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Author(s): Lionel W. McKenzie
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TURNPIKE THEORY

BY LIONEL W. MCKENZIE[1]

Support prices are derived for weakly maximal paths in an optimal growth model which is time dependent but without uncertainty. The notion of "reachable" stocks and paths is defined and used to derive turnpike theorems by the value loss method. The proofs do not depend on the presence of optimal balanced paths nor on the usual transversality conditions. The theorems are extended to the classical model which has a non-trivial von Neumann facet.

A TURNPIKE THEOREM was first proposed, at least in a way that came to wide attention, by Dorfman, Samuelson, and Solow in their famous Chapter 12 of *Linear Programming and Economic Analysis* [**11**], entitled "Efficient Programs of Capital Accumulation." This was in the context of a von Neumann model in which labor is treated as an intermediate product. I would like to quote the critical passage:

> "Thus in this unexpected way, we have found a real normative significance for steady growth—not steady growth in general, but maximal von Neumann growth. It is, in a sense, the single most effective way for the system to grow, so that if we are planning long-run growth, no matter where we start, and where we desire to end up, it will pay in the intermediate stages to get into a growth phase of this kind. It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end. The best intermediate capital configuration is one which will grow most rapidly, even if it is not the desired one, it is temporarily optimal" [**11**, p. 331].

It is due to this reference, I believe, that theorems on asymptotic properties of efficient, or optimal, paths of capital accumulation came to be known as "turnpike theorems."

For a long time the theory continued to be developed for the von Neumann model, in the strict sense of a model in which consumption appears as a necessary input to processes of production. In order to discuss efficient accumulation an objective must be introduced and in such a model the natural objective is to maximize the level of terminal stocks in some sense. The objective chosen by Dorfman, Samuelson, and Solow was to maximize the distance from the origin of the terminal stocks along a prescribed ray. The original proofs they used were only valid in a neighborhood of the turnpike. Also their arguments were incomplete and a slip occurred at one place.

[1] This paper is a revision of my Fisher-Schultz lecture to the European Meeting of the Econometric Society in Grenoble, France, in September, 1974. In preparing my lecture I benefited from a Guggenheim Fellowship and a Fellowship in the Center for Advanced Study in the Behavioral Sciences for 1973–1974. In the period of revision I benefited from participation in the 1975 summer seminar on the Structure of Dynamical Systems Arising in Economics and the Social Sciences sponsored by the Mathematical Social Science Board. I particularly appreciate the assistance of William Brock, David Cass, José Scheinkman, and Karl Shell.

The first complete proofs were provided by McKenzie [27], Morishima [31], and Radner [35]. McKenzie and Morishima proved global turnpike theorems in a simple Leontief-type model of accumulation, while Radner proved it in a model where all goods must be jointly produced at the turnpike. The Radner theorem was later strengthened by Nikaido [34] so that the periods near the turnpike are consecutive. As it turned out, it was Radner's method which led to a general style of proof, provided by McKenzie [26] and Tsukui [44] in the context of a Leontief model with durable capital goods and alternative processes in each industry. Also, a complete proof of the local theorem with many goods and differentiable production functions was given by McKenzie [25].

This development was severely criticized for its choice of objective and for its treatment of all goods as producible. One response was that the problem is so closely allied to the interesting case of maximizing consumption in the same production model that it seemed likely the theory would eventually prove useful for a multi-sector version of the traditional Ramsey problem too. This, indeed, did occur. For the one good Ramsey model, asymptotic theorems had been proved by Ramsey [36] and, more recently, by Cass [9], Koopmans [19], and Malinvaud [23]. Samuelson and Solow [41] sketched an extension of Ramsey's analysis to many capital goods. In these problems the objective adopted was the maximization of a utility sum over time, where utility is derived from current consumption and production is constrained by an exogenous labor supply. The first rigorous turnpike theorem for an economy with more than one sector was proved by Atsumi [2] in a two-good model using the method that Radner had introduced for the terminal objective in the von Neumann model. Independently, Romanovsky [39] in the Soviet Union solved a closely analogous problem in a dynamic programming format. Atsumi's method was extended to general multi-sector models by Gale [14], McKenzie [28], and Tsukui [45]. I should add that a significant role was played in this development by the Rochester Conference on Mathematical Models of Economic Growth held in the summer of 1964 under my direction with the support of the Social Science Research Council, and by a similar conference in the summer of 1965 at the Center for Advanced Study in the Behavioral Sciences under the direction of Kenneth Arrow with the support of the Mathematical Social Science Board. Finally, some results have been obtained by Atsumi [3] in a model which is intermediate between that of von Neumann and Ramsey, where labor is not a constraint, and the objective is to maximize a discounted sum of utility over time.

The results that have been listed all concern optimizing models, or efficient models, which require perfect foresight. There are also descriptive models in which optimization is replaced by *ad hoc* rules that govern the allocation of goods between consumption and accumulation, such as a constant savings ratio. However, we will not be concerned here with this large, and somewhat inconclusive, literature. For surveys you may consult Hahn and Matthews [15] and Burmeister and Dobell [8].

Until recently the matter stood thus for multi-sector models. There were global turnpike results for von Neumann models and Ramsey models where utility was undiscounted and for the special case of a Leontief-type model without scarce labor and with discounted utility. There were also local results for discounted utility with scarce labor by Levhari and Leviatan [**20**], but there were no global results for perhaps the most relevant case for decision making, the maximization of a discounted sum of utility over time with scarce labor. However, in the past two years the situation has changed significantly. First, Scheinkman [**42**] proved in a differentiable model that under the conditions leading to a global turnpike without discounting there will be a turnpike result when the discount factor is sufficiently near one. His theorem suffered somewhat from the lack of a criterion to indicate when the discount factor was sufficiently near one. Then Rockafellar [**38**] and Cass and Shell [**10**] provided criteria which can be interpreted in terms of the degree of concavity of the utility function. The proof of Cass and Shell, in effect, generalizes the method used by Radner in the von Neumann model. Next, Brock and Scheinkman [**7**] proved a closely related result in a differentiable model with continuous time in which the condition on the discount factor took an especially clear, local form. A careful analysis of the local problem has been provided by Magill [**22**]. Finally, Araujo and Scheinkman [**1**] have proved a turnpike theorem in a differentiable discounted model using a dominant diagonal condition which does not translate directly into the degree of concavity or the size of the discount factor. My own contribution to the recent development is a somewhat different order of proof, dispensing with the transversality condition, for the result of Cass and Shell, generalized to the case of a nonstationary utility function. In the preparation for this extension I derive prices to support simultaneously weakly maximal programs and their value functions. I must add the caution that this summary of the development of the turnpike theory in optimizing models is very cursory. In particular, we have omitted the literature that is now developing rapidly on models with stochastic utility and production.

1. KINDS OF TURNPIKES

The first turnpike theorem due to Dorfman, Samuelson, and Solow [**11**], was concerned with a finite accumulation path that swung toward an efficient balanced path in the middle phase of its history. In this paper I will be concerned with multi-sector Ramsey models, but there is a turnpike theorem in these models of the same kind. There is an assigned terminal capital stock and the objective is to maximize the sum of utility over the finite accumulation period. Then we show that if the accumulation period is long enough the optimal path will stay most of the time within an assigned small neighborhood of an infinite path that is optimal (using the term "optimal" vaguely at present). This kind of turnpike is illustrated in Figure 1, where the infinite path is balanced.

It should be mentioned that the use of a balanced path as the turnpike is incidental to the stationarity of the model. The real ground for the result is the tendency for finite optimal paths to bunch together in the middle time, and this tendency is preserved even in models which are time-dependent. Some theorems in a nonstationary context are proved by Keeler [**18**] for a simple Leontief model

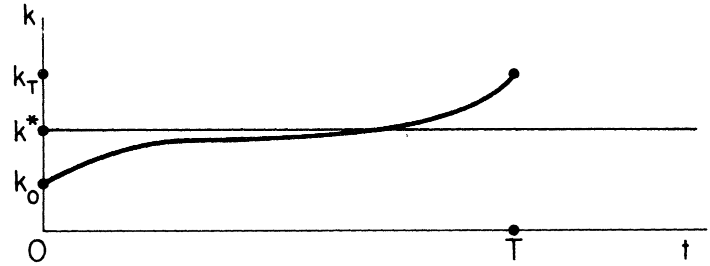

FIGURE 1.—The middle turnpike.

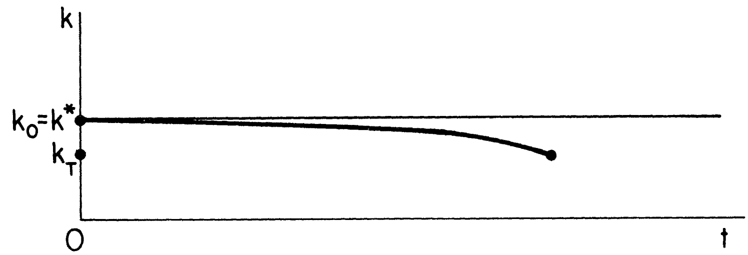

FIGURE 2.—The early turnpike.

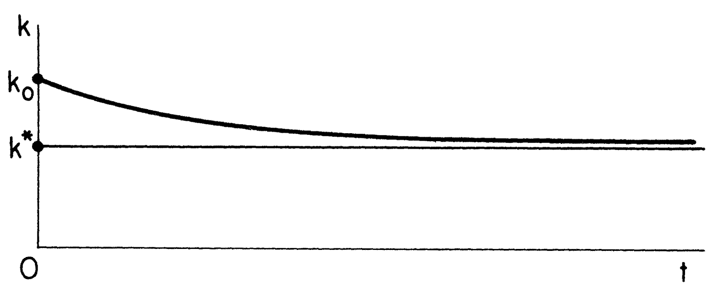

FIGURE 3.—The late turnpike.

with a terminal objective and by McKenzie in the multi-sector Ramsey model [**30**].

