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THE EQUITY RISK PREMIUM
A Solution

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In 'The Equity Risk Premium: A Puzzle', Mehra and Prescott (1985) developed an Arrow–Debreu asset pricing model. They rejected it because it could not explain high enough equity risk premia. They concluded that only non-Arrow–Debreu models would solve this 'puzzle'. Here, I re-specify their model, capturing the effects of possible, though unlikely, market crashes. While maintaining their model's attractive features, this allows it to explain high equity risk premia and low risk-free returns. It does so with reasonable degrees of time preference and risk aversion, provided the crashes are plausibly severe and not too improbable.

1. Introduction

In 'The Equity Risk Premium: A Puzzle', Rajnish Mehra and Edward C. Prescott (1985, p. 145) wrote:

> 'Restrictions that a class of general equilibrium models place upon the average returns of equity and Treasury bills are found to be strongly violated by the U.S. data in the 1889–1978 period. This result is robust to model specification and measurement problems. We conclude that, most likely, an equilibrium model which is not an Arrow–Debreu economy will be the one that simultaneously rationalizes both historically observed large average equity return (sic) and the small average risk-free return.'

I believe that their conclusion is too drastic and that their puzzle can be solved. The solution involves noting that, while some specification changes do not affect their results, other simple changes do. In particular, by specifying Mehra and Prescott's model to include a low-probability, depression-like third state, I can explain both high equity risk premia and low risk-free returns without abandoning the Arrow–Debreu paradigm.

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The motivation for adding the third state is simple. Risk-averse equity owners demand a high return to compensate for the extreme losses they may incur during an unlikely, but severe, market crash. To the extent that equity returns have been high with no crashes, equity owners have been compensated for the crashes that happened not to occur. High risk premia should not be puzzling in such a world.

2. The consumption asset pricing model

Mehra and Prescott model a frictionless pure exchange economy with a single representative agent and a single perishable consumption good produced by a single productive unit or 'tree'. There are two assets, an equity share in the tree and a risk-free asset. The tree yields a random dividend each period and the equity share entitles its owner to that dividend in perpetuity. The risk-free asset entitles its owner to one unit of the consumption good in the next period only.

Trading in competitive markets, the agent maximizes

$$E_0\left(\sum_{t=0}^{\infty}\beta^t U(c_t)\right), \tag{1}$$

subject to the budget constraint:

$$c_t = y_t e_{t-1} + p_t^{\mathrm{e}}(e_{t-1} - e_t) + f_{t-1} - p_t^{\mathrm{f}} f_t, \tag{2}$$

where c_t is the agent's consumption in period t, β is the agent's subjective time discount factor, $U(\cdot)$ is the agent's utility function, $E_0(\cdot)$ is the mathematical expectation operator conditional on information in period zero, y_t is the tree's dividend in period t, p_t^{e} and p_t^{f} are the prices of the equity and risk-free asset in period t, and e_t and f_t are the agent's equity and risk-free asset holdings in period t.

The first-order conditions for this problem are

$$\begin{aligned} p_t^{\mathrm{e}} U'(c_t) &= E_t \beta U'(c_{t+1})(y_{t+1} + p_{t+1}^{\mathrm{e}}), \\ p_t^{\mathrm{f}} U'(c_t) &= E_t \beta U'(c_{t+1}). \end{aligned} \tag{3}$$

Market clearing implies

$$c_t = y_t, \quad e_t = 1, \quad f_t = 0, \quad \text{for all } t. \tag{4}$$

The agent displays constant relative risk aversion as given by the utility function

$$U(c, \alpha) = \frac{c^{(1-\alpha)} - 1}{1 - \alpha}, \tag{5}$$

where α is the parameter of relative risk aversion. When $\alpha = 1$, the utility function is logarithmic.

The production levels, y_t, and, thus, the consumption levels, c_t, evolve through time according to

$$y_{t+1} = x_{t+1} y_t = c_{t+1}, \tag{6}$$

where $x_{t+1} \in \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R}^+$ is the gross growth rate, which follows an ergodic Markov process, i.e.,

$$\text{Prob}(x_{t+1} = \lambda_j \mid x_t = \lambda_i) = \phi_{ij}. \tag{7}$$

This process allows the apparent non-stationarity we observe in the per capita consumption stream over the sample period. It also implies that consumption is autocorrelated and realizations of the growth rate affect all later consumption levels.

