Learning Dynamics in Games with Stochastic Perturbations*

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Consider a generalization of fictitious play in which agents' choices are perturbed by incomplete information about what the other side has done, variability in their payoffs, and unexplained trembles. These perturbed best reply dynamics define a nonstationary Markov process on an infinite state space. It is shown, using results from stochastic approximation theory, that for 2 × 2 games it converges almost surely to a point that lies close to a stable Nash equilibrium, whether pure or mixed. This generalizes a result of Fudenberg and Kreps, who demonstrate convergence when the game has a unique mixed equilibrium. *Journal of Economic Literature* Classification Numbers: 000, 000, 000. © 1995 Academic Press, Inc.

1. A MODEL OF TECHNOLOGICAL ADOPTION

Consider two classes of agents who are deciding whether to adopt complementary technologies. Suppose, for example, that a new kind of gasoline (technology X) comes on the market. Filling stations must decide whether to stock X, and they will base their choice on an estimate of how many consumers have cars that run on X. Similarly, a consumer faced with the

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choice of whether to buy a car that runs on X will consider how many filling stations already offer X. In both cases the individual's decision depends on the proportion of people in the other class who have already adopted X, but these proportions are not precisely known. A consumer, in driving around, will notice that some stations offer X and some do not. Similarly, a filling station owner observes that some cars use X and some do not. From these casual observations they infer the proportions that are relevant to their decisions, but the information on which they base their decisions is quite incomplete. A similar story can be told for any technological innovation that complements another innovation, such as the arrangement of letters on a keyboard and the facility with which secretaries type on such keyboards (Arthur, 1989; David, 1985).

More generally we are interested in dynamical processes with the following four features. First, the current decision of each agent is affected by prior actions taken by other agents (*feedback effects*). Second, an agent may know *some* of the previous actions taken by others, but there is no reason to suppose that he actually knows *all* of them (*incomplete information*). Third, while a well-informed and highly sophisticated individual might, in theory, be able to forecast how the process is going to evolve over time, we do not want to assume that individuals are especially well-informed or highly sophisticated. We prefer to assume that they do more-or-less sensible things given a limited knowledge of the world around them (*bounded rationality*). Fourth, no matter how carefully we try to specify individuals' decision-making processes, there will inevitably be some random variation in their responses that arise from unmodeled factors (*stochastic perturbations*).

In this paper we examine a simple class of learning dynamics that incorporates these four features.[1] Specifically, we consider a stochastic version of fictitious play in which agents' information is incomplete, their payoff functions wobble, and their choices are sometimes random. As we show in Section 2, such a process can be represented as a generalized urn scheme of the type investigated by Arthur *et al.* (1987). Unfortunately their results rely on the existence of a Lyapunov function, which we have been unable to construct for the increasing returns case.

Another closely related paper is by Fudenberg and Kreps (1993). They showed that, when agents play a 2×2 game repeatedly with slightly perturbed payoffs, the frequency distribution of play converges with probability one to a neighborhood of the mixed strategy equilibrium *provided*

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[1] Other models of this general type include those given by Foster and Young (1990), Fudenberg and Harris (1992), Kandori *et al.* (1993), Young (1993a, 1993b), Eichberger *et al.* (1993), Jordan (1993), Ellison and Fudenberg (1994), Dosi and Kaniovski (1994), Posch (1994), and Bomze and Eichberger (1995). After completing this paper, we learned of independent work by Benaim and Hirsch (1994), who prove results similar to ours.

that the game has a unique equilibrium, which is completely mixed. Once again, however, this does not cover the increasing returns case, which has two pure equilibria.

The purpose of this paper is to establish, for a very general class of perturbed learning processes, that almost sure convergence holds for all 2×2 games. Moreover, when the game is nondegenerate, only the stable equilibria are attained with positive probability. In particular if the game has exactly three equilibria—two pure and one mixed—then the former are attained with probability one and the latter is attained with probability zero. To prove this result we do not rely on Lyapunov functions; instead we derive the relevant stability conditions for the associated system of differential equations using a geometric argument.

