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THE CLASSICAL THEOREM ON EXISTENCE OF COMPETITIVE EQUILIBRIUM[1]

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This paper presents the classical theorem on the existence of equilibrium as it was proved in the 1950's with the various improvements that have been made since then. In particular, the elimination of the survival assumption and of the requirement of transitive preferences are carried through with a proof that uses a mapping of social demand. This approach favors intuitive understanding and generalization of the results. Finally, the role of the firm and the introduction of external economies are critically viewed.

MY PURPOSE IS TO DISCUSS the present status of the classical theorem on existence of competitive equilibrium that was proved in various guises in the 1950's by Arrow and Debreu [**1**], Debreu [**5**, **6**], Gale [**8**], Kuhn [**14**], McKenzie [**17**, **18**, **19**], and Nikaido [**22**]. The earliest papers were those of Arrow and Debreu, and McKenzie, both of which were presented to the Econometric Society at its Chicago meeting in December, 1952. They were written independently. The paper of Nikaido was also written independently of the other papers but delayed in publication.

The major predecessors of the papers of the fifties were the papers of Abraham Wald [**31**, **32**] and John von Neumann [**30**], all of which were delivered to Karl Menger's Colloquium in Vienna during the 1930's. The paper of von Neumann was not concerned with competitive equilibrium in the classical sense but with a program of maximal balanced growth in a closed production model. However, he first used a fixed point theorem for an existence argument in economics and provided the generalization of the Brouwer theorem that was the major mathematical tool in the classical proofs. Wald achieved the first success with the general problem of the existence of a meaningful solution to the Walrasian system of equations. The proofs which were published used an assumption that later became known as the Weak Axiom of Revealed Preference. This axiom virtually reduces the set of consumers to one person, since it is equivalent to consistent choices under budget constraints. In a one consumer economy the existence of the equilibrium becomes a simple maximum problem and advanced methods are not needed. When many consumers with independent preference orders are present, it has been shown (Uzawa [**29**]) that fixed point methods are necessary. Wald also wrote a third paper whose main theorem was announced in a summary article [**33**], but which never reached publication in the disturbed

[1]This paper is a revision of my Presidential Address to the Econometric Society delivered in Ottawa and Vienna in 1977. I have benefited on several occasions when this paper was presented. I would especially like to recall the assistance I received from William Vickery in Ottawa, from Birgit Grodal in Vienna, and from Wayne Shafer in Princeton. I am also grateful to Kenneth Arrow, Gerard Debreu, Charles Wilson, and Makoto Yano for useful comments toward a revised draft, and to Martin Feinberg for the proof of the proposition in Appendix II.

conditions of Vienna of the period. This theorem does not make the Weak Axiom assumption and presumably fixed point methods were used in the proof. However, the paper apparently has not survived and did not directly influence the writers of the fifties.

The classical theorem is characterized above all by its use of assumptions of finiteness and convexity. That is, the economy comprises a finite number of economic agents or consumers who trade in a single market under conditions of certainty. The goods are finite in number and, as a consequence, the horizon is finite. Goods are divisible, and production is modeled either as a set of linear activities in the space of goods or as convex input-output sets belonging to a finite list of firms. Consumption sets and preference relations are also convex in an appropriate sense. Consumption and production activities are mutually independent.

In subsequent years the abstract model of an economy has been complicated for the existence theorems in many directions, principally weakening the crucial finiteness and convexity assumptions. However, somewhat surprisingly, in recent years the classical theorem itself has been improved in basic ways by Andreu Mas-Colell and James Moore. Mas-Colell **[16]** and Gale and Mas-Colell **[9]** showed that preferences need not be assumed transitive or complete. On the other hand, Moore **[21]** showed that the assumption that agents may survive without trade is superfluous for an irreducible economy.

In this paper I will introduce these innovations into an exposition based on the use of demand functions and production sets. I believe this order of proof is best for economic understanding and also for achieving the weakest assumptions. In particular, the Mas-Colell-Gale assumptions are weakened and a way is found to incorporate a version of the Moore result without returning to classical preferences. I shall also discuss three other themes that have been pursued in recent papers, the inclusion of external economies affecting production and consumption sets, the representation of firms as coalitions of economic agents, and the elimination of the free disposal assumption by new means.

1. THE CLASSICAL THEOREM

I will use the theorem of my paper of 1959 to represent the classical theorem on existence in fully developed form. The assumptions of this theorem fall naturally into three groups, assumptions on the consumption sets X_i, on the total production set Y, and on the relations between these sets. First, for the consumption sets, which lie in R^n, the Cartesian product of n real lines, we assume

(1) X_i is convex, closed, and bounded from below.

(2) X_i is completely ordered by a convex and closed preference relation.

X_i is interpreted as the set of feasible trades of the ith consumer. There are m consumers. That X_i is *bounded from below* means that there is ξ_i such that $x > \xi_i$ holds for all $x \in X_i$. *Convexity* of the preference relation $\succsim_i$ means that $x \succ_i x'$

implies $x'' \succ_i x'$ where $x'' = tx + (1-t)x'$, for $0 < t < 1$. *Closure* of $\succsim_i$ means that $x^s \to x$ and $x^{s'} \to x'$, where $x^s \succsim_i x^{s'}$, implies $x \succsim_i x'$.

For the total production set Y, which also lies in R^n, we assume

(3) Y is a closed convex cone.

(4) $Y \cap R^n_+ = \{0\}$.

R^n_+ is the nonnegative cone of R^n.

The assumption that Y is a cone recognizes the role of constant returns to scale as a basis for perfect competition. It may be defended as an approximation when efficient firm sizes are small, and in this sense was accepted by both Marshall and Walras. It may be argued that the error of this approximation is of the same order as the error introduced by the assumption of convexity in the presence of indivisible goods. In any case the assumption of convex production sets for firms may be shown to be mathematically equivalent to Assumption (3) (McKenzie [**19**, pp. 66–67]; also see McKenzie [**20**]). Assumption (4) is not a real restriction. It amounts to ignoring goods that are available in any desired quantities without cost. In this model the consumption sets X_i are net of initial stocks, that is, the elements of X_i are possible trades.

Finally, there are two assumptions on the relations between the X_i and Y. Let $X = \sum_1^m X_i$, where m is the number of consumers. Then the first assumption is:

(5) $X_i \cap Y \neq \emptyset$. Moreover, there is a common point $\bar{x}$ in the relative interiors of Y and X.

The first part of this assumption states that any consumer can survive without trade. The second part implies that we may choose the price space so that any price p that supports Y will have $p \cdot x < 0$ for some $x \in X$. In other words, if p is compatible with equilibrium in the production sector, there is a feasible trade for the group of all consumers with negative value. This may be interpreted as saying that some consumer has income, in the sense that he is not on the boundary of his consumption set.

Suppose there are m consumers. Let I_1 and I_2 be nonempty sets of indices for consumers such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1, \ldots, m\}$. Let $x_{I_k} = \sum_{i \in I_k} x_i$ and $X_{I_k} = \sum_{i \in I_k} X_i$, for $k = 1, 2$. The second relation between the X_i and Y is:

(6) However I_1 and I_2 may be selected, if $x_{I_1} = y - x_{I_2}$ with $x_{I_1} \in X_{I_1}$ $y \in Y$, and $x_{I_2} \in X_{I_2}$, then there is also $y' \in Y$, and $w \in X_{I_2}$, such that $x'_{I_1} = y' - x_{I_2} - w$ and $x'_i \succsim_i x_i$ for all $i \in I_1$, and $x'_i \succ_i x_i$ for some $i \in I_1$.

An alternative way of expressing the condition of (6), by substituting for x_{I_2}, is $x'_{I_1} - x_{I_1} = y' - y - w$. That is, I_1 may be moved to a preferred position by the addition of a vector $y' - y$ from the local cone of Y at y (see Koopmans [**13**, p. 83]) plus a feasible trade from I_2. The resource relatedness assumption of Arrow and Hahn [**2**, p. 117], implies Assumption (6), but the converse is not true. Since they assume that a household can survive with less of all the resources it holds (p.

77), they are able to take w equal to a small fraction of the resources held by I_2 consumers. Then it is supposed that I_1 consumers can be benefited with this w.

The purpose of Assumption (6) is to insure that everyone has income, if someone has income, at any price vector that supports the production set Y at y as well as the sets of consumption vectors at least as good as x_i, at the points x_i. Then if we choose I_1 to contain just the indices of the consumers with income, nonempty by Assumption (5), $p \cdot x'_{I_1} > 0$ must hold. Also Assumption (6) with Assumption (2) implies local nonsatiation within the feasible set $X \cap Y$ so that $p \cdot x_{I_1} = 0$. Since $p \cdot y = 0$ and $p \cdot y' \leqslant 0$, it follows that $p \cdot w < 0$. But $w \in X_{I_2}$ so some consumer in I_2 has income in contradiction to the choice of I_1. Thus I_2 must be empty, and the result follows.

Competitive equilibrium is defined by a price vector $p \in R^n$, an output vector y, and vectors $x_1, \ldots, x_m$ of consumer trades that satisfy

(I) $\quad y \in Y$ and $p \cdot y = 0$, and for any $y' \in Y$, $p \cdot y' \leqslant 0$.

(II) $\quad x_i \in X_i$ and $p \cdot x_i \leqslant 0$, and $x_i \succsim_i x'$ for any $x' \in X_i$ such that $p \cdot x' \leqslant 0$, $\quad i = 1, \ldots, m$.

(III) $$\sum_{i=1}^{m} x_i = y.$$

The first condition corresponds to Walras' requirement that in equilibrium there should be "ni bénéfice, ni perte" [**34**, p. 225]. It is not possible for a combination of resources to be formed that allows larger payments to some resource than those implied by p. Resources belonging to "entrepreneurs" are priced along with hired factors, and the entire income of a productive activity is imputed to the cooperating factors. This is the traditional picture of perfect competition in Marshall [**15**], as well as in Walras.