In this model, the expected utility in eq. (1) exists if, and only if, the matrix A, whose elements are given by $a_{ij} = \beta \phi_{ij} \lambda^{(1-\alpha)}$, is stable. This also establishes that a Debreu competitive equilibrium exists.[1]

The period t asset prices may be expressed as functions of the current state, (y_t, x_t), by

$$\begin{aligned} p^e(y_t, x_t) &= \mathrm{E}_t\{\beta(y_{t+1}^{-\alpha})[y_{t+1} + p^e(y_{t+1}, x_{t+1})](y_t^{\alpha})\}, \\ p^f(y_t, x_t) &= \mathrm{E}_t\{\beta(y_{t+1}^{-\alpha})(y_t^{\alpha})\}, \end{aligned} \tag{8}$$

where y_t and y_{t+1} have been substituted for c_t and c_{t+1}, and $y_t^{-\alpha}$ and $y_{t+1}^{-\alpha}$ have been substituted for $U'(c_t)$ and $U'(c_{t+1})$, respectively.

Note that y_t and x_t are sufficient for forecasting y_{t+1} and x_{t+1} and that the forecast depends only on the levels of y_t and x_t, not on the period. Therefore,

[1] See Mehra and Prescott (1984, 1985). If the $t = 0$ consumption level is c and its growth rate is λ_i, the discounted expected utility t periods in the future is given by

$$\mathrm{E}_0(U(c_t)) = (A_i A^{t-1} \iota c^{1-\alpha} - \beta^t)/(1-\alpha),$$

where A_i is the ith row of A and ι is a column vector of ones. Thus, the infinite sum in (1) will converge only when A is stable.

we can re-define the state as (c, i) when $y_t = c$ and $x_t = \lambda_i$, and then re-write (8) as

$$p^e(c,i) = \beta \sum_{j=1}^{n} \phi_{ij}(\lambda_j c)^{-\alpha}\left[p^e(\lambda_j c, j) + \lambda_j c\right]c^{\alpha},$$
$$p^f(c,i) = \beta \sum_{j=1}^{n} \phi_{ij}(\lambda_j c)^{-\alpha}(c^{\alpha}) = \beta \sum_{j=1}^{n} \phi_{ij}\lambda_j^{-\alpha}, \tag{9}$$

by substituting c and $\lambda_j c$ for y_t and y_{t+1} when the growth rate is λ_j, and then summing over transition probabilities.

Since the equity's price is homogeneous of degree one in c, it may be written as

$$p^e(c,i) = w_i c, \tag{10}$$

where w_i is an undetermined coefficient. Substituting into (9) yields

$$w_i = \beta \sum_{j=1}^{n} \phi_{ij}\lambda_j^{(1-\alpha)}(w_j + 1) \quad \text{for} \quad i = 1, \dots, n. \tag{11}$$

This may be re-written in matrix notation as

$$w = A(w + \iota), \tag{12}$$

where w is the column vector of w_i's, ι is a column vector of ones and A is the matrix defined above. The stability of A implies this system has a unique, positive solution. Lucas (1978) showed that there is only one equilibrium pricing equation in this context. Since we have found a solution for the asset pricing function in eqs. (10)–(12), we have found the unique asset pricing function.

If the current state is (c, i) and the next state proves to be $(\lambda_j c, j)$, the equity's return becomes

$$r_{ij}^e = \frac{p^e(\lambda_j c, j) + \lambda_j c - p^e(c,i)}{p^e(c,i)} = \frac{\lambda_j(w_j + 1)}{w_i} - 1. \tag{13}$$

Thus, when the current state is (c, i), the equity's expected return is

$$R_i^e = \sum_{j=1}^{n} \phi_{ij} r_{ij}^e, \tag{14}$$

and the risk-free asset's return is

$$R_i^f = \frac{1 - p^f(c,i)}{p^f(c,i)} = \frac{1}{p^f(c,i)} - 1. \tag{15}$$

The Markov chain's stationary probabilities, $\pi \in \mathbb{R}^{n+}$, exist and satisfy $\pi = \phi^T \pi$, where $\sum_i \pi_i = 1$ and $\phi^T = \{\varphi_{ji}\}$. Using these, the unconditional expected returns of the equity and the risk-free asset are

$$R^e = \sum_{i=1}^{n} \pi_i R_i^e \quad \text{and} \quad R^f = \sum_{i=1}^{n} \pi_i R_i^f. \tag{16}$$

The expected equity risk premium is $R^e - R^f$.