The paper proceeds as follows. In Section 2 we define a stochastic version of fictitious play in which the only noise arises from incomplete information (i.e., sampling variability). This stripped-down version of the model exhibits many of the key features mentioned above and is easy to grasp intuitively. Section 3 shows how to analyze the long-run behavior of such processes using stochastic approximation techniques (Nevelson and Hasminskii, 1976). We then broaden the framework in Sections 4 and 6 to include other sources of noise such as random perturbations in the players' choices and wobbles in their payoff functions. In Section 5 we introduce the concept of a "perturbed best reply dynamic," which covers all of the above sources of noise, as well as many others, and prove almost sure convergence to a stable Nash equilibrium. The rate of convergence is studied in Section 7.

2. Fictitious Play with Sampling

Fix a two-person game G with payoff matrix

$$\begin{pmatrix} \alpha_{11}, \beta_{11} & \alpha_{12}, \beta_{12} \\ \alpha_{21}, \beta_{21} & \alpha_{22}, \beta_{22} \end{pmatrix}.$$

Assume there are two populations of agents, row players (R) and column players (C). Each of these populations consists of one or more players. In every time period $t = 1, 2, \ldots$ one pair is drawn from $R \times C$ to play the game. The *state* at t is a vector of nonnegative integers $(a_1^t, a_2^t, b_1^t, b_2^t)$, where a_1^t, a_2^t are the numbers of row players who have chosen strategies 1 and 2, respectively, up to and including time t, and b_1^t, b_2^t are the numbers of column players who have chosen 1 and 2 respectively. We assume the agents selected to play the game in period $t + 1$ have incomplete information about the current state, which they gather by sampling previous actions

(see Young, 1993a). For notational simplicity we assume that all players have the same sample size s (a positive integer), though in fact our results extend to the case where players have different sample sizes. The sample size measures the extent of an agent's information, but we do not view it as the result of an optimal search. Rather, it reflects the extent to which the agent "gets around," i.e., is networked with other members of the population. We take this parameter to be exogenously given.

The process unfolds as follows. At time $t + 1$ one new row player and one new column player come forward. The row player draws a subset of s actions taken so far by the column players. The total number of such actions is $b^t = b_1^t + b_2^t$. For convenience we shall assume that all samples of size s are equally likely to be drawn.

Let the random variables B_1^t, B_2^t denote the actual numbers of previous actions by column players that Row draws at time $t + 1$. Row then adopts strategy 1 or 2 according to whether the criterion

$$\alpha_{11}B_1^t + \alpha_{12}B_2^t - \alpha_{21}B_1^t - \alpha_{22}B_2^t \tag{1}$$

is positive or nonpositive. Independently and simultaneously Column draws a subset of s previous actions by Row, the total number of such actions being $a^t = a_1^t + a_2^t$. The random variables A_1^t, A_2^t denote the number of actions of each type in the column player's sample. She then adopts strategy 1 or 2 according to whether the expression

$$\beta_{11}A_1^t + \beta_{21}A_2^t - \beta_{12}A_1^t - \beta_{22}A_2^t \tag{2}$$

is positive or nonpositive. (Thus we assume that ties are broken in favor of strategy 2.) These definitions yield a stochastic process of the form

$$(a_1^{t+1}, a_2^{t+1}, b_1^{t+1}, b_2^{t+1}) = (a_1^t, a_2^t, b_1^t, b_2^t) + \vec{I}^t(a_1^t, a_2^t, b_1^t, b_2^t), \qquad t \geq 1, \tag{3}$$

where $\vec{I}^t(\cdot, \cdot, \cdot, \cdot)$ are random vectors that take the values $(1, 0, 1, 0)$, $(0, 1, 1, 0)$, $(0, 1, 0, 1)$, and $(1, 0, 0, 1)$ with probabilities that depend on the current state and the time period t.