The second condition implies that consumers maximize preference over their budget sets. Debts and taxes are ignored in the classical theorems, though many writers have introduced them subsequently. The third condition says that consumer trades sum to the total production. Given $p \cdot y = 0$ and $p \cdot x_i \leqslant 0$ it follows from condition (III) that $p \cdot x_i = 0$.

We make the following definition.

DEFINITION: A *competitive equilibrium* is a set of vectors $(p, y, x_1, \ldots, x_m)$ that satisfy conditions (I), (II), and (III).

An *economy E* may be defined by $E = (Y, X_i, \succsim_i, i = 1, \ldots, m)$. One form of the classical theorem on the existence of a competitive equilibrium is:

THEOREM 1: *If an economy E satisfies the Assumptions* (1), (2), (3), (4), (5), *and* (6), *there is a competitive equilibrium for E.*

Debreu [**7**] has defined a weaker notion of equilibrium which he calls "quasi-equilibrium." A quasi-equilibrium in our setting satisfies (I) and (III), but in place of (II) there is:

(IIq) $x_i \in X_i$ and $p \cdot x_i \leqslant 0$, and $x_i \succsim_i x'$ for any $x' \in X_i$ such that $p \cdot x' \leqslant 0$, or $p \cdot x_i \leqslant p \cdot x'$ for all $x' \in X_i$, $i = 1, \ldots, m$.

Debreu's strategy for proving existence of equilibrium in this paper is to prove that a quasi-equilibrium exists and then introduce a further assumption which implies that a quasi-equilibrium is a competitive equilibrium. Arrow and Hahn **[2]**, and James Moore **[21]**, follow the same strategy using the closely related notion of a "compensated equilibrium." A compensated equilibrium replaces (II) by:

(IIc) $x_i \in X_i$ and $p \cdot x_i \leqslant 0$, and $p \cdot x_i \leqslant p \cdot x'$ for any $x' \in X_i$ such that $x' \succsim_i x_i$.

If indifference sets may be thick, (IIq) is a weaker condition than (IIc). In particular, when all consumers have income (IIc) implies Pareto optimality, while (IIq) does not. However, under our assumptions, these concepts are equivalent. The assumption that converts a proof that a quasi-equilibrium exists, given Assumption (2), into a proof of existence for competitive equilibrium is essentially irreducibility, our Assumption (6). These assumptions insure that all consumers have income at a quasi-equilibrium, so the second alternative of (IIq) does not occur and the condition of (IIq) implies (II). Since we are primarily interested in competitive equilibrium, we will not use this order of proof.

2. THE SURVIVAL ASSUMPTION

Perhaps the most dramatic innovation since 1959 is the discovery that the survival assumption, that is, the first part of Assumption (5), $X_i \cap Y \neq \emptyset$, can be dispensed with in the presence of the other assumptions, in particular in the presence of Assumption (6) that the economy is irreducible. In retrospect this seems a plausible result. However, it was hidden by the character of the mappings used in the early proofs. These mappings involve demand functions defined on price vectors that are normal to the production set. Then $p \cdot y \leqslant 0$ for all $y \in Y$, and in particular for $y \in X_i \cap Y$. This means that the budget set is never empty and the demand function is always well defined. The demand function may not be upper semi-continuous when the budget plane supports X_i, but the modified function defined by Debreu [7] even has this property. The modified function defines the demand set, when the budget plane supports X_i, as the intersection of X_i and the budget set. Then condition (IIq) of the quasi-equilibrium will be satisfied.

On the other hand, the mapping used by Arrow and Hahn to prove that a compensated equilibrium exists avoids mapping by means of a demand function by giving the Pareto frontier in the space of consumers' utilities a central role. Then the mapping can go forward even if prices are used for which the budget set, defined relative to X_i, is empty. Arrow and Hahn map a Cartesian product of a normalized price set, a set of normalized utility vectors from the Pareto efficient frontier, and a set of feasible allocations of goods into convex subsets. The Pareto efficient frontier is the set of feasible utility allocations $\{u_i\}$, $i = 1, \ldots, m$, such that there exists no feasible allocation $\{u_i'\}$ with $u_i' > u_i$ for all i. If indifference sets may be thick, this mapping need not be upper semi-continuous, so it cannot be used with Debreu's assumptions. Moreover, a compensated equilibrium is Pareto efficient if someone has income, though not necessarily Pareto optimal, while a quasi-equilibrium may not be even Pareto efficient. The key to this distinction is that at a quasi-equilibrium spending is maximized for the utility levels achieved. Therefore, it may be possible to increase everyone's utility without increasing total spending. But when spending is minimized for the utility levels achieved, and there is someone with income, this is no longer possible since to increase this consumer's utility his spending must go up, and no one can reduce spending without losing utility.

Arrow and Hahn still made the survival assumption, but James Moore [**21**] who uses the Arrow-Hahn mapping with small modifications dispenses with this assumption and replaces it for the purpose of compensated equilibrium by a weakened version of irreducibility. Moore also uses a slightly different Pareto frontier defined as the set of utility allocations $\{u_i\}$ such that $u_i \leqslant u_i'$ for all i, for some feasible $\{u_i'\}$, but there is no feasible $\{u_i''\}$ such that $u_i'' > u_i$ for all i. This allows him to drop the free disposal assumption implicitly made by Arrow and Hahn when they define equilibrium [**2**, p. 108].

I think there are advantages to the use of the demand function, or correspondence, in proofs of existence, both for mathematical power and for understanding the proof. I will show how the demand correspondence may be used in a mapping of the Cartesian product of the price simplex and the social consumption set into itself whose fixed points are competitive equilibria even in the absence of the survival assumption. At the equilibrium the budget sets will not be empty, the demand correspondences will be well defined and upper semi-continuous, but these conditions need not be satisfied for non-equilibrium prices. We will avoid the difficulties posed by this possibility by using an extension of the demand correspondence which reduces to the original correspondence whenever the original correspondence is well defined and nonempty. The extended demand correspondence will be well defined and nonempty for all price vectors.

We will first prove the special existence theorem in which an interiority assumption is made. Then the general theorem is proved by a limiting argument in which the interior is removed. This argument will be sketched later. Therefore, we make the six assumptions listed in Section 1 except that Assumption (6) is slightly weakened, and Assumption (5) is replaced by

(5′) $\quad X \cap \text{interior } Y \neq \emptyset.$

Assumption (5′) is used as we explained earlier to insure that someone has income at any price vector that supports Y. Then Assumption (6) will provide income to everyone in equilibrium. The interior point of (5′) is a temporary expedient for proving the special theorem. It is needed for the order of proof that we use since we project points of the social consumption set on the boundary of Y from an interior point of Y. The projection is then continous.

I will sometimes appeal to the results of my *Econometrica* paper of 1959 [**19**] and Debreu's paper of 1962 [**7**] in the subsequent argument. Let $\bar{X}_i$ be the convex hull of X_i and $\{0\}$. Since $0 \in Y$, the survival assumption is met in its original form of Assumption (5) in the economy with $\bar{X}_i$. We also introduce:

AUXILIARY ASSUMPTION: The X_i are bounded.

This assumption is innocuous since $X \cap Y$ is bounded as a consequence of Assumption (1), X_i bounded below, and Assumption (4), $Y \cap R^n_+ = \{0\}$. See Lemmas 8 and 9 of McKenzie [**19**].

Assumption (6) is weakened by choosing w from $\bar{X}_{I_2}$ rather than X_{I_2}.

(6′) However I_1 and I_2 may be selected, if $x_{I_1} = y - x_{I_2}$ with $x_{I_1} \in X_{I_1}$, $y \in Y$, and $x_{I_2} \in X_{I_2}$, then there is also $y' \in Y$, and $w \in \bar{X}_{I_2}$, such that $x'_{I_1} = y' - x_{I_2} - w$ and $x'_i \succsim_i x_i$ for all $i \in I_1$, and $x'_i \succ_i x_i$ for some $i \in I_1$.

This revision of (6) is a significant weakening unless $0 \in X_i$. If $0 \in X_i$ is assumed as in McKenzie [**20**], X_{I_2} and $\bar{X}_{I_2}$ are the same. The revision is in accord with Assumption (e.4′) used by Moore [**21**].

We will define an extension of the demand correspondence to the set $\bar{X}_i$. We first define a correspondence

$$\xi_i(p) = \{x \mid p \cdot x \leqslant 0 \text{ and } x \succsim_i z \text{ for all } z \in X_i \text{ such that } p \cdot z \leqslant 0\}.$$

This is the usual demand correspondence adapted to our case where income is zero, since entrepreneurial resources absorb profits and all resources are included as components of the vectors in X_i. The correspondence ξ_i will be upper semi-continous when there is a cheaper point in X_i, that is, a point x such that $p \cdot x < 0$. However, this property fails on the boundary of X_i. Therefore, we define a modification of the demand correspondence in the manner of Debreu by

$$\psi_i(p) = \xi_i(p) \quad \text{if there is } x \in X_i \text{ and } p \cdot x < 0,$$

$$\psi_i(p) = \left\{x \in \bar{X}_i \mid p \cdot x = 0\right\}, \quad \text{otherwise.}$$

Then $\psi_i(p)$ is well defined for all p, since $0 \in \bar{X}_i$ by the definition of $\bar{X}_i$. It is easily seen that $\psi_i(p)$ is upper semi-continuous (see Debreu [**7**, Lemmas 1 and 2]).

We will use a mapping of the Cartesian product of a normalized price set S with the extended consumption set $\bar{X}$, which can be interpreted as taking prices into demand sets, by means of the demand correspondences, and possible consumptions into the normalized price set by an inverse supply correspondence.