The second kind of turnpike theorem also concerns finite optimal paths but it compares them with an infinite path that is price-supported and starts from the same initial stocks. It asserts that a sufficiently long finite path will hug the infinite optimal path in its initial phase whatever terminal stocks are assigned. A strong theorem of this type was found by Brock in the one-sector case [**5**], and a multi-sector theorem was given by McKenzie [**30**]. A turnpike theorem of the first kind will usually imply a theorem of the second kind, but there are other cases as well. The second kind of turnpike is illustrated in Figure 2.

The third kind of turnpike deals with infinite paths that are optimal. It is the basic result that optimal paths converge to each other in appropriate circumstances. However, in stationary models it is convenient to describe this situation as convergence of infinite optimal paths to the optimal balanced path. Gale [**14**] and McKenzie [**28**] gave theorems of this kind. The critical property of optimal balanced paths in these models is that they can be supported by prices. This fact may be used to prove that infinite optimal paths exist from any initial stocks. The crucial role of price-supported paths for proofs of existence might be guessed from the early existence proof of C. C. von Weizsäcker [**46**] for one-sector Ramsey models. It was shown in a general setting by McKenzie [**30**]. The third kind of turnpike is illustrated in Figure 3.

It should be noted that in the first kind of turnpike theorem, as well as in the third kind, the converging paths may start from different initial capital stocks, while in the second kind of turnpike the converging paths must have the same initial stocks. Moreover, the finite paths in the first two cases show their turnpike tendencies independently of the assigned terminal stocks. However, the features of the model that allow the turnpike results to be reached are quite similar for the three cases, so their differences are sometimes a matter of form rather than substance.

It is worthwhile describing the practical utility of the three kinds of turnpike theorem. If the initial steps of a finite program of length T that is optimal must lie near the initial steps of the infinite optimal program from the same starting point, even though the target capital stock in period T ranges over a wide set of possibilities, it will not be necessary to know much about tastes and technology in periods beyond T in order to approximate an optimal program in the first period. Our models have a Markov property. The significance of facts beyond period T is fully allowed for in the choice of capital stocks for that period. To the degree that T period stocks can vary without substantial effect on choices in the first period, knowledge of tastes and technology beyond T is not needed.

On the other hand, if the capital stocks of finite optimal programs of length T must lie near together in period $\tau < T$ for widely differing initial and terminal stocks, it becomes possible to plan for an infinite program that is approximately optimal by aiming at the stock of period τ for whatever program of the set is easiest to compute. Once more, it is not necessary to know tastes and technology beyond T and, in addition, planning can be concentrated on the first τ periods. This assumes, of course, that the T period stock of the infinite optimal program belongs to the set of terminal stocks for which the theorem holds, and that an infinite optimal program exists.

Finally, the convergence to one another of the infinite optimal paths from different initial stocks means that infinite optimal paths may be approximated by computing finite optimal paths with the stock of any (within limits) optimal path in some period T as the target. This is useful if the infinite optimal path from a particular initial stock is easy to compute.

2. THE GENERAL MODEL

I will begin by describing a general model of which the models we will later use are special cases. In Ramsey fashion I will assume that the past influences the future only through the quantities of certain state variables at a point of time which we will identify with capital stocks. Then we suppose that the objective is given in the form of a sum of periodwise utilities that depend on events within the period, but in reduced form may be expressed as functions of initial and terminal stocks of the period. When our interest is an asymptotic property of the path of capital stocks, there is no need to show how utility depends on production and consumption during the period, for it is a necessary condition of an optimal program that these be chosen so that utility is maximized given the initial and terminal stocks of capital. Thus the significant choice from the viewpoint of the intertemporal maximization problem is the choice of terminal stocks given initial stocks. This fixes the contribution of the period to the optimal program.

The utility function may be allowed to depend on time where the dependence reflects changing technology, changing tastes, changing environment (so far as this is independent of path), and changing size and composition of population. Sometimes the effects of population size are recognized by using capital stocks per person as the arguments of the utility function. The utility function may also express a relative disinterest in the future. The changes must be thought of as foreseen and incorporated into a social evaluation function. I will not discuss the stochastic problem, except to remark that a turnpike theorem increases the interest of a model with certainty even if the world is known to be uncertain. If paths bunch together in the near future, it may not be necessary to know much about tastes and technology in the distant future.

Formally, let $u_t(x, y)$ be a function to the real line, defined on a set D_t contained in the nonnegative orthant of $E_{t-1} \times E_t$, an $n_{t-1} \cdot n_t$ dimensional Euclidean space. The vector $x \geqslant 0$ lies in E_{t-1}, and its components represent quantities of capital goods existing at time $t - 1$. The vector $y \geqslant 0$ lies in E_t and its components represent quantities of capital goods existing at time t. Then $u_t(x, y)$ represents the maximum utility realizable in the period from $t - 1$ to t when x is the initial capital stock and y is the terminal capital stock. Capital goods may be broadly construed to include a wide set of state variables, such as elements of pollution in the environment, properties and skills of the population, and remaining deposits of exhaustible natural resources. Thus we will describe a rather general nonlinear optimization problem in discrete time. The formalism and also the methods of proof are close to those which were pioneered by Romanovsky [**39**], but which unfortunately went unnoticed in the West.

We will say that a sequence of capital stocks $\{k_t\}$, $t \in I$, where I is a set of consecutive integers, is a *path of accumulation* if $(k_{t-1}, k_t) \in D_t$ when $t - 1$ and t are in I. Then a path of accumulation is *feasible* if it meets the assigned conditions on initial and terminal stocks. When the horizon is finite, an $\bar{x}$, $\bar{y}$ will be assigned and a feasible path of length s must satisfy $k_{t_0} = \bar{x}$, $k_{t_s} = \bar{y}$, where $I = \{t_0, \ldots, t_s\}$. When the horizon is infinite, the terminal requirement must be omitted.

3. SUPPORT PRICES

Turnpike profiles are characteristic of paths of accumulation that have certain optimal properties, that is, paths of accumulation which are feasible and in some sense maximize utility over finite or infinite horizons. Treat initial capital stocks as inputs and terminal capital stocks and utility for the period as outputs. Let the price of utility be one. Then associated with optimal paths under certain assumptions there are prices for the capital stocks at which input-output combinations along the optimal path achieve maximum value in each period for capital stock vectors in D_t. These prices also support the future utility sum in a similar way. The turnpike theorems will be proved by use of the support prices.

We will find price supports for optimal paths by a method due originally to Weitzman [**47**] and modified by McKenzie [**30**]. When feasible paths are infinite, the utility sum $\Sigma_t^T u_\tau(k_{\tau-1}, k_\tau)$ may diverge as $T \to \infty$. Thus, a straightforward definition of optimality is not available. Let $\{k_t\}$, $t = 0, 1, \ldots$, be an infinite path from $k_0 = \bar{x}$. Then $(k_t, k_{t+1}) \in D_{t+1}$ for all t. If $\{k_t'\}$ is a second path from $\bar{x}$ at time 0, let us say that $\{k_t\}$ *catches up to* $\{k_t'\}$ if

$$\limsup \sum_1^T (u_t(k_{t-1}', k_t') - u_t(k_{t-1}, k_t)) \leqslant 0$$

as $T \to \infty$. On the other hand, let us say that $\{k_t'\}$ *overtakes* $\{k_t\}$ if

$$\liminf \sum_1^T (u_t(k_{t-1}', k_t') - u_t(k_{t-1}, k_t)) \geqslant \varepsilon$$

for some $\varepsilon > 0$ as $T \to \infty$. An optimal path is a feasible path that catches up to every other feasible path from the same initial stocks. A weakly maximal path is a feasible path which is not overtaken by any other feasible path from the same initial stocks. This view of optimality was first proposed by von Weizsäcker [**46**] and later refined by Gale [**14**] and Brock [**4**].

In order to avoid trivial cases we make the following assumption:

ASSUMPTION 1: *For any given t and $\xi < \infty$, there is $\zeta < \infty$ such that $|x| < \xi$ implies $u_t(x, y) < \zeta$ and $|y| < \zeta$.*

We consider the weakly maximal path $\{k_t\}$, $t = 0, 1, \ldots$, from $k_0 = \bar{x}$. The addition of a constant μ_t to the utility function $u_t(x, y)$ has no effect on the optimality of a path. Then, by Assumption 1, we are free to choose $u_t(x, y)$ so that $u_t(k_{t-1}, k_t) = 0$ for all t. Let S_t be the set of paths $\{h_\tau\}$, $\tau = t, t+1, \ldots$, such that $\Sigma_{t+1}^T u_\tau(h_{\tau-1}, h_\tau)$ converges to a finite limit as $T \to \infty$. Define the value function $V_t(x) = \sup(\lim_{T\to\infty} \Sigma_{t+1}^T u_\tau(h_{\tau-1}, h_\tau))$ over all $\{h_\tau\} \in S_t$ with $h_t = x$. $V_t(x)$ is defined for x from which there is a path $\{h_\tau\} \in S_t$ with $h_t = x$. V_t is allowed to take the value ∞. Clearly the given path $\{k_t\}$ is an element of S_0 and $\Sigma_1^T u_\tau(k_{t-1}, k_t) = 0$ for all T. Then $V_t(k_t) = 0$ for all t since by weak maximality there cannot be a path in S_t starting from k_t whose finite utility sums exceed a positive ε for all large T, since this provides a path that overtakes $\{k_t\}$.

