ON THE CONVERGENCE OF THE LEARNING PROCESS IN A 2 x 2 NON-ZERO-SUM TWO-PERSON GAME

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Econometric Research Program
Research Memorandum No. 33
October 10, 1961

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I am deeply indebted to Professor Oskar Morgenstern for his helpful suggestions and encouragement. I am also very grateful to Professor Harold W. Kuhn for his kind comments. Finally I want to express my sincere thanks to every member of the Econometric Research Program who enabled me to work in such a friendly and stimulating atmosphere.

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§ 1. Introduction

Let the matrix

$$\| (a_{ij}, b_{ij}) \| , \quad i = 1, \ldots, m , \quad j = 1, \ldots, n ,$$

be the payoff matrix in the non-zero-sum two-person game Γ, where a_{ij} and b_{ij} are payoffs to players I and II respectively, when player I takes strategy i and player II takes strategy j. For notational convenience let us write

$$a_{ij} = \varphi_1(i, j) , \; b_{ij} = \varphi_2(i, j) .$$

Mixed strategies for players I and II will be written

$$x = (x_1, \ldots, x_m) \text{ and } y = (y_1, \ldots, y_n)$$

respectively, where

$$x_i \geq 0 , \quad \Sigma x_i = 1 , \quad \text{and} \quad y_j \geq 0 , \quad \Sigma y_j = 1 .$$

Then a pair of equilibrium strategies (x^*, y^*) is defined as follows:

$$\varphi_1(x^*, y^*) \geq \varphi_1(x, y^*) \qquad \text{for all } x ,$$

and

$$\varphi_2(x^*, y^*) \geq \varphi_2(x^*, y) \qquad \text{for all } y .$$

When both the equilibrium strategies x^* and y^* are pure strategies, let us call them pure equilibrium strategies. And we know the existence of at least one pair of equilibrium strategies in any non-zero-sum two-person game [1]. (In this paper we restrict ourselves to non-cooperative games.)

Let us consider the situation in which two players I and II play the non-zero-sum two-person game Γ infinitely many times, and define a "learning process" in the following way. For the first play of the game Γ, let players I and II arbitrarily choose pure strategies i_1 and j_1 respectively. Then let $v(1) \equiv V(1)$ be the j_1-th column of the payoff matrix $A = \| a_{ij} \|$ to player I and $u(1) \equiv U(1)$ be the i_1-th row of the payoff matrix $B = \| b_{ij} \|$ to player II, that is

$$(1.1) \qquad v(1) \equiv V(1) \equiv [V_1(1), \ldots, V_m(1)] = [\varphi_1(i, j_1) \mid i = 1, \ldots, m]$$

and

$$(1.2) \qquad u(1) \equiv U(1) \equiv [U_1(1), \ldots, U_n(1)] = [\varphi_2(i_1, j) \mid j = 1, \ldots, n] .$$

Next let us define i_2 and j_2 by the following:

$$(1.3) \qquad V_{i_2}(1) = \max_i V(1) ,$$

and

$$(1.4) \qquad U_{j_2}(1) = \max_j U(1) ,$$

where in general $\max_i V$ expresses one of the maximum components in the vector $V = (V_1, \ldots, V_n)$. Then for the second play of the game Γ, let players I and II choose strategies i_2 and j_2 respectively.

Now let

$$(1.5) \qquad v(2) = [\varphi_1(i, j_2) \mid i = 1, \ldots, m]$$

and

$$(1.6) \qquad u(2) = [\varphi_2(i_2, j) \mid j = 1, \ldots, n] .$$

We define $V(2)$ and $U(2)$ by

$$(1.7) \qquad V(2) = v(1) + v(2)$$

and

$$(1.8) \qquad U(2) = u(1) + u(2) .$$

Let i_3 and j_3 be such that

(1.9) $$V_{i_3}(2) = \max_i V(2)$$

and

(1.10) $$U_{j_3}(2) = \max_j U(2) .$$

Then for the third play of the game Γ, let players I and II choose strategies i_3 and j_3 respectively. In this way, after the strategy pairs (i_1, j_1), (i_2, j_2), ..., (i_{k-1}, j_{k-1}) are defined, the strategy pair (i_k, j_k) is defined by the following:

Let

(1.11) $$V(k-1) = v(1) + \ldots + v(k-1)$$

and

(1.12) $$U(k-1) = u(1) + \ldots + u(k-1) .$$

Then i_k and j_k are such that

(1.13) $$V_{i_k}(k-1) = \max_i V(k-1)$$

and

(1.14) $$U_{j_k}(k-1) = \max_j U(k-1) .$$

Definition 1.1 A sequence of strategy pairs (i_1, j_1), (i_2, j_2), ..., (i_k, j_k), ... defined as above is called a "learning process" in the repetition of the non-zero-sum two-person game Γ. Let $\alpha(i_k)$ be the unit vector in m-dimensional vector space whose non-zero component is the i_k-th one, and $\beta(j_k)$ be the unit vector in n-dimensional vector space whose non-zero component is the j_k-th one, $k = 1, 2, \ldots$. We define vectors x^k and y^k by

$$x^k = \frac{1}{k}[\alpha(i_1) + \ldots + \alpha(i_k)]$$

and

$$y^k = \frac{1}{k}[\beta(j_1) + \ldots + \beta(j_k)] .$$

Then x^k and y^k can be interpreted as mixed strategies for players I and II respectively. We call x^k and y^k mixed strategies of players I and II associated with a learning process $\{(i_k, j_k)\}$. Then it is clear that

$$(1.15) \qquad \frac{V(k)}{k} = [\varphi_1(1, y^k), \ldots, \varphi_1(m, y^k)]$$

and

$$(1.16) \qquad \frac{U(k)}{k} = [\varphi_2(x^k, 1), \ldots, \varphi_2(x^k, n)] .$$

Our problem is the following: Will a learning process give rise to an equilibrium point; that is, will a sequence of mixed strategies x^k, y^k associated with a learning process converge to equilibrium strategies x^*, y^*, of the game Γ, when k increases infinitely? This is one of the open problems in game theory. (In the case of zero-sum two-person games, the affirmative answer is given in [2] and [3].) It is our purpose in this paper to show that the answer to the above question is affirmative in the case of $m = n = 2$.

§ 2. A learning process in a 2 x 2 game

Let

	1	2
1	(a_{11}, b_{11}),	(a_{12}, b_{12})
2	(a_{21}, b_{21}),	(a_{22}, b_{22})

be the payoff matrix in the non-zero-sum two-person game Γ. We define the following notations:

$$(2.1) \qquad L_1 = a_{11} - a_{21}, \quad L_2 = a_{12} - a_{22},$$

$$(2.2) \qquad R_1 = b_{11} - b_{12}, \quad R_2 = b_{21} - b_{22},$$

$$(2.3) \qquad \ell_i = | L_i |, \quad r_i = | R_i |, \quad i = 1, 2 .$$

Then, in this 2 x 2 case, a learning process (i_k, j_k), $k = 1, 2, \ldots$ will be obtained as follows:

At first, we choose i_1 and j_1 arbitrarily among player I's strategies 1, 2 and player II's strategies 1, 2 respectively. Here we define two functions $\alpha(x)$ and $\beta(x)$ on the set of real numbers $\{x\}$ as follows:

(2.4)
$$\alpha(x) = \begin{cases} 1, & \text{when sign } (x) > 0, \\ 2, & \text{when sign } (x) < 0, \\ 1 \text{ or } 2 \text{ according to some rule,} & \\ \quad \text{when sign } (x) = 0. & \end{cases}$$

(2.5)
$$\beta(x) = \begin{cases} 1, & \text{when sign } (x) > 0, \\ 2, & \text{when sign } (x) < 0, \\ 1 \text{ or } 2 \text{ according to some rule which} & \\ \quad \text{may differ from the case of } \alpha, & \\ \quad \text{when sign } (x) = 0. & \end{cases}$$

Now let

(2.6)
$$A(1) \equiv L_{j_1} \quad \text{and} \quad B(1) \equiv R_{i_1} .$$

Then we define i_2 and j_2 by the following:

(2.7)
$$i_2 = \alpha(A(1)) \quad \text{and} \quad j_2 = \beta(B(1)) .$$

Let

(2.8)
$$A(2) \equiv L_{j_1} + L_{j_2}, \quad \text{and} \quad B(2) \equiv R_{i_1} + R_{i_2} .$$