The mapping is so defined that a fixed point will be an equilibrium. The mapping of prices into a social demand set is given by $f(p) = \sum_{i=1}^{m} \psi_i(p)$. Let $\bar{y}$ lie in the interior of Y. The normalized price set S is defined as $S = \{p \mid p \cdot y \leqslant 0$, for all $y \in Y$, and $p \cdot \bar{y} = -1\}$. The set S is convex and compact since Y has an interior and $Y \neq R^n$ (see McKenzie [**19**, Lemma 5]).

In order to map possible consumptions into the price set S, we define a projection on the boundary of Y from $\bar{y}$. Let $\pi(x)$ be the maximum number π such that $\bar{y} + \pi \cdot (x - \bar{y}) \in Y$. It is possible to choose $|\bar{y}|$ large enough so that the function $\pi(x)$ is well defined for $x \in \bar{X}$. See Appendix I for a proof. Then let $h(x) = \bar{y} + \pi(x)(x - \bar{y})$. Then define, for any $y \in$ boundary Y, $g(y) = \{p \in S \mid p \cdot y = 0\}$. It may be shown that h is continuous, and also that $g(y)$ is upper semi-continuous. We may think of $g \circ h$ as an inverse supply correspondence.

Now define the correspondence $F(p,x) = ((g \circ h)(x), f(p))$. F maps $S \times \bar{X}$ into the collection of subsets of $S \times \bar{X}$. The subsets $(g \circ h)(x)$ and $f(p)$ are convex and not empty. Also f is upper semi-continous, since ψ_i is for each i, h is continuous, and g is upper semi-continous. Since g and h are correspondences whose values lie in compact sets, their composition $g \circ h$ is upper semi-continuous, in the sense that $p^s \to p$, $p^{s'} \to p'$, and $p^{s'} \in (g \circ h)(p^s)$ for $s = 1, 2, \ldots,$ implies $p' \in (g \circ h)(p)$. Thus F is upper semi-continuous and convex valued. Since $S \times \bar{X}$ is convex, and compact by the Auxiliary Assumption, there is (p^*, x^*) such that $(p^*, x^*) \in F(p^*, x^*)$ by the fixed point theorem of Kakutani [**12**]. The mapping is illustrated in Figure 1.

To show that a fixed point of F leads to a competitive equilibrium for the economy $E = (Y, X_i, \succsim_i, i = 1, \ldots, m)$, we must show that Conditions (I), (II), and (III) are met by $x_i^* \in \psi_i(p^*)$, $y^* = h(x^*)$, and $p^* \in g(h(x^*))$. Consider $p^* \in g(h(x^*))$. From the definition of g it follows that $p^* \cdot y^* = 0$, where $y^* = h(x^*)$, and $p^* \cdot y \leqslant 0$ for all $y \in Y$. Thus Condition (I) is met for p^* and y^*. Now suppose that all consumers have income, that is, for all i there is $x_i \in X_i$ such that $p^* \cdot x_i < 0$. Then $\psi_i(p^*) = \xi_i(p^*)$ and $x_i^* \in X_i$. Then insatiability within $X \cap Y$ implies that $p^* \cdot x_i^* = 0$ if $x^* \in Y$. In any case $p^* \cdot x_i^* \leqslant 0$ must hold, and $x_i^* \succsim_i x_i$ for x_i such that $p \cdot x_i \leqslant 0$ by definition of ξ_i. Thus Condition (II) holds for p^* and x^*. To prove Condition (III), consider $y^* = h(x^*) = \alpha x^* + (1 - \alpha)\bar{y}$ where α is maximal for $y^* \in Y$. Then $p^* \cdot y^* = 0 = \alpha p^* \cdot x^* + (1 - \alpha) p^* \cdot \bar{y}$. Since $\bar{y} \in$ interior Y, $p^* \cdot \bar{y} < 0$, and by definition of f, $p^* \cdot x^* \leqslant 0$. If $\alpha > 1$ it follows that $x^* \in Y$. Then non-satiation in $X \cap Y$ implies $p^* \cdot x^* = 0$. This is incompatible with $p^* \cdot y^* = 0$. However, if $0 < \alpha \leqslant 1$, $p^* \cdot y^* = 0$ requires $\alpha = 1$ and $p^* \cdot x^* = 0$. Thus $x^* = y^*$ and Condition (III) of competitive equilibrium also holds.

This proves that a fixed point of F is a competitive equilibrium if all consumers have income at p^*. Some consumers must have income since $z \in$ interior Y implies $p^* \cdot z < 0$, and by Assumption (5′) there is $\bar{x} \in X \cap$ interior Y. Therefore, $p^* \cdot \bar{x}_i < 0$ holds for some i where $\bar{x}_i \in X_i$. Let $I_1 = \{i \mid p^* \cdot x_i < 0$ for some $x_i \in X_i\}$. Let $I_2 = \{1, \ldots, m\} - I_1$ and suppose I_2 is not empty. Suppose there is $x \in f(p^*)$ and $x \in X \cap Y$. By Assumption (6′), there is $y' \in Y$ and $w \in \bar{X}_{I_2}$ such that $x'_{I_1} = y' - x_{I_2} - w$ and $x'_i \succsim_i x_i$ for all $i \in I_1$, and $x'_i \succ_i x_i$ for some $i \in I_1$. These relations are well defined since $x_i \in X_i$ for $i \in I_1$ from the

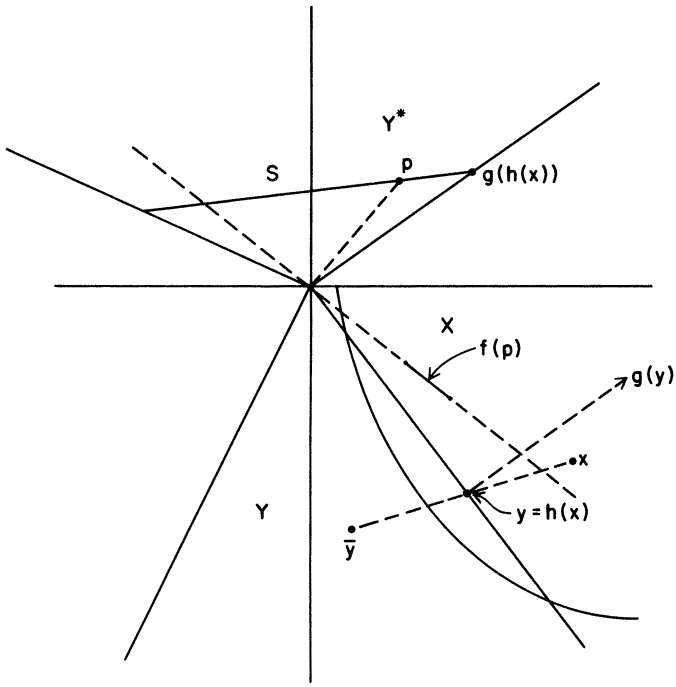


FIGURE 1—For $(p, x) \in S \times \bar{X}$, $F(p, x) = (g(h(x)), f(p)) \subset S \times \bar{X}$, where $\bar{X}$ is the convex hull of X and 0.

definition of ξ_i. Also non-satiation in $X \cap Y$ and the definition of ξ_i imply that $p^* \cdot x'_{I_1} > 0$ must hold. On the other hand, $p^* \cdot y' \leqslant 0$, and $p^* \cdot x_{I_2} = 0$ by definition of ψ_i. This implies $p^* \cdot w < 0$ contradicting $w \in \bar{X}_{I_2}$, since $p^* \cdot x \geqslant 0$ for all $x \in \bar{X}_{I_2}$, that is, no one in I_2 has income. Then it must be that $I_2 = \emptyset$ and all consumers have income. In other words, $(p^*, y^*, x_i^*, i = 1, \ldots, m)$ is an equilibrium of E.

On the other hand, if $f(p^*) \cap (X \cap Y) = \emptyset$, that is, $\xi_{I_1}(p^*) \subset Y - \bar{X}_{I_2}$, but $\xi_{I_1}(p^*) \cap (Y - X_{I_2}) = \emptyset$, the transitivity of preference for the members of I_1 will be violated on the boundary of $Y - X_{I_2}$ in the relative topology of $Y - \bar{X}_{I_2}$. That is:

LEMMA 1: *If (p^*, x^*) is a fixed point of F, $f(p^*)$ and $X \cap Y$ have a nonempty intersection.*

Define the feasible set F_1 for consumers in I_1 as $F_1 = Y - X_{I_2}$. The extended feasible set for I_1 is $\bar{F}_1 = Y - \bar{X}_{I_2}$. F_1 is properly contained in $\bar{F}_1$, and F_1 and $\bar{F}_1$ are convex and closed with nonempty interiors. Let B be the boundary of F_1 in the relative topology of $\bar{F}_1$, and let $B' = B \cap X_{I_1}$. Assume $f(p^*) \cap (X \cap Y) = \emptyset$. Then there is $x^*_{I_1} \in \psi_{I_1}(p^*)$ where $x^*_{I_1} \in \bar{F}_1 - F_1$. By Assumption (5′) there is a point $\bar{x}_{I_1} \in F_1$. Since X_{I_1}, $\bar{F}_1$, and F_1 are all convex, the line segment from $\bar{x}_{I_1}$ to $x^*_{I_1}$ must lie in $X_{I_1} \cap \bar{F}_1$ and intersect B'. Thus B' is not empty.

