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# Evolutionary Equilibria Resistant to Mutation

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A Nash equilibrium is a stationary point for a class of evolutionary dynamics. However, not all stationary points of these dynamics are Nash equilibria. An "evolutionary equilibrium" is the limit of stationary points of an evolutionary process as the proportion of the population that mutates goes to zero. The set of these evolutionary equilibria is a nonempty subset of the set of perfect equilibria (and thus of the set of Nash equilibria) and a superset of the set of regular equilibria and the set of ESS. *Journal of Economic Literature* Numbers: C72, C73. © 1994 Academic Press, Inc.

## 1. INTRODUCTION

The replicator model, developed by evolutionary biologists, formalizes the concept of "survival of the fittest." In this model, strategies are genetically determined and individuals have more offspring the more successful their strategies. Although individuals do not act as Bayesian maximizers, in equilibrium, individuals will be playing strategies that are best responses given the distribution of strategies of their opponents.

There is a very large literature in game theory which discusses different definitions of equilibrium. The most frequently used by economists is "Nash Equilibrium" although there are numerous examples where the equilibrium concept is not restrictive enough.[1] Attempts to strengthen the notion of Nash equilibrium led to equilibrium concepts such as "perfect

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[1] See for instance van Damme (1987).



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equilibrium" and "proper equilibrium."[2] This paper examines the relationship between the evolutionary process, as modeled according to the replicator model, and existing equilibrium concepts in game theory.[3]

The replicator model uses a differential (or difference) equation to describe the distribution of strategies in the population. In order to examine the dynamic equilibria of the replicator model, one must solve a system of nonlinear differential equations. This is usually done through numerical simulations. However, there are static equilibrium concepts that can be defined for the replicator model and which can be solved analytically. One particularly well known static equilibrium concept is the evolutionarily stable strategy (denoted in this paper by ESS) defined by Maynard Smith and Price (1973). This paper defines a different static equilibrium concept called "evolutionary equilibrium." The equilibrium concept is based on an arbitrarily small proportion of the population mutating towards an arbitrary strategy.

Section 2 describes the replicator model and its relationship with the concepts of Nash and perfect equilibrium. Section 3 defines and establishes formal properties of an evolutionary equilibrium, proving that an evolutionary equilibrium exists for a large class of payoff matrices. Section 4 analyzes the relationship between evolutionary equilibria and other equilibrium concepts in game theory. The set of evolutionary equilibria is shown to be a subset of the set of perfect equilibria and a superset of the set of regular equilibria. Throughout the paper the terms that are being defined are written in italics.

## 2. The Replicator Model

This section describes the replicator model and reviews Bomze's (1986) result regarding the connections between properties of the replicator model and game theoretic equilibria.

Suppose there is a large population[4] where each individual adopts a strategy $i \in \{1, \ldots, n\}$.[5] The number $x_i$ denotes the proportion of individuals adopting strategy $i$. The column vector $x = (x_1, \ldots, x_n)'$ describes the

[2] These equilibrium concepts were introduced in Selten (1975), and Myerson (1978).

[3] The following papers discuss similar issues: Crawford (1988), Friedman (1991), Nachbar (1990), and Samuelson (1988). The following papers discuss these issues in some more specialized contexts: Axelrod and Hamilton (1981), Boyd and Lorberbaum (1987), and Crawford (1989).

[4] Specifically suppose the set of individuals is isomorphic to the natural numbers.

[5] Hines (1980), Hines (1982), and Zeeman (1980), consider evolutionary dynamics where individuals adopt mixed strategies.

proportion of the population that adopts each possible strategy. Thus $x \in \Delta^n$ where

$$\Delta^n \equiv \left\{ x \in \mathbf{R}^n_+ : \sum_{i=1}^{n} x_i = 1 \right\}.$$

Individuals live for one period. During that period, each individual is randomly matched to one other individual. Individuals reproduce asexually, but the number of offspring that an individual has depends on (1) the strategy of the individual, and (2) the strategy of the randomly matched opponent. If an individual selects strategy $i$ and is matched with an individual that selects strategy $j$, the individual who adopts strategy $i$ has $a_{ij}$ offspring ($a_{ij} \geq 0$). An offspring selects the same strategy as the parent. The matrix $A$, where

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

is called the *payoff matrix for the evolutionary game.*

Given these assumptions, the proportion of the population adopting strategy $i$ at time $t + 1$, $x_i^{t+1}$, is

$$x_i^{t+1} = x_i^t \frac{(Ax^t)_i}{x^t \cdot Ax^t},$$

where $(Ax)_i = \sum_{j=1}^n a_{ij} x_j$ and $x \cdot Ax = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$.[6] The last expression can be rewritten as

$$\Delta x_i^t = x_i^{t+1} - x_i^t = x_i^t \frac{(Ax^t)_i - x^t \cdot Ax^t}{x^t \cdot Ax^t}.$$

or by dropping the time subscripts

$$\Delta x_i = x_i \frac{(Ax)_i - x \cdot Ax}{x \cdot Ax} \equiv R_i^D(x). \qquad (1)$$

[6] The validity of this statement is proven in Boylan (1992). Notice that by the definition of the inner product, $\Sigma_i x_i^t (Ax^t)_i / x^t \cdot Ax^t) = 1$. Since all $a_{ij}$ are nonnegative, if $x^t \in \Delta^n$, then $x_i^t ((Ax^t)_i / x^t \cdot Ax^t)$ is nonnegative, and thus $x^{t+1} \in \Delta^n$.

The system of difference equations $\Delta x = R^D(x)$ is called the *replicator model in discrete time*. When the payoff function, $A$, is not obvious, the replicator model in discrete time is denoted by $R^D_A$.

Let $\Delta t = \frac{\tau}{x \cdot Ax}$ be the time interval between periods (where $\tau \in \mathbf{R}_{++}$); then

$$\Delta x_i = x_i[(Ax)_i - x \cdot Ax] \, \Delta t$$

By letting $\Delta t \to 0$ the last expression can be written as

$$\dot{x}_i = x_i[(Ax)_i - x \cdot Ax] \equiv R^C_i(x).^7 \tag{2}$$

The system of differential equations $\dot{x} = R^C(x)$ is called the *replicator model in continuous time*. Again, when the payoff function, $A$, is not obvious, the replicator model in continuous time is denoted by $R^C_A$.[8]

Notice that if the vector of strategies $(x, x)$ is a Nash equilibrium of the normal game $(A, A^T)$ then the vector $x$ is a stationary point of the replicator model. However, there are stationary points of the replicator model that are not Nash equilibria of the normal form game. Bomze (1986), shows that such stationary points are not stable.

PROPOSITION 1 (Bomze). (i) *If the vector x is a stable stationary point*[9] *of the replicator model in continuous time, then the vector of strategies* $(x, x)$ *is a Nash equilibrium.* (ii) *If the vector x is an asymptotically stable*

[7] This derivation is entirely heuristic. Let $a = \max_{ij} |a_{ij} - 1|$. Akin and Losert (1984), use Euler's theorem to prove that as $a$ goes to 0 the solution of the replicator dynamics in discrete time converges to the solution of the replicator dynamics in continuous time. However, the convergence is pointwise, not uniform, and thus the limit of the two trajectories can be quite different. For constant sum games, for instance, the replicator dynamics in continuous time is a center while the replicator dynamics in discrete time is an unstable focus (for a definition of these terms see Arnold, 1973).