Then we define i_3 and j_3 by the following:

(2.9)
$$i_3 = \alpha(A(2)) \quad \text{and} \quad j_3 = \beta(B(2)) .$$

After strategy pairs $(i_1, j_1), \ldots, (i_{k-1}, j_{k-1})$ are defined in this way, we define i_k and j_k by the following:

Let

(2.10) $$A(k-1) \equiv L_{j_1} + \dots + L_{j_{k-1}} ,$$

and

(2.11) $$B(k-1) \equiv R_{i_1} + \dots + R_{i_{k-1}} ,$$

then

(2.12) $$i_k = \alpha(A(k-1)) \quad \text{and} \quad j_k = \beta(B(k-1)) \quad \text{for} \quad k = 2, 3, \dots .$$

In this case it is clear that a sequence of strategy pairs

$$\{(i_k, j_k)\} , \ k = 1, 2, \dots$$

obtained by the above rule constitutes a learning process in the sense of Definition 1.1. The above method for obtaining a learning process is shown in the following diagram:

	B	R	II	I	L	A	
$B(1)$	$= R_{j_1}$	R_{i_1}	j_1	i_1	L_{j_1}	L_{j_1}	$= A(1)$
$B(2)$	$= R_{j_1} + R_{j_2}$	R_{i_2}	j_2	i_2	L_{j_2}	$L_{j_1} + L_{j_2}$	$= A(2)$
	$\vdots$		$\vdots$	$\vdots$		$\vdots$	
$B(k-1)$	$= \sum_{\ell=1}^{k-1} R_{i_\ell}$	$R_{i_{k-1}}$	j_{k-1}	i_{k-1}	$L_{j_{k-1}}$	$\sum_{\ell=1}^{k-1} L_{j_\ell}$	$= A(k-1)$
			j_k	i_k			
			$\vdots$	$\vdots$			

<u>Definition 2.1:</u> When $A(k) = 0$ (or $B(k) = 0$), we say that a tie has occurred in A (or B) at the k-th play.

If, in the construction of a learning process, a tie occurs in A or B at some point k, the arbitrary situation which results must be settled by some rule for choosing 1 or 2 at such points.

Definition 2.2: A learning process $\{(i_k, j_k)\}$, is called a special learning process if, in the event of a tie, the rule for making the next move is defined as follows: If $A(k) = 0$ for some $k \geq 1$, then we define $\alpha(A(k))$ by

$$\alpha(A(k)) = \begin{cases} 1 \text{ , when } i_k = 2 \text{ ,} \\ 2 \text{ , when } i_k = 1 \text{ .} \end{cases} \tag{2.13}$$

If $B(k) = 0$, we define $\beta(B(k))$ according to the same rule for $\alpha(A(k))$. That is,

$$\beta(B(k)) = \begin{cases} 1 \text{ , when } j_k = 2 \text{ ,} \\ 2 \text{ , when } j_k = 1 \text{ .} \end{cases} \tag{2.14}$$

We may then proceed to prove the following theorem.

THEOREM: (2 x 2 non-zero-sum two-person games Γ) In a learning process $\{(i_k, j_k)\}$, $k = 1, 2, \ldots$, let

(2.15) $X_1(k)$ be the number of 1's in $i_1, i_2, \ldots, i_k$,

(2.16) $X_2(k)$ be the number of 2's in $i_1, i_2, \ldots, i_k$,

and $Y_1(k)$ be the number of 1's in $j_1, j_2, \ldots, j_k$, $Y_2(k)$ be the number of 2's in $j_1, j_2, \ldots, j_k$. We define

$$x_i(k) = \frac{1}{k} X_i(k) \text{ , } i = 1, 2, \tag{2.17}$$

and

$$y_i(k) = \frac{1}{k} Y_i(k) \text{ , } i = 1, 2 \text{ ;} \tag{2.18}$$

and let us write

$$x(k) = (x_1(k), x_2(k)) \text{ ,} \tag{2.19}$$

$$y(k) = (y_1(k), y_2(k)) \text{ .} \tag{2.20}$$

Then, in a special learning process starting with any strategies i_1 and

j_1 of players I and II, we have the following: $x(k)$ and $y(k)$ converge to equilibrium strategies of the game Γ as k increases infinitely.

At the first stage of a learning process, players I and II are allowed to choose any strategies i_1 and j_1, respectively; in this case they are 1 or 2. Therefore, we must prove our theorem for all the following possible starts:

A - start: $i_1 = 1$ and $j_1 = 1$,

B - start: $i_1 = 1$ and $j_1 = 2$,

C - start: $i_1 = 2$ and $j_1 = 1$,

D - start: $i_1 = 2$ and $j_1 = 2$.

The proof of this theorem will be described in the following sections, in which many of the different possible cases will be treated separately. It will be convenient to prove the theorem by dividing the possible cases into two major categories:

<u>Case I:</u> sign L_1 = sign L_2, and <u>Case II:</u> sign $L_1 \neq$ sign L_2.

Before we begin the proof, we make the following remarks: In the payoff matrix of a 2 x 2 non-zero-sum two-person game, the necessary and sufficient conditions for the (1, 1)-element, the (1, 2)-element, the (2, 1)-element, and the (2, 2)-element to be the pure equilibrium points are given by

$$L_1 \geq 0, \ R_1 \geq 0; \quad L_2 \geq 0, \ R_1 \leq 0;$$

$$L_1 \leq 0, \ R_2 \geq 0; \quad L_2 \leq 0, \ R_2 \leq 0;$$

respectively.

The proof of the theorem for the case in which no pure equilibrium point exists is given in § 5 below.

§ 3. Case I: sign L_1 = sign L_2

We will treat Case I in two parts:

Case I(A): sign L_1 = sign $L_2 \neq 0$; and

Case I(B): sign L_1 = sign $L_2 = 0$.

Case I(A): sign L_1 = sign $L_2 \neq 0$.

Let us consider the case in which $L_1 > 0$ and $L_2 > 0$. (The proof for $L_1 < 0$ and $L_2 < 0$ is omitted as it is essentially similar to the proof for $L_1 > 0$ and $L_2 > 0$.)

Case I(A)(i): $R_1 > 0$.

(5, 6)	(4, 2)
(3, b_{12})	(1, b_{22})

In this case, it is clear that the strategies $x^* = (1, 0)$ and $y^* = (1, 0)$ are the pure equilibrium strategies.

It is evident that we always have

$$A(1) < 0 \text{ , } A(2) > 0 \text{ ,..., } A(k) > 0 \text{ ,...,}$$

that is

$$i_k = 1 \text{ , for } k \geq 2 \text{ .}$$

Accordingly, $B(k)$ takes the following form:

$$B(k) = R_{i_1} + R_1 + \dots + R_1 \text{ .}$$

So, for a sufficiently large K , we have

$$B(k) > 0 \text{ , for } k \geq K \text{ ,}$$

since $R_1 > 0$. That is

$$j_k = 1 \text{ , for } k > K \text{ .}$$

Therefore it is clear that

$$\lim_{k\to\infty} x(k) = (1, 0) = x^* ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) = y^* .$$

Q.E.D.

<u>Case I(A)(ii):</u> $R_1 < 0$.

The proof is just the same as in Case I(A)(i).

<u>Case I(A)(iii):</u> $R_1 = 0$.

(5, 6)	(4, 6)
(3, b_{21})	(1, b_{22})

In this case (a_{11}, b_{11}) and (a_{12}, b_{12}) are two pure equilibrium points.

In A-start, because of our definition of a special learning process, we have the following diagram:

B	II	I	A
0	1	1	+
0	2	1	+
0	1	1	+
	2	1	
	.	.	
	.	.	

From the above diagram, it is clear that

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

and that in this case the strategies $x^* = (1, 0)$ and $y^* = (\frac{1}{2}, \frac{1}{2})$ constitute equilibrium strategies.

In B-start, it can be easily seen that we have the same result as in A-start.

Since $R_1 = 0$, C-start takes the following form:

B	II	I	A
R_2	1	2	+
R_2	.	1	+
.	.	.	.
.	.	.	.

Accordingly, if $R_2 > 0$, it is clear that

(5, 2)	(4, 2)
(3, 7)	(1, 6)

$$\lim_{k\to\infty} x(k) = (1, 0) , \lim_{k\to\infty} y(k) = (1, 0) .$$

Thus our theorem holds.