If no members of I_1 are satiated at p^*, Assumption (2) implies that $\{x_i^*\}_{I_1}$ is a Pareto optimal allocation for I_1 over all allocations feasible in $\bar{F}_1$. Assumptions (1) and (2) imply that the preference relation $\succsim_i$ may be represented on X_i by a real valued utility function that is continuous and positive valued. See Debreu [**6**, pp. 56–59]. We may take 0 to be the greatest lower bound of u_i on X_i for all i and 1 to be the least upper bound. Let $\bar{U}_1$ be the utility possibility set for I_1 consumers over $\bar{F}_1$, and let U_1 be the utility possibility set for I_1 consumers over F_1. Let $u_i^* = u_i(x_i^*)$. Since $u^* = \{u_i^*\}_{I_1}$ is undominated in $\bar{F}_1$, it is undominated in F_1. If there were a point $x = \{x_i\}_{I_1} \in F_1$ with $u_i(x_i) = u_i(x_i^*)$ for $i \in I_1$, $x = \sum_{i=1}^m x_i$ would lie in $f(p^*)$ with $x \in X \cap Y$. This contradicts the assumption that $f(p^*) \cap (X \cap Y) = \emptyset$. Therefore u^* is not an element of U_1.

Define the social utility function for I_1, $v(x) = \max \alpha$ such that $u_i(x_i) \geqslant \alpha u_i^*$, $i \in I_1$, where the maximum is taken over allocations $\{x_i\}_{I_1}$ such that $\sum_{i \in I} x_i = x$, $x_i \in X_i$. To see that v is continuous, consider $x_i^s \to x_i$, $\alpha^s \to \alpha$ where α^s is maximal for $x^s = \sum_{i \in I_1} x_i^s$. It is clear from the continuity of u_i that $u_i(x_i) \geqslant \alpha u_i^*$ must hold in the limit. But if there were $\epsilon > 0$ such that $u_i(x_i) \geqslant (\alpha + \epsilon) u_i^*$ held for $i \in I_1$, $u_i(x_i^s) \geqslant (\alpha^s + (\epsilon/2)) u_i^*$ would hold, for large s, contradicting the maximality of α^s. Also v inherits the quasi-concavity properties of u_i, derived from Assumption (2).

I claim v has no maximum on the set B'. Suppose b provided a maximum for v over B'. It must be that $v(b) < 1$, that is, $u_i(b_i) < u_i^*$ holds for some i, where $\{b_i\}_{I_1}$ is a maximizing allocation of b. Otherwise $u_i(b_i) \geqslant u_i(x_i^*)$, $i \in I_1$, and $u^* \in U_1$, which we have seen to be impossible. Since $b \in F_1$, it is feasible and by Assumption (2) and (6′) there is a point $w \in \bar{F}_i$ such that $c = b + w$ and $v(c) \geqslant v(b)$. Since c lies in the cone from the origin spanned by F_1 and $x_{I_1}^* \in \bar{F}_1 - F_1$, the line segment from c to $x_{I_1}^*$ cuts B in a point z. Indeed, c and $x_{I_1}^*$ both in X_{I_1} implies that $z \in B'$. It is shown in Appendix II that z may be chosen distinct from c. Therefore, Assumption (2) and the definition of v imply that $v(z) > v(b)$, so b does not provide a maximum for v over B'. This is a contradiction of the fact that B' is compact and v is continuous. Thus the assumption that $f(p^*) \cap (X \cap Y) = \emptyset$ must be rejected.

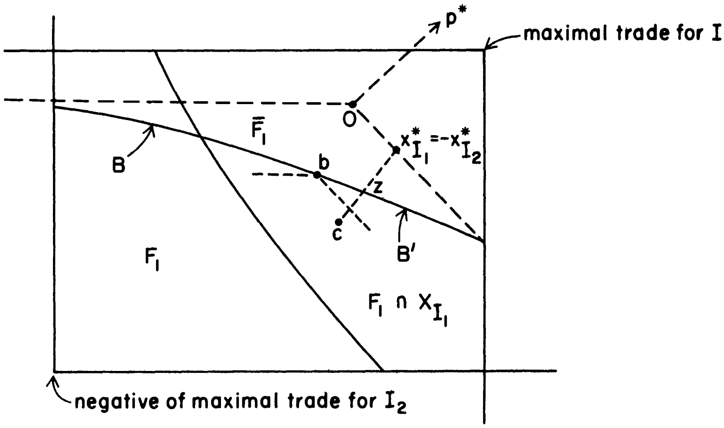
If some members of I_1 are satiated, it follows from the impossibility of satiation in the feasible set according to Assumption (6′) that u^* does not lie in U_1. Then the proof proceeds as before and Pareto optimality of $\{x_i^*\}_{I_1}$ in $\bar{F}_1$ is not needed. This completes the proof of Lemma 1. The argument is illustrated for the case where $Y = R_-^n$ in Figure 2.

Lemma 1 together with the previous argument implies that the case where all consumers have income at p^* is the only possible case. Therefore, the equilibrium proved to exist for this case must exist in general.

We have proved the following theorem.

THEOREM 2.: *If an economy* $E = (Y, X_i, \succsim_i, i = 1, \ldots, m)$ *satisfies the Assumptions* (1), (2), (3), (4), (5′), *and* (6′), *there is a competitive equilibrium for* E.

The interiority assumption of (5′) is easily removed. We will say more about this when recent remarks on free disposal are discussed later. However, two



$v(x^*_{I_1}) > v(b)$.
$v(c) \geqslant v(b)$.
$\therefore\ v(z) > v(b)$.
$\therefore$ b does not max v on B'.

I_2 cannot survive without trade.
$Y = -R^n$, $F_1 = R^n_- - X_{I_2}$.
No equilibrium exists in F_1, but $(p^*, x_i^*, i = 1, \ldots, m)$ is an equilibrium with $\sum_1^m x_i^* = x^* \in \bar{F}_1 - F_1$.
v has no maximum on $B' = B \cap X_{I_1}$.
This may be viewed as an Edgeworth box with the initial allocation at 0.

FIGURE 2.

assumptions are weakened following Moore. The survival assumption is removed and the assumption of irreducibility is weakened.

The proof of Lemma 1 allows us to see with clarity why a weaker irreducibility assumption used by James Moore [**21**, p. 272–273], suffices to establish the existence of a compensated equilibrium. His result requires that all the fixed points of F should have $x^* \in X \cap Y$. This is true if Lemma 1 holds. However, in the proof of Lemma 1 only the weak inequality $v(b + w) \geqslant v(b)$ was used, where $w \neq 0$ and $w \in Y - \bar{X}_{I_2}$.

3. A WEAKER PREFERENCE RELATION

The other direction in which the classical theorem has been substantially strengthened is to remove the requirement that the preference relation be transitive and complete. This process was begun by Sonnenschein [**28**] and brought to fruition by Mas-Colell [**16**] and Gale and Mas-Colell [**9**]. Later a more general theorem was proved in a very efficient way by Shafer and Sonnenschein [**26**] and applied to competitive equilibrium by Shafer [**25**]. Sonnenschein showed that the existence of a well defined demand function does not depend on the transitivity of preference. He also showed that the demand function would be upper semi-continuous if preferences are continuous. However, this still does not allow the demand correspondence to be used in the customary ways in proofs of existence of equilibrium since the value of the correspondence need not be a convex set, even though the preferred point set is convex. The proof of existence of equilibrium without transitivity was given first by Mas-Colell, and then in a slightly different form by Mas-Colell and Gale. Whether the preference relation $\succsim_i$ is complete seems to be a matter of definition, since x incomparable with y can be replaced harmlessly by x indifferent to y in the absence of transitivity.

Let $P_i(x)$ be the set of commodity bundles y such that $y \succ_i x$. Assumption (2) may be weakened to:

(2′) A preference correspondence P_i is defined on X_i into the collection of subsets of X_i, $i = 1, \ldots, m$. The correspondence P_i is open valued relative to X_i and lower semi-continuous. Also $x_i \notin$ convex hull $P_i(x_i)$.

Assumption (2′) is a major weakening of (2) since it does not require $\succ_i$ to be transitive, or even asymmetric, and no convexity assumption is made on the relation $\succsim_i$ (defined by $x \succsim_i y$ if and only if $\sim y \succ_i x$). As mentioned above, even though transitivity and asymmetry of $\succ_i$ were introduced and $P_i(x)$ were assumed convex, the non-convexity of $\succsim_i$ would still require a new proof of existence since the value of the demand correspondence need not be convex or even connected. On this point, see Mas-Colell [**16**].

Because Assumption (2′) does not include convexity and transitivity of preference in the sense of Assumption (2), the argument that excluded fixed points of F involving $x_i^* \in \bar{X}_i - X_i$ in the proof of Theorem 2 can no longer be made. Therefore, to prove existence with the preference correspondences P_i we introduce a stronger assumption of irreducibility. We assume:

(6″) However I_1 and I_2 may be selected, if $x_{I_1} = y - x_{I_2}$ with $x_{I_1} \in X_{I_1}$, $y \in Y$, and $x_{I_2} \in \bar{X}_{I_2}$, then there is also $y' \in Y$ and $w \in \bar{X}_{I_2}$, such that $x'_{I_1} = y' - x_{I_2} - w$ and $x'_i \in P(x_i)$ for all $i \in I_1$.

Assumption (6″) is stronger than (6′) in two respects. The point x_{I_2} may be in the enlarged possible consumption set $\bar{X}_{I_2}$, and it must be possible for an additional trade from $\bar{X}_{I_2}$ to improve the position of all consumers belonging to I_1. Then Lemma 1 is no longer needed.

A device introduced by Shafer [**24**] and exploited by Shafer and Sonnenschein [**26**] will allow us to use a mapping of the social consumption set on the boundary of the production possibility set, as before. Then the proof of existence for a production economy under quite weak conditions may be derived. The trick is to define a new preference relation that is transitive, by means of a preference correspondence R_i conditioned on $x \in \bar{X}_i$. First, define P'_i by $P'_i(x) =$ convex hull $P_i(x)$ for $x \in X_i$. It is easily seen that P'_i satisfies Assumption (2′) and, in addition, it is convex valued. Let G_i denote the graph of P'_i, and denote by $d((x,z), G_i)$ the Euclidean distance from $(x,z) \in \bar{X}_i \times X_i$ to G_i. Define $R_i(x, y)$

for $x \in \overline{X}_i$ and $y \in X_i$ by

$$R_i(x, y) = \{z \mid d((x,z), G_i) \leqslant d((x, y), G_i)\}, \quad \text{if } y \notin P_i'(x), \quad \text{and}$$

$$R_i(x, y) = \{z \mid d((x,z), G_i^c) \geqslant d((x, y), G_i^c)\}, \quad \text{if } y \in P_i'(x).$$

G_i^c is the complement of G_i in $X_i \times X_i$. For any given $x \in \overline{X}_i$, $R_i(x, y)$ defines a transitive and symmetric preference relation on X_i. Because $d((x,z), G_i)$ is a continuous function of (x,z), $R_i(x, y)$ is a continuous correspondence. This justifies the following lemma.

