AN ANALOG OF THE MINIMAX THEOREM FOR VECTOR PAYOFFS

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1. Introduction. The von Neumann minimax theorem [2] for finite games asserts that for every $r\times s$ matrix $M=\|m(i, j)\|$ with real elements there exist a number v and vectors

$$p=(p_1, \cdots, p_r),\ q=(q_1, \cdots, q_s), \quad p_i,\ q_j\geqq 0,\ \sum p_i=\sum q_j=1$$

such that

$$\sum_i p_i m(i, j)\geqq v\geqq \sum_j q_j m(i, j)$$

for all i, j. Thus in the (two-person, zero-sum) game with matrix M, player I has a strategy insuring an expected gain of at least v, and player II has a strategy insuring an expected loss of at most v. An alternative statement, which follows from the von Neumann theorem and an appropriate law of large numbers is that, for any $\varepsilon>0$, I can, in a long series of plays of the game with matrix M, guarantee, with probability approaching 1 as the number of plays becomes infinite, that his average actual gain per play exceeds $v-\varepsilon$ and that II can similarly restrict his average actual loss to $v+\varepsilon$. These facts are assertions about the extent to which each player can control the center of gravity of the actual payoffs in a long series of plays. In this paper we investigate the extent to which this center of gravity can be controlled by the players for the case of matrices M whose elements $m(i, j)$ are points of N-space. Roughly, we seek to answer the following question. Given a matrix M and a set S in N-space, can I guarantee that the center of gravity of the payoffs in a long series of plays is in or arbitrarily near S, with probability approaching 1 as the number of plays becomes infinite? The question is formulated more precisely below, and a complete solution is given in two cases: the case $N=1$ and the case of convex S.

Let

$$M=\|m(i, j)\|, \qquad 1\leqq i\leqq r,\ 1\leqq j\leqq s$$

be an $r\times s$ matrix, each element of which is a probability distribution over a closed bounded convex set X in Euclidean N-space. By a *strategy for Player* I is meant a sequence $f=\{f_n\}$, $n=0, 1, 2, \cdots$ of functions, where f_n is defined on the set of n-tuples $(x_1, \cdots, x_n)$, $x_i\in X$

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and has values in the set P of vectors $p=(p_1, \cdots, p_r)$ with $p_i \geqq 0$, $\sum_1^r p_i = 1$; f_0 is simply a point in P. A strategy $g=\{g_n\}$ for Player II is defined similarly, except that the values of g_n are in the set Q of vectors $q=(q_1, \cdots, q_s)$ with $q_j \geqq 0$, $\sum_1^s q_j = 1$. The interpretation is that I, II select i, j according to the distributions f_0, g_0 respectively, and a point $x_1 \in X$ is selected according to the distribution $m(i, j)$. The players are told x_1, after which they again select i, j, this time according to the distributions $f_1(x_1)$, $g_1(x_1)$, a point x_2 is chosen according to the $m(i, j)$ corresponding to their second choices, they are told x_2 and select a third i, j according to $f_2(x_1, x_2)$, $g_2(x_1, x_2)$, etc. Thus each pair (f, g) of strategies, together with M, determines a sequence of (vector-valued) random variables $x_1, x_2, \cdots\cdot$.

Let S be any set in N-space. We shall say that S is *approachable with f^* in M*, if for every $\varepsilon > 0$ there is an N_0 such that, for every g,

$$\text{Prob}\,\{\delta_n \geqq \varepsilon \text{ for some } n \geqq N_0\} < \varepsilon\,,$$

where δ_n denotes the distance of the point $\sum_1^n x_i/n$ from S and x_1, x_2, $\cdots$ are the variables determined by f^*, g. We shall say that S is *excludable with g^* in M*, if there exists $d > 0$ such that for every $\varepsilon > 0$ there is an N_0 such that, for every f,

$$\text{Prob}\,\{\delta_n \geqq d \text{ for all } n \geqq N_0\} > 1 - \varepsilon,$$

where x_1, x_2, $\cdots$ are the variables determined by f, g^*. We shall say that S is *approachable* (*excludable*) *in M*, if there exists f^* (g^*) such that S is approachable with f^* (excludable with g^*). Approachability and excludability are clearly the same for S and its closure, so that we may suppose S closed.