If $R_2 < 0$, then from the above diagram we have

(5, 2)	(4, 2)
(3, 6)	(1, 7)

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) ,$$

and the strategies $x^* = (1, 0)$ and $y^* = (0, 1)$ are pure equilibrium strategies in this case.

If $R_2 = 0$, then from our definition of a special learning process, we have

(5, 2)	(4, 2)
(3, 6)	(1, 6)

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (\tfrac{1}{2}, \tfrac{1}{2}) .$$

That is, our theorem also holds in this case.

It can be shown that the same results also hold for the case of D-start.

Q.E.D.

Case I(B): sign L_1 = sign L_2 = 0 .

That is

$$L_1 = L_2 = 0 .$$

Case I(B)(i): sign R_1 = sign $R_2 \neq 0$.

It is clear that in this case the same results hold as in Case I.

Case I(B)(ii): $R_1 > 0 > R_2$.

In this case two points (a_{11}, b_{11}) and (a_{22}, b_{22}) are pure equilibrium points.

Let us consider A-start. Then, because of our definition of a special learning process, we have the following diagram:

B	II	I	A
R_1	1	1	0
$R_1 + R_2$		2	0
$2R_1 + R_2$		1	0
$2R_1 + 2R_2$		2	0
⋮		⋮	⋮

(α) If $r_1 > r_2$, then, from the above diagram, it is clear that

(3, 7)	(5, 2)
(3, 6)	(5, 8)

$B(k) > 0$, for $k = 1, 2, \ldots$. That is, $j_k = 1$, for $k = 1, 2, \ldots$.

Accordingly, we have

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) .$$

And it can be easily seen that in this case $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (1, 0)$ are equilibrium strategies.

(β) If $r_1 < r_2$, we can write

(3, 8)	(5, 6)
(3, 2)	(5, 7)

$r_2 = ur_1 + \epsilon$,

where $u \geq 1$, $0 \leq \epsilon < 1$ and if $u = 1$ then $\epsilon > 0$.

In this case $B(k)$ takes the following values:

$$B(2k+1) = (k+1)R_1 + kR_2$$

$$= -kr_1(u-1) + r_1 - k\epsilon .$$

$$B(2k) = kR_1 + kR_2$$

$$= -kr_1(u-1) - k\epsilon .$$

As stated above, $u - 1$ and ϵ cannot be zero at the same time. Accordingly, for a sufficiently large K, we have

$$B(k) < 0 , \quad \text{for all } k \geq K ,$$

that is

$$j_k = 2 , \quad \text{for all } k > K .$$

Accordingly,

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) .$$

In this case it can be easily seen that $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (0, 1)$ are equilibrium strategies.

(γ) If $r_1 = r_2$, then, in the case of A-start, we have the following diagram:

B	II	I	A
R_1	1	1	0
0	1	2	0
R_1	2	1	0
0	1	2	0
.	.	.	.
.	.	.	.

Accordingly,

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

$$\lim_{k\to\infty} y(k) = (\tfrac{1}{2}, \tfrac{1}{2}) .$$

(3, 6)	(5, 2)
(3, 4)	(5, 8)

In this case, $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (\frac{1}{2}, \frac{1}{2})$ are equilibrium strategies.

In the case of B-start, C-start, and D-start, it can be easily seen that the same reasoning as for A-start also holds.

<u>Case I(B)(iii):</u> $R_1 > 0$, $R_2 = 0$ (or $R_1 = 0$, $R_2 < 0$).

In this case we always have

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) ,$$

and it can be easily seen that

(3, 4)	(5, 1)
(3, 2)	(5, 2)

$x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (1, 0)$ are equilibrium strategies.

Case I(B)(iv): $R_1 = R_2 = 0$.

(3, 2)	(5, 2)
(3, 7)	(5, 7)

In this case, from our definition of a special learning sequence, it follows that

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2})$$

$$\lim_{k\to\infty} y(k) = (\tfrac{1}{2}, \tfrac{1}{2}) .$$

And, in this case, the strategies $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (\frac{1}{2}, \frac{1}{2})$ are equilibrium strategies.

§ 4. Case II: sign $L_1 \neq$ sign L_2

Case II(A): $L_1 > 0$, $L_2 = 0$.

Case II(A)(i): If sign R_1 = sign $R_2 \neq 0$, we have the same reasoning as in Case I.

Case II(A)(ii): If $R_1 > 0 > R_2$, it is clear that A-start gives

(5, 3)	(4, 2)
(3, 6)	(4, 1)

$$i_k = 1 , \quad \text{for all} \quad k = 1, 2, \ldots .$$

$$j_k = 1 , \quad \text{for all} \quad k = 1, 2, \ldots .$$

That is, it gives

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) ,$$

and $x^* = (1, 0)$ and $y^* = (1, 0)$ are pure equilibrium strategies in this case.

In the cases of B-start and C-start, it can be easily seen that we have

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) ,$$

since

$$i_k = 1 \quad \text{for} \quad k = 3, 4, \ldots .$$

For D-start, the following two cases must be treated separately. If $r_1 \geq r_2$, we have

$$i_k = 1 , \text{ for } k = 4, 5, 6, \ldots .$$

Accordingly, for some $K > 4$,

$$j_k = 1 , \quad \text{for} \quad k \geq K .$$

That is,

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) .$$

If $r_1 < r_2$, we have

$$A(k) = 0 , \text{ for } k = 1, 2, \ldots ,$$

and

$$B(k) < 0 , \text{ for } k = 1, 2, \ldots .$$

Accordingly, in this case,

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) .$$

(5, 4)	(2, 3)
(3, 6)	(2, 9)

It can be easily seen that $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (0, 1)$ are equilibrium strategies in this case.

<u>Case II(A)(iii):</u> If $R_1 = 0$, $R_2 < 0$, then from our definition of a special learning process, in the case of A-start, it is clear that we have

(5, 2)	(4, 2)
(3, 1)	(4, 3)

$$A(k) = 1 \text{ , for } k = 1, 2, \ldots ,$$

and

$$B(2k+1) = 1 \text{ , for } k = 0, 1, \ldots ,$$

$$B(2k) = 2 \text{ , for } k = 1, 2, \ldots .$$

Accordingly, we have

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (\tfrac{1}{2}, \tfrac{1}{2}) .$$

And, in this case, $x^* = (1, 0)$ and $y^* = (\frac{1}{2}, \frac{1}{2})$ are equilibrium strategies.

In B-start, we have

$$A(k) = 1 \text{ , and } B(k) = 2 \text{ , for } k = 3, 4, \ldots .$$

Accordingly,

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) .$$

In this case, $x^* = (1, 0)$ and $y^* = (0, 1)$ are pure equilibrium strategies.

In C-start, we have

$$A(k) = 1 \text{ , and } B(k) = 2 \text{ , for } k = 2, 3, \ldots .$$

Accordingly, we have

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) .$$

In D-start, from our definition of a special learning process, it is clear that we have

$$A(2k) = 2 , \quad \text{for} \quad k = 1, 2, \ldots ,$$

$$A(2k+1) = 1 , \quad \text{for} \quad k = 0, 1, 2, \ldots ,$$

and

$$B(k) = 2 , \quad \text{for} \quad k = 1, 2, \ldots .$$

Accordingly, we have

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) .$$

In this case, $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (0, 1)$ are equilibrium strategies.

For all other cases which can occur with respect to the relations between R_1 and R_2 under the Case II(A) , it can be easily shown that the same reasoning as above holds. Also it is clear that in the cases of $L_1 < 0$, $L_2 = 0$; $L_1 = 0$, $L_2 > 0$; or $L_1 = 0$, $L_2 < 0$, we can obtain the same results as in Case II(A).

<u>Case II(B):</u> $L_1 > 0 > L_2$.

In this case we must also consider separately several subcases. But if at least one of the R_i is zero, then the same results hold as in Case II(A). Accordingly, we consider the following two cases:

<u>Case II(B)(i):</u> $L_1 > 0 > L_2$ and $R_1 > 0 > R_2$,

<u>Case II(B)(ii):</u> $L_1 > 0 > L_2$ and $R_1 < 0 < R_2$.