LEMMA 2: *R_i is a continuous correspondence mapping $\overline{X}_i \times X_i$ into the collection of subsets of X_i, $i = 1, \dots, m$.*

Now we use R_i to define a pseudo-demand correspondence $\xi_i'(p,x)$, where p is a price vector and $x \in \overline{X}_i$. Let $\xi_i(p,x) = \{y \in X_i \mid p \cdot y \leqslant 0$ and $p \cdot z \leqslant 0$ implies $y \in R_i(x,z)\}$. That is, $\xi_i(p,x)$ is the set of commodity bundles y such that (x, y) is as close to G_i (or as far from G_i^c) as (x,z) for any commodity bundle z in the budget set defined by p. Let ξ_i' be the convex hull of this set. It is clear that $\xi_i'(p,x)$ is contained in the budget set.

I claim that $\xi_i'(p,x)$ is upper semi-continuous at (p,x) if there is $z \in X_i$ such that $p \cdot z < 0$, that is, if the ith consumer has income. Suppose $p^s \to p$, $x^s \to x$, $y^s \to y$, $s = 1, 2, \dots,$ where $y^s \in \xi_i'(p^s, x^s)$ and $(x^s) \in \overline{X}_i$. Consider $w \in X_i$ where $p \cdot w \leqslant 0$. The existence of z implies that the budget set $B(p') = \{z \in X_i \mid p' \cdot z \leqslant 0\}$ is a continuous correspondence at $p' = p$. Then, for s large, there is $w^s \in X_i$ such that $p^s \cdot w^s \leqslant 0$ and $w^s \to w$. By definition of $\xi_i'(p^s, x^s)$, $y^s = \sum_1^{n+1} \alpha_j^s z_j^s$ where $z_j^s \in B(p^s) \cap R_i(x^s, w^s)$ and $\sum_1^{n+1} \alpha_j^s = 1$, $\alpha_j^s \geqslant 0$, all j. Since the X_i are bounded by the Auxiliary Assumption, z_j^s is bounded for each j, and a subsequence may be chosen which converges for z_j^s and α_j^s. Retaining notation, let $z_j^s \to z_j$, $\alpha_j^s \to \alpha_j$, all j. Then $y = \sum_1^{n+1} \alpha_j z_j$. But, by Lemma 2, R_i is continuous, so $z_j \in R_i(x, w)$. Since w is an arbitrary element of $B(p)$, this implies $z_j \in \xi_i(p,x)$ and, therefore, $y \in \xi_i'(p,x)$. Thus ξ_i' is upper semi-continuous. It is convex valued by definition. We have shown the following lemma to be true.

LEMMA 3: *The correspondence ξ_i' is upper semi-continuous and convex valued at (p,x) if there is $w \in X_i$ such that $p \cdot w < 0$.*

We define $f_i'(p,x)$ in a manner similar to the definition of $\psi_i(p)$, that is,

$$f_i'(p,x) = \xi_i'(p,x) \quad \text{if there is } w \in X_i \text{ and } p \cdot w < 0,$$

$$f_i'(p,x) = \{y \in \overline{X}_i \mid p \cdot y = 0\}, \quad \text{otherwise.}$$

We may prove the following lemma.

LEMMA 4: *The correspondence f_i' is upper semi-continuous and convex valued for any (p,x) with $p \in S$, $x \in \overline{X}_i$.*

The result follows from Lemma 3 if there is w such that $p \cdot w < 0$. If $p \cdot w \geqslant 0$ for all $w \in X_i$, $f_i'(p,x)$ is the intersection of the budget set with $\overline{X}_i$ and thus convex valued. Upper semi-continuity follows from the arguments of Debreu appealed to earlier in the case of $\psi_i(p)$.

We may now define a mapping F' whose fixed points will be competitive equilibria for an economy $E = (Y, X_i, P_i, i = 1, \ldots, m)$ that satisfies (1), (2′), (3), (4), (5′), and (6″). First, define f' on $S \times \prod_1^m \overline{X}_i$ by $f'(p, \prod_1^m x_i) = \prod_1^m f_i'(p, x_i)$. Thus $f'(p, \prod_1^m x_i) \subset R^{mn}$. Correspondences h and g may be defined as before in the definition of the mapping F. Let $F'(p, \prod_1^m x_i) = ((g \circ h)(\sum_1^m x_i), f'(p, \prod_1^m x_i))$. Then F' maps $S \times \prod_1^m \overline{X}_i$ into the collection of subsets of $S \times \prod_1^m \overline{X}_i$. Since the values of g, h, f', and f_i' lie in compact sets and each is continuous or upper semi-continuous, F is upper semi-continuous. Since $g \circ h$ is convex valued and not empty as before, and $\prod_1^m f_i'$ is also convex valued and not empty, F' has these properties. Then the Kakutani fixed point theorem provides a fixed point for F' on $S \times \prod_1^m \overline{X}_i$.

We must show that the fixed point of F' is a competitive equilibrium. However, first we will slightly modify condition II to read

(II′) $\quad x_i \in X_i$ and $p \cdot x_i \leqslant 0$, and $p \cdot z > 0$ for any $z \in P(x_i)$,

$i = 1, \ldots, m$.

This condition implies (II) whenever $\succsim_i$ can be consistently defined as a complete weak ordering over X_i. However, (II′) is not as restrictive as (II) since (II′) allows both $x \in P_i(y)$ and $y \in P_i(x)$.

Lower semi-continuity of P_i, and thus of P_i', plays a role like local non-satiation in the present context to imply the spending of all income for the pseudo-demand functions at a fixed point.

Lemma 5: *If $x \in f_i'(p,x)$, then $p \cdot x = 0$.*

The definition of $f_i'(p,x)$ implies $p \cdot x \leqslant 0$. Suppose $p \cdot x < 0$. Then $f_i'(p,x) = \xi_i'(p,x)$. Thus we only need consider the case $x \in \xi_i'(p,x)$. Suppose there is $z \in \xi_i(p,x)$ and $(x,z) \in G_i$, that is, $z \in P_i'(x)$. Then $y \in \xi_i(p,x)$ implies $y \in P_i'(x)$, that is, $(x,y) \in G_i$. This implies $x \in P_i'(x)$ by convexity of P_i', contradicting Assumption 2′. Therefore, $\xi_i(p,x) \cap P_i'(x) = \emptyset$. On the other hand, $p \cdot x < 0$ implies there is $y \in \xi_i(p,x)$ and $p \cdot y < 0$. Thus the result follows if $p \cdot y < 0$ implies that (x,y) cannot be a closest point to G_i of the form (x,z) for which $p \cdot z \leqslant 0$.

Suppose otherwise and let $(\bar{x}, \bar{z})$ be a point of G_i closest to (x,y). Let $x^s \to \bar{x}$, $s = 1, 2, \ldots$, where x^s lies on the line segment from x to $\bar{x}$. By lower semi-continuity of P_i' there is a sequence $z^s \to \bar{z}$ where $(x^s, z^s) \in G_i$. Choose a sequence $(x, w^s) \to (x,y)$, where $w^s = \alpha_s y + (1 - \alpha_s) z^s$ and $0 < \alpha_s < 1$. Convexity of X_i implies $w^s \in X_i$. Also $z^s - w^s = \alpha_s(z^s - y)$. We may choose α_s so that $\alpha_s |z^s - y| < |\bar{z} - y|$ for large s. This together with $|x^s - x| < |\bar{x} - x|$ implies for large s, $|(x^s, z^s) - (x, w^s)| < |(\bar{x}, \bar{z}) - (x,y)|$. But for s large $p \cdot w^s < 0$. Since $(x^s, z^s) \in G_i$, $y \notin R_i(x, w^s)$ for large s. This contradicts $y \in \xi_i(p,x)$, and the lemma is proved.

Consider $(p^*, x_1^*, \ldots, x_m^*) \in F'(p^*, x_1^*, \ldots, x_m^*)$. Let $x^* = \sum_1^m x_i^*$, where $x_i^* \in f_i'(p^*, x_i^*)$. Then by Lemma 5, $p^* \cdot x^* = 0$. Also $y^* \in h(x^*)$. Then $p^* \cdot \bar{y} < 0$, together with $y^* = \alpha x^* + (1 - \alpha)\bar{y}$, from the definition of h, and $p^* \cdot y^* = 0$ from the definition of g, gives $\alpha = 1$. Thus $x^* = y^*$. Since $p^* \cdot y \leqslant 0$ for $y \in Y$ follows from the definition of g, and $y^* \in Y$ from the definition of h, Conditions (I) and (III) are satisfied.