The derivation of support prices requires the following further assumption:

ASSUMPTION 2: *The utility functions $u_t(x, y)$ are concave and closed for all t. The set D_t is convex.*

For $u_t(x, y)$ to be closed means if $(x^s, y^s) \to (x, y)$ as $s \to \infty$, $s = 1, 2, \ldots$, where (x, y) lies on the boundary of D_t, then $u_t(x^s, y^s) \to u_t(x, y)$ if $(x, y) \in D_t$ and $u_t(x^s, y^s) \to -\infty$ otherwise.

Assumption 2 implies that $V_t(x)$ is a concave function. Note that $V_t(x)$ is well defined for any x from which the path $\{k_\tau\}$ can be reached by a path $\{k'_\tau\}$ where $k'_t = x$ and $k'_{t+n} = k_{t+n}$. Let K_t be the set over which V_t is well defined. K_t is convex from the concavity of u_τ and the convexity of D_τ for $\tau \geqslant t$. Then if $\bar{x}$ is in the relative interior of K_t, $V_t(\bar{x}) < \infty$ implies that $V_t(x)$ is finite valued over K_t from the concavity of V_t. Also, if $V_t(x) < \infty$ and y can be reached from x at time $t + n$, $V_{t+n}(y)$ is finite, since the intervening u_τ are finite over D_τ for each τ. Let P_t be the set of y such that $(x, y) \in D_t$ for some x. P_t is convex from the convexity of D_t. Let F_t be the smallest flat in E_t that contains P_t and K_t. Given initial stocks $\bar{x}$, we make the following assumption:

ASSUMPTION 3: *$\bar{x} \in$ relative interior K_0 and, for $t \geqslant 1$, interior $P_t \cap K_t \neq \emptyset$ relative to F_t.*

Let us call $\{k_t\}$ a relative interior path if k_0 is in the relative interior of K_0 and $k_t \in$ interior $P_t \cap K_t$ relative to F_t for $t > 0$. Then Assumption 3 is equivalent to the existence of a relative interior path from $\bar{x}$. Since $\bar{x} \in$ relative interior K_0, given any $x \in K_0$, there is x' such that $\bar{x} = \alpha x + (1 - \alpha)x'$ with $0 < \alpha < 1$ and $x' \in K_0$. Then, from concavity of u_t and $V_0(\bar{x}) = 0$, it follows that $V_0(x) < \infty$ for $x \in K_0$. But $V_t(x) < \infty$ and $(x, y) \in D_{t+1}$ implies $V_{t+1}(y) < \infty$. Since, by Assumption 3, y may be chosen in the interior of $P_{t+1} \cap K_{t+1}$ relative to F_t, and thus in the relative interior of K_{t+1}, $V_{t+1}(x) < \infty$ for all $x \in K_{t+1}$ as before. Thus we may treat V_t as finite valued in the subsequent argument. The first part of Assumption 3 may be weakened if we exclude goods from E_0 that are not initially held. We may also confine attention in subsequent periods to goods that can be produced on feasible paths from K_0. In any case, Assumption 3 depends on the particular optimal path both by way of the initial stocks and from the definition of K_t which is relative to the normalization of u_t.

Observe that $\{h_\tau\} \in S_t$ if and only if $(h_t, h_{t+1}) \in D_t$ and $(h_{t+1}, h_{t+2}, \ldots) \in S_{t+1}$, since membership in S_t only depends on the limiting behavior of the path. Thus the principle of optimality holds and

$$(1) \qquad V_t(x) = \sup\,(u_{t+1}(x, y) + V_{t+1}(y))$$

over all y such that $(x, y) \in D_t$ and $y \in K_{t+1}$. Let F_0 be the smallest flat in E_0 that contains K_0. Make the induction assumption that there exists $p_t \in F_t$, where $t \geqslant 0$, such that

$$(2) \qquad V_t(k_t) - p_t k_t \geqslant V_t(x) - p_t x$$

over all $x \in K_t$. Since the sup in (1) is attained at $y = k_{t+1}$ for $x = k_t$ by the assumption that $\{k_t\}$ is weakly maximal, we may substitute in (2) to obtain

$$(3) \qquad u_{t+1}(k_t, k_{t+1}) + V_{t+1}(k_{t+1}) - p_t k_t \geqslant u_{t+1}(x, y) + V_{t+1}(y) - p_t x,$$

for all $(x, y) \in D_{t+1}$ with $y \in K_{t+1}$. Denote the left side of (3) by v_{t+1}. Then

(4) $$v_{t+1} - u_{t+1}(x, y) + p_t x \geqslant V_{t+1}(y).$$

We define two sets for each $t \geqslant 0$,

$$A = \{(w, y) | y \in P_{t+1} \text{ and } w > v_{t+1} - u_{t+1}(x, y) + p_t x \text{ for all } x \text{ with } (x, y) \in D_{t+1}\},$$

and

$$B = \{(w, y) | y \in K_{t+1} \text{ and } w \leqslant V_{t+1}(y)\}.$$

By Assumption 3 $P_{t+1} \cap K_{t+1} \neq \emptyset$. Thus A and B are not empty. A and B are disjoint by the inequality (4). They are also convex. Thus A and B may be separated by a hyperplane contained in $R \times E_{t+1}$ defined by a vector $(\pi, -p_{t+1})$, whose inner product is not constant over $R \times F_{t+1}$. Then $\pi w - p_{t+1}y \geqslant \pi w' - p_{t+1}y'$ for all $(w, y) \in A$ and $(w', y') \in B$. From (4) and the definitions of w, w', and v_{t+1} this implies

(5) $$\pi\{u_{t+1}(k_t, k_{t+1}) + V_{t+1}(k_{t+1}) - p_t k_t - u_{t+1}(x, y) + p_t x\} - p_{t+1}y \geqslant \pi V_{t+1}(y') - p_{t+1}y',$$

for any (x, y) such that $(x, y) \in D_t$ and any $y' \in K_{t+1}$. Put $x = k_t$, $y = k_{t+1}$ and (5) becomes

(6) $$\pi V_{t+1}(k_{t+1}) - p_{t+1}k_{t+1} \geqslant \pi V_{t+1}(y') - p_{t+1}y',$$

for all $y' \in K_{t+1}$. Put $y' = k_{t+1}$ and we obtain

(7) $$\pi\{u_{t+1}(k_t, k_{t+1}) - p_t k_t\} + p_{t+1}k_{t+1} \geqslant \pi\{u_{t+1}(x, y) - p_t x\} + p_{t+1}y,$$

for all $(x, y) \in D_t$. If $\pi = 0$, (6) and (7) together would imply $p_{t+1}k_{t+1} = p_{t+1}y$ over a set W equal to all $y \in K_{t+1}$ such that there is x and $(x, y) \in D_{t+1}$. Since (π, p_{t+1}) does not have a constant inner product over $R \times F_{t+1}$, this equality is impossible if W contains an open set relative to F_{t+1}. However, $y \in P_{t+1}$ implies there is x for which $(x, y) \in D_{t+1}$. Thus W is equal to $P_{t+1} \cap K_{t+1}$ which has an interior relative to F_{t+1} by Assumption 3. Then we may set $\pi = 1$.

The induction is begun by supporting the value function $V_0(y)$ at $\bar{x}$ over K_0 in the smallest flat F_0 containing K_0. The concavity of $V_0(y)$ implies there is $(\pi, p_0) \neq 0$ such that $p_0 \in F_0$ and

(8) $$\pi V(\bar{x}) - p_0\bar{x} \geqslant \pi V(x) - p_0 x,$$

for all $x \in K_0$. If $\pi = 0$, $p_0(\bar{x} - x) \leqslant 0$ for all $x \in K_0$. Since $\bar{x} \in$ interior K_0 relative to F_0, the inequality (8) is impossible, and $\pi \neq 0$. We may choose $\pi = 1$.

Under some additional assumptions the capital value $p_t k_t$ along a weakly maximal path will be bounded. From (6), we obtain $V_t(k_t) - p_t k_t \geqslant V_t(x) - p_t x$ for any $x \in K_t$. Put $x = \alpha k_t$ for $1 - \varepsilon < \alpha < 1 + \varepsilon$. Suppose there is $\varepsilon > 0$ such that $\alpha k_t \in K_t$ for α in this range. Then

$$(9) \qquad (1-\alpha)p_t k_t \leqslant V_t(k_t) - V_t(\alpha k_t).$$

However, $V_t(k_t) = 0$ for all t. Thus if $V_t(\alpha k_t)$ is bounded over large t, $p_t k_t$ will be bounded above. Weitzman assumes that $\Sigma_t^\infty \tilde{u}_t(k_{t-1}, k_t)$ exists when the zero of the utility function is selected so that $\tilde{u}_t(0, 0) = 0$ for all t. Of course, this requires $(0, 0) \in D_t$. In this case, our normalization gives $V_t(0) = -\Sigma_{t+1}^\infty \tilde{u}_t(k_{t-1}, k_t)$, so $V_t(0) \to 0$ as $t \to \infty$. Then, using $\alpha = 0$, (9) implies $p_t k_t \to 0$, as $t \to \infty$. A less extreme assumption which will obtain the same end is that $V_t(\alpha k_t) \to 0$ as $t \to \infty$ for α near enough to 1. This may be expected to hold when $u_t(x, y) = \delta^t u(x, y)$ for some δ with $0 < \delta < 1$, that is, u_t is a stationary utility function discounted at a positive rate.