In terms of these concepts, von Neumann's theorem has the following analog.

For $N=1$, associated with every M are a number v and vectors $p \in P$, $q \in Q$ such that the set $S=\{x \geqq t\}$ is approachable for $t \leqq v$ with $f : f_n \equiv p$ *and excludable for $t > v$ with $g : g_n \equiv q$.*

A slightly more complete result for $N=1$, characterizing all approachable and excludable sets S for a given M, is given in § 4 below.

Obviously any superset of an approachable set is approachable, any subset of an excludable set is excludable, and no set is both approachable and excludable. Another obvious fact which will be useful is that if a closed set S is approachable in the $s \times r$ matrix M', the transpose of M, then any closed set T not intersecting S is excludable in M with any strategy with which S is approachable in M'. Thus any sufficient condition for approachability yields immediately a sufficient condition for excludability. A sufficient condition for approachability is given in § 2.

It turns out that every convex S satisfies either this condition for approachability or the corresponding condition for excludability, enabling us to give in § 3 a complete solution for convex S. For non-convex S, the problem is not solved except for $N=1$. An example of a set which is neither approachable nor excludable in a given M is given in § 5, the concepts of weak approachability and excludability are introduced, and it is conjectured that every set is either weakly approachable or weakly excludable.

2. A sufficient condition for approachability. If x, y are distinct points in N-space, H is the hyperplane through y perpendicular to the line segment xy, and z is any point on H or on the opposite side of H from x, then all points interior to the line segment xz and sufficiently near x are closer to y than is x. This fact is the basis for our sufficient condition for approachability.

For any matrix M, denote by $\overline{M}$ the matrix whose elements $\overline{m}(i, j)$ are the mean values of the distributions $m(i, j)$. For any $p \in P$ denote by $R(p)$ the convex hull of the s points $\sum_i p_i \overline{m}(i, j)$. The sufficient condition for approachability is given in the following theorem.

THEOREM 1. *Let S be any closed set. If for every $x \notin S$ there is a p $(=p(x)) \in P$ such that the hyperplane through y, the closest point in S to x, perpendicular to the line segment xy separates x from $R(p)$, then S is approachable with the strategy $f: f_n$, where*

$$f_n = \begin{cases} p(\bar{x}_n) \text{ if } n > 0 \text{ and } \bar{x}_n = \left(\frac{1}{n} \sum_1^n x_i\right) \notin S \\ \text{arbitrary if } n = 0 \text{ or } \bar{x}_n \in S. \end{cases}$$

Proof. Suppose the hypotheses satisfied, let I use the specified strategy, let II use any strategy, and let $x_1, x_2, \cdots$ be the resulting sequence of chance variables. For

$$\bar{x}_n = \left(\frac{1}{n} \sum_1^n x_i\right) \notin S ,$$

let y_n be the point of S closest to $\bar{x}_n$, and write $u_n = y_n - \bar{x}_n$. Then, for $\bar{x}_n \notin S$,

$$E((u_n, x_{n+1}) | x_1, \cdots, x_n) \geqq (u_n, y_n), \tag{1}$$

where $E(x|y)$ denotes the conditional expectation of x given y and (u, v) denotes the inner product of the vectors u and v.

Let δ_n denote the squared distance from $\bar{x}_n$ to S. If $\delta_n > 0$, then

$$\delta_{n+1} \leqq |\bar{x}_{n+1} - y_n|^2 = |\bar{x}_n - y_n|^2 + 2(\bar{x}_n - y_n, \bar{x}_{n+1} - \bar{x}_n) + |\bar{x}_{n+1} - \bar{x}_n|^2. \tag{2}$$

Since $\bar{x}_{n+1}-\bar{x}_n=(x_{n+1}-\bar{x}_n)/(n+1)$, we have

$$(3)\qquad (\bar{x}_n-y_n,\ \bar{x}_{n+1}-\bar{x}_n)=\frac{(\bar{x}_n-y_n,\ x_{n+1}-y_n)}{n+1}+\frac{(\bar{x}_n-y_n,\ y_n-\bar{x}_n)}{n+1}$$

and

$$(4)\qquad |\bar{x}_{n+1}-\bar{x}_n|^2\leqq c/(n+1)^2\ ,$$

where c depends only on the size of the bounded set X. From (2), using (1), (3), and (4), we obtain, replacing n by $n-1$,

$$(5)\qquad E(\delta_n|\delta_1,\ \cdots,\ \delta_{n-1})\leqq\left(1-\frac{2}{n}\right)\delta_{n-1}+\frac{c}{n^2}\qquad \text{if } \delta_{n-1}>0.$$

Moreover

$$(6)\qquad 0\leqq\delta_n\leqq a$$

and

$$(7)\qquad |\delta_n-\delta_{n-1}|\leqq\frac{b}{n}\ .$$

Thus it remains only to establish the following.