3. Mehra and Prescott's two- and four-state specifications

For testing, Mehra and Prescott specified their model with either two or four states. Using the method of moments, they estimated the parameters of the consumption process by matching its mean, variance and first-order autocorrelation with the corresponding sample moments from United States consumption data. Using these estimates and eqs. (10)–(16), they calculated each specification's predicted risk premium for given risk preference parameters (α's) and time preference parameters (β's). Finally, they searched for α values between zero and ten and β values between zero and one that gave both a reasonable risk-free return and a reasonable risk premium when compared to the United States economy.[2]

They found that the sample mean of the consumption level's gross growth rate was 1.018, its standard deviation was 0.036 and its first-order autocorrelation was -0.14. The economy's annual average risk-free return was 0.80 percent and the annual average equity return was 6.98 percent. The average risk premium was 6.18 percent with a standard error of 1.76 percent.[3]

In a two-state specification for the consumption growth process, they found that 0.35 percent was the largest risk premium corresponding to a risk-free return between 0 and 4 percent. Varying the parameters between reasonable limits raised the risk premium only to 0.39 percent [Mehra and Prescott (1985, pp. 154–160)]. They tested an alternate two-state specification in which equity

[2]Mehra and Prescott's data covered the period from 1889 to 1978. They used the Kuznets–Kendrick–USNIA measure on non-durables and services for real per capita consumption; the annual average Standard and Poor's Composite Stock Price Index, the annual dividends on this index and the consumption price deflator on non-durables and services to find the equity's return; and the same consumption deflator and the yields on ninety-day Treasury Bills, from 1931 to 1978, Treasury Certificates from 1920 to 1930 and sixty- to ninety-day prime commercial paper from 1889 to 1920 to find the risk-free asset's return.

[3]Here, Mehra and Prescott assume that U.S. Treasury bills are risk-free or, more correctly, that U.S. Treasury bill returns are a reasonable proxy for risk-free returns. I will make the same assumption. The evidence to date suggests this is reasonable. [The government did not default on its obligations during the sample period, which included the Great Depression. Further, Fama (1975, 1976) shows that inflation rate innovations have caused very little uncertainty in real returns.] If we allow the possibility of government default, the true risk premium will be greater than 6.18%, but the papers main results still hold. As the examples in section 4 will show, Mehra and Prescott's model can explain high risk premia when appropriately specified.

owners, who bear all the risk, receive only one tenth of the dividend on average. This increased the risk premium only 0.1 percent.[4] A four-state specification yielded risk premia ranging from 0.35 to 0.39 percent [Mehra and Prescott (1985, pp. 154–160)].

Thus, for acceptable risk-free returns and risk aversion parameters, these specifications predict risk premia that are much smaller than the risk premium actually observed. This is the puzzle posed by Mehra and Prescott.

4. The three-state specification

The Mehra–Prescott specifications always assume that consumption growth rates are symmetric about their mean and they fall above their mean as often as they fall below. Thus, in their two-state specification, times are always either good or poor, with the equity returns slightly higher or lower than average. In their four-state specification, times are good, poor or average, with average times twice as likely as either good or poor.

While equity returns vary little from the norm in good and poor times, we also observe rare bad times or crashes, when consumption falls drastically and equity returns are far below average. Incorporating a low-probability, depression-like third state in Mehra and Prescott's model not only captures the effects of these crashes, it also solves their puzzle.[5]

To specify a three-state version, assume $x_t \in \{\lambda_1, \lambda_2, \lambda_3\}$, where

$$\lambda_1 = 1 + \mu + \delta,$$

$$\lambda_2 = 1 + \mu - \delta,$$

$$\lambda_3 = \psi(1 + \mu),$$

and ψ is a fraction or a combination of the other parameters such that $\lambda_3 < \lambda_2 < \lambda_1$.

[4] Here, Mehra and Prescott used a security whose dividend in period $t+1$ is the tree's actual dividend minus a fraction of its expected dividend. This fraction is assumed to be committed as of time t. If θ is the fraction of the tree's expected dividend which is committed, the price of the security becomes

$$p^e(c,i) = \beta \sum_{j=1}^{n} \phi_{ij}(\lambda_j c)^{-\alpha} \left[p^e(\lambda_j c, j) + c\lambda_j - \theta \sum_{k=1}^{n} \phi_{ik} c \lambda_k \right] c^{\alpha}.$$

This can be written in the form $p^e(c,i) = w_i c$, where

$$w_i = \beta \sum_{j=1}^{n} \phi_{ij} \lambda_i^{-\alpha} \left[\lambda_j w_j + \lambda_j - \theta \sum_{k=1}^{n} \phi_{ik} \lambda_k \right].$$

In particular, Mehra and Prescott set $\theta = 0.9$ so that 90 percent of the expected dividend is committed, leaving all of the risk to the borne by the security owners who receive only 10 percent of the dividend on average. See Mehra and Prescott (1985, pp. 157–158).

