R_+^n. Analogously, each k_t associated with a feasible path of consumption lies in a compact subset of R_+^m. By Tychonoff's theorem, both $F(k)$ and the set of associated capital paths lie in product-compact sets. That $F(k)$ is closed in the product topology is then a simple consequence of the continuity and compact-valuedness of Φ.

[15] See Boyd (1990b).

What remains is to verify that lifetime utility is upper semicontinuous in the product topology on $F(k)$. The steps we follow are a 'partial summation' technique, adapted from Boyd (1990a). From A2 and the previous inequality, we obtain

$$u(c_t) \leq \nu + \mu\left[\eta + \theta b^{t-1}(\alpha + \|k\|)\right]^{\gamma}/\gamma$$

$$\leq \nu + M(b^{\gamma})^{t-1}$$

where $M \equiv \mu[\eta + \theta(\alpha + \|k\|)]^{\gamma}/\gamma$. The last inequality relies on the assumptions $\eta, \mu \geq 0$ and $b > \beta \geq 1$, and the fact that $(r)^{\gamma}/\gamma$ is an increasing function of $r \geq 0$.

For $T = 1, 2, \ldots$, consider the partial sums:

$$U_T(\{c_t\}_{t=1}^{\infty}) = \sum_{t=1}^{T} \delta^{t-1}\{u(c_t) - \nu - M(b^{\gamma})^{t-1}\}.$$

Given that u is upper semicontinuous on R_+^n, each U_T is upper semicontinuous in the product topology on $F(k)$. Moreover, given that the terms in the summations are nonpositive, the U_T's form a decreasing sequence, with infimum

$$U_{\infty}(\{c_t\}_{t=1}^{\infty}) = \sum_{t=1}^{\infty} \delta^{t-1}u(c_t) - \frac{\nu}{1-\delta} - \frac{M}{1-b^{\gamma}\delta},$$

since $\delta < 1$ and $b^{\gamma}\delta < 1$. As the infimum of any collection of upper semicontinuous functions is upper semicontinuous,[16] we conclude that $\sum_t \delta^{t-1}u(c_t)$ is upper semicontinuous in the product topology on $F(k)$. The result in the lemma then follows by the Weierstrass theorem. □

Note from the above arguments that the part of A3 which assumes $\delta < 1$ can be relaxed to state: either $\delta < 1$ or $\nu = 0$. This accommodates upcounting, though, as noted in Section 3, upcounting, consumption growth and utility unbounded above are not consistent with existence. When utility is bounded above by zero, but unbounded below, existence under upcounting presupposes consumption growth.

7.2. *Lemma* 4.1. Obviously, $k_0 \in K$ implies $k_t \in K$ for every t along an optimal path, since V must satisfy Bellman's equation. Also, $k_0 \in \text{int}(K)$ implies $\partial V(k_0) \neq \emptyset$, since V is proper, concave and bounded below on a neighborhood of k_0. The following steps set up an induction which, given $\partial V(k_0) \neq \emptyset$, show that $\partial V(k_t) \neq \emptyset$ for every t.

Suppose that $\partial V(k_{t-1}) \neq \emptyset$ for some $t \geq 1$, and consider the function W defined on $G = \text{Gr}(\Phi) \cap \{K \times R_+^n \times K\}$ as follows:

$$W(k, c, k') = u(c) + \delta V(k') - p_{t-1}k$$

[16] See Berge (1963, chapter IV, §8, Theorem 3).

where $p_{t-1} \in \partial V(k_{t-1})$. By definition of ∂V, we have:

$$V(k_{t-1}) - p_{t-1}k_{t-1} \geq V(k) - p_{t-1}k \ \forall k \in R^m_+.$$

Since (c_t, k_t) along the optimal path attains the maximum on the right-hand side of Bellman's equation at each date, given k_{t-1}, the left-hand side of the above inequality is simply $W(k_{t-1}, c_t, k_t)$. Meanwhile, by definition of $V(k)$, the right-hand side exceeds $u(c) + \delta V(k') - p_{t-1}k$ for any $(c, k') \in \Phi(k)$, for every k. In other words:

$$V(k) - p_{t-1}k \geq W(k, c, k') \ \forall (k, c, k') \in \mathrm{Gr}(\Phi),$$

and in particular $\forall (k, c, k') \in G$. Combining these inequalities, we have:

$$W(k_{t-1}, c_t, k_t) \geq W(k, c, k') \ \forall (k, c, k') \in G.$$

Since $k_\tau \in K$ at all dates τ along an optimal path, $(k_{t-1}, c_t, k_t) \in G$, and the above inequality may be stated as: (k_{t-1}, c_t, k_t) maximizes W over G. The function W is concave, and G is convex with nonempty interior. By the abstract Kuhn-Tucker theorem, a necessary condition for this maximization is that $\partial W(k_{t-1}, c_t, k_t)$ have a nonempty intersection with $\mathrm{supp}\{G, (k_{t-1}, c_t, k_t)\}$. But $\partial W(k_{t-1}, c_t, k_t)$ is clearly $\{-p_{t-1}\} \times \partial u(c_t) \times \delta\partial V(k_t)$. In other words, for some $q_t \in \partial u(c_t)$ and $p_t \in \partial V(k_t)$, we have $(-p_{t-1}, q_t, \delta p_t)$ supporting G at (k_{t-1}, c_t, k_t). The price vector p_t may be used to repeat this argument for the subsequent period. □

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