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Evolution of Smart$_n$ Players*

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To model the evolution of strategic intelligence, player types are drawn from a hierarchy of "smartness" analogous to the levels of iterated rationalizability. Nonrationalizable strategies die out, but when higher levels of smartness incur maintenance costs, being right is always as good as being smart. Moreover, if a manifest way to play emerges, then dumb players never die out, while smarter players with positive maintenance costs vanish. These results call to question the standard game-theoretic assumption of super-intelligent players. *Journal of Economic Literature* Classification Numbers: B40, C70, C72, C73. © 1993 Academic Press, Inc.

1. INTRODUCTION

The concept of "rationality" is problematic for multi-decision-maker problems (i.e., games). Many games have multiple Nash equilibria, but even when there is a unique Nash equilibrium, rationality alone (and even common knowledge of rationality) is often insufficient to predict the Nash equilibrium solution (Tan and Werlang, 1988; Brandenberger, 1992).

Recently fundamental questions have been raised about what it means to be a "rational and intelligent" player[1] in a game (Reny, 1986, 1992; Binmore, 1987; Basu, 1988, 1990). With only a handful of exceptions, game theory has taken as an implicit axiom that *all* players are *super-intelligent*, i.e., possessing omnicient powers beyond those supposed in single decision-maker problems. For example, in many games, all players'

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[1] This phase was coined by Myerson (1991, p. 4).

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priors and the selection rule for the solution concept must be common knowledge. The exceptions (e.g., Rosenthal, 1981; Kreps *et al.*, 1982; Fudenberg and Maskin, 1986) have found that super-intelligent behavior can be fundamentally different when there is even a small probability of irrational players and is very sensitive to the ad hoc specification of the irrational behavior.

To debate what it means to play "intelligently," we must also give meaning to "unintelligent" play and then be prepared to demonstrate just how intelligent play is superior to unintelligent play in an environment in which it is possible for some players to be unintelligent. Moreover, we should have a model that justifies the assumptions about unintelligent play that underpin the derived intelligent behavior.

The methods of evolutionary game theory are well-suited for this task.[2] To illustrate, consider a symmetric two-player game. We can conceive of a large class of behavioral rules, including constant-strategy rules as well as sophisticated multi-stage-reasoning rules. Suppose there is a very large population of potential players with an initial frequency distribution of behavioral types (rules). In the first period, players are randomly matched a large number of times so each behavior type receives its expected payoff against the population distribution. Between the first and second periods, each behavioral type reproduces at a growth rate proportional to its payoff, thus generating a new frequency distribution of behavioral types at the beginning of the second period. This process is repeated indefinitely. The obvious questions include: Which types survive and which die out, do the more intelligent types gain in population relative to the less intelligent types, does the frequency distribution converge, and if so to what?

Granted, we can reinterpret extant evolutionary game theory models as evolution *with* intelligence. That is, the growth of a strategy type can be reinterpreted as arising from conscious decisions of intelligent players to switch to better performing strategies. However, in this reinterpreted model, the intelligent trait itself (the switching behavior) is not subjected to evolutionary selection, and therefore the model tells us nothing about the evolution *of* intelligence. In contrast, the alternative model sketched above and to be developed in this paper is about the evolution *of* intelligence.

To implement this research program, we need to specify the class of behavioral rules used by players in the population. The specification of the unintelligent players is obvious. For each strategy of a specific game,

[2] For an introduction to the literature, see van Damme (1987, Chap. 9), Hofbauer and Sigmund (1988), Samuelson (1988), Friedman (1991), Selten (1991b), and Samuelson and Zhang (1992). For an innovative application, see Blume and Easley (1992).

we associate a player type who will always play that strategy in that game. The specification of intelligent players on the other hand is far from obvious, and we claim only to have a reasonable beginning model (see also Rosenthal, 1992). Henceforth, we use the term "smart" rather than "intelligent," and we use the term "smart_0" to refer to the unintelligent players.

Our approach is to model a hierarchy of evermore thoughtful and informed smart players, called smart_1, smart_2, . . . , who reason analogously to the iterative levels of "BP-rationalizability" (Bernheim, 1984; Pearce, 1984) and process successively more inclusive population information. A smart_1 player can reason about which strategies are first-level rationalizable conditional on his information about the population, but does not assume that other players can reach these first-level conclusions and does not reason further (the latter steps illustrate what we mean by higher-level reasoning). Similarly, a smart_2 player can reason about which strategies are first-level rationalizable conditional on her information; then given information about proportion of the population that is smart_1 or higher, she can reason about which strategies are second-level rationalizable conditional on her information, but she does not assume that other players can reach these second-level conclusions and does not reason further.