[8] Let the function $X$: $\mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ be such that for all $t \in \mathbf{R}_+$ and $x^0 \in \Delta^n$,

$$\frac{dX(t, x^0)}{dt} = R^C(X(t, x^0)) \qquad \text{and} \qquad X(0, x^0) = x^0.$$

(The existence and uniqueness of the function $X$ follow from the differentiability of the map $R^C$.) Notice that $(\Sigma_i \dot{x}_i) = \Sigma_i \dot{x}_i = 0$ and thus if $\Sigma_i x^0_i = 1$, then for all $i$, $\Sigma X_i(t, x^0) = 1$. Notice also that if $x_i = 0$, then $\dot{x}_i = 0$. Therefore if $x^0_i \geq 0$, then for all $t$, $X(t, x^0) \geq 0$. Consequently, if $x^0 \in \Delta^n$ and $t \in \mathbf{R}_+$, then $X(t, x^0) \in \Delta^n$.

[9] An equilibrium $\bar{x}$ is *stable* if given any positive scalar $\varepsilon$, there is a positive scalar $\delta$ such that for all strategies $x$ in the ball centered at $\bar{x}$ and with radius $\delta$, $x \in B(\bar{x}, \delta) \cap \Delta^n$, and for all positive $t$, $X(t, x) \in B(\bar{x}, \varepsilon) \cap \Delta^n$.

*stationary point*[10] *of the replicator model in continuous time then the vector of strategies* $(x, x)$ *is an isolated perfect equilibrium.*

Unfortunately, asymptotically stable stationary points are difficult to characterize and do not always exist. This paper further characterizes the relationship between the replicator model and game theory by examining stationary points of the replicator model that are resistant to mutation.

## 3. Evolutionary Equilibrium

This section defines the notions of generalized evolutionary system, evolutionary equilibrium, and nondegenerate payoff matrix. We prove that there exists an evolutionary equilibrium for all nondegenerate payoff matrices.

Consider the class of laws of motion, $\mathcal{H}$, such that strategies that are more successful are adopted more often. Specifically,

$$\mathcal{H}_A = \{H: \Delta^n \to T\Delta^n | H \text{ is continuously differentiable, and sign } H_i = \text{sign } x_i[(Ax)_i - xAx]\},$$

where $T\Delta^n$ is the tangent space of the simplex; i.e.,

$$T\Delta^n = \left\{x \in \mathbf{R}^n | \sum_i x_i = 0\right\}.$$

In order to make the notation less cumbersome, when no confusion can arise, the subscript $A$ in $\mathcal{H}_A$ is dropped. Note that $\mathcal{H}$ includes the replicator model in discrete and continuous time.[11]

Biologists have studied models where the frequency of types is affected not only by selection, as in the replicator model, but also by mutation. Mutation is the process through which a proportion of the population adopts a type independently of how well the type fared in previous periods.

[10] An equilibrium $\bar{x}$ is *asymptotically stable* if it is stable and if $\delta$ can be chosen such that

$$\forall x \in B(\bar{x}, \delta), \qquad \lim_{t\to\infty} X(x, t) = \bar{x}.$$

[11] One can also think of $H \in \mathcal{H}$ as a learning rule: suppose two individuals play a symmetric game for an infinite number of times. Each individual selects a mixed strategy $x$ for a large number of periods. After a while, each individual observes the average payoff, $x \cdot Ax$, and the payoff of all the pure strategies the individual adopted, $(Ax)_i$. The individual then selects a new mixed strategy $x + H(x)$. If a strategy $i$ is successful, $(Ax)_i > x \cdot Ax$, then strategy $i$ is played with greater frequency, $H_i(x) > 0$.

Denote by $m(x) = (m_1(x), \ldots, m_n(x))$ the change in population caused by mutation; this function has been modeled in different ways:

1. In Burger (1989), the proportion of the population of strategy $i$ that mutates to strategy $j$ is some constant $m_{ij}$. Thus,

$$m_j(x) = \sum_i x_i m_{ij} - x_j \sum_i m_{ji}.$$

2. Hofbauer and Sigmund (1988), page 23, and Burger (1988), consider a mutation function where, first the population changes through fitness (which is described by the payoff matrix) and then a certain proportion of the population mutates. In this case,

$$m_j(x) = \sum_i [x_i + H_i(x)]\, m_{ij} - [x_j + H_j(x)] \sum_i m_{ij}.$$

3. Hines (1982), considers a model where individuals select mixed strategies (instead of pure strategies). If an individual selects strategy $s$, then mutation leads an individual to select strategy $s + \varepsilon$ where $\varepsilon$ is a random variable independent from individual to individual.

4. Foster and Young (1990), consider a mutation function, $\Gamma$, which is continuous, satisfies $x \cdot \Gamma(x) = 0$, and is such that the boundaries are reflecting. The stochastic process $W$ is a continuous, white-noise process. In this case,

$$m(x) = \Gamma(x) W(t).$$

5. Maynard Smith and Price (1973), define evolutionary stable strategy using the mutation function

$$m(x) = \varepsilon y,$$

where $y \in \Delta$ can depend on $x$ and $\varepsilon > 0$.

This paper defines the set of equilibria that is consistent with a large class of possible mutation functions, $\mathcal{M}$, where

$$\mathcal{M}^n \equiv \Big\{ m\colon \Delta^n \to T\Delta^n \,|\, m \text{ is bounded, continuously differentiable and } (\forall S \subset \{1, \ldots, n\}) \sum_{i \in S} x_i = 1 \Rightarrow \sum_{i \in S} m_i(x) \leq 0 \Big\}.$$

Again, when it does not lead to confusion, the superscript $n$ is dropped. Notice that we can interpret members of $\mathcal{H}$ and $\mathcal{M}$ as vector fields. Then the last constraint implies that at the boundary the vector field points inwards. This class of mutation function includes the mutation functions (1), (2), and seems to be a mild restriction over the mutation functions considered in (5). The stochastic mutation rate functions considered in (3) and (4) are outside the deterministic model considered here.

For an evolutionary game with payoff matrix $A$, the generalized replicator model $H \in \mathcal{H}$, the mutation function $m \in \mathcal{M}$, and a scalar $\mu \in (0, 1)$, define an *evolutionary system* by the following differential equation:

$$\Delta x = (1 - \mu) H(x) + \mu m(x) \text{ (discrete version)}$$

$$\dot{x} = (1 - \mu) H(x) + \mu m(x) \text{ (continuous version)}.$$

We define an evolutionary equilibrium as a stationary point of the evolutionary system for all mutation function in $\mathcal{M}$. Formally, a vector $\bar{x}$ is an *evolutionary equilibrium* for the payoff function $A$ and the generalized replicator model $H \in \mathcal{H}$, if for every function $m$ in $\mathcal{M}$, there is a scalar $\mu' \in (0, 1)$ and a vector valued function $x$: $(0, \mu') \rightarrow \Delta^n$ such that for all $\mu \in (0, \mu')$,

$$(1 - \mu) H(x(\mu)) + \mu m(x(\mu)) = 0$$

and $\lim_{\mu \downarrow 0} x(\mu) = \bar{x}$.[12]

Note that in particular applications the set of mutations may be restricted. In such cases the set of evolutionary equilibria is a superset of the set of evolutionary equilibria when the set of mutation functions is $\mathcal{M}$.