[5] Two state specifications in which one state represents 'normal' growth and the other, low probability, state represents the crash were rejected because they allow no variance in 'normal' growth.

Let the transition probability matrix be

$$\phi = \begin{bmatrix} \phi & 1-\phi-\eta & \eta \\ 1-\phi-\eta & \phi & \eta \\ 1/2 & 1/2 & 0 \end{bmatrix}. \tag{17}$$

Interpret states 1 and 2 as the 'normal' states and state 3 as a one-time crash. A crash will only follow states 1 or 2, and then only with the low probability of η. It never occurs twice in a row.[6] For simplicity, states 1 and 2 follow a crash with equal probability. With λ_3 low, consumption falls drastically when a crash occurs. After a crash, the consumption growth rate returns to normal, though a crash will effect *all* future consumption levels.[7]

The corresponding stable probabilities are

$$\pi = \begin{bmatrix} 1/(2(1+\eta)) \\ 1/(2(1+\eta)) \\ \eta/(1+\eta) \end{bmatrix}. \tag{18}$$

The consumption growth rate's expected value is then

$$\mathrm{E}(x_t) = \frac{(1+\eta\psi)(1+\mu)}{(1+\eta)}, \tag{19}$$

its second moment about zero is

$$\mathrm{E}(x_t^2) = \frac{(1+\eta\psi^2)(1+\mu)^2+\delta^2}{(1+\eta)}, \tag{20}$$

[6]Setting $\phi_{33} = 0$ implies that, while consumption can return to its pre-crash level only after an extended period, its growth rate returns immediately to normal. This is certainly more plausible than consumption returning immediately to its pre-crash level, but may still be unrealistic. Setting $\phi_{33} > 0$ would allow extended periods of exceptionally low growth. This would make the equity riskier and, thus, allow lower (more reasonable) risk aversion parameters and lower crash probabilities to explain the observed risk premium.

[7]This is easily seen by picking a reference year and calling it year zero. Consumption in period $T > 0$ is given by

$$c_T = c_0 \prod_{t=1}^{T} x_t.$$

Thus, all past growth rates, including any past crashes, affect c_T. Further, one can determine how many 'average' growth years are required for consumption levels to recover from the crash. Set $x_1 = \lambda_3$, $c_T = c_0$ and $\lambda_t = M_1$ for $t > 1$ (where M_1 is the mean gross growth rate). Then solve the above equation for $T - 1$ to get the number of 'average' growth years for a complete recovery. For examples 1, 2 and 3, consumption recovers 38, 16 and 255 average growth years after a crash respectively.

and its moment with itself once lagged (again about zero) is

$$\mathrm{E}(x_t x_{t-1}) = \frac{\delta^2(2\phi - 1 + \eta) + (1+\mu)^2(1 - \eta + 2\eta\psi)}{(1+\eta)}. \tag{21}$$

Estimating the three-state process

To estimate risk premia, estimators of the parameters μ, δ and ϕ must be derived for given crash probabilities (η's). First, the specification chosen for the parameter ψ must be substituted into eqs. (19)–(21). If the resulting equations can be inverted to give real solutions for the parameters, the desired estimators can be derived using the method of moments.[8] Then, for any particular risk aversion and time preference parameters, the risk premium may be found from eqs. (11)–(16). Some examples follow. Each was chosen so that eqs. (19)–(21) can be inverted easily.

For each of these examples, I calculated risk premia and risk-free returns corresponding to various parameters. Using a grid search, I found that maximum risk premium subject to the constraints that $\eta \in [0.0001, 0.2]$, $\alpha \in (0, 10]$, $\beta \in (0, 1)$, the risk-free return is between 0 and 3 percent, and the matrix A is stable. I also found the parameter configurations that met these constraints while giving risk premia between 5 and 7 percent. My results are summarized in the tables with each example.