If we did not endow our smart players with some population information, then our smart players could deduce no more than a modified version of BP-rationalizability—modified to acknowledge the potential presence of smart_0 players, which prohibits the iterative elimination of never-best responses. But then a conclusion that smartness has limited survival fitness could be readily dismissed as an artifact of the dearth of information possessed by smart players. Part of the supposed advantage of being smart is the ability to use information, so the endowments of information are crucial to the ultimate performance; in contrast, an unintelligent player could not perform well even if provided vast amounts of information. To focus attention on the limits of reasoning in our hierarchial model rather than differential information, we make the extreme assumption that all players are perfectly informed about the entire distribution of smart types in the population.[3]

Given this information, smart_1 players confine their choices to the "first-level rationalizable" strategies: those that are best responses to some probability distribution over other players' strategies conditional on the

[3] Our subsequent results that show that smartness does not have superior survival fitness are strengthened by this assumption. Our results would also hold under the weaker assumption that smart_n players are perfectly informed about the distribution of less smart types, but uninformed about equal and smarter types; this follows because a smart_n player's limited reasoning faculties render the latter information useless.

information about smart$_0$ players. If there is a unique first-level rationalizable strategy, then smart$_1$ players choose that strategy.

However, more often than not, there will still be a large number of strategies to choose from, and the evolutionary dynamics can be quite sensitive to how this choice is made. For example, the "equal chance" rule will create a barrier that prevents convergence to nonuniform Nash equilibria. Rather than impose arbitrary fixed rules, we take a flexible approach. We suppose that, in addition to *primary* preferences over the consequences of the game, each smart$_1$ player is endowed with a *secondary* strict transitive preference ordering over the strategies and that this secondary preference is used to choose among the (first-level) rationalizable strategies. Each distinct secondary preference order distinguishes a smart$_1$ player's type, and there is an initial distribution of smart$_1$ players by type. Then, the proportion of smart$_1$ players who choose a particular first-level rationalizable strategy is the proportion of smart$_1$ players whose secondary preference (restricted to the first-level rationalizable set) is for that strategy. With this specification, evolutionary dynamics can operate on the (secondary preference) types of smart$_1$ players, thereby making the population of smart$_1$ players more adaptable than they would be with a fixed decision rule.[4]

We continue recursively defining smart$_n$ players for all $n \geq 2$. There are two senses in which a smart$_n$ player is smarter than a smart$_{n-1}$ player. First, the smart$_n$ player reasons that no smart$_{n-1}$ player will choose a strategy that is not $(n-1)$-level rationalizable conditional on the population information. Second, given this deduction, a smart$_n$ player's information allows him to predict the choice distribution of all less smart players. Nonetheless, the smart$_n$ player is not smart enough to anticipate the behavior of other equal or smarter players. To fully capture the notion of a "transcendentally smart" player requires an infinite hierarchy and a smart$_\infty$ player. (If this were not true, then the previously discussed problems in game theory would not have arisen.)

In Section 2, we formalize this model and derive several results about the structure. In Section 3, we specify the evolutionary dynamics, and in

[4] Another approach (e.g. Banerjee and Weibull, 1992) would be to assume that smart$_1$ players play a Nash equilibrium of the "modified" game (taking account of the known play of smart$_0$ players). We do not follow this approach for several reasons. First, there may be smarter players in the population who play differently. Second, there may be multiple Nash equilibria, in which case selecting any one presupposes a level of coordination (and super-intelligence) that should be explained by the model rather than assumed. Third, even when there is a unique Nash equilibrium, the principles of rationality alone do not compel that solution. Evolutionary models that do not permit players to anticipate the current response of other players can be soundly criticized (Selten, 1991a). However, our model does allow such anticipation within the hierarchy.

Section 4, we present our results. Section 5 concludes, and all proofs are relegated to the Appendix.

2. The Formal Structure of the Model

Let $G \equiv (A, \pi)$ be a symmetric finite two-player game, where A is a finite set of actions, and π is the payoff matrix. Let $M(X)$ denote the set of probability measures on a finite set X. Let $\mathcal{P}$ denote the set of all strict transitive orderings of A.

