CONTROLLED RANDOM WALKS

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1. *Introduction.* Let $M = ||m_{ij}||$ be an $r \times s$ matrix whose elements m_{ij} are probability distributions on the Borel sets of a closed bounded convex subset X of k-space. We associate with M a game between two players, I and II, with the following infinite sequence of moves, where $n = 0, 1, 2, \ldots$:

Move $4n + 1$: I selects $i = 1, \ldots, r$.
Move $4n + 2$: II selects $j = 1, \ldots, s$ not knowing the choice of I at move $4n + 1$.
Move $4n + 3$: a point x is selected according to the distribution m_{ij}.
Move $4n + 4$: x is announced to I and II.

Thus, a mixed strategy for I is a function f, defined for all finite sequences $a = (a_1, \ldots, a_n)$ with $a_k \in X$, $n = 0, 1, 2, \ldots$, with values in the set P_r of r-vectors $p = (p_1, \ldots, p_r)$, $p_i \geqq 0$, $\sum p_i = 1$: the ith coordinate of $f(a_1, \ldots, a_n)$ specifies the probability of selecting i at move $4n + 1$ when $a_1, \ldots, a_n$ are the X-points produced during the first $4n$ moves. A strategy g for II is similar, except that its values are in P_s. For a given pair f, g of strategies, the X-points produced are a sequence of random vectors $x_1, x_2, \ldots$, such that the conditional distribution of x_{n+1} given $x_1, \ldots, x_n$ is $\sum_{i,j} f_i(x_1, \ldots, x_n)\, m_{ij} g_j(x_1, \ldots, x_n)$, where f_i, g_j are the ith and jth coordinates of f, g.

The problem to be considered in this paper is the following: To what extent can a given player control the limiting behavior of the random variables $\bar{x}_n = (x_1 + \ldots + x_n)/n$? For a given closed nonempty subset S of X, we shall denote by $H(f,g)$ the probability that $\bar{x}_n$ approaches S as $n \to \infty$, i.e., the distance from the point $\bar{x}_n$ to the set S approaches zero, where $x_1, x_2, \ldots$ is the sequence of random variables determined by f, g. We shall say that S is *approachable by* I *with* f^* (II *with* g^*) if $H(f^*,g) = 1$ $(H(f, g^*) = 1)$ for all $g(f)$, and shall say that S is *approachable by* I (II) if there is an $f(g)$ such that S is approachable by I with f (II with g). We shall say that S is *excludable by* I *with* f if there is a closed T disjoint from S which is approachable by I with f. *Excludability by* II *with* g, *excludability by* I, and *excludability by* II are defined in the obvious way.

It is clear that no S can be simultaneously approachable by I and excludable by II. The main result to be described below is that every convex S is either approachable by I or excludable by II; a fairly simple necessary and sufficient condition for a convex S to be approachable by I is given, a specific f which achieves approachability is described, and an application is given. Finally, an example of a (necessarily nonconvex) S which is neither approachable by I nor excludable by II is given, and some unsolved problems are mentioned.

2. *The main result.* For any $p \in P_r (q \in P_s)$ denote by $R(p)$ $(T(q))$ the convex hull of the $s(r)$ points $\sum_i p_i \overline{m}_{ij}$, $j = 1, \ldots, s$ $(\sum_j \overline{m}_{ij} q_j, i = 1, \ldots, r)$ where $\overline{m}_{ij}$ is the mean of the distribution m_{ij}. By selecting i with distribution q at a given stage, I forces the mean of the vector x selected at that stage into $R(p)$, and no further control over the mean of x is possible. It is intuitively plausible, and true, that $R(p)$ $(T(q))$ is approachable by I (II) with $f \equiv p$ $(g \equiv q)$. Thus, unless S intersects every $T(q)$, it is excludable by II and hence not approachable by I. It turns out that any convex S which intersects every $T(q)$ is approachable by I; a more complete statement is

Theorem 1. *For any closed convex S, the following conditions are equivalent:*

(a) *S is approachable by* I.