We have proved a price support lemma for weakly maximal paths:

LEMMA 1: *Let* $\{k_t\}$, $t = 0, 1, \ldots, k_0 = \bar{x}$, *be a weakly maximal path of accumulation. If Assumptions* 1, 2, *and* 3 *are met, there exists a sequence of price vectors* $p_t \in F_t$, $p_t \cdot F_t \neq 0$, $t = 0, 1, \ldots,$ *which satisfy*

$$(10) \qquad V_t(k_t) - p_t k_t \geqslant V_t(y) - p_t y, \quad \textit{for all} \quad y \in K_t,$$

$$(11) \qquad u_{t+1}(k_t, k_{t+1}) + p_{t+1}k_{t+1} - p_t k_t \geqslant u_{t+1}(x, y) + p_{t+1}y - p_t x, \quad \textit{for all} \quad (x, y) \in D_{t+1}.$$

On reflection it will be clear that the argument leading to price supports for an infinite optimal path can be adapted to the finite case as well, indeed, with fewer complications since the finite feasible paths always have finite utility sums. Moreover, price supports can be found for infinite feasible paths whose finite subpaths are optimal by taking limits on the prices supporting initial segments of the infinite path as their lengths go to infinity. This requires that one bound the prices in each period, perhaps, by means of a feasible set of outputs attainable in that period which has an interior in F_t. This was the general method introduced by Malinvaud [**24**]. However, there seems no way to arrive at bounds on the asymptotic values of the capital stocks unless the prices support a value function such as $V_t(x)$, and this method of derivation of the prices does not provide such an implication.

The existence of bounds on capital values, so-called transversality conditions, has been a basic requirement of turnpike theorems in the past (see McKenzie [**30**], for example), but we will find in the sequel that what is needed is only that the value function exist and satisfy certain bounds. This will suffice to bound the difference in capital values which is critical in the arguments.

4. AN INSIGNIFICANT FUTURE

Oddly enough, the turnpike theorem of the second kind, where the approximation to the optimal path occurs in the early periods of accumulation, can best be proved under conditions that are most difficult for the proof of the other two kinds of theorem. This is where the distant future is insignificant for the utility sum. However, the second kind of theorem reaches strong conclusions on the basis of rather weak assumptions. As we have already seen it has an interest for the planner that compares favorably with the interest of other kinds of turnpike theorem.

A crucial role in the proof of our theorems will be played by the notion of a reachable stock or a reachable path. A path $\{k_\tau\}$ is said to be *reachable* from a given capital stock $\bar{y}$ at time t if there is a path $\{k'_\tau\}$, $\tau = t, \dots, t+n$, for some n, where $k'_t = \bar{y}$ and $k'_{t+n} = k_{t+n}$. We say that a capital stock $\bar{y}$ is *reachable* from a path $\{k_\tau\}$ if for any t during the path there is a path $\{k'_\tau\}$, $\tau = t, \dots, t+n$, for some n, where $k'_t = k_t$ and $k'_{t+n} = \bar{y}$, and moreover $\Sigma_{t+1}^{t+n} u_\tau(k'_{\tau-1}, k'_\tau) > U$. It is understood that n may depend on t while U is independent of t. We say that a path $\{k'_\tau\}$ is *reachable* from a path $\{k_\tau\}$ if for any t during the second path there is a path $\{k''_\tau\}$, $\tau = t, \dots, t+n$, for some n, where $k''_t = k_t$ and $k''_{t+n} = k'_{t+n}$, and $\Sigma_{t+1}^{t+n} u_\tau(k''_{\tau-1}, k''_\tau) > U$. Again n depends on t but U does not. A stronger notion is *uniform reachability* where n may also be chosen independently of t. We also speak of *free reachability* if U may be chosen to depend on t so that $U \to 0$ as $t \to \infty$.

The idea of reachability is natural in the turnpike context since there must be paths which approximate or attain capital stock objectives in some fashion if turnpikes are to be possible. In the earlier literature with constant tastes and technology the appropriate reachability was guaranteed by special assumptions on the technology, such as the existence of a capital stock that can be expanded in every component (see Gale [**14**] and McKenzie [**28**], for example). For variable models the assumption will be made directly.

In order to prove our turnpike theorem, we will need to strengthen the concavity assumption (Assumption 2). We assume the following:

ASSUMPTION 4: *The utility function $u_t(x, y)$ is strictly concave and closed.*

By use of Assumption 4 we can prove a preliminary lemma due to Radner [**35**] and applied to Ramsey problems by Atsumi [**2**]. First we define the notion of value loss. Given $(x, y) \in D_t$, let (p, q) satisfy

$$u_t(x, y) + qy - px = u_t(z, w) + qw - pz + \delta_t(z, w), \tag{12}$$

$$\text{for any} \quad (z, w) \in D_t, \quad \text{where} \quad \delta_t(z, w) \geqq 0.$$

Thus (p, q) are support prices for (x, y) in the tth period. Then $\delta_t(z, w)$ is the value loss associated with (z, w). We have the following lemma:

LEMMA 2: *If u_t satisfies Assumption* 4, *given $(x, y) \in D_t$ and (p, q) satisfying* (12), *there is $\delta > 0$ such that $|z - x| > \varepsilon$ implies $\delta_t(z, w) > \delta$ for any $(z, w) \in D_t$.*

If this lemma were false, there would be a sequence (z^i, w^i) such that $|z^i - x| > \varepsilon$ and $\delta_t(z^i, w^i) \to 0$. First, we note that by the concavity of u_t the value loss δ_t is falling as we move toward (x, y) along the line segment from (z, w). Thus it is just as well to put $|z^i - x| = \varepsilon$ for all i. Then w^i is bounded, and the sequence (z^i, w^i) has a point of accumulation $(\bar{z}, \bar{w})$ for which $\delta_t(\bar{z}, \bar{w}) = 0$ by continuity of u_t. But then by strict concavity of u_t, $\delta_t < 0$ would have to hold at points intermediate between (x, y) and $(\bar{z}, \bar{w})$, in violation of (12). This proves the lemma.

We want to compare an infinite path from $\bar{x}$ at time 0 that is weakly maximal with a finite optimal path from $\bar{x}$ that achieves a fixed objective $\bar{y}$ at time $T > 0$. We will examine these paths in the early periods as $T \to \infty$. Consider a finite optimal path $\{\bar{k}_t\}$ where $\bar{k}_0 = \bar{x}$ and $\bar{k}_T = \bar{y}$. Suppose that $\bar{y}$ is freely reachable from the weakly maximal path $\{k_t\}$. The definition of optimality implies, for $\{k_\tau^T\}$ a path from k_{T-n} to $\bar{y}$,

$$(13) \qquad \sum_1^T u_t(\bar{k}_{t-1}, \bar{k}_t) \geqslant \sum_1^{T-n} u_t(k_{t-1}, k_t) + \sum_{T-n+1}^{T} u_t(k_{t-1}^T, k_t^T),$$

for $T > n$. On the other hand, if we use the definition of δ_{t+1}, relative to (k_t, k_{t+1}) and (p_t, p_{t+1}) in (11), we obtain from Lemma 1,

$$(14) \qquad u_{t+1}(k_t, k_{t+1}) + p_{t+1}k_{t+1} - p_t k_t = u_{t+1}(\bar{k}_t, \bar{k}_{t+1}) + p_{t+1}\bar{k}_{t+1} - p_t\bar{k}_t + \delta_{t+1}(\bar{k}_t, \bar{k}_{t+1}).$$

Then summing (14) from $t = 0$ to $t = T - 1$ gives

$$(15) \qquad \sum_1^T u_t(\bar{k}_{t-1}, \bar{k}_t) = \sum_1^T u_t(k_{t-1}, k_t) + p_0(\bar{k}_0 - k_0) + p_T(k_T - \bar{y}) - \sum_1^T \delta_t(\bar{k}_{t-1}, \bar{k}_t).$$

Noting that $\bar{k}_0 = k_0$ and substituting (13) in (15), we obtain

$$(16) \qquad \sum_1^T \delta_t(\bar{k}_{t-1}, \bar{k}_t) \leqslant \sum_{T-n+1}^{T} (u_t - u_t^T) + p_T(k_T - \bar{y}).$$

Put $u_t(k_{t-1}, k_t) = 0$ for all t. Since $\bar{y}$ is freely reachable from $\{k_t\}$, the paths $\{k_t^T\}$ may be chosen so that $\Sigma_{T-n+1}^T(u_t - u_t^T) < U_1(T)$ where n depends on T and $U_1(T) \to 0$ as $T \to \infty$. Moreover, (10) implies that $p_T(k_T - \bar{y}) \leqslant V_T(k_T) - V_T(\bar{y})$. Since utility is normalized on $\{k_t\}$, $V_T(k_T) = 0$. Then the definition of V_T implies $V_T(\bar{k}_T) \geqslant \Sigma_{T+1}^{T+n} u_t^T(k_{t-1}^T, k_t^T)$ for any choice of a path $\{k_t^T\}$ with $k_T^T = \bar{k}_T$ and $k_{T+n}^T = k_{T+n}$. If we assume that $\{k_t\}$ is freely reachable from $\bar{y}$, the paths $\{k_t^T\}$ may be chosen so that $V_T(\bar{y}) \geqslant U_2(T)$ where $U_2(T) \to 0$ as $T \to \infty$. Substituting these results in (16) gives $\Sigma_1^T \delta_t(\bar{k}_{t-1}, \bar{k}_t) \leqslant -U_1(T) - U_2(T)$ where the right-hand side converges to 0 as $T \to \infty$.