The population of players consists of smart$_0$ players and smart$_n$ players for $n \geq 1$. Let y_{0a} denote the proportion of the whole population that consists of smart$_0$ players who always choose $a \in A$. Similarly, let y_{nk} denote the proportion of the whole population that consists of smart$_n$ players with secondary preference $k \in \mathcal{P}$. Further, let $y_0 \equiv \{y_a\}_{a \in A}$, $y_n \equiv \{y_{nk}\}_{k \in \mathcal{P}}$, and $y \equiv \{y_n\}_{n \geq 0}$. By definition, $\Sigma_{a \in A}\, y_{0a} + \Sigma_{n \geq 1} \Sigma_{k \in \mathcal{P}}\, y_{nk} = 1$, and the state variable y completely describes the player population.

We also want to know the distribution by smartness category. The proportion of the population consisting of smart$_0$ players is $s_0 \equiv \Sigma_{a \in A}\, y_{0a}$, and for $n \geq 1$, the proportion of the population consisting of smart$_n$ players is $s_n \equiv \Sigma_{k \in \mathcal{P}}\, y_{nk}$. Note that $\Sigma_{n \geq 0} s_n = 1$.

Given $s_0 > 0$, the subpopulation of smart$_0$ player types is distributed among the strategies according to $f_0 \equiv y_0/s_0 \in M(A)$, with f_{0a} denoting the proportion of smart$_0$ players who always play strategy $a \in A$. Similarly, given $s_n > 0$, the subpopulation of smart$_n$ players is distributed among the $\mathcal{P}$ orderings according to $f_n \equiv y_n/s_n \in M(\mathcal{P})$, with f_{nk} denoting the proportion of smart$_n$ players with secondary preference type k. Later we add a time index to these vectors.

2.1. *Player Behavior*

It is first convenient to introduce some notation-saving definitions. Let β: $M(A) \to A$ be the pure-strategy best-response correspondence. For each $b \in B \subseteq A$, let $P(b, B) \equiv \{k \in \mathcal{P} | k$ ranks b highest in $B\}$. In other words, $P(b, B)$ is the set of secondary preference types that rank strategy b highest among the strategies in B. Note that $\{P(b, B), b \in B\}$ is a partition of $\mathcal{P}$.

We let $\mu_n \equiv \{\mu_{na}, a \in A\}$ denote the distribution of smart$_n$ play. For example, a smart$_0$ player simply plays his strategy type, so the distribution of smart$_0$ play is f_0; hence, $\mu_0 \equiv f_0$.

Letting $R_0 \equiv A$, we recursively define (1) R_n, the set of n-level rationalizable strategies conditional on y, and (2) μ_n, for $n \geq 1$,

$$R_n \equiv \beta(Q_n), \qquad \text{where } Q_n \equiv \Sigma_{j<n}\, s_j\mu_j + (1 - \Sigma_{j<n}\, s_j)\, M(R_{n-1}), \quad (1)$$

and

$$\mu_{na} = \begin{cases} 0 \ \text{ for all } a \notin R_n, & \text{and} \\ \Sigma_{k\in P(a,R_n)} f_{nk} & \text{otherwise.} \end{cases} \quad (2)$$

Q_n is the set of probability distributions over A that are possible beliefs for a smart$_n$ player, given the information about all less smart players and the restriction that a smart$_n$ player believes that an equal or smarter player will never choose a strategy that is not $(n - 1)$-level rationalizable. The properties of the Q_n sets are illustrated in Fig. 1 and explained further in Section 2.2. R_n is then the set of all pure strategies that are best responses to some belief in Q_n. A smart$_n$ player then chooses the strategy in R_n most preferred according to his secondary preferences ordering; hence, we have Eq. (2). Since μ_n is a recursive function of y, Eqs. (1)–(2) define $\mu(y) \equiv \{\mu_n(y)\}_{n\geq 0}$. The aggregate distribution of strategy choices for the whole population is

$$p(y) \equiv \Sigma_n\, s_n\mu_n(y). \quad (3)$$

2.2. *Properties of the n-Level Rationalizable Sets.*

Figure 1 illustrates the construction of the Q_n and R_n sets. First, partition the simplex $M(A)$ into the pure-strategy best-response regions. Next, locate f_0, and construct Q_1 as a $(1 - s_0)$ scaling of $M(A)$. [Note that the