In this section we will take up Case II(B)(i) . (Case II(B)(ii) will be treated separately in § 5.) In Case II(B)(i) it is clear that in A-start we have

$$i_k = j_k = 1 , \quad \text{for} \quad k = 1, 2, \ldots ,$$

that is

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) .$$

In D-start, we have

$$i_k = j_k = 2 , \quad k = 1, 2, \ldots ,$$

that is

$$\lim_{k\to\infty} x(k) = (0, 1) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) .$$

In this case both $x^* = (1, 0)$, $y^* = (1, 0)$ and $x^{**} = (0, 1)$, $y^{**} = (0, 1)$ are equilibrium strategies.

Consequently, the proof of the theorem for B-start and C-start remains. For that proof, we consider the following two cases separately:

<u>Case II(B)(i)(α):</u> sign $(L_1 + L_2)$ = sign $(R_1 + R_2)$.

First we consider the following case:

$$L_1 + L_2 > 0 \text{ and } R_1 + R_2 > 0 ,$$

that is

$$\ell_1 > \ell_2 > 0 \text{ and } r_1 > r_2 > 0 .$$

Then for B-start it can be easily seen that we have

$$i_k = j_k = 1 , \quad \text{for } k \geq 3 .$$

Accordingly,

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) .$$

Next, in the case of

$$L_1 + L_2 = 0 \text{ and } R_1 + R_2 = 0 ,$$

that is,

$$\ell_1 - \ell_2 = 0 \text{ and } r_1 - r_2 = 0 ,$$

we have

$$\lim_{k\to\infty} x(k) = (\tfrac{1}{2}, \tfrac{1}{2}) ,$$

$$\lim_{k\to\infty} y(k) = (\tfrac{1}{2}, \tfrac{1}{2}) .$$

(5, 4)	(4, 1)
(3, 2)	(6, 5)

In this case, $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (\frac{1}{2}, \frac{1}{2})$ are equilibrium strategies.

Case II(B)(i)(β): sign $(L_1 + L_2) \neq$ sign $(R_1 + R_2)$.

Let us consider the case $\ell_1 > \ell_2$. Then we have $r_1 \leq r_2$. We can write

$$\ell_1 = u\ell_2 + \epsilon , \quad r_2 = vr_1 + \delta$$

where u, v are integers greater than or equal to 1 and we assume that $0 < \epsilon < \ell_2$, $0 < \delta < r_1$. (The proof for the case $\epsilon = 0$ and|or $\delta = 0$ will be given in the same way as below.)

1) If $u < v$, in B-start, it can be seen that we have

(6, 5)	(2, 3)
(1, 2)	(5, 7)

$i_k = j_k = 2$, for $k = u + 2 , u + 3 , \ldots$.

Accordingly,

$$\lim_{k\to\infty} x(k) = (0, 1) ,$$

$$\lim_{k\to\infty} y(k) = (0, 1) ,$$

and $x^* = (0, 1)$ and $y^* = (0, 1)$ are pure equilibrium strategies.

2) If $u = v$,

$$\ell_1 = u\ell_2 + \epsilon , \quad r_2 = ur_1 + \delta .$$

In this case we must treat the several subcases separately.

(a) $\epsilon \neq 0$, $\delta \neq 0$, and $a\epsilon < \ell_2 < (a+1)\epsilon$, and

$$r_1 > (a+1)\delta ,$$

where a is an integer. In this case

(9, 7)	(2, 4)
(1, 3)	(5, 10)

$A(k(u+1)) = k\epsilon$ and $B(k(u+1)) = -k\delta$ for $k = 1, 2, \ldots, a+1$.

Accordingly, we have

$$A((a+1)(u+1)+1) = -\ell_2 + (a+1)\epsilon > 0 ,$$

and

$$B((a+1)(u+1)+1) = r_1 - (a+1)\delta > 0 .$$

This means that

$$i_k = j_k = 1 , \quad \text{for} \quad k \geq u(a+1) + 2 ,$$

and we have

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) .$$

(b) $\epsilon \neq 0$, $\delta \neq 0$, and

$$\ell_2 = a\epsilon , \quad b\delta < r_1 < (b+1)\delta , \quad a < b .$$

In this case it can be shown that

(10, 8)	(9, 1)
(7, 3)	(11, 12)

$A(a(u+1)+1) = -\ell_2 + a\epsilon = 0$,

$B(a(u+1)+1) = r_1 - a\delta > 0$.

Thus, from the definition of a special learning process, we can check that

$$A(a(u+1)+1+a+1) = \epsilon > 0$$

and

$$B(a(u+1)+1+a+1) = r_1 - (a+1)\delta > 0 .$$

Accordingly, in this case we have

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) .$$

(c) $e \neq 0$, $\delta \neq 0$, and $\ell_2 = a\epsilon$, $r_1 = a\delta$.

Thus we have

(2, 1)	(-1, -1)
(-1, -1)	(1, 2)

$$\ell_1 = (ua + 1)\epsilon , \quad r_2 = (ua + 1)\delta .$$

We are now considering B-start. Accordingly,

$$A(1) = -\ell_2 = -a\epsilon < 0 , \quad B(1) = r_1 = a\delta > 0 ,$$

and

$$i_2 = 2 , \quad j_2 = 1 .$$

Therefore, we have

$$A(2) = [(u-1)a + 1]\epsilon > 0 , \quad B(2) = -[(u-1)a + 1]\delta < 0 .$$

In this way we have

$$i_k = 1 , \quad j_k = 2 , \quad \text{for} \quad k = u + 1 ,$$

and

$$A(u + 1) = \epsilon > 0 , \quad B(u + 1) = - \delta < 0 .$$

Accordingly we have the following reasoning:

$$i_k = 1 , \quad j_k = 2 , \quad \text{for} \quad k = u + 2 ,$$

$$A(u + 2) = -(a-1)\epsilon < 0 , \quad B(u + 2) = (a-1)\delta > 0 ,$$

$$i_k = 2 , \quad j_k = 1 , \quad \text{for} \quad k = u + 3 ,$$

$$A(u+ 3) = [(u-1)a + 2]\epsilon > 0 , \quad B(u+ 3) = -[(u-1)a + 2]\delta < 0 ,$$

$$i_k = 1 , \quad j_k = 2 , \quad \text{for} \quad k = u + 4 .$$

In this way we have

$$i_k = 1 \,,\ j_k = 2 \,, \quad \text{for} \quad k = u + 4,\ u + 5,\ \ldots,\ 2u + 2 \,,$$

and

$$A(2u + 2) = 2\epsilon > 0 \,, \quad B(2u + 2) = -2\delta \,.$$

Accordingly,

$$i_k = 1 \,,\ j_k = 2 \,, \quad \text{for} \quad k = 2u + 3 \,,$$

and

$$A(2u + 3) = -(a-2)\epsilon < 0 \,,$$

$$B(2u + 3) = (a-2)\delta > 0 \,.$$

Proceeding in this way, at the $a(u+1)$-th stage we have

$$A(a(u+1)) = 0 \,, \quad B(a(u+1)) = 0 \,.$$

Then, because of our definition of a special learning process, we have

$$i_k = 2 \,,\ j_k = 1 \,, \quad \text{for} \quad k = a(u+1) + 1 \,,$$

and at the $a(u+1) + u + 2$ -th stage we have

$$A = \epsilon \quad \text{and} \quad B = -\delta \,.$$

After this point, we repeat the same phase as from the $(u+1)$-th stage to $a\ (u+1) + u + 1$-th stage. In this interval, player I uses strategy 1

$$2 + u(a-1) + u - 1 = ua + 1$$

times and strategy 2

$$a$$

times; and then it is clear that in this interval player II uses strategy 2

$$ua + 1$$

times and strategy 1

$$a$$

times.

Therefore, in this case we conclude that:

$$\lim_{k\to\infty} x(k) = \left(\frac{ua + 1}{(u+1)a+1} , \frac{a}{(u+1)a+1} \right) \equiv x' ,$$

$$\lim_{k\to\infty} y(k) = \left(\frac{a}{(u+1)a+1} , \frac{ua+1}{(u+1)a+1} \right) \equiv y' .$$

From Lemma 4.1 to be stated below, it is clear that in this case x' and y' are equilibrium strategies. The same result holds in the case of C-start.