Example 1: $\psi = k, \quad 0 < k < 1 - \delta/(1+\mu)$

In this example, $\lambda_3 = k(1+\mu) < \lambda_2 < \lambda_1$ (given $\delta > 0$) and, therefore, is a crash state. Substituting $\psi = k$ into (19)–(21) yields the following estimators:

$$\hat{\mu} = (1+\eta)M_1/(1+k\eta) - 1, \tag{22}$$

$$\hat{\delta} = \left[(1+\eta)M_2 - (1+\eta k^2)(1+\hat{\mu})^2\right]^{1/2}, \tag{23}$$

$$\hat{\phi} = \left[(1+\eta)M_3 - (1-\eta+2\eta k)(1+\hat{\mu})^2 + (1-\eta)\hat{\delta}^2\right]\Big/2\hat{\delta}^2, \tag{24}$$

provided that (23) has real solutions. In these equations, M_1 and M_2 are the sample mean and second moment (about 0) of the consumption growth rate; M_3 is the sample moment of consumption with itself once lagged (again about 0).

[8]Estimators in terms of the mean, variance and first-order autocorrelation may also be found, but the algebra using the moments about zero is more convenient.

Table 1

Example 1: $\lambda_3 = (1+\mu)/2$.[a] (Output falls to one-half of its normal expected value during a crash.)

Maximum risk premia for valid crash probabilities				
Crash probability (η)	Maximum risk premium (annual %)	Corresponding risk aversion parameter (α)	Corresponding time preference parameter (β)	Corresponding risk-free return (annual %)
0.0001	7.25	10.00	0.997	2.98
0.0002	11.98	10.00	0.920	2.75
0.0003	15.45	9.85	0.890	1.49
0.0004	20.96	10.00	0.800	2.02
0.0005	25.23	10.00	0.750	1.88
0.0006	29.53	10.00	0.700	2.61
0.0007	30.29	9.85	0.700	0.67
0.0008	35.61	9.95	0.650	0.14
0.0009	40.69	10.00	0.600	1.53
0.0010	40.70	9.85	0.600	1.40
0.0020	70.56	9.95	0.400	0.48
0.0030	91.33	9.95	0.300	2.23
0.0040	104.34	9.90	0.250	2.41

[a]Here λ_3 is the gross growth rate in output during a crash year and $(1+\mu)$ is the average gross growth rate during 'normal' years.

For $k = 0.5$, output falls about as much in one crash year as it did in the first three years of the Great Depression.[9] Table 1 gives the maximum risk premia and the corresponding α's, β's and risk-free returns for all valid η's in the grid search when $k = 0.5$ [i.e., all η's for which (23) has real solutions]. For each η, table 2 shows the α and β ranges that gave risk premia between 5 and 7 percent and risk-free returns between 0 and 3 percent. Notice a risk aversion parameter as low as 4.7 can lead to both a reasonable risk premium and a reasonable risk-free return. Some particularly interesting parameter sets are given the table 3. They are some of the α and β combinations that give risk-free returns and risk premia that are very near those found in the economy. Notice that, as expected, the risk aversion parameter needed to explain the risk premium decreases as the probability of a crash increases.

[9]To put these disasters in perspective, consider how they would compare to the worst disaster in the sample period, the Great Depression. If, in one year, *production* were to fall as much as it did during the entire Great Depression, the resulting disaster would be similar to example 1. [Real per capita industrial production fell to 52% of its original value between 1929 and 1932. It surpassed its 1929 value in 1940. The gross growth rate during the recovery period (1932–40) averaged 1.1.] Similarly, if, in one year, *consumption* were to fall as much as it did during the entire great depression, the resulting disaster would be similar to example 2. [Real per capita personal consumption expenditures fell to 78% of its original level between 1929 and 1933. It surpassed its 1929 value in 1939 with an average gross growth rate of 1.04 over the recovery period (1933–39).] See the United States Department of Commerce (1973).

Table 2

Example 1: $\lambda_3 = (1+\mu)/2$.[a] (Output falls to one-half of its normal expected value during a crash.)

Risk aversion and time preference parameter ranges that yield risk premia between 5 and 7 percent with risk-free returns under 3 percent for valid crash probabilities		
Crash probability (η)	Risk aversion parameter (α) range	Time preference parameter (β) range
0.0001	—	—
0.0002	8.85–9.00	0.991–0.999
0.0003	8.20–8.50	0.989–0.999
0.0004	7.70–8.10	0.989–0.999
0.0005	7.30–7.80	0.980–0.999
0.0006	7.00–7.60	0.980–0.999
0.0007	6.80–7.35	0.980–0.999
0.0008	6.65–7.20	0.980–0.999
0.0009	6.50–7.05	0.970–0.999
0.0010	6.35–6.85	0.970–0.999
0.0020	5.50–6.00	0.970–0.999
0.0030	5.00–5.45	0.960–0.999
0.0040	4.70–5.15	0.960–0.980

[a]Here λ_3 is the gross growth rate in output during a crash year and $(1+\mu)$ is the average gross growth rate during 'normal' years.