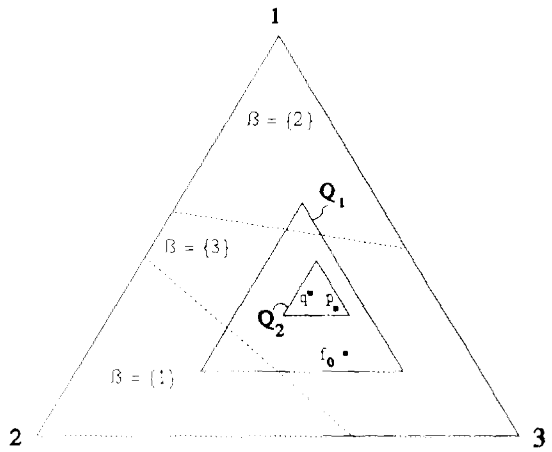

FIG. 1.

image of f_0 in $M(A)$ coincides with the image of f_0 in Q_1.] Then, R_1 consists of the associated best responses that intersect Q_1: {1, 2, 3} in Fig. 1. By construction, $q \equiv s_0\mu_0 + s_1\mu_1$ lies in Q_1. Then, Q_2 is a $(1 - s_0 - s_1)$ scaling of $M(R_1)$, and R_2 consists of the associated best responses that intersect Q_2: {3} in Fig. 1. Hence, $R_n = \{3\}$ for all $n \geq 2$. Consequently, p lies on the straight line connecting q and the {3}-vertex of Q_2. It is easy to see that $R_n \subseteq R_{n-1}$ for all $n \geq 1$, and hence, $\{R_n, n \geq 1\}$ is a nonincreasing sequence of nested sets.

Observe that if $s_0 > 0$ and f_0 is an interior distribution, then R_1 cannot contain any weakly dominated strategies. Thus, smart$_n$ players will never choose a weakly dominated strategy given an interior distribution of smart$_0$ players.

As a correspondence from y to A, R_n for $n > 1$ is necessarily neither upper or lower hemicontinuous, because the distribution of smart$_{n-1}$ play, μ_{n-1}, can change discontinuously. These potential discontinuities create technical problems for the existence of a well-defined solution path of a continuous-time dynamical system [e.g., Champsaur *et al.* (1977) require upper hemicontinuity]. While these problems could perhaps be handled, they are beyond the scope of this paper, so we opt for the simpler environment of discrete-time dynamics, in which unique solution paths always exist.

3. The Evolutionary Dynamics

To represent the strategy choice of each type of smart$_n$ player, let $\sigma(n, k) \equiv \{\sigma \in M(A) | \sigma_a = 1$ iff a is most preferred relative to R_n by smart$_n$ type $k\}$. We also let $\sigma(0, k)$ denote the strategy choice of the smart$_0$ players by letting the k index range over A (instead of $\mathcal{P}$). For $n \geq 1$, note that $\sigma(n, k)$ depends on R_n, which is a deterministic function of y.

Given payoff matrix π and aggregate play p, then πp is the vector of expected payoffs to each strategy when matched with an opponent randomly drawn from the population of players. The expected payoff averaged over the population is $p \cdot \pi p$. The expected payoff to a smart$_n$ player of type k is $\sigma(n, k) \cdot \pi p$.

Typically, evolutionary models assume that the growth rate of a species type is proportional to its expected payoff (Friedman, 1991; Nachbar, 1990). It follows then that the growth rate of the population share of a species type is proportional to the difference between its expected payoff and the population average payoff. We adopt this approach and assume for all $n \geq 0$ that

$$\frac{y_{nk}(t + l) - y_{nk}(t)}{y_{nk}(t)} = \nu(t) [\sigma(n, k; t) \cdot \pi p(t) - p(t) \cdot \pi p(t)], \tag{4}$$

where $\nu(t) > 0$ is the adjustment speed parameter, and t denotes the temporal period. The right-hand side of the Eq. (4) is a deterministic function of the state variable $y(t)$, so given an initial condition $y(0)$, Eq. (4) defines a unique dynamic path. While the adjustment speed parameter has no affect on the direction of the path at any point, it does affect the length of each step. We hereby assume that $\nu(t)$ is always sufficiently small that $y_{nk}(t) \geq 0$ for all t.[5] Moreover, it is well-known that high adjustment speeds can severely destabilize a system because of overshooting. To reduce the overshooting problems, we are interested in the behavior of Eq. (4) for low adjustment speeds.