Lemma 4.1 In 2 x 2 non-zero-sum two-person games Γ, if

(4.1) $$L_1 > 0 > L_2 \quad \text{or} \quad L_1 < 0 < L_2$$

and

(4.2) $$R_1 > 0 > R_2 \quad \text{or} \quad R_1 < 0 < R_2 ,$$

hold, then one of the equilibrium strategy pairs (x^*, y^*) is given by

(4.3) $$x^* = \left(\frac{R_2}{R_2 - R_1} , \frac{-R_1}{R_2 - R_1} \right)$$

and

(4.4) $$y^* = \left(\frac{L_2}{L_2 - L_1} , \frac{-L_1}{L_2 - L_1} \right) .$$

Proof of Lemma 4.1: Let $x = (\alpha, 1-\alpha)$, $y = (\beta, 1-\beta)$ be mixed strategies for players I and II respectively. Then the following equation

(4.5) $$\varphi_1(1, y) = \varphi_1(2, y)$$

becomes

$$\beta\, a_{11} + (1-\beta)\, a_{12} = \beta\, a_{21} + (1-\beta)\, a_{22} .$$

That is

$$\beta(L_1 - L_2) = -L_2 .$$

If condition (4.1) holds, it is clear that

$$0 < \beta = \frac{L_2}{L_2 - L_1} < 1 .$$

In the same way, from equation

$$(4.6) \qquad \varphi_2(x, 1) = \varphi_2(x, 2)$$

we have

$$\alpha(R_2 - R_1) = R_2 .$$

If condition (4.2) holds, it is clear that

$$0 < \alpha = \frac{R_2}{R_2 - R_1} < 1 .$$

This proves Lemma 4.1. Q.E.D.

3) If $u > v$, in B-start, we have

$$i_k = 1 , \; j_k = 2 , \quad \text{for} \quad k = v ,$$

and

(8, 5)	(7, 3)
(1, 2)	(9, 7)

$$A(v+1) = (u-v)\ell_2 + \epsilon > 0 ,$$

$$B(v+1) = - \delta .$$

We are assuming that $\delta > 0$. Accordingly, we have

$$i_k = 1 , \; j_k = 2 , \quad \text{for} \quad k = v + 1$$

and

$$A(v+2) = (u-v-1)\ell_2 + \epsilon ,$$

$$B(v+2) = r_1 - \delta > 0 .$$

Accordingly, under the assumption $\epsilon > 0$, we have

$$i_k = 1 , \; j_k = 1 , \quad \text{for} \quad k = v+2, v+3, \ldots .$$

This means that we have

$$\lim_{k\to\infty} x(k) = (1, 0) ,$$

$$\lim_{k\to\infty} y(k) = (1, 0) .$$

In other possible cases, we can also check the truth of our theorem. In the case of C-start, we have the same results as in B-start.

§ 5. Case II(B)(ii)

$L_1 > 0 > L_2$ and $R_1 < 0 < R_2$

In all the cases treated above, there was at least one pure equilibrium point; but in this case, there is no pure equilibrium point.

(5, 4)	(2, 7)
(3, 6)	(8, 1)

According to Lemma 4.1, we know in this case that the equilibrium strategies x^* and y^* are given by

(5.1) $$x^* = \left(\frac{r_2}{r_1 + r_2}, \frac{r_1}{r_1 + r_2}\right)$$

and

(5.2) $$y^* = \left(\frac{\ell_2}{\ell_1 + \ell_2}, \frac{\ell_1}{\ell_1 + \ell_2}\right),$$

where as before

(5.3) $$\ell_i = |L_i| \text{ and } r_i = |R_i|, \quad i = 1, 2 .$$

In order to prove the theorem for this case, let us assume that we can write

(5.4) $$\ell_2 = u\ell_1 + \epsilon, \quad 0 \leq \epsilon < \ell_1$$

and

(5.5) $$r_1 = vr_2 + \delta, \quad 0 \leq \delta < r_2,$$

where u, v are integers greater than or equal to 1. We will prove the theorem only for B-start. (We remark that the same reasoning holds for the other possible cases concerning the magnitudes of ℓ_1, ℓ_2, r_1 and r_2 and for A-, C-, and D-start.)

According to the definition of B-start, we have

$$i_1 = 1 \, , \; j_1 = 2$$

and

(5.6) $$A(1) = -\ell_2 \equiv G(0) < 0 \, , \; B(1) = -r_1 \equiv H(0) < 0 \, .$$

Then it can be easily seen that for some time we have the equalities

$$2 = i_2 = i_3 = \ldots$$
$$2 = j_2 = j_3 = \ldots$$

and

(5.7) $$A(2) = -2\ell_2 \, , \; A(3) = -3\ell_2 \, , \; \ldots$$

(5.8) $$B(2) = H(0) + r_1 \, , \; B(3) = H(0) + 2r_2 \, , \; \ldots \, .$$

This means that A remains negative for some time, and that B changes its sign from negative to positive or from negative to zero at some point. Let f(1) be the number of plays from the second play to the play at which B finishes its sign change. Let us call this period phase I.

Phase I can be shown in the following diagram:

B	II	I		A
($0 > H(0) \equiv -r_1$	2	1		$-\ell_2 \equiv G(0) < 0$)
$0 > H(0) + r_2$	2	2	↑	$G(0) - \ell_2 < 0$
⋮	2	2		⋮
	⋮	⋮	f(1)	⋮
$0 > H(0) + vr_2 = -\delta$	2	2		⋮
$0 \leq H(0) + (v+1)r_2$	2	2	↓	$G(0) - (v+1)\ell_2 < 0$

Let us call the $1 + f(1)$-th value of A and B , G(1) and H(1) respectively. Then it is clear that

$$i_k = 2 \, , \; j_k = 1 \, , \text{ for } k = 1 + f(1) + 1 \, , \; 1 + f(1) + 2 \, , \ldots ,$$

since even in the case of $H(1) = 0$, our definition of a special learning process indicates that the $1 + f(1)$-th strategy of player II will be 1.

A takes on the values

$$G(1) + \ell_1 < 0 , \quad G(1) + 2\ell_1 < 0 , \ \ldots$$

and B takes on the values

$$H(1) + r_2 > 0 , \quad H(1) + 2r_2 , \ \ldots .$$

Accordingly, after A continues to be negative for some time, its sign will change from negative to positive or from negative to zero. Let us call this positive or zero value $G(2)$ and let $f(2)$ be the number of plays during this period. This period will be called phase II. It is clear that, in phase II, B takes on positive values. Let $H(2)$ be the value of B corresponding to $G(2)$. Phase II is shown in the following diagram:

	B	II	I		A	
$0 <$	$H(1) + r_2$	1	2	↑	$G(1) + \ell_1$	< 0
$0 <$	$H(1) + 2r_2$	1	2	$f(2)$	$G(1) + 2\ell_1$	< 0
	⋮	⋮	⋮		⋮	
$0 <$	⋮	1	2		⋮	< 0
$0 <$	$H(2)$	1	2	↓	$G(2)$	≥ 0

Accordingly, we have

$$i_k = j_k = 1 , \text{ for } k = 1 + f(1) + f(2) + 1 , \ 1 + f(1) + f(2) + 2 , \ldots,$$

and A takes on the values

$$G(2) + \ell_1 > 0 , \quad G(2) + 2\ell_1 > 0 , \ \ldots ,$$

and B takes on the values

$$H(2) - r_1 > 0 , \quad H(2) - 2r_1 > 0 , \ \ldots .$$

Accordingly, after B continues to be positive for some time, it will change its sign from positive to negative or from positive to zero. Let us call this negative or zero value $H(3)$ and let $f(3)$ be the number of plays during this period. This period will be called phase III. It is clear that, in phase III, A takes on positive values. Let $G(3)$ be the value of A corresponding to $H(3)$. Phase III is shown in the following diagram:

	B	II	I		A	
$0 <$	$H(2) - r_1$	1	1	↑	$G(2) + \ell_1$	> 0
$0 <$	$H(2) - 2r_1$	1	1	$f(3)$	$G(2) + 2\ell_1$	> 0
	⋮	⋮	⋮			
$0 <$						
$0 \geq$	$H(3)$	1	1	↓	$G(3)$	> 0

Accordingly, we have

$$i_k = 1 \, , \; j_k = 2 \, , \quad \text{for} \quad k = 1 + f(1) + f(2) + f(3) + 1 \, , \; 1 + f(1) + f(2) + f(3) + 2 \, , \; \ldots \, .$$

A takes on the values

$$G(3) - \ell_2 > 0 \, , \quad G(3) - 2\ell_2 > 0 \, , \; \ldots$$

and B takes on the values

$$H(3) - r_1 < 0 \, , \quad H(3) - 2r_1 < 0 \, , \; \ldots \, .$$

A will change its sign from positive to negative or from positive to zero at some point. Let us call this negative or zero value $G(4)$ and let $f(4)$ be the number of plays during this period.