The verification of Condition (II′) like that of Condition (II) is less simple than the verification of (I) and (III), in this instance because the maximization of preference in $f_i'(p^*, x_i^*)$ is relative to x_i^* by way of a pseudo-preference ordering. As before, make the provisional assumption that all consumers have income at p^*, that is, for each i there is $w \in X_i$ and $p^* \cdot w < 0$. Then $f_i' = \xi_i'$ and $x_i^* \in \xi_i'(p^*, x_i^*)$. By definition of ξ_i' it must hold that $x_i^* \in X_i$ and $p^* \cdot x_i^* \leqslant 0$. Also $x_i^* = \sum_1^{n+1} \alpha_j z_j$, $\alpha_j \geqslant 0$, $\sum_1^{n+1} \alpha_j = 1$, where $z_j \in \xi_i(p^*, x_i^*)$, that is, $z_j \in R_i(x_i^*, y)$ for all j and for any $y \in X_i$ such that $p^* \cdot y \leqslant 0$, and $p \cdot z_j \leqslant 0$. Let F_i be the smallest affine subspace containing X_i. We will need the following lemma:

LEMMA 6: *If $y \in$ interior $P_i'(x)$ relative to F_i, then $(x, y) \in$ interior G_i relative to $X_i \times X_i$.*

The proof follows from the fact that P_i' is lower semi-continuous and convex valued. Suppose $y \in$ interior $P_i'(x)$ relative to F_i. Then there is a neighborhood N of y in F_i such that $N \subset P_i'(x)$. Indeed, we may choose N to be the interior, relative to F_i, of the convex hull of points $w_j \in P_i'(x)$, $j = 1, \ldots, n_i + 1$, where n_i is the dimension of F_i. Let y be an arbitrary element of N. Then $y = \sum_1^{n_i+1} \alpha_j w_j$ for unique α_j with $\sum \alpha_j = 1$, and $\alpha_j > 0$, all j. We must show that any point (x^s, y^s) sufficiently near (x, y) belongs to G_i. Suppose not. Then there is a sequence $(x^s, y^s) \to (x, y)$ and $y^s \notin P(x^s)$. However, by lower semi-continuity for any sequence $x^s \to x$, there is a sequence $w_j^s \in P_i'(x^s)$ with $w_j^s \to w_j$, for all j. The affine independence of the w_j implies the affine independence of w_j^s for large s. Therefore, $y^s = \sum_1^{n_i+1} \alpha_j^s w_j^s$ for unique α_j^s for large s, where $\sum \alpha_j^s = 1$. Since $y^s \to y$ and $w_j^s \to w_j$, it must be that $\alpha_j^s \to \alpha_j$. Otherwise, a subsequence of α_j^s could be chosen converging to α_j', where $\alpha_j' \neq \alpha_j$ for some j, and $y = \sum_1^{n_i+1} \alpha_j' w_j$, in contradiction to the uniqueness of the α_j. Thus $\alpha_j^s > 0$ for s large, and $y^s \in P_i'(x^s)$ by the convexity of $P_i'(x^s)$. This proves the lemma.

To verify Condition (II′), we must show that $y \in P_i(x_i^*)$ implies $p^* \cdot y > 0$. Assume the contrary, that is, $y \in P_i(x_i^*)$ and $p^* \cdot y \leqslant 0$. However, $y \in P_i(x_i^*)$ implies $y \in P_i'(x_i^*)$. Since all consumers have income, there is $w \in X_i$ satisfying $p^* \cdot w < 0$. It is clear that w may be chosen interior to X_i relative to F_i. Let $S_\epsilon(w)$ be an ϵ-ball in F_i about w contained in X_i. Consider $N_\alpha = \alpha y + (1 - \alpha) S_\epsilon(w)$. N_α is an open set relative to F_i and contained in X_i for all α with $0 \leqslant \alpha < 1$. Since $y \in P_i'(x_i^*)$ and $P_i'(x_i^*)$ is open relative to X_i, for α near 1, $N_\alpha \subset P_i'(x_i^*)$. Let $y' \in N_\alpha$, so that $y' \in$ interior $P'(x_i^*)$ relative to F_i. Then $(x_i^*, y') \in$ interior G_i relative to $X_i \times X_i$ by Lemma 6, and $d((x_i^*, y'), G_i^c) > 0$. But x_i^* is a convex combination of $n + 1$ points $z_j \in \xi_i(p^*, x_i^*)$. Thus $p^* \cdot y' < 0$ implies that $z_j \in R_i(x_i^*, y')$. Then, by definition of R_i, $d((x_i^*, z_j), G_i^c) > 0$ must hold for all j. In other words, $(x_i^*, z_j) \in G_i$, or $z_j \in P_i'(x_i^*)$, all j. Since P_i' is convex valued, it follows that $x^* \in P_i'(x_i^*)$. But $P_i'(x_i^*)$ is the convex hull of $P_i(x_i^*)$, which contradicts Assumption (2′) and establishes Condition (II′) of competitive equilibrium. This argument is analogous to an argument of Shafer and Sonnenschein **[26]**, where G_i was assumed to be an open graph.

An argument parallel to that in the case of Theorem 2 shows that on the basis of (5′) and (6″) every consumer does in fact have income for any fixed point of F'. We have shown for a fixed point (p^*, x^*) of F' that $x^* = y^*$ where $x^* \in f'(p^*, \prod x_i^*)$ and $y^* \in Y$. Thus by the definition of f' we find $x^* \in \overline{X} \cap Y$. As before, Assumption (5′) implies that someone has income at p^*, so the set I_1 of consumers with income is not empty. Suppose the set I_2 of consumers without income is not empty. Then by Assumption (6″), there is $y' \in Y$ and $w \in \overline{X}_{I_2}$ such that $x'_{I_1} = y' - x^*_{I_2} - w$ and $x_i' \in P_i(x_i^*)$ for all $i \in I_1$. Then $x_i' \in P_i'(x_i^*)$ also holds and $(x_i^*, x_i') \in G_i$. Since $(x_i^*, x_i^*) \notin G_i$ by Assumption (2′), $x_i' \in R_i(x_i^*, x_i^*)$. Therefore, $p^* \cdot x_i' > p^* \cdot x_i^*$ by definition of $\xi_i'(p^*, x^*)$. Then, by Lemma 5, $p^* \cdot x_i' > 0$ must hold for all i. However, $p^* \cdot x_i^* = 0$ for $i \in I_2$ by definition of f_i'. Also $p^* \cdot y' \leqslant 0$ since p^* supports Y, and $p \cdot w \geqslant 0$ since $w \in X_{I_2}$, which is also supported by p^*. Thus $p^* \cdot x'_{I_1} = p^* \cdot y' - p^* \cdot x^*_{I_2} - p^* \cdot w \leqslant 0$ in contradiction to $p^* \cdot x_i' > 0$. Therefore, $I_2 = \emptyset$ and all consumers have income. Thus we have the following theorem:

THEOREM 3: *If an economy* $E = (Y, X_i, P_i,\ i = 1, \ldots, m)$ *satisfies the Assumptions* (1), (2′), (3), (4), (5′), *and* (6″), *there is a competitive equilibrium for* E.

Except for the interiority assumption of (5′), this is the modern form of the classical theorem on existence of competitive equilibrium that was promised. Its major improvements are the removal of the survival assumption based on the work of Moore and the discard of transitivity of the preference relation based on the work of Sonnenschein, Shafer, Mas-Colell, and Gale. The interiority assumption is made only for simplicity. A method for its removal will be described in the next section.

However, the greatly weakened assumption on preferences for Theorem 3 required that the irreducibility assumption be significantly stronger than Moore's. Irreducibility was assumed for an economy with expanded consumption sets $\overline{X}_i$ and the improvement for I_2 consumers was positive for all $i \in I_2$. This stronger form of irreducibility would actually be needed even for preferences of the type assumed by Debreu [7] where thick indifference sets were allowed and transitivity was retained. Otherwise, the contradiction that establishes $I_2 = \emptyset$ is not available, since $x_i' \succsim_i x_i^*$ does not imply $p \cdot x_i' \geqslant p \cdot x_i^*$ when indifference sets may be thick, that is, when $x \succ_i y$ and $z = \alpha x + (1 - \alpha) y$ for $0 < \alpha < 1$ only implies $z \succsim_i y$.

4. FREE DISPOSAL

The question of free disposal, or more generally, the question of interiority, for existence of equilibrium was essentially settled in McKenzie **[19]**. However, the explicit proof offered in the original paper does not cover the case of production sets that are linear subspaces, in particular, the case where $Y = \{0\}$. A hint was given in the *Econometrica* reprint volume [**19**, p. 350, fn. 13] on how the gap should be repaired, but the original omission may still lead to confusion. There have been recent articles, in particular Hart and Kuhn [**10**], and Bergstrom [**3**], providing alternative proofs that "free disposal" is not needed as an assumption. These proofs are, of course, valuable to have, but the original proof can be recommended for simplicity and intuitive appeal. Thus it may be worthwhile to provide the missing steps from the 1959 argument, particularly since the recent proofs have interiority assumptions stronger than the second part of (5). The final form of the interiority assumption is

(5″) relative interior $X \cap$ relative interior $Y \neq \emptyset$.

This is the second part of (5), from which the survival assumption has been removed. The interior is taken relative to the smallest linear subspace containing X, or Y, respectively.

The technique for reducing (5′) to (5″) is to expand Y so that it acquires an interior in such a way that a relative interior point given by (5″) is interior to Y, thus re-establishing (5′). Then an equilibrium exists for the modified system. The modification is made to depend continuously on a parameter $\epsilon \geqslant 0$ in such a way that $\epsilon = 0$ corresponds to the original model. The equilibria for E^s as $\epsilon_s \to 0$, $s = 1, 2, \ldots,$ lie in a compact set, and the limit of the equilibria $(p^s, y^s, x_i^s,$ $i = 1, \ldots, m)$ for a converging subsequence is an equilibrium for E. Conditions (I) and (III) hold in the limit from the continuity of sum and inner product. Condition (II) of Theorems 1 and 2 follows from the upper semi-continuity of the demand correspondence, once it is shown that all consumers have income at the limit price vector p, as a consequence of Assumption (5″) and (6) or (6′). The argument of McKenzie [**19**, p. 64] may be used for this purpose. However, for the case of Theorems 2 and 3, X_i should be replaced by $\overline{X}_i$ and X by $\overline{X}$ in the argument. Assumption (5″) implies that relative interior $\overline{X} \cap$ relative interior Y $\neq \emptyset$. To establish (II′) in the case of Theorem 3 note that $p \cdot x_i \leqslant 0$ holds by continuity when x_i is a limit point of x_i^s and apply the same argument used to establish (II′) for E^s. The proof is simpler than in McKenzie [**19**] since it is no longer necessary to move the X_i. Formerly, X_i was moved to provide a point interior to $X_i \cap Y$ that would guarantee income for the ith consumer in the modified economy E^s. This role is now played in the same way in E and E^s by Assumption (6′) or (6″).