The following theorem by Jiang (1963) (which generalizes the better known theorem by Fort, 1950) is used in the proof of the existence of evolutionary equilibria. Let $X$ be a compact convex subset of a normed space, let $d$ be the metric defined on $X$ defined by the norm, and let $C(X, X)$ be the set of continuous function with domain and range in $X$. Then $(C(X, X), \rho)$ is a metric space where

$$\rho(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Finally, let $F$: $C(X, X) \rightarrow X$ be the fixed point correspondence; i.e., for all $f \in C(X, X)$,

[12] A different stability notion is structural stability (see Hirsch and Smale, 1974, p. 312) which requires that trajectories of the perturbed system to be homeomorphic to the original system. Note that evolutionary equilibria need not be dynamically stable.

$$F(f) = \{x \in X | f(x) = x\}.$$

A set $D$ is said to be *totally disconnected* if all the connected subsets of $D$ are singletons.

THEOREM 1 (Jiang). *Suppose $F(f)$ is a totally disconnected set. Then there is a vector $p$ in $F(f)$ that satisfies the following property. For every neighborhood $U$ of $p$ there is an $\varepsilon > 0$ such that:*

$$g \in C(X, X) \quad \text{and} \quad \rho(f, g) < \varepsilon \Rightarrow F(g) \cap U \neq \emptyset.$$

The vector $p$ described in the theorem is called an essential fixed point.

For all subsets of the strategy set $I \subset \{1, \ldots, n\}$, let $A|_I$ be the matrix $(a_{ij})_{i \in I, j \in I}$. A matrix $A$ is *nondegenerate* if for all $I \subset \{1, \ldots, n\}$ such that $\#(I) \geq 2$, the matrix $A|_I$ is nonsingular.[13]

LEMMA 1. *If $A$ is nondegenerate then $H \in \mathcal{H}_A$* has finitely many stationary points.

*Proof.* Let $A$ be a nondegenerate payoff matrix. Let $S = \{x \in \Delta^n | i \in I \Leftrightarrow x_i > 0\}$ be a face of the simplex. Then $(x_S, 0_{-S}) \in S$ is a stationary point of $R_A^C$ if and only if there is a scalar $\lambda$ such that $A_S x_S = \lambda \mathbf{1}$. Since the matrix $A_S$ is invertible, there is a unique stationary point $(x_S, 0_{-S})$. Furthermore, since the simplex has finitely many faces there are finitely many stationary points of $R_A^C$. By the assumptions on $\mathcal{H}$, this implies that $H$ has finitely many fixed points. ■

PROPOSITION 2. *For all nondegenerate payoff matrices $A$ there is an evolutionary equilibrium.*

*Proof.* Fix a payoff matrix $A$ and a generalized replicator function $H \in \mathcal{H}_A$. We first construct a function, $H'$, whose fixed points correspond to the stationary points of $H$. Then we use Theorem 1 to prove the existence of an essential fixed point. Finally we show that essential fixed points of $H'$ are evolutionary equilibria for $A$ (where the law of motion is $H$). Let

[13] The only property used in the paper is that there are finitely many symmetric equilibria in all the submatrices $A|_S$. I think that a necessary and sufficient condition for the latter property is

$$\forall (i, j) \subset \{1, \ldots, n\}, \qquad a_{ij} = a_{ii} \Rightarrow a_{ji} \neq a_{jj}.$$

Notice that either assumption is much weaker than the Lemke and Howson nondegeneracy condition (for a definition of the Lemke and Howson nondegeneracy condition see van Damme, 1987, p. 52).

$$H'\colon \Delta^n \to \mathbf{R}^n \text{ be defined by } H' \equiv I + H.$$

Notice that fixed points of $H'$ are stationary points for $H$. Unfortunately the function $H'$ does not map its domain, $\Delta^n$, into itself. In order to remedy this problem we extend the function $H'$ to a domain $E\Delta^n$ which is invariant under the extension, $\tilde{H}$. Specifically let

$$M = \max_{x\in\Delta^n, i\in\{1,\ldots,n\}} |H_i'(x)| + 1;$$

$$E\Delta^n = \left\{x \in \mathbf{R}^n \Big| \sum_{i=1}^{n} x_i = 1, \quad \forall i \in \{1, \ldots, n\}\, x_i \in [-M, M]\right\};$$

$$\alpha\colon E\Delta^n \to [0, 1)$$

$$\alpha(x) \equiv \min\{\alpha \in [0, 1) | \alpha/n\mathbf{1} + (1 - \alpha)x \in \Delta^n\};$$

$$\tilde{H}\colon E\Delta^n \to E\Delta^n$$

$$\tilde{H}(x) = H'(\alpha(x)/n\mathbf{1} + (1 - \alpha(x))x).$$

Notice that fixed points of $H'$ are stationary points for $H$ and that all the fixed point of $\tilde{H}$ are in $\Delta^n$ (and thus are fixed points of $H'$).[14] By assumption $A$ is nondegenerate; thus, by Lemma 1 there are finitely many stationary points of $H$ and Theorem 1 is applicable. Let $g = (1 - \mu)\, \tilde{H} + \mu(m + I)$. Then for small enough $\mu$, $g\colon E\Delta^n \to E\Delta^n$ and $\rho(\tilde{H}, g) < \varepsilon$. A fixed point of $g$ corresponds to a stationary point of $(1 - \mu)\, H + \mu m$. Thus the set of perturbations allowed in the theorem includes the ones in the definition of evolutionary equilibrium.