This period will be called phase IV. In phase IV, B takes on negative values. Let $H(4)$ be the value of B corresponding to $G(4)$.

Phase IV is shown in the following diagram:

	B	II	I		A	
$0 >$	$H(3) - r_1$	2	1	↑	$G(3) - \ell_2$	> 0
$0 >$	$H(3) - 2r_1$	2	1	$f(4)$	$G(3) - 2\ell_2$	> 0
	⋮	⋮	⋮		⋮	
		2	1			> 0
$0 >$	$H(4)$	2	1	↓	$G(4)$	≤ 0

After phase IV, four phases which are similar to phases I, II, III and IV respectively, will appear cyclically.

From the above reasoning, it is clear that there is a general cycle, which is shown in the following diagram:

		II	I			
$(0 >$	$H(4k)$	2	1		$G(4k)$	$\leq 0\ )$
$0 >$		2	2	↑		< 0
⋮			⋮	$f(4k+1)$		⋮
$0 \leq$	$H(4k+1)$	2	2	↓	$G(4k+1)$	< 0
$0 <$		1	2	↑		< 0
⋮		⋮	⋮	$f(4k+2)$		⋮
$0 <$	$H(4k+2)$	1	2	↓	$G(4k+2)$	≥ 0
$0 <$		1	1	↑		> 0
⋮		⋮	⋮	$f(4k+3)$		⋮
$0 \geq$	$H(4k+3)$	1	1	↓	$G(4k+3)$	> 0
$0 >$		2	1	↑		> 0
⋮		⋮	⋮	$f(41+4)$		⋮
$0 >$	$H(4k+4)$	2	1	↓	$G(4k+4)$	≤ 0

for $k = 0, 1, 2, \dots$.

In this way, if players I and II continue to play the game Γ according to a special learning process, then the number of times that player I uses his strategy 1 is given by

$$1 + [f(3) + f(4)] + [f(7) + f(8)] + \dots \tag{5.9}$$

and the number of times player I uses his strategy 2 is given by

$$[f(1) + f(2)] + [f(5) + f(6)] + \dots . \tag{5.10}$$

The number of times player II uses his strategy 1 is given by

$$[f(2) + f(3)] + [f(6) + f(7)] + \dots \tag{5.11}$$

and the number of times player II uses his strategy 2 is given by

$$1 + f(1) + [f(4) + f(5)] + [f(8) + f(9)] + \dots . \tag{5.12}$$

Now let us write

$$| G(t) | = g(t) \text{ and } | H(t) | = h(t) . \tag{5.13}$$

From the diagram above, we have the following equations:

In the period $(4k + 1)$:

$$h(4k) = r_2 f(4k+1) - h(4k+1) , \tag{5.14}$$

$$0 \leq h(4k+1) < r_2 , \quad \text{and} \tag{5.15}$$

$$g(4k+1) = g(4k) + \ell_2 f(4k+1) . \tag{5.16}$$

In the period $(4k + 2)$:

$$g(4k+1) = \ell_1 f(4k+2) - g(4k+2) , \tag{5.17}$$

$$0 \leq g(4k+2) < \ell_1 , \quad \text{and} \tag{5.18}$$

$$h(4k+2) = h(4k+1) + r_2 f(4k+2) . \tag{5.19}$$

In the period $(4k+3)$:

$$h(4k+2) = r_1 f(4k+3) - h(4k+3) , \tag{5.20}$$

(5.21) $$0 \leq h(4k+3) < r_1 , \qquad \text{and}$$

(5.22) $$g(4k+3) = g(4k+2) + \ell_1 f(4k+3) .$$

In the period $(4k+4)$:

(5.23) $$g(4k+3) = \ell_2 f(4k+4) - g(4k+4) ,$$

(5.24) $$0 \leq g(4k+4) < \ell_2 , \qquad \text{and}$$

(5.25) $$h(4k+4) = h(4k+3) + r_1 f(4k+4) .$$

From (5.14) we have

(5.26) $$f(4k+1) = \frac{1}{r_2}[h(4k) + h(4k+1)] .$$

From (5.16) and (5.17) we have

(5.27) $$f(4k+2) = \frac{\ell_2}{\ell_1} f(4k+1) + \frac{1}{\ell_1}[g(4k) + g(4k+2)] .$$

From (5.19) and (5.20) we have

(5.28) $$f(4k+3) = \frac{r_2}{r_1} f(4k+2) + \frac{1}{r_1} [h(4k+1) + h(4k+3)] .$$

From (5.22) and (5.23) we have

(5.29) $$f(4k+4) = \frac{\ell_1}{\ell_2} f(4k+3) + \frac{1}{\ell_2} [g(4k+2) + g(4k+4)] .$$

From (5.25) and (5.14) we have

(5.30) $$f(4k+1) = \frac{r_1}{r_2} f(4k) + \frac{1}{r_2} [h(4k-1) + h(4k+1)] .$$

If we put

(5.31) $$\alpha(4k+1) = \frac{1}{r_2} [h(4k-1) + h(4k+1)] ,$$

(5.32) $$\alpha(4k+2) = \frac{1}{\ell_1} [g(4k) + g(4k+2)] ,$$

(5.33) $$\alpha(4k+3) = \frac{1}{r_1} [h(4k+1) + h(4k+3)] ,$$

(5.34) $$\alpha(4k+4) = \frac{1}{\ell_2} [g(4k+2) + g(4k+2)] ,$$

then from (5.26) ~ (5.30) we have

$$(5.35) \qquad f(1) = \frac{1}{r_2}[h(0) + h(1)] ,$$

$$(5.36) \qquad f(4k+1) = \frac{r_1}{r_2} f(4k) + \alpha(4k+1) ,$$

$$(5.37) \qquad f(4k+2) = \frac{\ell_2}{\ell_1} f(4k+1) + \alpha(4k+2) ,$$

$$(5.38) \qquad f(4k+3) = \frac{r_2}{r_1} f(4k+2) + \alpha(4k+3) ,$$

$$(5.39) \qquad f(4k+4) = \frac{\ell_1}{\ell_2} f(4k+3) + \alpha(4k+4) .$$

Here (5.36) holds for $k = 1, 2, 3, \ldots,$ and (5.37), (5.38), and (5.39) holds for $k = 0, 1, 2, 3, \ldots$.

We derive here the following expressions from (5.36) ~ (5.39) for $k = 1$, which will be used below:

$$(5.40) \begin{cases} f(5) = \frac{r_1}{r_2} f(4) + \alpha(5) , \\ f(6) = \frac{\ell_2}{\ell_1}\frac{r_1}{r_2} f(4) + \frac{\ell_2}{\ell_1}\alpha(5) + \alpha(6) , \\ f(7) = \frac{\ell_2}{\ell_1} f(4) + \frac{r_2}{r_1}\frac{\ell_2}{\ell_1}\alpha(5) + \frac{r_2}{r_1}\alpha(6) + \alpha(7) , \\ f(8) = f(4) + \frac{r_2}{r_1}\alpha(5) + \frac{\ell_1}{\ell_2}\frac{r_2}{r_1}\alpha(6) + \frac{\ell_1}{\ell_2}\alpha(7) + \alpha(8) . \end{cases}$$

Now we define the functions Φ_i and Ψ_i , $i = 1, 2,$ as follows:

$$(5.41) \qquad \Phi_1(4k+1) = \frac{r_1}{r_2}\left(1 + \frac{\ell_2}{\ell_1}\right)\alpha(4k) + \left(1 + \frac{\ell_2}{\ell_1}\right)\alpha(4k+1) ,$$

$$(5.42) \qquad \Phi_2(4k+3) = \left(1 + \frac{\ell_1}{\ell_2}\right)\alpha(4k+2) + \frac{r_1}{r_2}\left(1 + \frac{\ell_1}{\ell_2}\right)\alpha(4k+3) ,$$

$$(5.43) \qquad \Psi_1(4k+1) = \frac{r_2}{r_1}\Phi_1(4k+1) ,$$

$$(5.44) \qquad \Psi_2(4k+3) = \frac{r_2}{r_1}\Phi_2(4k+3) ,$$

for $k = 1, 2, \ldots$, but we take $\alpha(4) \equiv f(4)$.