If Y is not a linear subspace, let $\bar{x} \in$ relative interior $X \cap$ relative interior Y. The existence of $\bar{x}$ results from (5″). Let $S_\epsilon(\bar{x}) = \{y \,|\, |y - \bar{x}| \leqslant \epsilon\}$, where $\epsilon > 0$ is chosen sufficiently small so that Assumptions (3) and (4) are protected. Let $Y(\epsilon) = \{y \,|\, y = \alpha w + \beta z$, where $\alpha \geqslant 0$, $\beta \geqslant 0$, $w \in Y$, and $z \in S_\epsilon(\bar{x})\}$. $Y(\epsilon)$ is the convex cone spanned from the origin by Y and $S_\epsilon(\bar{x})$. Then $\bar{x} \in \operatorname{int} Y(\epsilon)$ and Assumption (5′) is satisfied. Thus there is an equilibrium by Theorem 3 for $E(\epsilon) = (Y(\epsilon), X_i, P_i,\ i = 1, \ldots, m)$, and similarly for Theorem 2. We must show that the equilibria $(p^s, y^s, x_i^s,\ i = 1, \ldots, m)$ lie in a compact set as $\epsilon^s \to 0$. Eventually $Y(\epsilon^s) \subset Y(\epsilon)$ for an $\epsilon > 0$, and $Y(\epsilon) \cap R_+^n = \{0\}$. Therefore, X_i

bounded below implies that y^s and x_i^s are bounded over s by the usual argument (McKenzie [**19**, p. 62]). To bound p^s, renormalize the prices given by Theorem 3 by setting $|p^s| = 1$. By letting ϵ tend to 0, the existence of an equilibrium for E is established in the way described above. Note that Y may contain a linear subspace, although it is not equal to a linear subspace.

If Y is a linear subspace, the construction is slightly more complicated. An $n+1$st pseudo-good is introduced into the model. Let $Y' = \{y \in R^{n+1} | (y_1, \ldots, y_n) \in Y \text{ and } y_{n+1} \leqslant 0\}$. Since $(0, -1) \in Y'$, the pseudo-good is freely disposable. Let $\bar{x} \in$ relative interior $X \cap$ relative interior Y as before. Define $S_\epsilon(\bar{x}, -1) = \{y \in R^{n+1} | |y - (\bar{x}, -1)| \leqslant \epsilon, \text{ where } \epsilon < |\bar{x}, -1|\}$. Define $X_i' = \{x \in R^{n+1} | (x_1, \ldots, x_n) \in X_i \text{ and } x_{n+1} = -1/m\}$. Define $Y'(\epsilon)$ as before and (5″) is satisfied. Then by the previous argument an equilibrium exists for $E'(\epsilon) = (Y'(\epsilon), X_i', P_i'', i = 1, \ldots, m)$, where $y \in P_i''(x)$ if and only if $(y_1, \ldots, y_n) \in P_i(x_1, \ldots, x_n)$. As $\epsilon \to 0$, and the productivity of the pseudo-good disappears, any limit of the p^s has $p_{n+1}^s = 0$. Therefore, a limit $(p, y, x_i, i = 1, \ldots, m)$ of the equilibria for $E'(\epsilon)$, as $\epsilon \to 0$, is an equilibrium for E when the coordinate $n+1$ is ignored. Since $Y = \{0\}$ is a possibility, the pure trade economy without disposal is covered by this argument.

5. ROLE OF THE FIRM

It is an unusual characteristic of my contributions to the theory of existence of equilibrium that the social production set has been represented by a convex cone. I would claim that this properly represents the Walrasian system where production processes rather than firms are featured, but also it is a fair representation of a Marshallian economy of competitive industries where firm size is small relative to the market and firms operate in a small neighborhood of the minimum cost points on their U-shaped cost curves. The representation of the competitive economy as a fixed collection of disparate firms maximizing profit over concave production functions probably dates from Hicks' *Value and Capital* [**11**], but it was taken up by Arrow and Debreu [**1**]. Wald [**31, 32**] had used a simplified Walrasian model. My own initial contribution [**17**] was in the context of Graham's model of world trade which was also linear. However, I have continued to regard the linear process model to be the appropriate ideal type for the competitive economy.

It should be remarked that in a strict mathematical sense the models of Walras and Hicks are equivalent, without resort to approximations. In one direction this is obvious since linear processes may be assigned to Hicksian firms leading to a social production set that is a cone. This is a Hicksian model without scarce unmarketed resources. In the other direction an artificial construction is needed. An entrepreneurial factor is introduced for each firm which is divided among the owners in proportion to their ownership shares. This factor is always supplied and freely disposable. The production set Y_j of the jth firm is displaced by appending minus one unit of the jth entrepreneurial factor to each of its input-output vectors and setting all other entrepreneurial inputs equal to zero. The new production set $\tilde{Y}^j$, for a unit input of entrepreneurship, lies in a space R^{n+l} when there are l firms. Then the social production set $\tilde{Y}$ is taken to be the closure of the set, $\sum_i^m \alpha_j \tilde{Y}_j$, $\alpha_j \geqslant 0$. The only part of the set $\tilde{Y}$ that contributes to an equilibrium is its intersection with the hyperplane defined by setting entrepreneurial components equal to -1. This set is not affected by taking the closure. The pricing of the entrepreneurial factors will provide for the distribution of profits by the firms and the order of proof is the same as before.

On the other hand, my own preference is to regard the entrepreneurial factors as no different from other goods, suffering indeed from some indivisibilities in the real world, but approximated in the competitive model by divisible goods just as in the case of television sets and steel mills. Viewed this way the use of the production cone approximates the basic competitive notion of free entry more closely than Hicks' unmarketed factors. In the firm model of Hicks and Arrow-Debreu a firm becomes active whenever its profit becomes nonnegative. However, the list of firms is given and for Hicks, at least, identified with the list of consumers. Trouble immediately arises if firms are run by coalitions of people, that is, entrepreneurial factors can be supplied to a firm by more than one consumer. Merely earning a positive return does not then activate a firm, as a potential coalition, unless the profit is sufficient to match the earnings that the coalition members already receive elsewhere. Suppose entrepreneurial factors do not affect preferences directly. Then given the prices of traded goods, an equilibrium would be an allocation of profits in the core of a profit game, in which the firms are coalitions of entrepreneurs rather than single entrepreneurs as in the Hicks theory, or unique unreproducible resources as in Arrow-Debreu.

Let e_j be a point in R^m, where $e_{ij} = 1$ if the ith entrepreneur is in the jth coalition and $e_{ij} = 0$ otherwise. Let e_j be the jth column of the matrix $[e_{ij}]$ which has m rows and 2^m columns. Let π be a point in R^m representing a distribution of profits, and let $v(e_j)$ be the maximum profits attainable by the jth coalition at the ruling prices. Let $e \in R^m$ be the vector all of whose components equal 1. Then π is in the core if $\pi \cdot e_j \geqslant v(e_j)$, for all j, and there exists $\delta \in R^{2^m}$ such that $\delta_j = 0$ or 1, all j, $\sum_j \delta_j e_j = e$, and $\pi \cdot e_j = v(e_j)$ if $\delta_j = 1$. Under these conditions there is a collection of coalitions accommodating all entrepreneurs with sufficient profits to pay their members at the rate π and there is no coalition with enough profits to better these rates of pay.

Unfortunately the conditions that will imply a nonempty core are onerous. Scarf **[23]** has shown that a game has a nonempty core if it is balanced. A game with transferable utility is *balanced* if, for any $\delta \in R_+^{2^m}$ and $\pi \in R^m$, $\sum_j \delta_j e_j = e$, all i, and $\pi \cdot e_j \leqslant v(e_j)$, all j with $\delta_j > 0$, implies that $\pi \cdot e \leqslant v(e)$. There seems no reason why these conditions should be met. In particular, they imply that the problems of coordination within firms are overcome by economies of scale in production, no matter how large the firms grow. There can always be a single firm embracing all entrepreneurs to realize the core allocation. The role of the market is unimportant.

There seems no way out of this difficulty except to allow the distribution of effort implied by δ to be realized in fact, either by a distribution of time over coalitions by individuals or by the presence of many individuals of each of m types who may be spread over coalitions. Then the managerial structure of the firm appears like a linear activity. The whole set of firms generates Y, a convex cone from the origin in R^{m+1}, spanned by $(v(e_j), -e_j)$, $j = 1, \ldots, 2^m$, with $v(0) = -1$, to allow free disposal of profit. A competitive equilibrium will exist for the reduced economy, implied by given prices for goods and marketed factors, where Y is the production set. The ith consumer supplies one unit of his entrepreneurial factor while demanding π_i units of the output (profit) when the price vector is $(1, \pi)$. In equilibrium prices for entrepreneurial factors may be normalized on the price of profit $= 1$, since every vector in the dual cone is nonnegative by virtue of $v(0) = -1$. Let $\delta^i \in R^m$ have $\delta_j^i = 0$ for $j \neq i$ and $\delta_i^i = 1$. Let $\delta \in R^{2^m}$ satisfy $\delta \geqslant 0$. The equilibrium $((1, \pi), \bar{y}, \bar{x}_1, \ldots, \bar{x}_m)$ will satisfy $(1, \pi) \cdot y \leqslant 0$ for $y \in Y$, $(1, \pi) \cdot \bar{y} = 0$, $\bar{y} = \sum_j \delta_j (v(e_j), -e_j)$, $\sum_j \delta_j e_j = e$, $\sum_j \delta_j v_j = \pi \cdot e$, $\bar{x}_i = (\pi_i, -\delta^i)$, $\bar{y} = \sum_i \bar{x}_i$. Thus π is in the core when δ is allowed to vary continuously with no further assumptions.