We have adopted a discrete-time dynamic model to avoid technical problems due to discontinuities on the right-hand side of Eq. (4). Doing so guarantees the existence of a well-defined solution path. However, the reader may wonder whether the technical problems of the continuous-time version might manifest themselves in some other form (such as instabilities) in our discrete-time version. We believe not because we can formulate a continuous-time approximation to Eq. (4) with unique solution paths.[6]

An artifact of an evolutionary dynamic system specified in terms of growth rates is that if $y_{nk}(t_0) = 0$, then $y_{nk}(t) = 0$ for all $t > t_0$. In other words, player types that do not exist or that die out can never reemerge. Therefore, initial conditions that have $y_{nk}(0) = 0$ for some (j, k) are of limited interest. We henceforth limit attention to "semi-interior" initial conditions of the form: for all $k \in \mathcal{P}$, $y_{nk}(0) > 0$ for all $n < n^* + 1$ and $y_{nk}(0) = 0$ for all $n > n^*$, where $n^* \in \{0, 1, \ldots, \infty\}$ is the maximum level of smartness in the population.

4. Results

There are two types of results: one concerning the evolution of aggregate play p, and the other concerning the evolution of the population by smartness, s_n. The first result identifies a set of strategies that will eventually never be played. This result does not require convergence of the solution path.

[5] Define $K_\pi \equiv \sup\{(p - e_a) \cdot \pi p | p \in M(A)$ and $a \in A\}$, where e_a puts probability one on strategy a. Then, from Eq. (4), as long as $\nu(t) < 1/K_\pi$, $y_{nk}(t) \geq 0$.

[6] To do so, suppose that every smart$_n$ of type k ($n \geq 1$) unknowingly receives slightly distorted information about y. Further, suppose the information received is, say, uniformly distributed over an ε-ball around true state and that each player's distortion is independent of all other player's distortions. Then, integrating over this uncertainty, μ_n and hence p would be Lipschitzian continuous functions of y. In addition, the definition of $\sigma(n, k)$ would be modified, and it too would be a Lipschitzian continuous function of y.

PROPOSITION 1. *If $a \in A$ is not BP-rationalizable, then starting from any semi-interior $y(0)$, for sufficiently slow adjustment speeds, $p_a(t) \to 0$.*

An analogous result for standard continuous-time evolutionary dynamics has been shown by Samuelson and Zhang (1992). It does not necessarily hold for discrete-time dynamics with arbitrary adjustment speeds (see Dekel and Scotchmer, 1992; Cabrales and Sobel, 1992).

An immediate implication of Proposition 1 is that all smart_0 types associated with non-BP-rationalizable strategies die out. On the other hand, Proposition 1 does not imply that smart_n types whose secondary preference ordering ranks the non-BP-rationalizable strategies highest die out. Since eventually a non-BP-rationalizable strategy is not in R_n, the position of these strategies in a smart_n player's secondary preference ordering is irrelevant to the evolutionary dynamics.

We consider next the class of games that have a unique BP-rationalizable strategy, say a*.

PROPOSITION 2. *If game G has a unique BP-rationalizable strategy, a^*, then for sufficiently slow adjustment speeds (i) $p_{a^*}(t) \to 1$ and $f_{0a^*}(t) \to 1$, and (ii) there is a $\delta > 0$ such that $\delta < s_n(t) < 1 - \delta$ for all t and all $n < n^* + 1$.*

In other words, all smart_0 player types *except* the a* types die out. Smart players do not die out, but neither do they dominate. Intuitively, the smart_0 player who happens to be genetically disposed toward a* is just as "fit" as any smart_n player; i.e., "*being right is just as good as being smart.*"

We now generalize Proposition 2 to cases where G has multiple BP-rationalizable strategies. When $\liminf \beta[p(t)] \neq \emptyset$, there is at least one strategy that is always a best response after some finite time, so we say that a "manifest way to play the game emerges."

PROPOSITION 3. *If $\liminf \beta[p(t)] \neq \emptyset$, then there is a $\delta > 0$ such that $\delta < s_n(t) < 1 - \delta$ for all t and all $n < n^* + 1$.*

In other words, if a manifest way to play the game emerges, then no smart_n or smart_0 player has a superior (or inferior) survival fitness. Again, we have the principle that being right is just as good as being smart.