Then we will prove the following two relations (5.45) and (5.46):

(5.45) $f(4t+1) + f(4t+2)$

$$= \Phi_1(5) + \sum_{k=2}^{t} [\Phi_2(4k-1) + \Phi_1(4k+1)] + \alpha(4t+2), \text{ for } t = 2,3, \ldots$$

(5.46) $f(4t+3) + f(4t+4)$

$$= \sum_{k=1}^{t} [\psi_1(4k+1) + \psi_2(4k+3)] + \alpha(4t+4), \text{ for } t = 1, 2, 3, \ldots .$$

For $t = 2$, we have:

The right hand side of (5.45)

$$= \Phi_1(5) + \Phi_2(7) + \Phi_1(9) + \alpha(10)$$

$$= \frac{r_1}{r_2}\left(1 + \frac{\ell_2}{\ell_1}\right) f(4) + \left(1 + \frac{\ell_2}{\ell_1}\right)\alpha(5)$$

$$+ \left(1 + \frac{\ell_1}{\ell_2}\right)\alpha(6) + \frac{r_1}{r_2}\left(1 + \frac{\ell_1}{\ell_2}\right)\alpha(7)$$

$$+ \frac{r_1}{r_2}\left(1 + \frac{\ell_2}{\ell_1}\right)\alpha(8) + \left(1 + \frac{\ell_2}{\ell_1}\right)\alpha(9) + \alpha(10)$$

$$= \left[\frac{r_1}{r_2} f(4) + \alpha(5) + \frac{\ell_1}{\ell_2}\alpha(6) + \frac{r_1}{r_2}\frac{\ell_1}{\ell_2}\alpha(7)\right.$$

$$\left. + \frac{r_1}{r_2}\alpha(8) + \alpha(9)\right] + \left[\frac{\ell_2}{\ell_1}\frac{r_1}{r_2} f(4) + \frac{\ell_2}{\ell_1}\alpha(5)\right.$$

$$\left. + \alpha(6) + \frac{r_1}{r_2}\alpha(7) + \frac{\ell_2}{\ell_1}\frac{r_1}{r_2}\alpha(8) + \frac{\ell_2}{\ell_1}\alpha(9) + \alpha(10)\right]$$

$$= \left[\frac{r_1}{r_2} f(8) + \alpha(9)\right] + \left[\frac{\ell_2}{\ell_1}\frac{r_1}{r_2} f(8) + \frac{\ell_2}{\ell_1}\alpha(9) + \alpha(10)\right]$$

(because of (5.40))

$$= f(9) + \left[\frac{\ell_2}{\ell_1} f(9) + \alpha(10)\right]$$

(because of (5.36))

$$= f(9) + f(10)$$

(because of (5.37)).

This proves that (5.45) holds for $t = 2$.

Next we will prove (5.45) for $(t+1)$, under the assumption that it holds for t . From this inductive assumption and from (5.37), (5.42) and (5.41), we have the following transformation:

$$(5.47)\qquad \Phi_1(5) + \sum_{k=2}^{t+1} [\Phi_2(4k-1) + \Phi_1(4k+1)] + \alpha(4(t+1)+2)$$

$$= \Phi_1(5) + \sum_{t=2}^{t} [\Phi_2(4k-1) + \Phi_1(4k+1)] + \alpha(4t+2)$$

$$- \alpha(4t+2) + \Phi_2(4t+3) + \Phi_1(4(t+1)+1) + \alpha(4(t+1)+2)$$

$$= f(4t+1) + f(4t+2) - \alpha(4t+2) + \Phi_2(4t+3) + \Phi_1(4(t+1)+1)$$

$$+ \alpha(4(t+1)+2)$$

$$= (1 + \frac{\ell_2}{\ell_1}) f(4t+1)$$

$$+ (1 + \frac{\ell_1}{\ell_2}) \alpha(4t+2) + \frac{r_1}{r_2} (1 + \frac{\ell_1}{\ell_2}) \alpha(4t+3)$$

$$+ \frac{r_1}{r_2} (1+ \frac{\ell_2}{\ell_1}) \alpha(4t+4) + (1+ \frac{\ell_2}{\ell_1}) \alpha(4(t+1)+1) + \alpha(4(t+1)+2) .$$

On the other hand, using the relations (5.36) ~ (5.39), we have the following transformation:

$$(5.48)\qquad f(4(t+1)+1) = \frac{r_1}{r_2} f(4t+4) + \alpha(4(t+1)+1)$$

$$= \frac{r_1}{r_2} \frac{\ell_1}{\ell_2} f(4t+3) + \frac{r_1}{r_2} \alpha(4t+4) + \alpha(4(t+1)+1)$$

$$= \frac{\ell_1}{\ell_2} f(4t+2) + \frac{r_1}{r_2} \frac{\ell_1}{\ell_2} \alpha(4t+3) + \frac{r_1}{r_2} \alpha(4t+4) + \alpha(4(t+1)+1)$$

$$= f(4t+1) + \frac{\ell_1}{\ell_2} \alpha(4t+2) + \frac{r_1}{r_2} \frac{\ell_1}{\ell_2} \alpha(4t+3) + \frac{r_1}{r_2} \alpha(4t+4)$$

$$+ \alpha(4(t+1)+1)$$

and

$$(5.49)\quad f(4(t+1)+2) = \frac{\ell_2}{\ell_1} f(4(t+1)+1) + \alpha(4(t+1)+2)$$

$$= \frac{\ell_2}{\ell_1}\frac{r_1}{r_2} f(4t+4) + \frac{\ell_2}{\ell_1}\alpha(4(t+1)+1) + \alpha(4(t+1)+2)$$

$$= \frac{r_1}{r_2} f(4t+3) + \frac{\ell_2}{\ell_1}\frac{r_1}{r_2}\alpha(4t+4) + \frac{\ell_2}{\ell_1}\alpha(4(t+1)+1) + \alpha(4(t+1)+2)$$

$$= f(4t+2) + \frac{r_1}{r_2}\alpha(4t+3) + \frac{\ell_2}{\ell_1}\frac{r_1}{r_2}\alpha(4t+4) + \frac{\ell_2}{\ell_1}\alpha(4(t+1)+1)$$

$$+ \alpha(4(t+1)+2)$$

$$= \frac{\ell_2}{\ell_1} f(4t+1) + \alpha(4t+2) + \frac{r_1}{r_2}\alpha(4t+3) + \frac{\ell_2}{\ell_1}\frac{r_1}{r_2}\alpha(4t+4)$$

$$+ \frac{\ell_2}{\ell_1}\alpha(4(t+1)+1) + \alpha(4(t+1)+2) .$$

It is clear that the sum of the last expressions in (5.48) and (5.49) is equal to the last expression in (5.47). This proves that equation (5.45) holds for $t+1$ when it holds for t. This completes the proof of (5.45).

Next we will prove the equality (5.46). For $t = 1$, using (5.41) ~ (5.44) and (5.40) we have the following:

The right hand side of (5.46)

$$= \psi_1(5) + \psi_2(7) + \alpha(8)$$

$$= (1 + \frac{\ell_2}{\ell_1}) f(4) + \frac{r_2}{r_1}(1 + \frac{\ell_2}{\ell_1})\alpha(5)$$

$$+ \frac{r_2}{r_1}(1 + \frac{\ell_1}{\ell_2})\alpha(6) + (1 + \frac{\ell_1}{\ell_2})\alpha(7) + \alpha(8)$$

$$= f(7) + f(8) ,$$

that is, the equality (5.46) holds for $t = 1$.