By the assumption of strict concavity (Assumption 4) and Lemma 2, $|\bar{k}_t - k_t| > \varepsilon > 0$ implies there is $\delta(t) > 0$ such that $\delta_t(\bar{k}_t, \bar{k}_{t+1}) > \delta(t)$. Thus $\max |\bar{k}_t - k_t| > \varepsilon$ for $1 \leqslant t \leqslant \tau_1$ implies $\Sigma_2^{\tau_1} \delta_t(\bar{k}_{t-1}, \bar{k}_t) > \delta = \min \delta(t)$ over $2 \leqslant t \leqslant \tau_1$. But there is τ_2 such that $T \geqslant \tau_2$ implies $\Sigma_1^T \delta_t(\bar{k}_{t-1}, \bar{k}_t) \leqslant \delta$. This means that $|\bar{k}_t - k_t| \leqslant \varepsilon$ must hold for $t \leqslant \tau_1$ when $T \geqslant \tau_2$. We have proved the following theorem:

THEOREM 1: *Let $\{k_t\}$ be a weakly maximal path. Suppose Assumptions* 1, 3, *and* 4 *are met. Let $\bar{y}$ be a stock vector that is freely reachable from $\{k_t\}$ and from which $\{k_t\}$ is freely reachable. For any $\varepsilon > 0$ and any τ_1 there is τ_2 such that if $\{\bar{k}_t\}$, $t = 0, \ldots, T, T \geqslant \tau_2$, $\bar{k}_0 = k_0$, $\bar{k}_T = \bar{y}$, is an optimal path, then $|\bar{k}_t - k_t| < \varepsilon$ for $t \leqslant \tau_1$.*

An example of a utility function that satisfies the conditions of Theorem 1 is a stationary current utility function that is strictly concave where the objective is to maximize a utility sum discounted at a positive rate, that is,

$$\sum_{1}^{T} u_t(k_{t-1}, k_t) = \sum_{1}^{T} \rho^t u(k_{t-1}, k_t)$$

for $0 < \rho < 1$. Say that a stock x is *expansible* if there is (x, y) in the domain D of u such that $y > x$. Assume *free disposal*, that is, $x' \geqslant x, y' \leqslant y$, implies $(x', y') \in D$ and $u(x', y') \geqslant u(x, y)$. In this model there is an optimal stationary path, $k_t = k$, if there is an expansible stock $\bar{x}$; see Sutherland [**43**]. Suppose that k is expansible and $\bar{y}$ is expansible. Then, in particular, Theorem 1 holds for the optimal stationary path and this $\bar{y}$.

We may derive additional optimality properties for a weakly maximal path under the assumptions of Theorem 1. Let $\{k_t'\}$ be any path with $k_0' = k_0$ that $\{k_t\}$ does not overtake, that is, $\limsup (u_t' - u_t) \geqslant 0$. Replacing $\bar{k}_t$ by k_t' in (14) and summing, we obtain

$$\sum_{1}^{T} (u_t(k_{t-1}', k_t') - u_t(k_{t-1}, k_t)) = p_T(k_T - k_T') - \sum_{1}^{T} \delta_t(k_{t-1}', k_t'). \tag{17}$$

Assume that $\{k_t\}$ is freely reachable from any path from k_0 that it does not overtake. Then $p_T(k_T - k_T') \leqslant -V_T(k_T') \to 0$ as $T \to \infty$. Thus (17) implies

$$\limsup \sum_{1}^{T} (u_t' - u_t) \leqslant 0,$$

and $\{k_t\}$ must catch up to any path. This means $\{k_t\}$ is optimal. Our result is the following proposition:

PROPOSITION 1: *Let $\{k_t\}$ be a weakly maximal path and let Assumptions* 1, 3, *and* 4 *hold. Suppose $\{k_t\}$ is freely reachable from any path from k_0 that it does not overtake. Then $\{k_t\}$ is optimal.*

5. UNIFORMLY CONCAVE UTILITY

We next prove a turnpike theorem of the first kind where finite optimal paths that are sufficiently long spend most of the time near a weakly maximal path. The crucial fact that underlies the turnpike property is uniform concavity of the utility functions. As a consequence, paths that do not converge to the turnpike suffer value losses that are unbounded as the length of the paths increases.

According to Lemma 2, if u_t is strictly concave its graph may be supported at any point (x, y) of its domain of definition D_t by prices (p, q) such that $|z - x| > \varepsilon > 0$ implies $\delta_t(z, w) > \delta$ for some $\delta > 0$. The value loss $\delta_t(z, w)$ is defined by (12). *Uniform concavity* requires, in addition, that δ may be chosen independently of $(x, y) \in D_t$ and of t. In the case of $u_t(x, y) = \rho^t u(x, y)$, where $u(x, y)$ is strictly concave this condition fails for $0 < \rho < 1$ since the corresponding value loss can be given the form $\rho^t \delta(z, w)$ which converges to 0 as $t \to \infty$. Thus in a *quasi-stationary* model ($u_t = \rho^t u$) it is necessary to choose $\rho \geqq 1$ to apply the results of this section. As we will see, $\rho > 1$ must also be excluded. We assume the following:

ASSUMPTION 5: *The utility functions $\{u_t\}$ are uniformly concave, that is, the δ of Lemma 2 may be chosen independently of (x, y) and t.*

Uniform concavity is a condition analogous to the assumptions made by C. C. von Weizsäcker [**46**] and used to prove the existence of an optimal path of accumulation in a one-sector model. It is used in a more general context by McKenzie [**28**]. With Assumption 5 it is possible to establish a turnpike theorem for a wide class of finite optimal paths.

Let $\{k_t\}$, $t = 0, 1, \ldots,$ be a weakly maximal path. Assume Assumptions 1, 3, and 5 are valid. Let $\bar{x}$ be a vector of capital stocks at $t = 0$ from which $\{k_t\}$ is reachable. Let $\bar{y}$ be a vector of capital stocks from which $\{k_t\}$ is uniformly reachable, and which is uniformly reachable from $\{k_t\}$, that is, from any $\tau \geqq 0$ there is a path departing from k_τ and arriving at $\bar{y}$ after n periods with a utility sum over these periods bounded below by a number U. We recall that n and U are independent of the choice of τ. These reachability assumptions can be met in a quasi-stationary model where $\rho \geqq 1$, only if $\rho = 1$, since $\rho > 1$ implies U must become infinite with t. Also utility is normalized so that $u_t(k_{t-1}, k_t) = 0$ along $\{k_t\}$.

Consider any finite optimal path $\{\bar{k}_t\}$, starting with $\bar{x}$ at $t = 0$ and terminating with $\bar{y}$ at $t = T$. Suppose $\{k_t\}$ is reachable from $\bar{x}$ at $t = 0$ in n_1 periods and $\bar{y}$ is uniformly reachable from $\{k_t\}$ in n_2 periods. Choose $T > n_1 + n_2$. Let $\{k'_t\}$, $t = 0, \ldots, T$, satisfy, $k'_0 = \bar{x}$, $k'_t = k_t$ for $n_1 \leqq t \leqq T - n_2$, and $k'_T = \bar{y}$. The existence of such a path is guaranteed by the reachability assumptions. Write u'_t for $u_t(k'_{t-1}, k'_t)$. The definition of optimality implies that

$$\sum_{1}^{T} u_t(\bar{k}_{t-1}, \bar{k}_t) \geqq \sum_{1}^{n_1} u'_t + \sum_{n_1+1}^{T-n_2} u_t(k_{t-1}, k_t) + \sum_{T-n_2+1}^{T} u'_t. \tag{18}$$

On the other hand, Lemma 1 implies that $\{p_t\}$, $t = 0, 1, \ldots,$ exist that satisfy (14) for the present paths $\{k_t\}$ and $\{\bar{k}_t\}$, with $\delta_{t+1}(\bar{k}_t, \bar{k}_{t+1}) \geqq 0$. Summing as before gives

$$\sum_{1}^{T} u_t(\bar{k}_{t-1}, \bar{k}_t) = \sum_{1}^{T} u_t(k_{t-1}, k_t) + p_0(\bar{x} - k_0) + p_T(k_T - \bar{y}) - \sum_{1}^{T} \delta_t(\bar{k}_{t-1}, \bar{k}_t). \tag{19}$$

Substituting (18) in (19), putting $\Sigma_1^{n_1} u_t' = U_1, \Sigma_{T-n_2+1}^T u_t' = U_2$, and using $u_t(k_{t-1}, k_t) = 0$, we obtain

$$(20) \qquad \sum_1^T \delta_t(\bar{k}_{t-1}, \bar{k}_t) \leqslant p_0(\bar{x} - k_0) + p_T(k_T - \bar{y}) - U_1 - U_2.$$

Since $\{k_t\}$ is reachable from $\bar{y}$, $\bar{y} \in K_t$ and we may apply (10) of Lemma 1. Noting that $V_t(k_t) = 0$ for all t, we have $p_T(k_T - \bar{y}) \leqslant -V_T(\bar{y})$. By uniform reachability of $\{k_t\}$ from $\bar{y}$, there exists a path from $\bar{y}$ with utility sum bounded below by U. Thus $V_T(\bar{y})$, which is the supremum of utility sums over infinite paths from $\bar{y}$ at T, is bounded below by U, or $p_T(k_T - \bar{y}) \leqslant -U$. This shows that the right-hand side of (20) is bounded above independently of T, or $\Sigma_1^T \delta_t(\bar{k}_{t-1}, \bar{k}_t) \leqslant M$.