LEMMA. *A sequence of chance variables* $\delta_1,\ \delta_2,\ \cdots$ *satisfying* (5), (6), *and* (7) *converges to zero with probability* 1 *at a rate depending only on* a, b, c, *that is, for every* $\varepsilon>0$ *there is an* N_0 *depending only on* ε, a, b, c *such that for auy* $\{\delta_n\}$ *satisfying* (5), (6), *and* (7), *we have*

$$\text{Prob}\ \{\delta_n\geqq\varepsilon \text{ for some } n\geqq N_0\}<\varepsilon.$$

Proof of Lemma. Let n_0 be any integer. There exists $n_1>n_0$, depending only on n_0, ε, a, c such that

$$\text{Prob}\ \{\delta_n\geqq\varepsilon/2 \text{ for } n_0\leqq n\leqq n_1\}<\varepsilon/2.$$

To see this, define, for $n\geqq n_0$, $\alpha_n=\delta_n$ if $\delta_i>0$ for $n_0\leqq i\leqq n$, and $\alpha_n=0$ otherwise. Then $\alpha_n<\varepsilon/2$ implies $\delta_i<\varepsilon/2$ for some i with $n_0\leqq i\leqq n$. Also $\alpha_{n_0}\leqq a$ and, for $n>n_0$,

$$E(\alpha_n|\alpha_{n_0},\ \cdots,\ \alpha_{n-1})\leqq\left(1-\frac{2}{n}\right)\alpha_{n-1}+\frac{c}{n^2}\ ,$$

so that

$$E(\alpha_n)\leqq\left(1-\frac{2}{n}\right)E(\alpha_{n-1})+\frac{c}{n^2}\ .$$

Thus $E(\alpha_n)\to 0$ at a rate depending only on n_0, a, c, and there is an n_1 depending only on n_0, ε, a, c for which $E(\alpha_{n_1})$ is so small that

$$\text{Prob}\ \{\alpha_{n_1}<\varepsilon/2\}>1-(\varepsilon/2)\ .$$

For every n, k with $n\leqq k$ we define variables z_{nk} as follows. Unless $\delta_{n-1}<\varepsilon/2$ and $\delta_n\geqq\varepsilon/2$, $z_{nk}=0$ for all k. If $\delta_{n-1}<\varepsilon/2$ and $\delta_i\geqq\varepsilon/2$ for $n\leqq i\leqq k$, then $z_{nk}=\delta_k$. If $\delta_{n-1}<\varepsilon/2$, $\delta_i\geqq\varepsilon/2$ for $n\leqq i<k_0$ and $\delta_{k_0}<\varepsilon/2$, then $z_{nk}+z_{nk_0}=\delta_{k_0}$ for $k\geqq k_0$. If $\delta_n\geqq\varepsilon$ for some $n\geqq n_1$, either $\delta_n\geqq\varepsilon/2$ for all n such that $n_0\leqq n\leqq n_1$ or $z_{nk}\geqq\varepsilon$ for some $n\geqq n_0$. The former event has already been shown to have probability less than $\varepsilon/2$; it remains to show that the probability of the latter event can be made less than $\varepsilon/2$ by choosing n_0 sufficiently large.