Fix a mutation function $m \in \mathcal{M}$. Since all the conditions are satisfied, we use Theorem 1 to prove the existence of an evolutionary equilibrium. Thus there is an $\bar{x}$ such that for every $\varepsilon > 0$ there is a $\mu_\varepsilon$ and a function

$$x_\varepsilon(\mu)\colon (0, \mu_\varepsilon) \to \Delta^n$$

such that $\forall \mu \in (0, \mu_\varepsilon)$

$$(1 - \mu)\, H(x(\mu)) + \mu m(x_\varepsilon(\mu)) = 0$$

and $\sup_{\mu\in(0,\mu_\varepsilon)} |x_\varepsilon(\mu) - \bar{x}| < \varepsilon$. Then since there are finitely many fixed points of $H$ there is an $\varepsilon' > 0$ such that $B(\bar{x}, \varepsilon') \cap F(H) = \{\bar{x}\}$. For $\mu \in (0, \mu_{\varepsilon'})$ let $x(\mu) = x_{\varepsilon'}(\mu)$. Then $x\colon (0, \mu_{\varepsilon'}) \to \Delta$ is such that

[14] *Proof.* Suppose that $\tilde{H}(x) = x$ and $x \notin \Delta^n$. Since $\tilde{H}\cdot\mathbf{1} = 0$, there exists a strategy $i$ such that $x_i < 0$. Let $y \in \partial\, \Delta_n$ be such that $y = \alpha(x)/n\mathbf{1} + (1 - \alpha(x))x$. Then $y_i = 0$. But this implies that $\tilde{H}_i(x) = H_i'(y) \geq 0$. Contradiction.

$$(1 - \mu)\, H(x(\mu)) + \mu m(x(\mu)) \equiv 0,$$

and $x(\mu) \to \bar{x}$. Therefore, $\bar{x}$ is an evolutionary equilibrium. ■

The next game shows that nondegeneracy is not a necessary condition for the existence of an evolutionary equilibrium. The game $A$, where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is degenerate ($|A_{12}| = 0$) and has $(0, 0, 1)$ as an evolutionary equilibrium. However, some degenerate payoff matrices do not have evolutionary equilibria; the game $B$, where

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

is degenerate since $|B_{12}| = 0$. Let $m$ be such that for all $i$, $m_i(x) = a_i - x_i$, where $\sum_i a_i = 1$ and $a_i > 0$. Then for all positive $\mu$, the only stationary points of the generalized replicator with mutation are points such that $x_2 = a_2$. Thus there are no evolutionary equilibria.

We have shown that if the payoff matrix $A$ is nondegenerate then every law of motion $H \in \mathcal{H}$ has an evolutionary equilibrium. However, it is possible that different laws of motion give different set of evolutionary equilibria. The next proposition proves that all laws of motions $H \in \mathcal{H}_R \subset \mathcal{H}$ have the same set of evolutionary equilibria, where

$$\mathcal{H}_R = \{H: \Delta^n \to T\Delta^n | \exists f: \Delta \to \mathbf{R}_{++} \text{ such that } f \text{ is continuously differentiable and } H = f \cdot R^C\}.$$

THEOREM 2. *Let $H, H' \in \mathcal{H}_R$ and suppose $\bar{x}$ is an evolutionary equilibrium for $H$. Then $\bar{x}$ is an evolutionary equilibrium for $H'$.*

*Proof.* Without loss of generality suppose that $H' = f \cdot H$, where $f$ is strictly positive and continuously differentiable. Fix $m' \in \mathcal{M}$ and let $m = m'/f \in \mathcal{M}$. Since $\bar{x}$ is an evolutionary equilibrium for $\mathcal{H}$ there is a scalar $\mu' \in (0, 1)$ and a vector valued function $x$: $(0, \mu') \to \Delta^n$ such that for all $\mu \in (0, \mu')$,

$$(1 - \mu)\, H(x(\mu)) + \mu m(x(\mu)) = 0$$

and $\lim_{\mu \downarrow 0} x(\mu) = \bar{x}$. Consequently, we have that for all $\mu \in (0, \mu')$,

$$(1 - \mu)\, H'(x(\mu)) + \mu m'(x(\mu)) = f(x)[(1 - \mu)\, H(x(\mu)) + \mu m(x(\mu))] = 0$$

and $\lim_{\mu \downarrow 0} x(\mu) = \bar{x}$. Since $m'$ was arbitrary, $\bar{x}$ is an evolutionary equilibrium for $H'$. ∎

The rest of this section gives a local definition of evolutionary equilibrium which is then shown to be equivalent to the first definition.

For all $O \subset \Delta^n$ let $\mathcal{M}^O$ be the set of mutation functions that are equal to zero outside $O$; i.e.,

$$\mathcal{M}^O = \{m \in \mathcal{M} | m|_{\Delta^n \setminus O} = 0\}.$$

A vector $\bar{x}$ is an *O-evolutionary equilibrium* for the payoff function $A$ and the generalized replicator model $H \in \mathcal{H}$ if for every function $m$ in $\mathcal{M}^O$, there is a scalar $\mu' \in (0, 1)$ and a vector valued function $x$: $(0, \mu') \to \Delta^n$ such that for all $\mu \in (0, \mu')$,

$$(1 - \mu)\, H(x(\mu)) + \mu m(x(\mu)) = 0$$

and $\lim_{\mu \downarrow 0} x(\mu) = \bar{x}$. A vector $\bar{x}$ is a *local evolutionary equilibrium* if there exists a neighborhood $O$ of $\bar{x}$ such that $\bar{x}$ is an $O$-evolutionary equilibrium.

Clearly, an evolutionary equilibrium is a local evolutionary equilibrium. The converse is also true.

THEOREM 3. *A local evolutionary equilibrium is an evolutionary equilibrium.*

*Proof.* Let $\bar{x}$ be a local evolutionary equilibrium and thus an $O$-evolutionary equilibrium. Let $m \in \mathcal{M}$ and let $m^O$ be equal to $m$ on $O$ and identically zero outside $O$. Since $\bar{x}$ is an $O$-evolutionary equilibrium there exists a function $x$: $(0, \mu') \to \Delta^n$ such that $x(\mu) \to \bar{x}$ and for all $\mu \in (0, \mu')$

$$(1 - \mu)\, H(x(\mu)) + \mu m^O(x(\mu)) = 0.$$

Let $\mu'' \in (0, \mu')$ be such that for all $\mu \in (0, \mu'')$, $x(\mu) \in O$. Then for all $\mu \in (0, \mu'')$,

$$(1 - \mu) H(x(\mu)) + \mu m(x(\mu)) = 0.$$

Therefore $\bar{x}$ is an evolutionary equilibrium. ■

## 4. Relationship between Evolutionary Equilibrium and Other Equilibrium Concepts

The rest of the paper relates evolutionary equilibria to other game theoretic equilibria, i.e., equilibrium concepts that are derived from assuming that individuals are Bayesian maximizers and by making assumptions about the individuals' beliefs. There are three reasons for being interested in these relationships: (i) showing that an evolutionary equilibrium corresponds to a game theoretic equilibrium allows us to argue that individuals act "as if" they are Bayesian maximizers; (ii) there are ways of computing game theoretic equilibria that can be used to compute evolutionary equilibria; (iii) evolutionary stability acts as a refinement of the set of Nash equilibria.

The following nine equilibrium concepts are analyzed: Nash equilibrium, undominated Nash equilibrium, perfect equilibrium, strict dominance solvability, regular equilibrium, proper equilibrium, strictly proper equilibrium, ESS, and essential equilibrium.

The main results in this section are: an evolutionary equilibrium is a symmetric perfect equilibrium (Section 4.3); a symmetric regular equilibrium is an evolutionary equilibrium (Section 4.5); an ESS is an evolutionary equilibrium (Section 4.8); not all symmetric proper equilibria are evolutionary equilibria (Section 4.6); not all evolutionary equilibria are strictly perfect equilibria (Section 4.7).