On the other hand, by uniform concavity there is $\delta > 0$ such that $\bar{\delta}_{t+1} = \delta_{t+1}(\bar{k}_{t-1}, \bar{k}_t) > \delta$ whenever $|\bar{k}_t - k_t| > \varepsilon$. Since $\bar{\delta}_t \geqslant 0$ for all t, $\Sigma_1^T \bar{\delta}_t \leqslant M$ places the upper bound M/δ in the number of times that $|\bar{k}_t - k_t| > \varepsilon$ can occur. We have proved the next theorem:

THEOREM 2: *Let $\{k_t\}$, $t = 0, 1, \ldots$, be a weakly maximal path. Suppose Assumptions* 1, 3, *and* 5 *are satisfied. Let $\bar{x}$ be a stock vector at $t = 0$ from which $\{k_t\}$ is reachable. Let $\bar{y}$ be a stock vector that is uniformly reachable from $\{k_t\}$ and from which $\{k_t\}$ is uniformly reachable. Then for any $\varepsilon > 0$ there is $N(\varepsilon)$ such that a finite path $\{\bar{k}_t\}$ with $\bar{k}_0 = \bar{x}$ and $\bar{k}_T = \bar{y}$ must satisfy $|\bar{k}_t - k_t| < \varepsilon$ for all except, at most, $N(\varepsilon)$ values of t. $N(\varepsilon)$ is independent of T.*

We may also prove in a similar manner and on similar assumptions a turnpike theorem of the third kind. Suppose $\{\bar{k}_t\}$, $t = 0, 1, \ldots$, is a second weakly maximal path and assume that $\{k_t\}$ is uniformly reachable from $\{\bar{k}_t\}$. Construct the comparison path $\{k_t'\}$, $k_0' = \bar{k}_0$, $k_t' = k_t$ for $t \geqslant n$. Applying the definition of weakly maximal to $\{\bar{k}_t\}$,

$$(21) \qquad \liminf \sum_1^T (u_t' - \bar{u}_t) \leqslant 0, \quad \text{as} \quad T \to \infty,$$

where $u_t' = u_t(k_{t-1}', k_t')$ and $\bar{u}_t = u_t(\bar{k}_{t-1}, \bar{k}_t)$. On the other hand, summing (15) for $\{\bar{k}_t\}$ gives, using $u_t = 0$ for all t and $\bar{\delta}_t = \delta_t(\bar{k}_{t-1}, \bar{k}_t)$,

$$(22) \qquad \sum_1^T \bar{u}_t = p_0(\bar{k}_0 - k_0) + p_T(k_T - \bar{k}_T) - \sum_1^T \bar{\delta}_t,$$

and, deriving U from uniform reachability,

$$(23) \qquad \sum_1^T u_t' = \sum_1^n u_t' \geqslant U.$$

Substituting (22) and (23) in (21) gives

$$(24) \qquad \liminf \left(U - p_0(\bar{k}_0 - k_0) - p_T(k_T - \bar{k}_T) + \sum_1^T \bar{\delta}_t \right) \leqslant 0,$$

as $T \to \infty$. Since $\{\bar{k}_t\}$ is an infinite path, $\bar{k}_T \in K_T$. Thus (10) applies and $p_T(k_T - \bar{k}_T) \leqslant -V_T(\bar{k}_T) \leqslant -U$, so (24) becomes

$$\text{(25)} \qquad \liminf \sum_1^T \bar{\delta}_t \leqslant -2U - p_0(\bar{k}_0 - k_0), \qquad T \to \infty.$$

Then $\bar{\delta}_t \geqslant 0$ implies $\bar{\delta}_t \to 0$, and by uniform concavity $\bar{k}_t \to k_t$. We have the following theorem:

THEOREM 3: *Let $\{k_t\}$, $\{\bar{k}_t\}$, $t = 0, 1, \ldots,$ be weakly maximal paths. Suppose Assumptions 1 and 5 are satisfied, and Assumption 3 holds for $\{k_t\}$. Assume that $\{k_t\}$ is uniformly reachable from $\{\bar{k}_t\}$. Then $\bar{k}_t \to k_t$ as $t \to \infty$.*

We may use an argument like that for Proposition 1 to prove a similar optimality result here. Suppose $\{k_t\}$ is weakly maximal and $\{k'_t\}$, $t = 0, 1, \ldots$, is any other path with $k'_0 = k_0$ that satisfies $\limsup \Sigma_1^T (u'_t - u_t) \geqslant 0$. This means that $\{k_t\}$ does not overtake $\{k'_t\}$. Assume that $\{k_t\}$ is uniformly reachable from any path that it fails to overtake. Replacing $\bar{k}_t$ by k'_t in (14) and summing we have

$$\text{(26)} \qquad \sum_1^T (u'_t - u_t) = p_T(k_T - k'_T) - \sum_1^T \delta_T,$$

where $\delta_t = \delta_t(k'_{t-1}, k'_t)$ relative to a price sequence $\{p_t\}$ supporting $\{k_t\}$. Under the hypothesis of Theorem 3, $\{p_t\}$ can be found from Lemma 1. Also from Lemma 1 and uniform reachability $p_T(k_T - k'_T) \leqslant -V(k'_T) \leqslant -U$ for some number U. Since $\{k_t\}$ does not overtake $\{k'_t\}$, (26) implies that $\delta_t \to 0$, so $k'_t \to k_t$ as $t \to \infty$. Now add the assumption that $\{p_t\}$ is bounded, and $\limsup \Sigma_1^T (u'_t - u_t) \leqslant 0$ follows. Thus for any $\{k'_t\}$ with $k'_0 = k_0$ we have $\limsup \Sigma_1^T (u'_t - u_t) \leqslant 0$. Thus $\{k_t\}$ catches up to $\{k'_t\}$. Since $\{k_t\}$ catches up to any path from k_0 that if fails to overtake, it is optimal.

PROPOSITION 2: Assume that $\{k_t\}$, $t = 0, 1, \ldots$, is a weakly maximal path and Assumptions 1, 3, and 5 hold. Assume that $\{k_t\}$ is uniformly reachable from any path from k_0 that it does not overtake, and that it is supported by a bounded price sequence $\{p_t\}$. Then $\{k_t\}$ is optimal.

6. WEAKLY MAXIMAL PATHS

From Theorem 3 we know that two weakly maximal paths will converge under the assumption of uniformly concave utility if one of them is uniformly reachable from the other. However, we have noted that uniform concavity does not apply to the stationary utility function subject to discounting which is the basis for many frequently used models. However, recent work of Cass and Shell [**10**] and Rockafellar [**38**] has shown a way to weaken the demand for uniformity by putting a lower bound, in effect, on the degree of concavity. The nature of this bound has been explored further by Brock and Scheinkman [**6** and **7**] for the case of differentiable utility.

The Cass–Shell argument is made in the framework of Hamiltonian theory as developed by Rockafellar [**37**]. We will derive our results using the method of our earlier arguments, a method which may be referred to as "value loss". This approach is mentioned briefly in the Cass–Shell paper, but it is not carried out there. The new move that permits assumptions to be weakened is simply to sum the value losses $\delta_t(k'_{t-1}, k'_t)$ and $\delta'_t(k_{t-1}, k_t)$ of two weakly maximal paths, the loss along each path evaluated at the price supports of the other. This provides a Liapounov function, and summing the inequalities (10) derived from supports of the value functions provides an upper bound to the Liapounov function.

Let us consider two paths $\{k_t\}$ and $\{k'_t\}$, $t = 0, 1, \ldots$, that are weakly maximal among paths from k_0 and k'_0, respectively. Adopting Assumptions 1 and 4, and 3 for each path, price supports $\{p_t\}$ and $\{p'_t\}$ are provided by Lemma 1. We will say that two paths commute if each is uniformly reachable from the other. Note that if Assumption 3 is satisfied for one of a pair of commuting paths, it is satisfied for the other. The definition of the value losses and formula (11) provide symmetrical expressions from the viewpoint of the two paths,

$$(27)\qquad u_t(k_{t-1}, k_t) + p_t k_t - p_{t-1}k_{t-1} = u_t(k'_{t-1}, k'_t) + p_t k'_t - p_{t-1}k'_{t-1} + \delta_t,$$

$$(28)\qquad u_t(k_{t-1}, k_t) + p'_t k_t - p'_{t-1}k_{t-1} = u_t(k'_{t-1}, k'_t) + p'_t k'_t - p'_{t-1}k'_{t-1} - \delta'_t.$$

In these formulae, $\delta_t = \delta_t(k'_{t-1}, k'_t)$, and $\delta'_t = \delta'_t(k_{t-1}, k_t)$. The prices, the validity of formula (11), and thus the size of value losses do not depend on the normalization of u_t. Subtracting (28) from (27) gives

$$(29)\qquad (p'_t - p_t)(k'_t - k_t) - (p'_{t-1} - p_{t-1})(k'_{t-1} - k_{t-1}) = \delta_t + \delta'_t.$$

Note that the utilities that featured in previous value loss formulae cancel out. Let $L_p(t) = (p'_t - p_t)(k'_t - k_t)$. In order for $L_p(t)$ to serve as a Liapounov function it is necessary that $L_p(t)$ be bounded and uniformly increasing for $|k'_t - k_t| > \varepsilon > 0$. The latter condition is certainly provided if the utility functions are uniformly concave. (This means, of course, that assumptions are not weakened.)

In order to bound $L_p(t)$, assume that the two paths commute. Let V_t be the value function when the origin of utility is assigned so that $u_t(k_{t-1}, k_t) = 0$, and V'_t correspondingly for $u'_t(k'_{t-1}, k'_t) = 0$. Since the paths commute, $V_t(k'_t)$ and $V'_t(k_t)$ are well defined and finite. By (10) we have

$$(30)\qquad V_t(k_t) - p_t k_t \geqslant V_t(k'_t) - p_t k'_t,$$

$$(31)\qquad V'_t(k_t) - p'_t k_t \leqslant V'_t(k'_t) - p'_t k'_t.$$

The shift of normalization implies that $u_t(k'_{t-1}, k'_t) = -u'_t(k_{t-1}, k_t)$, and therefore $V_t(k'_t) = -V'_t(k_t)$. Subtracting (31) from (30), and using $V_t(k_t) = 0$, $V'_t(k'_t) = 0$, gives

$$(32)\qquad (p'_t - p_t)(k'_t - k_t) \leqslant 0.$$

In other words, $L_p(t)$ is bounded above by zero, and the conditions for it to serve as a Liapounov function are met. This Liapounov function was first applied to a stationary model by Samuelson [**41**] in dealing with a local problem. Of course, so far we have not advanced beyond Theorem 3.