To analyze this process, we project it into the space of proportions of the two populations. Let $X^t = a_1^t/a^t$ and $Y^t = b_1^t/b^t$. Then there exist independent Bernoulli random variables $\xi^t(y) = 0$ or 1 and $\psi^t(x) = 0$ or 1 such that, for $t \geq 1$,

$$X^{t+1} = X^t + (1/a^{t+1})[\xi^t(Y^t) - X^t],$$
$$Y^{t+1} = Y^t + (1/b^{t+1})[\psi^t(X^t) - Y^t], \tag{4}$$

These equations define two parallel or *co-evolving* processes on the space $[0, 1] \times [0, 1]$. The two-dimensional process (X^t, Y^t) is Markov, but nonstationary because the denominators a^{t+1} and b^{t+1} depend on t. In fact we have the simple relations $a^{t+1} = t + a^1$ and $b^{t+1} = t + b^1$ because the number of actions already taken grows by one for each player in each period. Note that the distributions of $\xi^t(\cdot)$ and $\psi^t(\cdot)$ depend on the *number* of agents in the other class (not just their *proportions*) because the sampling is without replacement. We call this process *fictitious play with sampling*.

The process can also be represented as an urn scheme. Imagine two urns R and C of infinite capacity. Each contains two colors of balls—red for strategy 1 and white for strategy 2. Initially there are a_1^1 red balls and a_2^1 white balls in Row's urn. Similarly, there are b_1^1 red balls and b_2^1 white balls in Column's urn. In the first period, a representative row player reaches into Column's urn and pulls out s balls at random. Then he adds a red ball to his own urn if the criterion (1) is positive and adds a white ball if it is nonpositive. Simultaneously and independently a representative column player reaches into Row's urn and pulls out s balls at random. He then applies criterion (2) to determine what color of ball to add to his own urn. We call this a *co-evolving urn scheme*.

The process can also be represented (in a more complicated way) by a single urn containing four colors of balls. At each stage $t = 1, 2, \ldots$ two balls of various colors are added according to a probability distribution that depends on t and the proportions of balls currently in the urn. Let us identify a ball of the first color with a red ball in the first urn, a ball of the second color with a white ball in the first urn, a ball of the third color with a red ball in the second urn, and a ball of the fourth color with a white ball in the second urn. Designate by x_i^t the current proportion of balls of the ith color, $i = 1, 2, 3$. (The value x_4^t is determined by these.) Then

$$X^t = \frac{x_1^t}{x_1^t + x_2^t} \quad \text{and} \quad Y^t = \frac{x_3^t}{1 - x_1^t - x_2^t}, \quad t \geq 1.$$

Now we can characterize the process as follows. Add one ball of the first color and one ball of the third color if both (1) and (2) are positive. If (1) is nonpositive but (2) is positive, add one ball of the second color and one of the third color, and so on. This is an example of a generalized single-urn scheme with multiple additions (Arthur *et al.*, 1987). Unfortunately, proving convergence for such processes using the approach of Arthur *et al.* (1987) requires the construction of a Lyapunov function, which poses difficulties in this case. Instead, we shall develop a new approach that exploits the geometry of the situation together with the qualitative theory of ordinary differential equations.

3. ASYMPTOTIC BEHAVIOR OF FICTITIOUS PLAY WITH SAMPLING

We begin by analyzing the situation when the only source of noise is sampling variability. Players always choose best replies given the information in their samples; there is no variability in their payoffs and no trembling. This model is easy to grasp and contains almost all of the essential features of the more general case.

Consider the following example:

$$\begin{pmatrix} 2,2 & 4,0 \\ 1,3 & 6,4 \end{pmatrix}.$$

(5)