### 4.1. Nash Equilibrium

Let $\bar{x} \in \Delta^n$. A vector of strategies $(\bar{x}, \bar{x})$ is a *symmetric Nash equilibrium* for the symmetric game $(A, A^T)$ if for all strategies $y \in \Delta^n$, $\bar{x} \cdot A\bar{x} \geq y \cdot A\bar{x}$. The concept of Nash equilibrium is the most widely used equilibrium concept in game theory although it is often considered to be too weak (see, however, Bernheim, 1984 and Pearce, 1984).

Proposition 3. *An evolutionary equilibrium, $\bar{x}$, is a symmetric Nash equilibrium $(\bar{x}, \bar{x})$.*

*Proof.* Let $\bar{x}$ be an evolutionary equilibrium. (i) Suppose that there is a strategy, say 1, such that $\bar{x}_1 = 0$. In order to prove that $\bar{x}$ is a Nash equilibrium we need to show that $(A\bar{x})_1 \leq \bar{x}A\bar{x}$. Let $m$ be such that $m_1(x) > 0$ for all $x$ in a neighborhood of $\bar{x}$. Then since $\bar{x}$ is an evolutionary equilibrium there is a $\mu' > 0$ and a function $x$: $(0, \mu') \to \Delta$ such that for every $\mu$ in $(0, \mu')$,

$$(1 - \mu) H(x(\mu)) + \mu m(x) = 0.$$

This implies that for all $\mu$ in $(0, \mu')$,

$$(Ax(\mu))_1 - x(\mu) Ax(\mu) < 0.$$

Thus $(A\bar{x})_1 \leq \bar{x}A\bar{x}$. (ii) Suppose that $\bar{x}_i > 0$ and $\bar{x}_j > 0$. Then, $(A\bar{x})_i - \bar{x}A\bar{x} = 0$ and $(A\bar{x})_j - \bar{x}A\bar{x} = 0$. Thus $(A\bar{x})_i = (A\bar{x})_j$. Therefore $\bar{x}$ is a Nash equilibrium. ■

### 4.2. *Undominated Nash Equilibrium*

In this subsection we show that not all perfect equilibria are evolutionary equilibria and thus that not all Nash equilibria are evolutionary equilibria.

A strategy $i$ is *weakly dominated* if there exists a mixed strategy $y$ such that the payoff for using $y$ is at least as great as the payoff for using $i$ regardless of the other players' strategy and strictly better for some strategy; i.e., for all $x \in \Delta^n$, $y \cdot Ax \geq (Ax)_i$ and there is a $z \in \Delta^n$ such that $y \cdot Az > (Az)_i$.

The principle that strategies that are weakly dominated should not be played is very intuitive, although when used repeatedly it can give results that are surprisingly strong.[15] The next lemma characterizes the relationship between evolutionary equilibria and weakly dominated strategies.

LEMMA 2. *Let $\bar{x}$ be an evolutionary equilibrium. Then $\bar{x}_i = 0$, if strategy $i$ is weakly dominated.*

*Proof.* Suppose that mixed strategy $y$ weakly dominates strategy 1 and suppose, without loss of generality, that $1 \notin$ support $(y)$. Suppose that $\bar{x}$ is an evolutionary equilibrium and $\bar{x}_1 > 0$. Let $m \in \mathcal{M}$ be such that in a neighborhood of $\bar{x}$ (where $x_1 > 0$)

$$m_1(x) \equiv -1, \qquad \text{and for all } i \in \text{support } (y),\ m_i(x) \equiv 1.$$

Then since $\bar{x}$ is an evolutionary equilibrium there exists a constant $\mu'$ and a function $x$: $(0, \mu') \to \Delta$ such that for $\mu \in (0, \mu')$ and for all $i \in$ support $(y)$,

[15] van Damme shows that in a game where player one first decides whether to discard \$1 and then plays a battle of the sexes game, repeated elimination of weakly dominated strategies results to player one getting the highest possible payoff.

$$(1-\mu)H_i(y)+\mu=0, \tag{3}$$

$$(1-\mu)H_1(y)-\mu=0, \tag{4}$$

and $\lim_{\mu\to 0} x(\mu) = \bar{x}$. Condition (3.3) implies that

$$x(\mu)\, Ax(\mu) > (Ax(\mu))_i;$$

condition (3.4) implies that for all $\mu \in (0, \mu')$

$$(Ax(\mu))_1 > x(\mu)\, Ax(\mu).$$

Thus (3.3) and (3.4) combined give

$$(Ax(\mu))_1 > \sum_i y_i(Ax(\mu))_i$$

which contradicts the assumption of weak domination. ■

Thus an evolutionary equilibrium is an undominated Nash equilibrium (a Nash equilibrium where dominated strategies are given zero weight).

*4.3. Perfect Equilibrium*

There are several ways in which perturbations have been introduced in solution concepts. Evolutionary equilibria consider perturbations in the law of motion; essential equilibria (which are analyzed later in this section) consider perturbation in the payoff function; finally, perfect equilibria consider equilibria that are "resistant" to some perturbation of the strategy set.

Let $\mathbf{R}^n_{++} = \{x \in \mathbf{R}^n | (\forall i)\ x_i > 0\}$, $\Delta^n_{++} = \Delta^n \cap \mathbf{R}^n_{++}$. Let $\bar{x} \in \Delta^n$. A vector of strategies $(\bar{x}, \bar{x})$ is a *symmetric perfect equilibrium* if there exist sequences $\{\varepsilon^t\}$ and $\{x^t\}$, where $\varepsilon^t \in \mathbf{R}_{++}$ and $x^t \in \Delta^n_{++}$, such that: (i) for all $t$, $x^t_i > \varepsilon^t$ only if $i \in \operatorname{argmax}_j (Ax^t)_j$; (ii) $\lim_{t\to\infty} \varepsilon^t = 0$; (iii) $\lim_{t\to\infty} x^t = \bar{x}$.

PROPOSITION 4. *An evolutionary equilibrium is a symmetric perfect equilibrium.*

*Proof.* The result follows from Lemma 2 and the result (van Damme (1987), Theorem 3.2.2) that for a two person finite normal game an equilibrium is perfect if and only if every weakly dominated strategy is played with probability 0. ■

The next example shows that not all symmetric perfect equilibria are evolutionary equilibria. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The vector $(\bar{x}, \bar{x})$ where $\bar{x} = (0, 1, 0)$ is a symmetric perfect equilibrium (since strategy 2 is not weakly dominated) but $\bar{x}$ is not an evolutionary equilibrium.[16] Incidentally, all trajectories of the replicator model in continuous time that start in the interior of the strategy simplex converge to (1, 0, 0).

### 4.4. *Strict Dominance Solvability*

The next proposition establishes that an evolutionary equilibrium is resistant to the elimination of dominated strategies. Thus, restricting the replicator model to rationalizable strategies will not reduce the set of evolutionary equilibria.