Table 3

Example 1: $\lambda_3 = (1+\mu)/2$.[a] (Output falls to one-half of its normal expected value during a crash.)

Parameter configurations that give risk-free returns and risk premia very near the economy's sample values				
Crash probability (η)	Risk aversion parameter (α)	Time preference parameter (β)	Corresponding risk-free return (annual %)	Corresponding risk premium (annual %)
0.0008	7.05	0.997	0.77	6.36
0.0008	7.00	0.999	0.83	6.18
0.0009	6.90	0.994	0.87	6.38
0.0009	6.90	0.995	0.77	6.38
0.0009	6.85	0.997	0.83	6.19
0.0009	6.85	0.998	0.73	6.19
0.0010	6.75	0.993	0.88	6.34
0.0010	6.75	0.994	0.78	6.33
0.0010	6.70	0.996	0.84	6.15
0.0010	6.70	0.997	0.74	6.14
0.0010	6.65	0.999	0.79	5.96
0.0020	5.75	0.989	0.83	5.92
0.0020	5.75	0.990	0.73	5.92
0.0030	5.30	0.980	0.89	6.15

[a]Here λ_3 is the gross growth rate in output during a crash year and $(1+\mu)$ is the average gross growth rate during 'normal' years.

Table 4

Example 2: $\lambda_3 = 0.75$.[a] (Output falls to three-fourths of its previous value during a crash.)

Maximum risk premia for valid crash probabilities				
Crash probability (η)	Maximum risk premium (annual %)	Corresponding risk aversion parameter (α)	Corresponding time preference parameter (β)	Corresponding risk-free return (annual %)
0.0001	0.23	1.75	0.999	2.99
0.0002	0.23	1.75	0.999	2.99
0.0003	0.23	1.75	0.999	2.99
0.0004	0.23	1.75	0.999	2.99
0.0005	0.23	1.75	0.999	2.98
0.0006	0.23	1.75	0.999	2.98
0.0007	0.23	1.75	0.999	2.98
0.0008	0.23	1.75	0.999	2.98
0.0009	0.23	1.75	0.999	2.98
0.0010	0.23	1.75	0.999	2.98
0.0020	0.24	1.75	0.999	2.97
0.0030	0.25	1.75	0.999	2.97
0.0040	0.25	1.75	0.999	2.96
0.0050	0.26	1.75	0.999	2.95
0.0060	0.26	1.75	0.999	2.94
0.0070	0.27	1.75	0.999	2.94
0.0080	6.37	10.00	0.992	2.97
0.0090	6.89	10.00	0.989	2.14
0.0100	7.45	10.00	0.980	1.96
0.0110	8.08	10.00	0.960	2.97
0.0120	8.65	10.00	0.950	2.98
0.0130	9.14	10.00	0.950	1.95
0.0140	9.66	10.00	0.940	2.08

[a] Here λ_3 is the gross growth rate in output during a crash year.

Table 5

Example 2: $\lambda_3 = 0.75$.[a] (Output falls to three-fourths of its previous value during a crash.)

Risk aversion and time preference parameter ranges that yield risk premia between 5 and 7 percent with risk-free returns under 3 percent for valid crash probabilities		
Crash probability (η)	Risk aversion parameter (α) range	Time preference parameter (β) range
0.0008	9.75–10.0	0.992–0.999
0.0009	9.40–10.0	0.989–0.999
0.0010	9.10–9.80	0.980–0.999
0.0110	8.80–9.60	0.980–0.999
0.0120	8.55–9.40	0.980–0.999
0.0130	8.30–9.20	0.980–0.999
0.0140	8.15–9.05	0.980–0.999

[a] Here λ_3 is the gross growth rate in output during a crash year.

Table 6

Example 2: $\lambda_3 = 0.75$.[a] (Output falls to three-fourths of its previous value during a crash.)