This game has three equilibria: $((0, 1), (0, 1))$, $((1, 0), (1, 0))$, and $((\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, \frac{1}{3}))$. To simplify notation we shall refer to these equilibria as $(0, 0)$, $(1, 1)$, and $(\frac{1}{3}, \frac{2}{3})$ respectively, that is, we write only the probabilities of choosing strategy 1 by each player. The direction of motion of ordinary fictitious play is deterministic, as shown in Fig. 1. Moreover, each of the pure equilibria is *dynamically stable* in the sense that it is the unique limit of ordinary fictitious play (which is deterministic) whenever the process
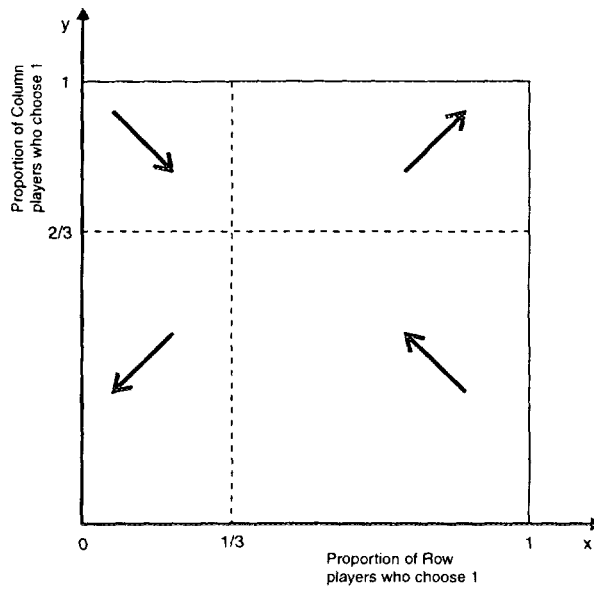

FIG. 1. Direction of motion of fictitious play for the game in (5).

starts in a sufficiently small neighborhood of that equilibrium. The mixed strategy equilibrium, by contrast, is not dynamically stable.

Consider now the stochastic process defined by (4) when agents have sample information with sample size s. Let the process begin in an arbitrary state $(a_1^1, a_2^1, b_1^1, b_2^1)$, $a_i^1 \geq 1$, $b_i^1 \geq 1$. (We have to assume here that $a_1^1 + a_2^1 \geq s$ and $b_1^1 + b_2^1 \geq s$ to be sure that the samples are feasible.) Sampling changes fictitious play in two ways: (i) it creates variability around the best-reply path; (ii) it creates bias in the replies because of the finiteness of the sample.

To state our result precisely, let us say that a 2×2 game G is *nondegenerate* if it has exactly three Nash equilibria (two pure and one mixed) or exactly one mixed Nash equilibrium. The situation in which G has exactly three equilibria will be called "case 1" and the situation where it has a unique equilibrium will be termed "case 2."

In both cases the formula for the mixed equilibrium (β, α) is (see, for example, Vorob'ev, 1977, pp. 99–103)

$$\beta = \frac{\beta_{22} - \beta_{21}}{\beta_{11} - \beta_{21} - \beta_{12} + \beta_{22}}, \qquad \alpha = \frac{\alpha_{22} - \alpha_{12}}{\alpha_{11} - \alpha_{21} - \alpha_{12} + \alpha_{22}}.$$

Without loss of generality in case 1 we have

$$\alpha_{11} - \alpha_{21} - \alpha_{12} + \alpha_{22} > 0 \qquad \text{and} \qquad \beta_{11} - \beta_{21} - \beta_{12} + \beta_{22} > 0,$$

while in case 2 we may assume that

$$\alpha_{11} - \alpha_{21} - \alpha_{12} + \alpha_{22} > 0 \qquad \text{and} \qquad \beta_{11} - \beta_{21} - \beta_{12} + \beta_{22} < 0.$$

THEOREM 1. *Let G be a nondegenerate 2×2 game. For all sufficiently large sample sizes, fictitious play with sampling converges almost surely. For case 1 games, the limit (X^*, Y^*) takes exactly two values $(0, 0)$ and $(1, 1)$ with probabilities that depend on the initial state. For case 2 games the limit (X^*, Y^*) is deterministic, and for every $\varepsilon > 0$ there exists a positive integer s_ε such that whenever $s \geq s_\varepsilon$, $\|(X^*, Y^*) - (\beta, \alpha)\| < \varepsilon$.*