Proposition 3 implies that if we are to find cases for which, say, smart_0 players die out, we must focus on the nonconvergent (often chaotic) cases or convergent cases for which the best-response correspondence cycles. Even if we succeed, the victory will be soured by the observation that despite the superior survival fitness of some smart_n players, a manifest way to play the game does not emerge.

When the maximum level of smartness in the population is finite, we have a more remarkable result that does not require any conditions on the dynamic path.

PROPOSITION 4. *Given* $n^* < \infty$, *there exists a* $\delta > 0$ *such that* $s_0(t) \geq \delta$ *infinitely often.*

The intuition behind this result is that because the R_n sets are nested, some of the smart$_n$ types are mimicking the smart$_{n+1}$ players and hence must do as well. Thus, if $s_0(t) \to 0$, then $s_n(t) \to 0$ for all $n \geq 1$, but this is clearly impossible since we always have $\Sigma_{n\geq 0} s_n = 1$. Thus, with a finite upper bound on smartness, smart$_0$ players will never be driven out.

One way to rationalize a finite n^* would be to assume there are maintenance costs, c_n, for smart$_n$ players, with $c_n > c_{n-1}$ for all $n \geq 1$ and $\lim_{n\to\infty} c_n \gg \max \pi$. Letting $\bar{c}(t) \equiv \Sigma_{n\geq 0} s_n(t)\, c_n$, we would add $[\bar{c}(t) - c_n]$ inside the brackets of Eq. (4). Then, for sufficiently large n, $\Delta y_{nk}(t)/y_{nk}(t) < 0$ for all $k \in \mathcal{P}$; hence, $s_n(t) \to 0$. The consequences of costly maintenance is more dramatic for a wide class of cases.

PROPOSITION 5. *For any* $n \geq 1$, *if* $p(t) \to p^*$ *and/or* lim inf $\beta[p(t)] \neq \emptyset$, *and if* $c_n > 0$, *then* $s_n(t) \to 0$.

Thus, if aggregate play converges and/or a manifest way to play emerges, then costly maintenance drives smart$_n$ players to extinction.[7] In other words, being right is strictly better than being smart when smartness carries maintenance costs.

Notwithstanding the above remarks, Proposition 4 appears to leave open the possibility that, when $n^* = \infty$ *and* there are no maintenance costs, all mass may escape to infinity: i.e., there may exist an increasing divergent sequence $\{m(t)\}$ such that $\Sigma_{n>m(t)} s_n(t) \approx 1$. In other words, a dominant transcendentally smart player would evolve—a result that would vindicate traditional game theory, albeit a manifest way to play the game would not emerge.

It also remains an open question whether or not smart$_n$ players can be driven out. The difficulty in trying to prove that a smart$_n$ player will not be driven out (or that mass can escape to infinity) is that the $R_n(t)$ correspondence is discontinuous, and when a new strategy becomes n-level rationalizable, a relatively large proportion of the smart$_n$ players may switch to this strategy, but it could actually perform worse than average.

5. DISCUSSION

We have developed a hierarchial model in which players have varying degrees of reasoning abilities or smartness and for which an infinitely

[7] Conlisk (1980) in a related model with "optimizers" and "imitators" showed that imitators survive iff maintenance costs are sufficiently large.

smart player possesses all the reasoning abilities of the super-intelligent player assumed in game theory. We supposed that smartness is subject to evolutionary selection pressures and asked whether smartness has superior survival fitness. Our findings were negative: Generally, being right is just as good as being smart.[8] Smart_0 players are never driven out by smart players whenever (i) a manifest way to play the game emerges, (ii) when there are increasing maintenance costs for smartness, or (iii) when there is a finite upper bound on smartness.

We have focused our analysis on a fixed (albeit arbitrary) finite, symmetric, two-player game. Given our general definition of smartness, we could be criticized for this focus on a fixed game rather than on a distribution of games in this class. For example, we could consider a set of M × M symmetric games defined by a diverse but finite set of payoff matrices π and a probability distribution on these games. In each period, players are randomly matched and a payoff matrix is drawn. The evolutionary dynamics would depend on the average performance of all player types over all possible games in this class.