In the case of a stationary utility function with discounting, where $u_t = \rho^t u$ for $0 < \rho < 1$, the condition of uniform concavity over time can be restored by replacing u_t with $\tilde{u}_t = \rho^{-t} u_t = u$. Cass and Shell showed that one could define a current value Liapounov function in terms of a price support of u that might be effective. Brock and Scheinkman [**6**] gave a sufficient condition on the degree of concavity of u in the differentiable case to insure that the Liapounov function works. This suggests that we seek positive numbers ρ_t such that $\tilde{u}_t = \Pi_1^t \rho_\tau^{-1} u_t$ are uniformly concave.

Define a current value Liapounov function by $L_c(t) = \Pi_1^t \rho_\tau^{-1} L_p(t)$. We may write (29) as

$$L_p(t) - L_p(t-1) = \delta_t + \delta_t'. \tag{33}$$

Multiply (33) through by $\Pi_1^t \rho_\tau^{-1}$. Put $\rho_\tau = 1/(1 + r_\tau)$ and $\beta_t = \Pi_1^t \rho_\tau$, and simplify to give

$$L_c(t) - L_c(t-1) - r_t L_c(t-1) \geqslant \beta_t^{-1}(\delta_t + \delta_t'). \tag{34}$$

Note that $\beta_t^{-1}(\delta_t + \delta_t') + r_t L_c(t-1) \geqslant 0$ implies $L_c(t) \geqslant L_c(t-1)$. β_t is the discount factor that converts t-period current prices into t-period present prices, as realized at time zero. Thus $\beta_t^{-1} \to \infty$ with t if $r_\tau > \varepsilon > 0$ for all τ. We will use the following concavity assumption:

ASSUMPTION 6: For any $\varepsilon > 0$, there is $\delta > 0$ such that $|x' - x| > \varepsilon$ implies $\beta_t^{-1}(\delta_t + \delta_t') + r_t L_c(t-1) > \delta$, independently of t, where δ_t and δ_t' are derived from a support of u_t at (x, y) and (x', y'), respectively, and $\delta_t = \delta_t(x', y')$, $\delta_t' = \delta_t'(x, y)$.

When $r_t = 0$ for all t, Assumption 6 reduces to Assumption 5. Moreover, if the utility functions $\beta_t^{-1} u_t$ are uniformly concave, it may be expected that Assumption 6 will continue to hold if r_t is near enough to zero for all t.

Let $\{k_t\}$ and $\{k_t'\}$ be weakly maximal paths for $t = 0, 1, \ldots$. Let u_t be a normalization of utility so that $u_t(k_{t-1}, k_t) = 0$ for all $t \geqslant 1$, and let u_t' be a normalization satisfying $u_t'(k_{t-1}', k_t') = 0$ for all $t \geqslant 1$. Note that $\Sigma_t^T u_{\tau+1}(k_\tau', k_{\tau+1}')$ convergent as $T \to \infty$ implies that the limit of this sum is $V_t(k_t')$ and, *mutatis mutandis*, for $V_t'(k_t)$. Otherwise there would be a path with a convergent utility sum whose limit lay closer to $V_t(k_t')$, and this path would overtake $\{k_t'\}$ in contradiction to weak maximality. From (32) we have $L_p(t) \leqslant 0$. Thus $L_c(t) = \Pi_1^t \rho_\tau^{-1} L_p(t) \leqslant 0$, or the current Liapounov function is also bounded above. On the other hand, if Assumption 6 holds, $|k_t' - k_t| > \varepsilon > 0$ for an infinite number of periods implies $L_c(t)$ is not bounded above. Thus $|k_t' - k_t| \to 0$, as $t \to \infty$. We have proved the following:

THEOREM 4: *Let $\{k_t\}$, $\{k_t'\}$, $t = 0, 1, \ldots$, be weakly maximal paths that commute. Assume Assumptions* 1, 2, *and* 3 *are met. If discount factors β_t can be chosen so that Assumption* 6 *holds, then $|k_t' - k_t| \to 0$ as $t \to \infty$.*

We are able to use Lemma 1 to obtain the price sequences $\{p_t\}$ and $\{p_t'\}$ that are needed to state Assumption 6. We may note that since the paths commute, the sets K_t are the same for the two paths so that Assumption 3 is satisfied for both paths if it is satisfied for either. Also value losses will only be needed in the proof for weakly maximal paths. This means that F_t can be taken as the smallest flat containing K_t, for capital stocks in P_t and not in K_t cannot appear on a weakly maximal path and so are irrelevant for value loss calculations. Then the interiority assumption (Assumption 3) is much weaker. Finally, the assumption that the paths commute can be replaced by the assumption that $\Sigma_1^T u_t(k'_{t-1}, k'_t)$ converges to a finite limit as $t \to \infty$, when $u_t(k_{t-1}, k_t) = 0$ for all t.

The situation is simplest for application of Theorem 4 in the model with a present utility that equals a discounted stationary current utility, $u_t(x, y) = \rho^t u(x, y)$, where $0 < \rho < 1$. It is obvious for bounded current utilities that $\Sigma_1^T u_t$ will converge. However, it is a common assumption in these models that *sustainable* stocks (all x such that there is $y \geqslant x$ and $(x, y) \in D$), and thus current utilities, are bounded above. If the utility function $u(x, y)$ is also bounded below on D, or if k_0 is expansible, the convergence of $\Sigma_1^T u_t$ along weakly maximal paths follows. Assume free disposal. Assume that an expansible stock exists. Let

$$W = \{y' | y > y' > x\}$$

for x expansible and $(x, y) \in D$. Then $W \subset P$ (the set of outputs) and $W \subset K$ (the set of inputs that start infinite paths with finite utility sums). Thus the second part of Assumption 2 is satisfied. Then $k_0, k'_0 \in$ interior K completes Assumption 2. If Assumption 6 also holds, Theorem 4 may be applied to derive convergence of k_t and k'_t as $t \to \infty$. However, under fairly weak conditions when ρ is near one Assumption 6 may be shown to hold. Brock and Scheinkman [**6**] treat the differentiable case.

Consider (27) for the utility function $u_t = \rho^t u$. Multiplying through by ρ^{-t} gives

$$(35) \qquad u + \rho^{-t} p_t k_t - \rho^{-t} p_{t-1} k_{t-1} = u' + \rho^{-t} p_t k'_t - \rho^{-t} p_{t-1} k'_{t-1} + \rho^{-t} \delta_t,$$

or, putting $q_t = \rho^{-t} p_t$, we have

$$(36) \qquad u + q_t k_t - \rho^{-1} q_{t-1} k_{t-1} = u' + q_t k'_t - \rho^{-1} q_{t-1} k'_{t-1} + \rho^{-t} \delta_t.$$

Then (29) becomes

$$(37) \qquad (q'_t - q_t)(k'_t - k_t) - \rho^{-1}(q'_{t-1} - q_{t-1})(k'_{t-1} - k_{t-1}) = \rho^{-t}(\delta_t + \delta'_t).$$

If we assume the concavity of u is uniform over D, $|k'_{t-1} - k_{t-1}| > \varepsilon > 0$ implies there is $\delta > 0$ such that $\rho^{-t}\delta_t > \delta$. Thus the right-hand side of (37) is larger than δ. If we assume further that capital values are bounded over $t \geqslant 0$, it follows that ρ may be chosen near enough to one, so that

$$(38) \qquad (q'_t - q_t)(k'_t - k_t) - (q'_{t-1} - q_{t-1})(k'_{t-1} - k_{t-1}) > \delta/2$$

holds when $|k'_t - k_t| > \varepsilon$. The key to this argument is the boundedness of the capital values for ρ near one, independently of ρ. The condition $\rho^{-t}\delta_t > \delta$ is independent of ρ from uniform concavity, since uniform concavity provides the inequality for (q_{t-1}, q_t) as a support of u. Brock and Scheinkman assumed that the paths are contained in a compact set D' in the interior of D. In that case the boundedness condition is immediate. We may state our result as the following proposition:

PROPOSITION 3: Assume uniform concavity of u. Let $\{k_t\}$, $\{k'_t\}$ be weakly maximal paths for the utility functions $\rho^t u$, $t = 1, 2, \ldots$. Let $\{p_t\}$, $\{p'_t\}$ be corresponding sequences of support prices. Then if $k_t, k'_t, \rho^{-t}p_t, \rho^{-t}p'_t$ are bounded with respect to t. Assumption 6 holds for these paths for ρ sufficiently near one.

Under certain additional assumptions, principally free disposal and boundedness of sustainable stocks ($y \geqslant x$ and $(x, y) \in D$ implies $|x| < \zeta$), it may be shown that an optimal stationary path exists [**43**]. Then under the assumption of Theorem 4, all weakly maximal paths will converge to the optimal stationary path. Indeed, the weakly maximal paths will be optimal paths. This is the traditional context in which Ramsey turnpike theorems have been proved.

7. THE VON NEUMANN FACET

All of our turnpike arguments have used value losses as Liapounov functions. The sum of the shortfalls of the values of input-output combinations along an alternative path from these values along a given weakly maximal path is bounded using assumptions of reachability and the optimality properties of the alternative path. The bound forces the paths together to reduce the shortfall toward zero. However, simple concavity of the utility function does not imply such a strong condition. That is to say, $\delta_t(z, w) = 0$ in (12) does not imply $z = x$ on the assumption that u_t is concave, although if u_t is strictly concave, $(z, w) = (x, y)$ is implied. The facet notion was introduced in McKenzie [**26**] for a generalized Leontief model with terminal objective. It was extended by Makarov [**21**] to a von Neumann model with terminal objective. Also see Drandakis [**12**]. The idea is implicit in Romanovsky [**39**] in a context like the present, but the facet definition was formally adapted to the multi-sector Ramsey problem by McKenzie [**28**].