Now we will prove (5.46) for $t+1$ under the assumption that it holds for t. From this inductive assumption and (5.39), (5.41 ~ (5.44), we have the following transformation:

$$(5.50) \quad \sum_{k=1}^{t+1} [\psi_1(4k+1) + \psi_2(4k+3)] + \alpha(4(t+1)+4)$$

$$= f(4t+3) + f(4t+4) - \alpha(4t+4)$$

$$+ \psi_1(4(t+1)+1) + \psi_2(4(t+1)+3) + \alpha(4(t+1)+4)$$

$$= (1 + \frac{\ell_1}{\ell_2}) f(4t+3)$$

$$+ (1 + \frac{\ell_2}{\ell_1}) \alpha(4t+4) + \frac{r_2}{r_1} (1 + \frac{\ell_2}{\ell_1}) \alpha(4(t+1)+1)$$

$$+ \frac{r_2}{r_1} (1 + \frac{\ell_1}{\ell_2}) \alpha(4(t+1)+2) + (1 + \frac{\ell_1}{\ell_2}) \alpha(4(t+1)+3)$$

$$+ a(4(t+1)+4) .$$

On the other hand, using (5.36) ~ (5.39), we have the following transformation:

$$(5.51) \quad f(4(t+1)+3) = \frac{r_2}{r_1} f(4(t+1)+2) + \alpha(4(t+1)+3)$$

$$= \frac{r_2}{r_1}\frac{\ell_2}{\ell_1} f(4(t+1)+1) + \frac{r_2}{r_1} \alpha(4(t+1)+2) + \alpha(4(t+1)+3$$

$$= \frac{\ell_2}{\ell_1} f(4t+4) + \frac{r_2}{r_1}\frac{\ell_2}{\ell_1} \alpha(4(t+1)+1) + \frac{r_2}{r_1} \alpha(4(t+1)+2)$$

$$+ \alpha(4(t+1)+3)$$

$$= f(4t+3) + \frac{\ell_2}{\ell_1} \alpha(4t+4) + \frac{r_2}{r_1}\frac{\ell_2}{\ell_1} \alpha(4(t+1)+1)$$

$$+ \frac{r_2}{r_1} \alpha(4(t+1)+2) + \alpha(4(t+1)+3) .$$

Following the same process as in (5.51), it can be easily shown that we have

$$(5.52)\quad f(4(t+1)+4) = \frac{\ell_1}{\ell_2} f(4t+3) + \alpha(4t+4) + \frac{r_2}{r_1}\alpha(4(t+1)+1)$$

$$+ \frac{\ell_1}{\ell_2}\frac{r_2}{r_1}\alpha(4(t+1)+2) + \frac{\ell_1}{\ell_2}\alpha(4(t+1)+3) +$$

$$+ \alpha(4(t+1)+4) .$$

It is clear that the sum of the last expressions in (5.51) and (5.52) is equal to the last expression of (5.50). This proves that (5.46) holds for $t+1$. This completes the proof of (5.46).

Now, let $\xi_1(n)$ and $\xi_2(n)$ $[\eta_1(n)$ and $\eta_2(n)]$ be the number of plays in which player I[II] uses his strategy 1 and 2, respectively, during the first

$$1 + \sum_{k=0}^{n} [f(4k+1) + f(4k+2) + f(4k+3) + f(4k+4)]$$

plays in a learning process which is B-start.

Then, using (5.46), we have

$$(5.53)\quad \xi_1(n) = 1 + [f(3) + f(4)] + \sum_{t=1}^{n} [f(4t+3) + f(4t+4)]$$

$$= 1 + [f(3) + f(4)]$$

$$+ \sum_{t=1}^{n}\sum_{k=1}^{t} [\psi_1(4k+1) + \psi_2(4k+3)] + \sum_{t=1}^{n}\alpha(4t+4)$$

$$= 1 + [f(3) + f(4)]$$

$$+ n[\psi_1(5) + \psi_2(7)]$$

$$+ (n-1)[\psi_1(9) + \psi_2(11)]$$

$$\vdots$$

$$+ 2[\psi_1(4n-3) + \psi_2(4n-1)]$$

$$+ 1[\psi_1(4n+1) + \psi_2(4n+3)]$$

$$+ \sum_{t=1}^{n}\alpha(4t+4) .$$

Now we put

$$(5.54)\qquad \rho(n) = n[\psi_1(5) + \psi_2(7)] + (n-1)[\psi_1(9) + \psi_2(11)] + \cdots + 2[\psi_1(4n-3) + \psi_2(4n-1)] + 1[\psi_1(4n+1) + \psi_2(4n+3)] ,$$

and

$$(5.55)\qquad \theta(n) = \sum_{t=1}^{n} \alpha(4t+4) ,$$

then from (5.53) we have

$$(5.56)\qquad \xi_1(n) = 1 + [f(3) + f(4)] + \rho(n) + \theta(n) .$$

Using (5.45), we have

$$(5.57)\qquad \xi_2(n) = f(1) + f(2) + f(5) + f(6) + \sum_{t=1}^{n-1} [f(4(t+1)+1) + f(4(t+1)+2)]$$

$$= f(1) + f(2) + f(5) + f(6) + \sum_{t=1}^{n-1} [\Phi_1(5) + \sum_{k=2}^{t+1} (\Phi_2(4k-1) + \Phi_1(4k+1)) + \alpha(4(t+1)+2)]$$

$$= f(1) + f(2) + f(5) + f(6) + \sum_{t=1}^{n-1} [\sum_{k=1}^{t} \{\Phi_1(4k+1) + \Phi_2(4k+3)\} + \Phi_1(4t+5)] + \sum_{t=1}^{n-1} \alpha(4(t+1)+2)$$

$$= f(1) + f(2) + f(5) + f(6) + (n-1)[\Phi_1(5) + \Phi_2(7)] + (n-2)[\Phi_1(9) + \Phi_2(11)] + \vdots + 2[\Phi_1(4n-7) + \Phi_2(4n-5)] + 1[\Phi_1(4n-3) + \Phi_2(4n-1)]$$

$$+ [\Phi_1(9) + \Phi_1(13) + \cdots + \Phi_1(4n+1)]$$

$$+ \sum_{t=1}^{n-1} \alpha(4(t+1)+2)$$

$$= f(1) + f(2) + f(5) + f(6)$$

$$+ n[\Phi_1(5) + \Phi_2(7)]$$

$$+ (n-1)[\Phi_1(9) + \Phi_2(11)]$$

$$+$$

$$\vdots$$

$$+ 2[\Phi_1(4n-3) + \Phi_2(4n-1)]$$

$$+ 1[\Phi_1(4n+1) + \Phi_2(4n+3)]$$

$$- [\Phi_1(5) + \Phi_2(7)]$$

$$+ \sum_{t=1}^{n-1} [\alpha(4t+6) - \Phi_2(4t+7)] .$$

If we put

$$\delta(n) = \sum_{t=1}^{n-1} [\alpha(4t+6) - \Phi_2(4t+7)] , \tag{5.58}$$

then from the last expression in (5.55) and (5.43), (5.44), we have

$$\xi_2(n) = f(1) + f(2) + f(5) + f(6) - [\Phi_1(5) + \Phi_2(7)] + \frac{r_1}{r_2} \rho(n) + \delta(n) . \tag{5.59}$$

Now from our definition of the function $\alpha(t)$, it is clear that there exists a finite number K such that

$$0 \leq \alpha(t) < K , \quad 0 \leq \Phi_i(t) < K , \quad 0 \leq \psi_i(t) < K , \quad i = 1, 2,$$

for all possible values of t. And

$$\Phi_1(4k+1) + \Phi_2(4k+3)$$

is a linear combination of $\alpha(4k)$, $\alpha(4k+1)$, $\alpha(4k+2)$, and $\alpha(4k+3)$, with constant coefficients.

Therefore, we can say in general that the order of magnitude of $\rho(n)$ is that of

$$1 + 2 + \dots + n = \frac{n(n-1)}{2} ,$$

that is, $\rho(n)$ is of the order n^2. On the other hand, $\theta(n)$ and $\delta(n)$ are of the order n.

Therefore, from (5.56) and (5.59), we have

$$\lim_{n\to\infty} \frac{\xi_1(n)}{\xi_2(n)} = \frac{r_2}{r_1} . \tag{5.60}$$

From (5.60) and from the fact that the number of plays in each period is finite, we have

$$\lim_{k\to\infty} x(k) = \left(\frac{r_2}{r_1 + r_2} , \frac{r_1}{r_1 + r_2} \right) . \tag{5.61}$$

In the same way, it can be shown that

$$\lim_{k\to\infty} y(k) = \left(\frac{\ell_2}{\ell_1 + \ell_2} , \frac{\ell_1}{\ell_1 + \ell_2} \right) . \tag{5.62}$$

But from Lemma 4.1, we know that these limit strategies are surely equilibrium strategies. This proves the theorem.

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