It is reasonable to conjecture that smartness would have superior survival fitness over smart_0 players in this environment. However, this result would be due to the unreasonable limitations of smart_0 players. The fundamental characteristic that distinguishes any smart player from a smart_0 player is not the ability to know the game variables, but the ability to think about how other players will behave. That a "dumb" player who cannot even discriminate between different games might die out is hardly a victory for the super-intelligence axiom. To permit a smart_0 player to discriminate between the alternative games, we can introduce more complex "genes." Given N possible games, and M strategies, there would be M^N possible genetic types. (The secondary preferences of the smart_n players should also be expanded analogously.) Now the apparent inferiority of smart_0 players disappears, and the principle that *being right is just as good as being smart* would seem to hold.

Of course for an infinite set of possible games, we would need an infinity of types, and it may be reasonable to restrict smart_0 players to a finite set of types. However, it is not obvious that these smart_0 players would be driven out. The smart_0 players' discrimination abilities could partition the space of payoff matrices in a way that minimizes the consequences of the incomplete information. (For example, for 2 × 2 games it may suffice to have a three-part partition that recognizes when each of the strategies is strictly dominant.) In general, it may be adequate to have the cardinality

[8] A referee pointed out that a similar model in which all players have unbounded reasoning abilities but possess hiearchial sets of population information would yield analogous results with the interpretation that being right is just as good as being *informed*.

of the partition equal to $2^M - 1$ corresponding to each possible nonempty subset of A.

For the sake of argument, suppose in this infinitely diverse environment of games and finite smart$_0$ types that the smart$_0$ types are driven out. Our celebration will be tempered by the observation that after some finite time, the smart$_1$ players will be virtually indistinguisable from the smart$_0$ types confined to the R_1 set and hence no more fit. Moreover, for any given game, these smart$_1$ players will be indistinguishable from the smart$_0$ players of this paper. Therefore, we will not find that "the smarter, the better."

Thus, our model does not provide an evolutionary foundation for the usual assumption in game theory that all players are super-intelligent, and it seems unlikely that any other model will satisfactorily meet this goal. Future research should develop more realistic models of intelligence subject to evolutionary selection with the goal of developing a theory of "intelligent" play in an evolutionary context.

Appendix: Proofs

Proof of Proposition 1. Let BP_n denote the n^{th} level BP-rationalizable set: i.e., the subset of A that survives n rounds of elimination of never-best responses (or equivalently, since we are focusing on two-player games, strictly dominated strategies).

(1) Take any $\hat{a} \notin BP_1$, so $\mu_{n\hat{a}} = 0$ for all $n \geq 1$, and $p_{\hat{a}} = s_0 f_{0\hat{a}}$. Since $\hat{a}$ is not first-level BP-rationalizable, $\exists\ q \in M(A)$ such that $q \cdot \pi p$ for all $p \in M(A)$. Let $\varepsilon \equiv \min_p\{(q - e_{\hat{a}}) \cdot \pi p\} > 0$. Define

$$V(t) \equiv f_{0\hat{a}}(t)/(\Pi_{a \in A} [f_{0a}(t)]^{q_a}), \tag{A1}$$

Note that $V > 0$ *iff* $f_{0\hat{a}} > 0$. Define $\Delta V(t) \equiv V(t + 1) - V(t)$. Then,

$$\frac{\Delta V(t)}{V(t)} = \frac{d \ln[V(t)]}{dt} + o(\nu^2) = \left[\frac{\Delta f_{0\hat{a}}(t)}{f_{0\hat{a}}(t)} - \sum_{a \in A} q_a \frac{\Delta f_{0a}(t)}{f_{0a}(t)}\right] + o(\nu^2). \tag{A2}$$

Note that, by virtue of the dynamic specification, Eq. (4), the expression in square brackets in Eq. (A1) is equal to $\nu[e_{\hat{a}} \cdot \pi p - q \cdot \pi p] < -\nu\varepsilon/2$. Therefore, for sufficiently small $\nu > 0$, $V(t)$ is a Liapounov function; so by Liapounov's Direct Method [see, e.g. LaSalle (1986)], $f_{0\hat{a}}(t) \rightarrow 0$.

(2) Next take $\hat{a} \in BP_1/BP_2$, so $\exists\ q \in M(A)$ such that $q \cdot \pi p > e_{\hat{a}} \cdot \pi p$ for all $p \in M(BP_1)$. By virtue of (1), there exists a t_1 and an $\varepsilon > 0$ such that for all $t > t_1$ and all $p \in M(R_1(t))$, $(q - e_{\hat{a}}) \cdot \pi p \geq \varepsilon$. Now $p_{\hat{a}} = s_0 f_{0\hat{a}} + s_1 \mu_{1\hat{a}}$. By the same methods used in (1), we can show that $f_{0\hat{a}}(t) \rightarrow 0$. Further, since $f_{0\hat{a}}(t) \rightarrow 0$, for sufficiently large t, $\hat{a} \notin R_1(t)$; hence, $\mu_{1\hat{a}} = 0$.