We now give the intuition behind this theorem. (A formal proof of a more general result from which this one follows is given in the Appendix.) We can think of the process as fictitious play with a small noise or "wobble." There are two kinds of wobble. One is sampling variability. In fact, since the sample size is fixed and the sample frequencies take the values i/s, $i = 0, 1, \ldots, s$, the expected direction of motion is not quite the same as fictitious play; that is, there is a slight bias due to the finiteness of s. The second source of noise is the stochastic variation in the choices that agents make

each period. As time runs on, each new choice counts for less and less relative to the total number of choices that have already been made. The incremental changes in the population proportions decreases as $1/t$, and so does the variability in these increments. Thus we have an annealing process in which the level of noise damps down over time. Our result says that the state of the system—projected into the space of proportions—converges with probability one. Moreover, the limit of the process is precisely a fixed point of the expected motion, which is equal to (or close to) a Nash equilibrium of the game when s is large. If this equlibrium is not dynamically stable, however, then because of the perpetual wobble the process will not converge to it (except with probability zero).

While these statements make sense intuitively, they are not easy to establish rigorously. The difficulty is that the expected motion of the process (projected into the space of population proportions) is slowing down in tandem with the size of the stochastic perturbations. If the perturbations die out too slowly, the process will keep bouncing around and may fail to converge. If the perturbations die out too quickly, the process could conceivably converge to a nonequilibrium point. Without the benefit of a Lyapunov function, we are therefore forced to examine the behavior of the process and the related dynamic system in considerable detail.

The argument proceeds along the following lines. Let G be a nondegenerate 2×2 game. Let us derive analytic expressions for the distributions of the Bernoulli random variables involved in (4). Assume we are in case 1. At time t, as above, X^t stands for the proportion of strategy 1 chosen so far by the row players, and Y^t the proportion of strategy 1 chosen so far by the column players. Let $\xi^t(Y^t)$ be the indicator of the random event that Row plays strategy 1 in period $t + 1$ and let $\psi^t(X^t)$ be the indicator of the random event that Column plays strategy 1 in period $t + 1$. Let s be the sample size. The random variable B_1^t denotes the number of 1's that appear in Row's sample, while $B_2^t = s - B_1^t$ denotes the number of 2's in Row's sample. Define A_1^t and A_2^t similarly. Then we have the relations[2]

$$P\{\xi^t(Y^t) = 1\} = P\{B_1^t > \alpha s\}, \tag{6}$$

$$P\{\psi^t(X^t) = 1\} = P\{A_1^t > \beta s\}. \tag{7}$$

In case 2, (6) still applies but (7) must be replaced by

$$P\{\psi^t(X^t) = 1\} = P\{A_2^t > (1 - \beta)s\}. \tag{8}$$

[2] From these formulae it is clear why we require G to be nondegenerate. If, for example, $\alpha \notin (0, 1)$ then the event $\{B_1^t > \alpha s\}$ becomes deterministic and the dynamic for X^t is also deterministic. Hence sampling does not create anything interesting for degenerate games: the theorem holds *a fortiori*.

Consider the inequality $B_1^t > \alpha s$. The probability of this event is equal to

$$\sum_{i > \alpha s}^{s} H(i; b^t, s, b_1^t),$$

where $H(i; b^t, s, b_1^t)$ is the hypergeometric distribution

$$H(i; b^t, s, b_1^t) = \frac{\dbinom{b_1^t}{i} \dbinom{b^t - b_1^t}{s - i}}{\dbinom{b^t}{s}}. \tag{9}$$

To avoid the trivial case when this probability is identically 0 or 1, we shall assume that initially

$$b_1^1 > \alpha s \qquad \text{and} \qquad b_2^1 > s - \alpha s.$$

The analogous condition for the row players is

$$a_1^1 > \beta s \qquad \text{and} \qquad a_2^1 > s - \beta s.$$

We say that the initial state is *rich* if the above inequalities hold. We shall henceforth assume this condition to avoid the less interesting case where the process is deterministic.