However, if this way out is chosen there is no advantage over simply treating entrepreneurial resources like other resources, in particular, without the restriction that the amounts of different entrepreneurial resources in a given activity be used in the same amounts, as the coalition model requires, or the restriction that entrepreneurial resources do not affect preferences. I conclude that whatever resources are brought together to comprise the "unmarketed" resource base of the firm are most reasonably treated symmetrically with other resources. Most goods in the real world are indivisible, so the competitive model is an approximation to reality, but the entrepreneurial resources, or firms' special resources, seem to be no more nor less subject to these reservations than other goods or resources.

6. EXTERNAL ECONOMIES

Some authors, in particular, McKenzie [**18**], Arrow and Hahn [**2**], and Shafer and Sonnenschein [**26**, **27**] have relaxed the assumption that producer and consumer actions are independent except for the balance between demand and supply in equilibrium. My early model made consumer preferences depend on the choices of other consumers, the activity levels, and prices. However, feasible sets for consumers or producers were kept independent. On the other hand, Arrow and Hahn, and Shafer and Sonnenschein, as well as others, have allowed feasibility effects as well. The early model for their analyses was a paper of Debreu [**4**] which preceded the existence theorem of Arrow and Debreu. However, in the use by Arrow and Debreu of Debreu's results no external economies were allowed.

My omission of feasibility effects from external economies was not an oversight. I did not succeed in formulating such effects in a satisfactory way. It is my view that this question remains unresolved. The difficulties can be illustrated by external economies between firms in an Arrow-Debreu model. It is usual to assume that a set Y_j exists for the jth firm that is closed and convex and includes every input-output combination that this firm could achieve under any conditions that can arise. Then there is a continuous correspondence $\mathscr{Y}_j$ for the jth firm that maps $Y = \prod_1^l Y_j$ into the collection of subsets of Y_j. The value of $\mathscr{Y}_j$ on an element y of Y is interpreted as the set of feasible outputs for the jth firm given that the ith firm, $i \neq j$, is producing y_i. Since y_i is chosen from Y_i there is no guarantee that y is feasible. The technologically feasible outputs are among the fixed points of the product correspondence $\mathscr{Y} = \prod_1^l \mathscr{Y}_j$ which maps Y into the collection of subsets of Y. Since $\mathscr{Y}_j$ is assumed to be convex valued, fixed points do exist. Outside the set of fixed points the points of Y cannot be realized as inputs and outputs even when resource supplies are adequate.

The competitive equilibria are a subset of the fixed points of $\mathscr{Y}$, since technological feasibility is a necessary condition for equilibrium. However, which fixed points of $\mathscr{Y}$ are equilibria will depend on, among other things, what alternative outputs are allowed to the firms, that is, on the precise content of $\mathscr{Y}_j$ for the jth firm. Given the price vector and the output vector y, a necessary condition for equilibrium is that $p \cdot y_j$ be a maximum over $\mathscr{Y}_j(y)$. But $\mathscr{Y}_j(y)$ has no empirical correlates except for the constraints imposed by the set of feasible points of $\mathscr{Y}$, that is, $\mathscr{Y}_j(y)$ contains y_j if y is a feasible point. Moreover, the set of fixed points of $\mathscr{Y}$ may not coincide with the set of technically feasible outputs of the economy, unless the $\mathscr{Y}_j$ can be designed so that they have their assumed properties without introducing new fixed points.

In these circumstances the maps $\mathscr{Y}_j(y)$ are artificial constructions except for the feasible set, that is, the set of fixed points of well defined $\mathscr{Y}_j$'s. What kinds of feasible sets would admit an appropriate set of $\mathscr{Y}_j$'s is unknown. Given that otherwise appropriate $\mathscr{Y}_j$'s exist, the equilibria may then depend on the choice of the correspondences. Since the papers in the literature do not address these problems it is not clear to what extent the subject has been advanced since 1955.

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APPENDIX I

Let $\bar{y} = \alpha z$ for $z \in$ interior Y. Suppose $x \in \bar{X}$ and no matter how large α is chosen the number π is unbounded. Then $(\alpha(1-\pi)/\pi)z + x \in Y$ for all large π. Let $w(\alpha,\pi) = (\alpha(1-\pi)/\pi)z + x$. As $\pi \to \infty$, $w(\alpha,\pi) \to w(\alpha) = -\alpha z + x$, which lies in Y since Y is closed. Also $w(\alpha)/\alpha = -z + (x/\alpha) \in Y$, and, as $\alpha \to \infty$, $-z + (x/\alpha) \to -z$, which also lies in Y since Y is closed. Thus z and $-z$ are in Y, and since z is interior to Y, $Y = R^n$ in contradiction to Assumption (4). This shows that α exists for any particular $x \in \bar{X}$.

Suppose $w(\alpha,\pi) \notin Y$. Then $(\alpha(1-\pi)/\pi)z + x' \notin Y$ will also hold for x' near enough to x. In other words, α is effective for a sufficiently small open neighborhood of x. Let $U(x)$ be such a neighborhood, relative to $\bar{X}$, for $x \in \bar{X}$. Since $\bar{X}$ is compact, by the Auxiliary Assumption, there is a finite set $\{x_i\}$, $i = 1, \ldots, N$, such that $\bigcup_{i=1}^N U(x_i) = \bar{X}$. Therefore, we may choose $\alpha = \max \alpha_i$, where α_i is effective for x_i.

APPENDIX II

We must show that z may be chosen distinct from c. Let $C \subset R^n$ be a closed convex set that does not contain 0. Define cone $(C) = \{\alpha x \mid x \in C, \alpha \geqslant 0\}$, the cone spanned by C from the origin. Let $D = \{x \in C \mid \alpha x \notin C, \text{ for } 0 \leqslant \alpha < 1\}$. D is the set of points of first contact with C of rays from the origin. $D \subset C$ and D is contained in the boundary of C in cone (C), perhaps properly. We have the following proposition:

PROPOSITION: *Let $x \in D$ and y be any other point of C. Then $x - y \notin$ cone (C).*

Suppose $x - y \in$ cone (C). Then there exists $\alpha > 0$ such that $\alpha(x - y) \in C$. Since y also lies in C and C is convex, any convex combination of y and $\alpha(x - y)$ must lie in C. In particular,

$$\frac{\alpha}{\alpha+1} y + \frac{1}{\alpha+1}[\alpha(x-y)] = \frac{\alpha}{\alpha+1} x \in C.$$

Since $\alpha/(\alpha + 1) < 1$, this contradicts the assertion that $x \in D$.

The Proposition must be applied to the closed convex set F_1 and the cone with vertex at $x^*_{I_1}$ spanned by F_1, denoted cone $(x^*_{I_1}, F_1)$. Cone $(x^*_{I_1}, F_1) = \{y \mid y = \alpha x + (1-\alpha)x^*_{I_1}), x \in F_1, \alpha \geqslant 0\}$. Assume that c is a point of first contact of a ray from $x^*_{I_1}$ with F_1, that is, $c \in F_1$ and $x^*_{I_1} + \alpha(c - x^*_{I_1}) \notin F_1$ for $0 \leqslant \alpha < 1$. By choosing $x^*_{I_1}$ to be the origin, we see that the Proposition implies $x^*_{I_1} + c - b = x^*_{I_1} + w \notin$ cone $(x^*_{I_1}, F_1)$. We will show that this contradicts the choice of w to lie in cone $(F_1) =$ cone $(0, F_1)$.

Since $x^*_{I_1} + w \notin$ cone $(x^*_{I_1}, F_1)$, $x^*_{I_1} + w \neq \alpha x + (1-\alpha)x^*_{I_1}$ for $\alpha \geqslant 0$, $x \in F_1$, or $(1/\alpha)w + x^*_{I_1} \notin F_1$ for any $\alpha \geqslant 0$. If $x^*_{I_1} = 0$ the contradiction follows from $w \in \bar{F}_1$. In any case, $y = (1/\alpha)w + x^*_{I_1} \in$ cone (F_1) for all $\alpha \geqslant 0$ since F_1 and thus cone (F_1) are convex. However, $x^*_{I_1} \in \bar{F}_1 - F_1$ and $x^*_{I_1} \neq 0$ imply $\beta x^*_{I_1} \in F_1$ for some $\beta > 1$, and $w \in$ cone (F_1) implies $\gamma w \in F_1$ for some $\gamma > 0$. Thus $\delta\gamma w + (1-\delta)\beta x^*_{I_1} \in F_1$ for all δ with $0 \leqslant \delta \leqslant 1$. Choose δ such that $(1-\delta)\beta = 1$. Then $x \notin F_1$ is contradicted for the choice of $\alpha = 1/\delta\gamma$, so $x^*_{I_1} + w \in$ cone $(x^*_{I_1}, F_1)$ must hold. This implies that c is not a point of first contact of the ray from $x^*_{I_1}$ with F_1. Since $c \in F_1$ and $x^*_{I_1} \notin F_1$, there must be α where $0 < \alpha < 1$ and $x^*_{I_1} + \alpha(c - x^*_{I_1}) \in F_1$. Let $\bar{\alpha}$ be the smallest value of α satisfying these conditions. Choose $z = \bar{\alpha}c + (1-\bar{\alpha})x^*_{I_1}$. Then $z \neq c$, and $z \in$ boundary F_1 in $\bar{F}_1$, as required.

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