(3) Repeating these steps for all levels of BP-rationality, we conclude that if $a \notin \cap_{n>0} BP_n$, then $p_a(t) \rightarrow 0$.

Proof of Proposition 2. If G has a unique BP-rationalizable strategy a^*, then it follows immediately from Proposition 1 that $f_{0a^*}(t) \to 1$. To prove the second part, first observe that for all $n \geq 1$, $R_n(t)$ converges to $\{a^*\}$ in finite time, say t^*. Thus, for all $t > t^*$, all smart$_n$ types play a^* and, therefore, have identical growth rates, which is also the growth rate of y_{0a^*}. The populations of all these player types remain in constant ratio to each other for all $t > t^*$, and hence none die out. Further, since $s_0 \geq y_{0a^*}$, smart$_0$ types do not die out either. In other words, for all $j < n^* + 1$, $\exists\ \delta > 0$ such that $\delta < s_j(t) < 1 - \delta$.

Proof of Proposition 3. By the premise, $\exists\ a^* \in \beta(p^*) \cap \beta(p(t)) \subseteq R_j(t)$ for all $j \geq 1$ and t sufficiently large. Therefore, all player types who rank a^* highest, do not die out and grow at the same strictly positive rate as y_{0a^*}. Therefore, $\exists\ \delta > 0$ such that $s_0(t) \geq y_{0a^*}(t) \geq \delta$ for t sufficiently large. For $n \geq 1$, let k^* denote the secondary preference type that ranks a^* highest. Then, similarly, $\exists\ \delta > 0$ such that $s_n(t) \geq y_{0k^*}(t) \geq \delta$ for t sufficiently large.

Proof of Proposition 4. (a) Suppose $s_0(t) \to 0$. Then, observe that, for sufficiently large, t, $R_1(t)$ differs from the first-level BP-rationalizable set only by deleting all strategies (if any) that are never perfect best responses to some $g \in M(A)$; call this set PBP$_1$. For each $a \in$ PBP$_1$ and each $k \in P(a, \text{PBP}_1)$, y_{1k} and y_{0a} have the same growth rate; hence, $s_0(t) \to 0$ implies $s_1(t) \to 0$.

Next, for sufficiently large t, $R_2(t)$ differs from the second-level BP-rationalizable set only by deleting all strategies (if any) that are never perfect best responses to some $g \in M(\text{PBP}_1)$; call this set PBP$_2$. For each $a \in$ PBP$_2$ and each $k \in P(a, \text{PBP}_2)$, y_{2k} and y_{0a} have the same growth rate; hence, $s_0(t) \to 0$ implies $s_2(t) \to 0$. Therefore, by induction, $s_0(t) \to 0$ implies $s_n(t) \to 0$ for all $n \geq 1$.

(b) Given $n^* < \infty$ and $\Sigma_{n \geq 0}\, s_n(t) = 1$ for all t, we cannot have $s_0(t) \to 0$, so there must be a $\delta > 0$ such that $s_0(t) > \delta$ infinitely often.

Proof of Proposition 5. (a) If $\beta^* \equiv \liminf \beta[p(t)] \neq \emptyset$, let $a^* \in \beta^*$. Then, $\Delta y_{0a^*}(t)/y_{0a^*}(t) - \Delta y_{nk}(t)/y_{nk}(t) \geq \nu c_n > 0$. Therefore, $y_{0a^*}(t)/y_{nk}(t) \to \infty$; hence, $y_{nk}(t) \to 0$.

(b) If $p(t) \to p^*$, let $a^* \in \text{supp}(p^*)$. Then, for all $\varepsilon \in (0, c_n)$, there exists a t_ε such that for all $t > t_\varepsilon$, $\{\beta[p(t)] - p(t)\} \cdot \pi p(t) < \varepsilon$. It follows that $\Delta y_{0a^*}(t)/y_{0a^*}(t) - \Delta y_{nk}(t)/y_{nk}(t) \geq \nu(c_n - \varepsilon) > 0$. Therefore, $y_{0a^*}(t)/y_{nk}(t) \to \infty$; hence, $y_{nk}(t) \to 0$.

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