Assume that the "extensive" model has neo-classical production functions without net joint products. (If (x, y) is an input-output vector for the jth industry, for $i \neq j$, $x_i \geqslant y_i$ and if $x_i > 0$, $x_i > y_i$.) Let output be divided between consumption and terminal stocks and utility be a strictly concave function of consumption. Then the "reduced" model *cannot* have a strictly concave utility function in terms of initial and terminal stocks. A flat piece of the graph of $u_t(x, y)$ will be generated by all the variations in activity levels which are consistent with the consumption vector that underlies a particular value of $u_t(x, y)$. The corresponding variations of input and output will be absorbed by changes in the arguments x and y of u_t without a change in utility level. If the technology is also irreducible in the sense that n activities must be used to obtain $y \geqslant x$ for $(x, y) \in D_t$, the dimension of the flat pieces of the graph of u_t cannot fall below $n - 1$ when stocks are being maintained.

We will define $N_t(p, q)$ as the set of triples (u_t, x, y) such that $u_t = u_t(x, y)$ and $\delta_t(x, y) = 0$ when the price supports are (p, q). Then concavity of u_t implies that $N_t(p, q)$ is a closed, convex subset of the graph of u_t. If $\{k_t\}$, $t = 0, 1, 2, \ldots$, is a path supported in the sense of Lemma 1 by $\{p_t\}$, we call the set $N_t(p_{t-1}, p_t)$ the von Neumann facet for this path in the tth period. This sequence of facets is the general turnpike provided by value loss arguments. Let

$$d((z, w), N_t) = \min |(z, w) - (x, y)|$$

for $(x, y) \in N_t$. In order to derive the turnpike results in terms of von Neumann facets we may replace Lemma 2 by the following lemma:

LEMMA 3: *Let u_t satisfy Assumptions 1 and 2. Let (p, q) be support prices for $(x, y) \in D_t$ in the sense of (12). Let $N_t(p, q)$ be a facet of the graph of u_t. For any $\eta > 0$, $\varepsilon > 0$ there is $\delta > 0$ such that $|z| < \eta$ and $(z, w) \in D_t$ implies $\delta_t(z, w) > \delta$ for $d((z, w), N_t) > \varepsilon$.*

The proof comes from considering a sequence (z^s, w^s) that violates the conclusion, that is, $|z^s| < \eta$, $d((z^s, w^s), N_t) > \varepsilon$, but $\delta_t(z^s, w^s) < \delta^s$ where $\delta^s \to 0$. Using Assumption 1, there are convergent subsequences whose limits $(\bar{z}, \bar{w})$ and $\bar{x}$ would satisfy $\delta(\bar{z}, \bar{w}) = 0$, $|\bar{z}| \leqslant \eta$, and $d((\bar{z}, \bar{w}), N_t) \geqslant \varepsilon$. However $\delta(\bar{z}, \bar{w}) = 0$ implies $(\bar{z}, \bar{w}) \in N_t$, so we have arrived at a contradiction that proves the lemma.

It is not unreasonable, in the light of bounded labor services, to suppose the relevant facets to be bounded. Suppose, in fact, that N_t is uniformly bounded for $t \geqslant 1$; then Lemma 3 may be applied uniformly, that is, η, ε, and δ may be selected independently of t. Then a turnpike result corresponding to Theorem 2 may be proved. In this theorem convergence of finite optimal paths is to the sequence of facets $\{N_t\}$ rather than to the sequence of capital stocks $\{k_t\}$, where $(k_{t-1}, k_t) \in N_t$ in each period. Of course, if strict concavity should hold, $N_t = (k_{t-1}, k_t)$. Similarly, the conclusion of Theorem 1, that a long finite optimal path begins with initial input-outputs $\bar{k}_t$ near the initial input-outputs k_t of an infinite price-supported path that starts from the same initial input $k_0 = \bar{k}_0$, is replaced by the condition that $(\bar{k}_t, \bar{k}_{t-1})$ lies near N_t for small values of t. The conclusion of Theorem 3 is changed in a similar way so that convergence to facets replaces convergence to paths.

However, this is not the end of the story. Depending on the character of the facets it may be that paths which remain close to a sequence of facets for a long time must approach each other. This can be studied most effectively for quasi-stationary models ($u_t = \rho^t u$, $\rho < 1$) where one price-supported path is an optimal stationary path so that the facet sequence is $N_t = N^*$ for all t. Choose points in the facet N^* that (linearly) span the facet N^*, say (u^i, x^i, y^i), $i = 1, \ldots, r$, where the dimension of N^* is $r - 1 \leqslant 2n$. Then a point on N^* satisfies

$$(u, x, y) = \sum_{1}^{r} \alpha_i (u^i, x^i, y^i)$$

for some real numbers α_i, $\Sigma_1^r \alpha_i = 1$. If $\{k_t\}$ is a path on N^*, we have $(k_{t-1}, k_t) =$

$\Sigma_1^r \alpha_i^t(x^i, y^i)$, and $(k_t, k_{t+1}) = \Sigma_1^r \alpha_i^{t+1}(x^i, y^i)$, or $\Sigma_1^r \alpha_i^t y^i = \Sigma_1^r \alpha_i^{t+1} x^i$. Suppose for simplicity that $r = n + 1$ and A and B are square matrices with columns

$$\begin{pmatrix} x^i \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y^i \\ 1 \end{pmatrix},$$

respectively. Then for each $t \geqslant 0$, the equation $B\alpha^t = A\alpha^{t+1}$ must be satisfied for some vectors α^t if (k_{t-1}, k_t) lies on N^*. If A is nonsingular, this may be written

$$\alpha^{t+1} = A^{-1}B\alpha^t. \tag{39}$$

Now suppose $A^{-1}B$ has only one characteristic root λ with absolute value one and this root is simple. Then $\lambda = 1$, since α^* must solve (1), where

$$\sum_1^r \alpha_i^*(x^i, y^i) = k^*,$$

the capital stock vector of the optimal stationary path. If we make the assumption described earlier of bounded sustainable stocks, $|k_t|$ is bounded by a number ζ. Then for any path $\{k_t\}$ on N^*, $k_t \to k^*$ must hold. This is easily seen if the characteristic roots λ_i, $i = 1, \ldots, r$, are all simple, so the characteristic vectors span the complexification of the r-dimensional Euclidean space (see Hirsh and Smale [**16**, pp. 64–65]). Then $k_t = \Sigma_1^r \alpha_i \lambda_i^t z^i$ where z^i is the characteristic vector associated with λ_i and α_i is a given number, possibly complex. If $|\lambda_i| > 1$, $\alpha_i = 0$ must hold, or else the path is unbounded as $t \to \infty$. If $|\lambda_i| < 1$, $\lambda_i^t \to 0$ as $t \to \infty$. Thus $k_t \to \alpha_1 z^1 = \alpha_1 k^*$, where $\lambda_1 = 1$. We will presently see by an extension of this argument that the same convergence property will hold for any path that converges to N^*. Thus we return to path convergence once more.

The conditions needed for convergence of $\{k_t\}$ to k^* when optimal stationary states exist and the conditions of our theorems are less stringent than a requirement of nonsingularity of A would suggest. Indeed, it would seem that this result for quasi-stationary models would fail only in a set of models of "measure zero". Gale [**13**, Theorem 5] shows that it will always be possible to express k^* as a convex combination of no more than $n + 1$ processes. Moreover, small perturbations of the model will eliminate characteristic roots of absolute value one except for the root one which is present by construction. Morishima [**32**, Ch. 10 and 13] has a careful analysis of the case of a unique stationary state for a polyhedral model.

Let us say that the technology of the von Neumann facet N^* is regular if, for any $\varepsilon > 0$, there is T such that every solution α^t of the difference equation $A\alpha^{t+1} = B\alpha^t = k_{t+1}$ for which $(k_t, k_{t+1}) \in N^*$ for $t \geqslant 0$, satisfies $|k_t - k^*| < \varepsilon$ for all $t > T$. This type of formulation was introduced by Inada [**17**] and developed by McKenzie [**29**]. Also see Movshovich [**33**]. Suppose that a bounded path $\{k_t\}$ converges to N^*. We may then show that $\{k_t\}$ also converges to k^* if N^* is regular, or else we will reach a contradiction. Choose a sequence of neighborhoods U^s of N^* defined by $U^s = \{(x, y) | d((x, y), N^*) < \varepsilon^s > 0\}$ where $\varepsilon^s \to 0$, and a sequence of times t_s such that $(k_t, k_{t+1}) \in U^s$ for $t \geqslant t_s$. Since $\{k_t\}$ is bounded, the sequence of paths $\{k_\tau^s\}$, $\tau = 0, 1, \ldots$, where $k_\tau^s = k_{t_s+\tau}$, will have a subsequence converging to a path $\{k_\tau'\}$, where $(k_\tau', k_{\tau+1}') \in N^*$ for all $\tau \geqslant 0$. But if $\{k_t\}$ does not converge to k^*, for any number n the paths $\{k_\tau^s\}$ may be chosen so that $|k_{\tau+n}^s - k^*| > \varepsilon > 0$ for all s, where n may be any number greater than or equal to zero independently of ε. This implies that there exist paths beginning at time $t = 0$ on N^* that lie outside an ε-neighborhood of k^* at time $t = T$ where T may be chosen arbitrarily large. This contradicts the regularity of N^*. Thus we may state a final result from the turnpike literature.

PROPOSITION 4: If a path $\{k_t\}$ in a quasi-stationary model, satisfying Assumption 2, and boundedness of sustainable stocks, converges to the von Neumann facet N^* and the technology of N^* is regular, then $k_t \to k^*$ where k^* is the capital stock vector of the unique optimal stationary path.

The facet N^* where A is nonsingular and $A^{-1}B$ has a unique characteristic root with absolute value one, which is simple and equal to one, gives a particular case of this proposition. A condition which is equivalent to the regularity condition of Proposition 4 is that the optimal stationary path is unique and there are no cyclic paths with constant amplitude [**28**].

The University of Rochester

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