PROPOSITION 5. *Suppose that strategy j dominates strategy i and suppose that $\bar{x}$ is an evolutionary equilibrium for the payoff matrix A. Let $A_{-i}$ be the payoff matrix where the ith row and column have been deleted. Then for all $H \in \mathcal{H}_R$, $\bar{x}_i$ is an evolutionary equilibrium for the payoff matrix $A_{-i}$.*

*Proof.* Let $m' \in \mathcal{M}^{n-1}$ be a mutation function. Let $m \in \mathcal{M}^n$ be such that

$$m_j(x) = \begin{cases} m_j'(x) & \text{if } j \neq i \\ 0 & \text{if } j = i. \end{cases}$$

Since $\bar{x}$ is an evolutionary equilibrium, then for small enough $\mu' > 0$ there is a function $x$: $(0, \mu') \to \Delta^n$ such that $x(\mu) \to \bar{x}$ and for all $\mu \in (0, \mu')$,

$$(1 - \mu)\, H_A(x(\mu)) + \mu m(x(\mu)) = 0.$$

Since $\bar{x}$ is a Nash equilibrium and since strategy $i$ dominates strategy $j$, then $\bar{x}A\bar{x} \geq (A\bar{x})_j > (A\bar{x})_i$. Then there is a small enough $\mu'' > 0$ such that for all $\mu \in (0, \mu'')$,

[16] *Proof.* Suppose $\bar{x}$ is an evolutionary equilibrium. Let $m \in \mathcal{M}$ be such that $m(x) = (1, -1, 0)$ for every $x$ in a neighborhood of $\bar{x}$ and let $x(\mu)$ be the corresponding sequence of stationary points for the generalized replicator model. Since for all $x$ in a neighborhood of $\bar{x}$, $x \cdot Ax > (Ax)_3$, for small enough $\mu$ we must have $x_3(\mu) = 0$. For such $\mu$, $H_1(x(\mu)) > 0$ and thus $(1 - \mu)\, H_1(x(\mu)) + \mu m_1(x(\mu)) > 0$. Thus, $\bar{x}$ cannot be an evolutionary equilibrium.

$$x(\mu)\, Ax(\mu) > (Ax(\mu))_i.$$

Since $m_i(x) = 0$ then $x_i(\mu) = 0$. Therefore, for all $\mu \in (0, \mu'')$,

$$(1 - \mu)\, H_{A_{-i}}\, (x_{-i}(\mu)) + \mu m'\, (x_{-i}(\mu)) = 0,$$

and $\bar{x}_{-i}$ is an evolutionary equilibrium for $A_{-i}$. ■

An equilibrium is *strictly dominance solvable* if it can be obtained by reducing the game to a single cell by iterated deletion of dominated strategies.

PROPOSITION 6. *A strictly dominance solvable equilibrium is an evolutionary equilibrium.*

*Proof.* Suppose that $\bar{x}$ is an evolutionary equilibrium, strategy 1 dominates strategy 2 in the normal game $A_{\{1,\ldots,n\}-\{4\}}$, and strategy 3 dominates strategy 4 in the game $A$. By Lemma 2, $\bar{x}_4 = 0$. Suppose $\bar{x}_2 > 0$. Choose the function $m$ such that for every $x$ in a neighborhood of $\bar{x}$, $m_1(x) = 1$, $m_2(x) = -1$. Then for small enough $\mu$, $(Ax(\mu))_2 > (Ax(\mu))_1$. But this is impossible since $x_4(\mu) \to 0$. Thus $\bar{x}_2 = 0$. ■

*4.5. Regular Equilibrium*

The concept of regular equilibrium was introduced by Harsanyi (1973). The following description of the equilibrium is based on van Damme (1987) and is simplified to two-person symmetric games. Let $z = (x, y)$ be a vector of strategies for the game $(A, A^T)$. Let $k \in \text{supp}(x)$, let $l \in \text{supp}(y)$ and let $m = (k, l)$. Let $F(x|k)$ be such that

$$(\forall i \neq k)\, F_i(x|k) = x_i[(Ay)_i - (Ay)_k] \qquad \text{and} \qquad F_k(x|k) = \sum_{i=1}^{n} x_i - 1.$$

Similarly, let $F(y|l)$ be such that

$$(\forall i \neq l)\, G_i(y|l) = y_i[(Ax)_i - (Ax)_l] \qquad \text{and} \qquad G_l(y|l) = \sum_{i=1}^{n} y_i - 1.$$

Finally, let

$$H(z|m) = (F(x|k), G(y|l))^T \qquad \text{and} \qquad J(\bar{z}|m) = \frac{\partial H(z|m)}{\partial z}\Big|_{z=\bar{z}}.$$

A vector $\bar{z}$ is a *regular equilibrium* if for some $m \in \text{supp}(x) \times \text{supp}(y)$, $H(z|m) = 0$ and $\det J(z|m) \neq 0$.

Intuitively, a regular equilibrium is one for which the best response mapping is continuously differentiable at a neighborhood of the Nash equilibrium.

PROPOSITION 7. *A symmetric regular equilibrium is an evolutionary equilibrium for the law of motion $H \in \mathcal{H}_R$.*

*Proof.* Without loss of generality, suppose $H = R^C$. Theorem 9.4.3 in van Damme (1987), states that a Nash equilibrium $(\bar{x}, \bar{x})$ is regular if and only if $dR^C/dx|_{x=\bar{x}}$ is nonsingular. Notice that if $dR^C\backslash dx|_{x=\bar{x}}$ is nonsingular and $\mu$ is small enough then

$$\frac{d}{dx}[(1-\mu)\,L(R^C(x)) + \mu m(x)]|_{\mu=0,x=\bar{x}}$$

is nonsingular. Therefore, if $(\bar{x}, \bar{x})$ is a regular equilibrium then by the implicit function theorem, $\bar{x}$ is an evolutionary equilibrium. ■

The next example shows that not all games with nondegenerate payoff matrices $A$ have a regular equilibrium.

Let

$$(A, A^T) = \begin{pmatrix} 2,2 & 2,2 \\ 2,2 & 1,1 \end{pmatrix}.$$

Clearly the matrix is nondegenerate and the only perfect equilibrium is "top," "left." The Jacobian of the best response function (as defined by Harsanyi, 1973) at the equilibrium point is

$$\det\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 0$$

and thus the game has no regular equilibria. It is clear that all the trajectories of the replicator model that start in the interior of the simplex converge to (1, 0).

### 4.6. *Proper Equilibrium*

In this subsection we show that not all proper equilibria are evolutionary equilibria.

Let $\bar{x} \in \Delta^n$. A vector of strategies $(\bar{x}, \bar{x})$ is a *symmetric proper equilibrium*

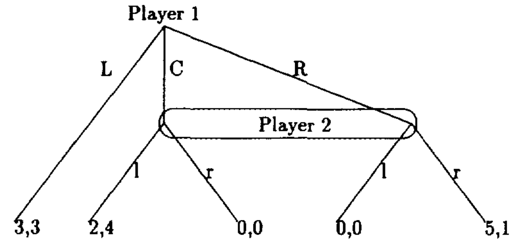


FIG. 1. Game with a proper equilibrium which is eliminated by forward induction.

if there exist sequences $\{\varepsilon^t\}$ and $\{x^t\}$, where $\varepsilon^t \in \mathbf{R}_{++}$ and $x^t \in \Delta^n_{++}$, such that: (i) for all $t$, $x_i^t \leq \varepsilon^t x_j^t$ if $(Ax^t)_i < (Ax^t)_j$; (ii) $\lim_{t\to\infty} \varepsilon^t = 0$; and (iii) $\lim_{t\to\infty} x^t = \bar{x}$.