Since $b^t = b^1 + t - 1$ and $Y^t = b_1^t/b^t$, we can write (9) in the form

$$\binom{s}{i} \frac{Y^t \left(Y^t - \frac{1}{b^1 + t - 1} \right) \cdots \left(Y^t - \frac{i - 1}{b^1 + t - 1} \right)(1 - Y^t)\left(1 - Y^t - \frac{1}{b^1 + t - 1} \right) \cdots \left(1 - Y^t - \frac{s - i - 1}{b^1 + t - 1} \right)}{1\left(1 - \frac{1}{b^1 + t - 1} \right) \cdots \left(1 - \frac{s - 1}{b^1 + t - 1} \right)}.$$

Consequently we have

$$\sum_{i > \alpha s}^{s} H(i; b^t, s, b_1^t) = f_\alpha^s(Y^t) + \delta_\alpha^s(b^t, Y^t),$$

where

$$f_\alpha^s(y) = \sum_{i > \alpha s}^{s} \binom{s}{i} y^i (1 - y)^{s-i}, \tag{10}$$

and

$$\sup_{y \in [0,1]} |\delta_\alpha^s(b^t, y)| \le c_{\alpha,s}/b^t. \tag{11}$$

We can sum up these observations in the following lemma.

LEMMA 1. *Let G be a nondegenerate* 2×2 *game with mixed Nash equilibrium* (β, α). *For all* $(x, y) \in [0, 1] \times [0, 1]$,
Case 1.

$$P\{\xi^t(y) = 1\} = f_\alpha^s(y) + \delta_\alpha^s(b^t, y), \qquad P\{\psi^t(x) = 1\} = f_\beta^s(x) + \delta_\beta^s(a^t, x),$$

Case 2.

$$P\{\xi^t(y) = 1\} = f_\alpha^s(y) + \delta_\alpha^s(b^t, y),$$

$$P\{\psi^t(x) = 1\} = f_{1-\beta}^s(1 - x) + \delta_{1-\beta}^s(a^t, 1 - x),$$

where the functions involved are as in (10) *and* (11).

Define $\Xi^t(y) = \xi^t(y) - E\xi^t(y)$ and $\Psi^t(y) = \psi^t(y) - E\psi^t(y)$, where E designates mathematical expectation, and rewrite (4) in the following way:
Case 1.

$$X^{t+1} = X^t + (1/a^{t+1})\{[f_\alpha^s(Y^t) + \delta_\alpha^s(b^t, Y^t) - X^t] + \Xi^t(Y^t)\},$$

$$Y^{t+1} = Y^t + (1/b^{t+1})\{[f_\beta^s(X^t) + \delta_\beta^s(a^t, X^t) - Y^t] + \Psi^t(X^t)\},$$
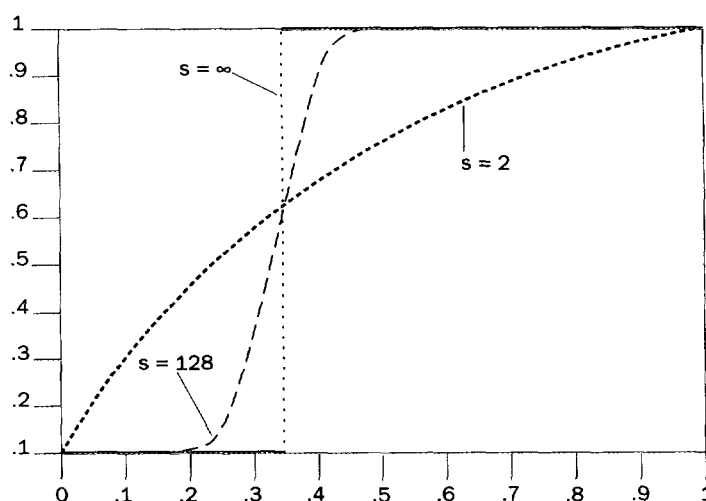
Case 2.

$$X^{t+1} = X^t + (1/a^{t+1})\{[f_\alpha^s(Y^t) + \delta_\alpha^s(b^t, Y^t) - X^t] + \Xi^t(Y^t)\},$$

$$Y^{t+1} = Y^t + (1/b^{t+1})\{[f_{1-\beta}^s(1 - X^t) + \delta_{1-\beta}^s(a^t, 1 - X^t) - Y^t] + \Psi^t(X^t)\}.$$

These equations define a two-dimensional stochastic approximation procedure (see, for example, Nevelson and Hasminskii, 1976).