(b) *S intersects every $T(q)$.*

(c) *For every supporting hyperplane H of S, there is a p such that $R(p)$ and S are on the same side of H.*

If S is approachable by I, it is approachable by I with f defined as follows. For any $a = (a_1, \ldots, a_n)$ for which $\bar{a} = (a_1 + \ldots + a_n)/n \in S$, $f(a)$ is arbitrary. If $\bar{a} \notin S$, $f(a)$ is *any $p \in P_r$ such that $R(p)$ and S are on the same side of H, where H is the supporting hyperplane of S through the closest point s_0 of S to $\bar{a}$ and perpendicular to the line segment joining $\bar{a}$ and s_2.*

Theorem 1 is proved in [1]; equivalence of (b) and (c) is an immediate consequence of the von Neumann minimax theorem [2], while the proof of the rest of the theorem is complicated in detail, though the main idea is simple.

3. *An application.* As an application of Theorem 1, we deduce a result of Hannan and Gaddum. This result concerns the repeated playing of a zero-sum two person game with $r \times s$ payoff matrix $A = ||a_{ij}||$. If the game is to be played N times (N large), and I knows in advance that the number of times II will choose j is Nq_j, $j = 1, \ldots, s$, he can achieve the average amount $h(q) = \max_i \sum a_{ij} q_j$. Hannan and Gaddum show that, without knowing q in advance I can play so that, for any q, I's averge income is almost $h(q)$; in our terminology, this result is the following:

Let M be the $r \times s$ matrix with $m_{ij} = (\delta_j, a_{ij})$, where δ_j is the jth unit vector in s-space. The set S consisting of all (q, y) such that $y \geqq h(q)$ is approachable by I.

This follows immediately from condition (b) of Theorem 1, for $T(q)$ is the convex hull of the r points $(q, \sum a_{ij}q_j)$, and one of these is the point $(q, h(q))$, so that $T(q)$ intersects S.

4. *An example.* If $k = 1$, every closed S is either approachable by I or excludable by II. For $k = 2$, there are sets which are neither; an example is:

$$M = \left\| \begin{matrix} (0, 0) & (0, 0) \\ \\ (1, 0) & (1, 1) \end{matrix} \right\|,$$

$S = A\ B$, where A is the line segment joining $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{4})$ and B is the line segment joining $(1, \frac{1}{2})$ and $(1,1)$. The strategy g with $g(a_1, \ldots, a_n) = 1$ for $u_{2n} \leqq n < u_{2n+1}$, $g = 2$ otherwise, where $\{u_n\}$ is a sequence of integers becoming infinite so fast that $(u_1 + \ldots + u_n)/u_{n+1} \to 0$ forces $\bar{x}_n$ to oscillate between the lines $y = 0$ and $y = x$, so that $\bar{x}_n$ cannot converge to S, and S is not approachable by I. On the other hand, I can force $\bar{x}_n$ to come arbitrarily near S infinitely often as follows. By choosing 2 successively a number of times large in comparison with the number of previous trials, I forces an $\bar{x}_n$ near $(1, a)$ for some a, $0 \leqq a \leqq 1$. If $a \geqq \frac{1}{2}$, $\bar{x}_n$ is near S; if $a < \frac{1}{2}$, by choosing 1 n times in succession, I forces $\bar{x}_{2n}$ to be approximately $\left(\frac{1}{2}, \frac{a}{2}\right)$, which is in S. Thus S is neither approachable by I nor excludable by II.

5. *Some unsolved problems.*

A. Find a necessary and sufficient condition for approachability. This problem has not been solved even for the example of section 4.

B. Call a closed S *weakly approachable* by I if there is a sequence of strategies f_n such that for every $\varepsilon > 0$,

$$\sup_g \text{Prob}\ \{\varrho(\bar{x}_n(f_n, g), S) > \varepsilon\} \to 0$$

as $n \to \infty$, where $\varrho(x, S)$ is the distance from x to S.

Define weak approachability by II similarly, and call S *weakly excludable* by II if there is a closed T disjoint from S which is weakly approachable by II. Is every S either weakly approachable by I or weakly excludable by II? For the example of section 4, the answer is yes.

C. Does the class of (weakly) approachable sets for a given M depend only on the matrix of mean values of M?

References.

[1] David Blackwell, "An analog of the minimax theorem for vector payoffs," to appear in the *Pacific Journal of Mathematics*.

[2] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton, 1944.

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