The game in Fig. 1 is used by Tan and Werlang (1988), to show the insufficiency of the concept of proper equilibrium. There are two proper equilibria in the game: Rr and Ll. By a forward induction argument Tan and Werlang argue that since C is dominated by L, C should never be played and therefore l should never be employed.[17] Therefore properness allows unreasonable equilibria, such as Ll.

Are Rr and Ll evolutionary equilibria for the law of motion $R^C$? We can construct a symmetric game by assuming that two individuals are randomly assigned to the roles of player 1 and player 2. Figure 2 shows the extensive form for such a game.

An evolutionary game is constructed by normalizing the symmetric

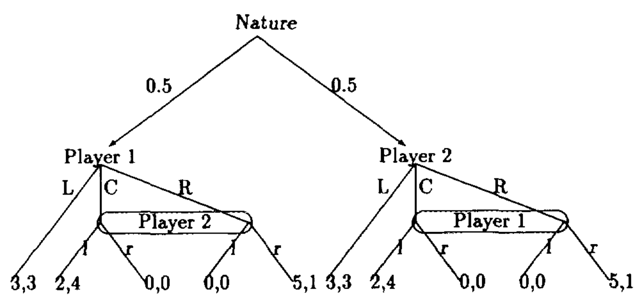


FIG. 2. Symmetrization of extensive game in Fig. 1.

[17] Ll corresponds to the equilibrium where player 2 warns player 1 that he will play l. Then player 1 has the choice of playing L and receiving 3, playing C and receiving 2, and playing R and receiving 0. If player 2 gets to move he realizes that player 1 did not believe in his bluff. Then player 2 is better off not to follow with his threat and play r.

| | | | | | | |
|---|---|---|---|---|---|---|
| Rr | 6 | 1 | 5 | 0 | 8 | 3 |
| Rl | 5 | 0 | 9 | 4 | 8 | 3 |
| Cr | 1 | 3 | 0 | 2 | 3 | 5 |
| Cl | 0 | 2 | 4 | 6 | 3 | 5 |
| Lr | 4 | 3 | 3 | 3 | 6 | 6 |
| Ll | 3 | 7 | 7 | 7 | 6 | 6 |

FIG. 3. Symmetrization of the game in Fig. 2.

extensive game. Figure 3 shows the symmetrization of the game in Fig. 2. Notice that the matrix is degenerate. The strategy Rr is clearly a strict Nash equilibrium and thus a regular equilibrium (van Damme, 1987, Theorem 2.3.3) and thus an evolutionary equilibrium (Proposition 7).

Notice that $\bar{x} = (0, 0, 0, 0, 0, 1)$ is a symmetric proper equilibrium.[18] Suppose that $\bar{x}$ is an evolutionary equilibrium. Then for all $x$ in a neighborhood of $\bar{x}$ let

$$m(x) = (1, 0, 0, 0, 1, -2).$$

Choose $\mu$ small enough so that $x(\mu)\ Ax(\mu) > 5.5$ and $x_6 > 0.9$. Then $x_2(\mu) = x_3(\mu) = x_4(\mu) = 0$. The assumptions on $m$ also give that $x_1(\mu) > 0$ and $(Ax)_6 > xAx > (Ax)_5$ which is impossible given the payoff function. Thus $\bar{x}$ is not an evolutionary equilibrium.[19]

### 4.7. *Strictly Perfect Equilibrium*

A Nash equilibrium is strictly perfect if it is resistant to all perturbations of the strategy set. Formally, $(\bar{x}, \bar{x})$ is a *symmetric strictly perfect equilibrium* if there exists a vector $\eta \in \mathbf{R}^n_{++}$ such that for all sequences $\{\eta^t\}$, where $\eta^t_i \in (0, \eta_i)$, there exists a sequence $\{x^t\}$, where $x^t \in \Delta^n$, such that: (i) for all $i$ and $t$, $x^t_i \geq \eta^t_i$; (ii) $x^t_i \geq \eta^t_i$ implies that $i \in \text{argmax}_j(Ax^t)_j$; (iii) $\lim_{t\to\infty} \eta^t = 0$; (iv) $\lim_{t\to\infty} x^t = \bar{x}$.

The concept of strict perfect equilibrium resembles the concept of evolutionary equilibrium but as the following example illustrates not every nondegenerate game has a symmetric strictly perfect equilibrium.

Let

[18] Just set $x_i = \varepsilon^2/(1 + \varepsilon + 4\varepsilon^2)$ if $i \neq 5, 6$, $x_5 = \varepsilon/(1 + \varepsilon + 4\varepsilon^2)$, and $x_6 = 1/(1 + \varepsilon + 4\varepsilon^2)$.

[19] Selten (1983) extends the concept of ESS to extensive games. The first extension is called direct ESS. In this game the only direct ESS is *Rr*. (The intuition for this is that the strategy *Lr* generates the same payoffs as the strategy *Ll*.) The second extension is called limit ESS. In this game both *Ll* and *Rr* are direct ESSs. (The intuition for this is as follows: if the population adopts *Ll*, the mutants that adopt *Rr* get a strictly lower payoff when there are trembles.)

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then the game $(A, A^T)$ has no symmetric strictly perfect equilibrium but has a unique evolutionary equilibrium $\bar{x} = (1, 0, 0)$.[20] Similarly, all orbits of the replicator model in continuous time that start in the interior of the simplex converge to (1, 0, 0).

*4.8. ESS*

The most widely used equilibrium concept in evolutionary game theory is the concept of evolutionary stable strategy (ESS). While evolutionary equilibria consider dynamic perturbations, ESS considers stable perturbations. A strategy $\bar{x} \in \Delta^n$ is an ESS if for any other strategy $y \in \Delta^n - \{\bar{x}\}$ there is an $\varepsilon'$ such that for all $\varepsilon \in (0, \varepsilon')$

$$\bar{x} \cdot A(\varepsilon y + (1 - \varepsilon)\bar{x}) > y \cdot A(\varepsilon y + (1 - \varepsilon)\bar{x}).$$

Thus contrary to the notion of evolutionary equilibrium, ESS considers mutation in a static framework. This condition can be rewritten in the following way. Strategy $\bar{x}$ is an ESS if for all strategies $y$ different than $\bar{x}$, one of the two conditions holds

(i) $\bar{x} \cdot A\bar{x} > y \cdot A\bar{x}$

[20] *Proof.* The vector $(\bar{x}, \bar{x})$ is the unique symmetric perfect equilibrium, $A$ is nondegenerate and thus $\bar{x}$ is an evolutionary equilibrium. Consider the perturbation

$$\frac{1}{2\varepsilon + \varepsilon^2}(\varepsilon, \varepsilon^2, \varepsilon).$$

Suppose $x(\varepsilon) \rightarrow (1, 0, 0)$. Then

$$x_1(\varepsilon) + x_2(\varepsilon) \geq x_1(\varepsilon) + x_3(\varepsilon);$$

i.e., $x_2(\varepsilon) \geq x_3(\varepsilon)$. This implies that

$$x_1(\varepsilon) + x_3(\varepsilon) \geq x_1(\varepsilon) + x_2(\varepsilon),$$

or $x_2(\varepsilon) = x_3(\varepsilon)$. But this is possible only if

$$x_1(\varepsilon) - x_3(\varepsilon) \geq x_1(\varepsilon) + x_3(\varepsilon).$$

Contradiction.