Suppose that at time t the process is at the point (x, y). Then the *expected motion*, given the current state (x, y), is to the state
Case 1.

$$(x + (1/a^{t+1})[f_\alpha^s(y) + \delta_\alpha^s(b^t, y) - x], y + (1/b^{t+1})[f_\beta^s(x) + \delta_\beta^s(a^t, x) - y]);$$

Fig. 2. The function $f_{1/3}^s(\cdot)$ for $s = 2, 128, \infty$.

Case 2.

$$(x + (1/a^{t+1})[f_\alpha^s(y) + \delta_\alpha^s(b^t, y) - x],$$
$$y + (1/b^{t+1})[f_{1-\beta}^s(1 - x) + \delta_{1-\beta}^s(a^t, 1 - x) - y]).$$

Since $a^{t+1} = a^1 + t$ and $b^{t+1} = b^1 + t$, it seems reasonable to suppose that, as $t \to \infty$, this process behaves like the system of ordinary differential equations

Case 1. $\dot{x} = f_\alpha^s(y) - x,$ $\dot{y} = f_\beta^s(x) - y;$

Case 2. $\dot{x} = f_\alpha^s(y) - x,$ $\dot{y} = f_{1-\beta}^s(1 - x) - y.$ (12)

(By (11) we have neglected terms of order higher than t^{-1}.) This supposition turns out to be correct, though it requires a detailed argument, which is given in the Appendix. From this it follows that the singular points of (12) are the only possible limits of the process, and they are precisely the solutions of

Case 1. $x = f_\alpha^s(y),$ $y = f_\beta^s(x);$

Case 2. $x = f_\alpha^s(y),$ $y = f_{1-\beta}^s(1 - x).$ (13)

The next step is to show that the solutions of (13) are close to the Nash equilibria of the game. Consider $f_\alpha^s(\cdot)$ for a fixed α and variable s. As s increases, $f_\alpha^s(\cdot)$ becomes more and more S-shaped and approaches the step function (see Fig. 2)

$$f_\alpha^\infty(x) = \begin{cases} 1 & \text{if } x \in (\alpha, 1), \\ \frac{1}{2} & \text{if } x = \alpha, \\ 0 & \text{if } x \in (0, \alpha). \end{cases}$$

We may state this result more exactly as follows. Say that a function $f(\cdot)$: $[0, 1] \mapsto [0, 1]$ is *convex–concave* if for some $z \in (0, 1)$, $f(\cdot)$ is convex on $[0, z)$ and concave on $(z, 1]$.

LEMMA 2.   *For every* $\alpha \in (0, 1)$

(i) $\lim_{s\to\infty} f_\alpha^s(x) = f_\alpha^\infty(x)$ *for all* $x \in [0, 1]$;

(ii) $f_\alpha^s(x)$ *is strictly increasing in* x, *continuously differentible, and for all sufficiently large* s *it is convex–concave.*

Proof.   The value $f_\alpha^s(x)$ can be represented as

$$P\left\{ \sum_{k=1}^{s} a_k > \alpha s \right\},$$

where a_k are independent Bernoulli random variables satisfying

$$a_k = \begin{cases} 1 & \text{with probability } x, \\ 0 & \text{with probability } 1 - x. \end{cases}$$

Set $b_k = a_k - x$. Then the above probability equals

$$P\left\{ \sum_{k=1}^{s} b_k > (\alpha - x)s \right\}.$$

From the law of large numbers it follows that, in probability,

$$\lim_{s\to\infty} \frac{1}{s} \sum_{k=1}^{s} b_k = 0.$$

Consequently,

$$\lim_{s\to\infty} f_\alpha^s(x) = \lim_{s\to\infty} P\left\{ \frac{1}{s} \sum_{k=1}^{s} b_k > \alpha - x \right\} = 0 \text{ for } \alpha > x,$$

and

$$\lim_{s \to \infty} f_\alpha^s(x) = \lim_{s \to \infty} P\left\{\frac{1}{s} \sum_{k=1}^{s} b_k > \alpha - x\right\} = 1 \text{ for } \alpha < x.$$

Finally, $f_\alpha^\infty(\alpha) = \frac{1}{2}$ by symmetry. This proves statement(i).