Parameter configurations that give risk-free returns and risk premia very near the economy's sample values				
Crash probability (η)	Risk aversion parameter (α)	Time preference parameter (β)	Corresponding risk-free return (annual %)	Corresponding risk premium (annual %)
0.010	9.80	0.999	0.74	6.95
0.011	9.50	0.999	0.82	6.80
0.012	9.25	0.999	0.84	6.69
0.013	9.15	0.995	0.84	6.86
0.013	9.10	0.997	0.81	6.74
0.013	9.05	0.999	0.78	6.63
0.014	9.00	0.994	0.82	6.83
0.014	8.95	0.996	0.80	6.72
0.014	8.90	0.998	0.77	6.60
0.014	8.85	0.999	0.84	6.49

[a] Here λ_3 is the gross growth rate in output during a crash year.

Example 2: $\psi = k/(1+\mu)$, $0 < k < 1 - \mu - \delta$

Here $\lambda_3 = k < \lambda_2 < \lambda_1$ (given $\delta > 0$). Substituting $\psi = k/(1+\mu)$ into (19)–(21) yields the following estimators:

$$\hat{\mu} = (1+\eta)M_1 - \eta k - 1, \tag{25}$$

$$\hat{\delta} = \left[(1+\eta)M_2 - (1+\hat{\mu})^2 - \eta k^2\right]^{1/2}, \tag{26}$$

$$\hat{\phi} = \left[(1+\eta)M_3 - (1-\eta)\left[(1+\hat{\mu})^2 + \hat{\delta}^2\right] - 2\eta k(1+\hat{\mu})\right] \Big/ 2\hat{\delta}^2, \tag{27}$$

provided that (26) has real solutions.

For $k = 0.75$, consumption falls in one crash year about as much as it did in the first four years of the Great Depression. Table 4 gives the maximum risk premia and the corresponding α's, β's and risk-free returns for all valid η's when $k = 0.75$. For each η, table 5 shows the α and β ranges that give risk premia between 5 and 7 percent and risk-free returns between 0 and 3 percent. Table 6 gives some parameter sets that correspond to risk-free returns and risk premia that are very near those found in the economy. Again the risk aversion parameter needed to explain the risk premium decreases as the probability of a crash increases.

Table 7

Example 3: $\lambda_3 = \mu$.[a] (All output is lost except its normal net growth during a crash year.)

Maximum risk premia for valid crash probabilities				
Crash Probability (η)	Maximum risk premium (annual %)	Corresponding risk aversion parameter (α)	Corresponding time preference parameter (β)	Corresponding risk-free return (annual %)
0.0001	186.83	2.45	0.35	1.42
0.0002	294.22	2.40	0.25	1.09
0.0003	293.19	2.30	0.25	1.69
0.0004	382.95	2.30	0.20	2.65
0.0005	794.45	2.45	0.10	2.96
0.0006	1443.85	2.60	0.05	1.71
0.0007	291.98	2.10	0.25	1.67
0.0008	790.84	2.35	0.10	0.22
0.0009	525.46	2.20	0.15	2.59
0.0010	789.10	2.30	0.10	0.40

[a]Here λ_3 is the gross growth rate in output during a crash year and μ is the expected normal net growth rate.

Table 8

Example 3: $\lambda_3 = \mu$.[a] (All output is lost except its normal net growth during a crash year.)

Risk aversion and time preference parameter ranges that yield risk premia between 5 and 7 percent with risk-free returns under 3 percent for valid crash probabilities		
Crash probability (η)	Risk aversion parameter (α) range	Time preference parameter (β) range
0.0003	1.30–1.35	0.940–0.960
0.0004	1.20–1.25	0.940–0.960
0.0005	1.15–1.20	0.940–0.960
0.0006	1.15	0.940–0.960
0.0007	1.10–1.15	0.930–0.960
0.0008	1.05–1.10	0.940–0.960
0.0009	1.05	0.940–0.960
0.0010	1.00–1.05	0.940–0.960

[a]Here λ_3 is the gross growth rate in output during a crash year and μ is the expected normal net growth rate.

Example 3: $\psi = \mu/(1+\mu)$

Here $\lambda_3 = \mu < \lambda_2 < \lambda_1$. The estimators are given by

$$\hat{\mu} = M_1 - 1/(1+\eta), \tag{28}$$

$$\hat{\delta} = \left[(1+\eta)M_2 - (1+\hat{\mu})^2 - \eta\hat{\mu}^2\right]^{1/2}, \tag{29}$$

$$\hat{\phi} = \left[(1+\eta)M_3 - (1+\hat{\mu})^2 - \eta(\hat{\mu}^2 - 1) + (1-\eta)\hat{\delta}^2\right]\Big/2\hat{\delta}^2, \tag{30}$$

provided that (29) has a positive solution.