(ii) $\bar{x} \cdot A\bar{x} = y \cdot A\bar{x}$ and $\bar{x} \cdot Ay > yAy$.

The next proposition relates ESS to the replicator model in continuous time.

PROPOSITION 8 (Zeeman 1980). *An ESS as an asymptotically stable stationary point of the replicator model* $R^C$.[21]

Unfortunately the requirements of ESS and asymptotically stable stationary points seem too strict as demonstrated by the following example. Let

$$A = \begin{pmatrix} \varepsilon & 1 & -1 \\ -1 & \varepsilon & 1 \\ 1 & -1 & \varepsilon \end{pmatrix},$$

where $\varepsilon \in (0, \frac{1}{3})$. The only Nash equilibrium is $\bar{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The Hessian of the law of motion is positive definite, therefore $\bar{x}$ is not asymptotically stable and hence $\bar{x}$ is not an ESS. Furthermore, since $\bar{x}$ is the only Nash equilibrium, the game has no asymptotically stable stationary points and thus no ESS. Finally, since the determinant of the Hessian is nonzero, $\bar{x}$ is a regular equilibrium and thus an evolutionary equilibrium.

Suppose that $\varepsilon < 0$. Then $\bar{x}$ is an ESS and is thus an asymptotically stable stationary point for the replicator model. The replicator model in discrete time is not stable at $\bar{x}$ since one of the eigenvalues of the linearized system is greater than one. Thus ESS are not necessarily asymptotically stable points of the replicator model in discrete time.

PROPOSITION 9. *A hyperbolic stationary point of the replicator model in continuous time is an evolutionary equilibrium for the law of motion* $R^C$.

*Proof.* A stationary point is *hyperbolic* if none of the eigenvalues of $dR^C/dx$ have zero real parts. Thus the proposition follows from the implicit function theorem. ∎

[21] The proposition does not hold for $R^D$. Let

$$A = \begin{pmatrix} 0 & 1+\varepsilon & -1 \\ -1 & 0 & 1+\varepsilon \\ 1+\varepsilon & -1 & 0 \end{pmatrix},$$

where $\varepsilon > -1$. Then $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is an ESS but not an asymptotically stable equilibrium for $R^D$.

PROPOSITION 10. *An ESS is an evolutionary equilibrium for the law of motion $H \in \mathcal{H}_R$.*

*Proof.* Without loss of generality take $H = R^C$. Let $\bar{x}$ be an ESS. In the proofs of Theorem 9.2.8, 9.4.8 van Damme (1987) shows that there is an open ball $U$ centered at $\bar{x}$ such that the function

$$V: U \to \mathbf{R}$$
$$V(x) \equiv \prod_i x_i^{\bar{x}_i}$$

is a Lyapunov function and such that $\bar{x}$ is the only fixed point of $R^C$ in $U$. Take $c$ to be large enough so that $V^{-1}(c) \subset U$. Let $X$ be the solution of the differential equation $\dot{x} = R^C$. For $x \in U$, let $F(x) \equiv X(1, x)$. Then $F$ is continuous and maps $V^{-1}(c)$ into $V^{-1}(c)$. Then by an argument similar to the one in Proposition 2 one can show that $\bar{x}$ is an evolutionary equilibrium for $R^C$. ∎

Finally notice that not all ESS are regular equilibria. For example, the game matrix discussed in Section 4.5 has no regular equilibria but has (1, 0) as the unique ESS.

### 4.9. *Essential Equilibrium*

A Nash equilibrium $(x, y)$ is essential[22] for a game $(A, B)$ if for an arbitrarily small perturbation of the payoff matrix $(A', B')$ there is a Nash equilibrium to $(A', B')$ close to $(x, y)$. This notion predates the concepts of hyperstable equilibrium introduced by Kohlberg and Mertens (1986).[23]

A symmetric Nash equilibrium $(\bar{x}, \bar{x})$ for the game $(A, A^T)$ is *symmetric essential* if for any symmetric game with payoffs close enough to $A$ there is a symmetric Nash equilibrium close to $(\bar{x}, \bar{x})$. The next propositions characterize the set of symmetric essential equilibria.

PROPOSITION 11 (Bomze). *A regular equilibrium is an essential equilibrium.*

PROPOSITION 12 (Bomze). *An ESS is a symmetric essential equilibrium.*

PROPOSITION 13. *Restrict the set of mutations to functions of the form*

[22] The equilibrium concept is defined by Wu and Jiang (1962).

[23] A subset, $H$, of the set of Nash equilibria for the game $(A, B)$ is *hyperstable* if it is minimal according to the following condition: Given any small perturbation of the payoff matrix, $(A', B')$, there is a Nash equilibrium to $(A', B')$, $(x', y')$, close to the set $H$.

$$m_i(x) = x_i[(Cx)_i - xCx].$$

*Then a symmetric essential equilibrium is an evolutionary equilibrium.*

*Proof.* Suppose $\bar{x}$ is a symmetric essential equilibrium of the game $(A, A^T)$. Let $m_i(x) = x_i[(Cx)_i + xCx]$ and let $A_\mu = (1 - \mu) A + \mu C$. Then

$$\begin{aligned}\dot{x}(\mu) &= (1 - \mu) x_i(\mu) [(Ax(\mu))_i - x(\mu) Ax(\mu)] \\ &\quad + \mu x_i(\mu) [(Cx(\mu))_i - x(\mu) Cx(\mu)] \\ &= x_i(\mu) [(A_\mu x(\mu))_i - x(\mu) A_\mu x(\mu)] = 0\end{aligned}$$

if $(x(\mu), x(\mu))$ is a Nash equilibrium of the game $(A_\mu, A_\mu^T)$. But since $\bar{x}$ is a symmetric essential equilibrium then for any perturbation of the payoff there is a Nash equilibrium arbitrarily close to $\bar{x}$. Therefore $\bar{x}$ is an evolutionary equilibrium. ■

## 5. Conclusion

The paper establishes the relationship between evolutionary notions of equilibrium and game theoretic notions of equilibrium. In the evolutionary model: (i) the proportion of the population that adopts a strategy is dependent on the distribution of strategies in the population and (ii) mutation affects the dynamics. An evolutionary equilibrium is a mix of strategies which is a stationary point for the law of motion of the replicator model with arbitrarily small levels of mutation. We show that an evolutionary equilibrium exists for all nondegenerate payoff matrices and that evolutionary selection, as described by the concept of "evolutionary equilibrium," can result in elimination of some unintuitive equilibria.

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