Fix $n\geqq n_0$ and write $\beta_k=z_{nk}-z_{n\,k-1}$, $k>n$, $\beta_n=0$. Then, if $z_{n\,k-1}\geqq\varepsilon/2$

$$E(\beta_k|z_{nn},\beta_n,\cdots,\beta_{k-1})\leqq-\frac{2}{k}z_{n\,k-1}+\frac{c}{k^2}\leqq-\frac{\varepsilon}{2k}$$

for sufficiently large n_0 depending on c and ε, and $|\beta_k|\leqq b/k$. If $z_{n\,k-1}<\varepsilon/2$, $\beta_k=0$ so that, in any case

$$E(\beta_k|\beta_n,\cdots,\beta_{k-1})\leqq-\frac{\varepsilon}{2b}\max(|\beta_k|\|\beta_n,\cdots,\beta_{k-1})\ . \tag{8}$$

We now apply the following form of the strong law of large numbers, recently proved by the writer [1].

THEOREM 2. *If z_1, z_2, $\cdots$ is a sequence of random variables such that $|z_k|\leqq 1$ and*

$$E(z_k|z_1,\cdots,z_{k-1})\leqq-u\max(|z_k|\|z_1,\cdots,z_{k-1}),\qquad u>0,$$

then for all t,

$$\text{Prob}\ \{z_1+\cdots+z_k\geqq t\ \text{for some}\ k\}\leqq\left(\frac{1-u}{1+u}\right)^t .$$

The variables $z_k=(n/b)\beta_{k-n+1}$ satisfy the hypotheses of Theorem 2, with $u=(\varepsilon/2b)$, so that

$$\text{Prob}\ \{z_{nk}-z_{nn}>t\ \text{for some}\ k\geqq n\}\leqq r^{tn},\qquad r=\left(\frac{1-u}{1+u}\right)^{1/b},$$

For large n_0, $z_{nn}<3\varepsilon/4$, so that $z_{nk}\geqq\varepsilon$ for some k implies $z_{nk}-z_{nn}>\varepsilon/4$. Thus

$$\text{Prob}\ \{z_{nk}\geqq\varepsilon\ \text{for some}\ k\geqq n\}\leqq s^n,$$

where $s=r^{\varepsilon/4}$, so that

$$\text{Prob}\,\{z_{nk}\geqq\varepsilon \text{ for some } n\geqq n_0,\ k\geqq n\}\leqq\sum_{n_0}^{\infty}s^n,$$

which will be less than $\varepsilon/2$ for n_0 sufficiently large. This completes the proof.

3. The case of convex S.

THEOREM 3. *Let $T(q)$ denote the convex hull of the r points $\sum_1^s q_j\overline{m}(i, j)$. A closed convex set S is approachable if and only if it intersects every set $T(q)$. If it fails to intersect $T(q_0)$, it is excludable with $g: g_n\equiv q_0$.*

Proof. Suppose S intersects every $T(q)$, let $x_0\notin S$, let y be the point of S closest to x_0, and consider the game with matrix $A=\|a(i, j)\|$, where $a(i, j)=(y-x_0, \overline{m}(i, j))$. Its value is

$$\min_q \max_i\,(y-x_0, \sum_j q_j\overline{m}(i, j))=\min_q \max_{t\in T(q)}(y-x_0, t)\geqq\min_{s\in S}(y-x_0, s).$$

Consequently there is a $p\in P$ such that

$$(y-x_0, \sum_i p_i\overline{m}(i, j))\geqq\min_{s\in S}(y-x_0, s)$$

for all j, that is,

$$(y-x_0, r)\geqq(y-x_0, y)$$

for all $r\in R(p)$. Since $(y-x_0, x_0)<(y-x_0, y)$, the hyperplane $(y-x_0, x)=(y-x_0, y)$ separates x_0 from $R(p)$, completing the proof.

On the other hand, any $T(q_0)$ satisfies the hypotheses of Theorem 1 in M' with $f: f_n\equiv q_0$, and so is approachable in M' with this f. Consequently, if S fails to intersect $T(q_0)$, S is excludable in M with $g: g_n\equiv q_0$.

COROLLARY 1. *The sets $R(p)$ are approachable with $f: f_n\equiv p$.*

COROLLARY 2. *A closed convex set S is approachable if and only if for every vector u,*

$$v(u)\geqq\min_{s\in S}(u, s),$$

where $v(u)$ is the value of the game with matrix $\|(u, \overline{m}(i, j))\|$.