Table 9

Example 3: $\lambda_3 = \mu$.[a] (All output is lost except its normal net growth during a crash year.)

Parameter configurations that give risk-free returns and risk premia very near the economy's sample values				
Crash probability (η)	Risk aversion parameter (α)	Time preference parameter (β)	Corresponding risk-free return (annual %)	Corresponding risk premium (annual %)
0.0001	1.590	0.960	0.86	6.17
0.0001	1.595	0.960	0.75	6.29
0.0001	1.585	0.962	0.75	6.04
0.0002	1.430	0.956	0.82	6.37
0.0002	1.420	0.958	0.83	6.12
0.0002	1.425	0.958	0.72	6.24
0.0002	1.415	0.960	0.73	6.00
0.0003	1.325	0.956	0.83	6.18
0.0003	1.330	0.956	0.72	6.30
0.0003	1.320	0.958	0.73	6.06
0.0004	1.260	0.954	0.85	6.28
0.0004	1.250	0.956	0.86	6.03
0.0004	1.255	0.956	0.75	6.15
0.0005	1.205	0.954	0.82	6.22
0.0005	1.195	0.956	0.82	5.98
0.0005	1.200	0.956	0.71	6.09
0.0006	1.160	0.954	0.80	6.17
0.0007	1.130	0.952	0.82	6.32
0.0007	1.120	0.954	0.83	6.07
0.0007	1.125	0.954	0.72	6.19
0.0008	1.095	0.952	0.86	6.22
0.0008	1.100	0.952	0.75	6.34
0.0008	1.090	0.954	0.76	6.09
0.0009	1.065	0.952	0.89	6.15
0.0009	1.070	0.952	0.77	6.27
0.0009	1.060	0.954	0.78	6.02
0.0010	1.050	0.950	0.86	6.38
0.0010	1.040	0.952	0.87	6.12
0.0010	1.045	0.952	0.76	6.24
0.0010	1.035	0.954	0.77	6.00

[a]Here λ_3 is the gross growth rate in output during a crash year and μ is the expected normal net growth rate.

This is an unprecedented, but conceivable crash. Because this crash is so extreme, its probability must be very low to be consistent with the consumption series' observed moments (η's greater than 0.001 are inconsistent). For each η, table 7 shows the maximum risk premium and the corresponding α, β and risk-free return. Table 8 shows the α and β ranges that give risk premia between 5 and 7 percent and risk-free returns between 0 and 3 percent. Table 9 gives some parameter sets that correspond to risk-free returns and risk premia that are very near those found in the economy. Again the risk aversion parameter needed to explain the risk premium decreases as the probability of a crash increases. Note risk aversion parameters as low as 1 can explain both the high risk premium and low risk-free returns.

4. Conclusions

In an attempt to explain the equity risk premium, Mehra and Prescott (1985) developed a frictionless, pure exchange Arrow–Debreu economy. They also rejected it on the grounds that it seemed inconsistent with the data. Further, they concluded that we will need to abandon Arrow–Debreu type asset pricing models to explain both high equity risk premia and low risk-free returns.

In this paper, I specified their model to capture the effects of possible, though unlikely, crashes. This specification does not alter the attractive features of their model. The economy is still a finite state version of Lucas' (1978) model and still has a Debreu competitive equilibrium with non-stationary consumption levels. There are no frictions and no closed markets. But, with the addition of a crash state, the model explains both high equity risk premia and low risk-free returns; it solves the Mehra–Prescott puzzle. Further, it does so with reasonable degrees of time preference and risk aversion provided the crash is plausibly severe and not too improbable.

References

Fama, E.F., 1976, Inflation uncertainty and expected returns on Treasury bills, Journal of Political Economy 84, 427–448.

Fama, E.F., 1975, Short-term interest rates as predictors of inflation, American Economic Review 65, 269–282.

Lucas, R.E., 1978, Asset prices in an exchange economy, Econometrica 46, 1429–1445.

Mehra, R. and E.C. Prescott, 1984, Asset prices with nonstationary consumption, Working paper (Graduate School of Business, Columbia University, New York).

Mehra, R. and E.C. Prescott, 1985, The equity premium: A puzzle, Journal of Monetary Economics 15, 145–161.

United States Department of Commerce, 1973, Long term economic growth: 1860–1970 (United States Government Printing Office, Washington, DC).