Proof of Corollary 2. If for some u_0 the inequality fails, then $T(q_0)$ is disjoint from S, where q_0 is a good strategy for II in the game with matrix $\|(u_0, \overline{m}(i, j))\|$, and conversely if any $T(q_0)$ is disjoint from S and u_0 is a vector with

$$\max_{t \in T(q_0)} (u_0, t) < \min_{s \in S} (u_0, s) ,$$

then

$$v(u_0) < \min_{s \in S} (u_0, s).$$

4. The case $N=1$.

THEOREM 4. *For $N=1$, let v, v' be the values of the games with matrices M, M'. If $v' \leqq v$, a closed set S is approachable if it intersects the closed interval $v'v$ and excludable otherwise. If $v' \geqq v$, a closed set S is approachable if it contains the closed interval vv' and excludable otherwise.*

Proof. Application of Corollary 2 to the closed interval AB, $A < B$ with $u = \pm 1$ yields that AB is approachable if and only if $v \geqq A$ and $-v' \geqq -B$. If $v' \leqq v$, these are simply the conditions that AB intersect the closed interval $v'v$, and if $v' \geqq v$, they are the conditions that AB contain vv'. Thus if $v' \leqq v$ every point in $v'v$ is approachable, so that any set S intersecting $v'v$ contains an approachable subset and is hence approachable, while if $v' \geqq v$, the interval vv' and hence any set containing it, is approachable. The last sentence, applied to M', yields that if $v' \leqq v$, the interval $v'v$ is approachable in M', so that any closed set not intersecting $v'v$ is excludable in M, and that if $v' \geqq v$, any point in vv' is approachable in M' so that any closed set not containing vv' is disjoint from a point approachable in M' and consequently is excludable in M. This completes the proof.

5. An example. We saw in the last section that for $N=1$ every set is approachable or excludable. This is false for $N=2$ as is shown by the following example. Let

$$r = s = 2,\ m(1, 1) = m(1, 2) = (0, 0),\ m(2, 1) = (1, 0),\ m(2, 2) = (1, 1),$$

let I_1 be the set of points $(\frac{1}{2}, y)$, $0 \leqq y \leqq \frac{1}{4}$, let I_2 be the set of points $(1, y)$, $\frac{1}{4} \leqq y \leqq 1$, and let $S = I_1 \cup I_2$. For every n, player I has a strategy which guarantees that $\bar{x}_{2n} \in S$, as follows: $f_j \equiv (0, 1)$ for $j < n$, so that $\bar{x}_n = (u, 1)$; if $u \geqq \frac{1}{2}$, $f_j \equiv (0, 1)$ for $j \geqq n$, and if $u < \frac{1}{2}$, $f_j \equiv (1, 0)$ for $j \geqq n$. Then for $u \geqq \frac{1}{2}$, $\bar{x}_{2n} \in I_2$, and for $u < \frac{1}{2}$, $\bar{x}_{2n} \in I_1$. However S is not approachable, since the following strategy for II does permit $\bar{x}_n$ to remain near either I_1 or I_2. Let $\bar{x}_n = (a_n, b_n)$, if $a_n \geqq \frac{3}{4}$, $g_n = (1, 0)$; if $a_n < \frac{3}{4}$, $g_n = (0, 1)$. Thus S is neither approachable nor excludable.

In the above example, S is weakly approachable, where a set S is said to be *weakly approachable in M* if for every $\varepsilon > 0$ there is an N_0 such that for every $n \geqq N_0$ there is a strategy f for I such that, for all g,

$$\text{Prob}\,\{\delta_n > \varepsilon\} < \varepsilon\,,$$

where δ_n is the distance from $\bar{x}_n$ to S. Similarly S is *weakly excludable in M* if there is a $d>0$ such that for every $\varepsilon>0$ there is an N_0 such that for every $n \geqq N_0$ there is a strategy g for II such that, for all f,

$$\text{Prob}\,\{\delta_n < d\} < \varepsilon\,.$$

Clearly no S is both weakly approachable and weakly excludable, we conjecture that every S is one or the other. In the above example, it is not hard to show that a closed S is weakly approachable if it intersects the graph of every function h defined for $0 \leqq t \leqq 1$ which satisfies

$$h(0)=0, \quad 0 \leqq (h(t_2)-h(t_1))/(t_2-t_1) \leqq 1 \qquad \text{for } 0 \leqq t_1 < t_2 \leqq 1,$$

and is weakly excludable if there is such an h whose graph it fails to intersect.

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