To prove statement(ii) let us observe that $f_\alpha^s(x)$ can be expressed in another way. Namely, $f_\alpha^s(x)$ is the probability that at least $k = [\alpha s] + 1$ of the v_i, $i = 1, 2, \ldots, s$, are less than or equal to x. Here v_i are independent random variables that are uniformly distributed over $[0, 1]$ and $[\alpha s]$ designates the integer part of αs. Thus $f_\alpha^s(\cdot)$ is the distribution function of the kth order statistic of the v_i, $i = 1, 2, \ldots, s$. It follows from standard arguments (see, for example, David, 1981, p. 9) that the corresponding density is given by

$$\frac{d}{dx} f_\alpha^s(x) = \frac{s!}{(k-1)!(s-k)!} x^{k-1}(1-x)^{s-k}.$$

It is easily checked that its derivative is positive on $(0, (k-1)/(s-1))$ and negative on $((k-1)/(s-1), 1)$. Hence $f_\alpha^s(\cdot)$ is convex–concave; moreover it is continuously differentiable and increasing. This completes the proof of Lemma 2.

LEMMA 3.   *Let G be nondegenerate with unique mixed equilibrium* (β, α).

Case 1.   *For all sufficiently large* s, (13) *has exactly three solutions*: $(0, 0)$, $(1, 1)$, (x_β, y_α), *and* $(x_\beta, y_\alpha) \to (\beta, \alpha)$ *as* $s \to \infty$.

Case 2.   *For each* s, (13) *has exactly one solution* (x_β', y_α'), *and* $\lim_{s \to \infty} (x_\beta', y_\alpha') = (\beta, \alpha)$.

Proof.   We shall give the argument for case 1; the other case is similar. Fix a small $\varepsilon > 0$. By statement(i) of Lemma 2 there is an integer s_ε such that, for all $s \geq s_\varepsilon$,

$$f_\beta^s(\beta - \varepsilon) \leq \varepsilon, \qquad f_\beta^s(\beta + \varepsilon) \geq 1 - \varepsilon, \tag{14}$$

and

$$f_\alpha^s(\alpha - \varepsilon) \leq \varepsilon, \qquad f_\alpha^s(\alpha + \varepsilon) \geq 1 - \varepsilon. \tag{15}$$

The curves $x = (f_\beta^s)^{-1}(y)$ and $x = f_\alpha^s(y)$ intersect at $(0, 0)$ and $(1, 1)$, where $(f_\beta^s)^{-1}(\cdot)$ stands for the inverse function. By choice of ε the only other points of intersection (if any) must be inside the box $B_\varepsilon = \{(x, y): \beta - \varepsilon \leq x \leq \beta + \varepsilon, \alpha - \varepsilon \leq y \leq \alpha + \varepsilon\}$. Moreover, since the functions are

continuous, by statement(i) of Lemma 2 they have at least one point of intersection in B_ε for $s \geq s_\varepsilon$. We shall show that the intersection is unique.

Let x_f be the first x such that $(x, f_\beta^s(x))$ is in the box B_ε. Then there exists $x' \in (\beta - \varepsilon, x_f)$ such that

$$f_\beta^s(x_f) - f_\beta^s(\beta - \varepsilon) = \frac{d}{dx} f_\beta^s(x')(x_f - \beta + \varepsilon),$$

and hence by (14),

$$\frac{d}{dx} f_\beta^s(x') \geq \frac{\alpha - 2\varepsilon}{2\varepsilon}.$$

Similarly,

$$\frac{d}{dx} f_\beta^s(x'') \geq \frac{1 - \alpha - 2\varepsilon}{2\varepsilon}$$

for some $x'' \in (x_l, \beta + \varepsilon)$, where x_l is the last x such that $(x, f_\beta^s(x))$ is in the box.

By statement (ii) of Lemma 2, $(d/dx) f_\beta^s(\cdot)$ is first increasing, then decreasing. Hence

$$\frac{d}{dx} f_\beta^s(x) \geq \min\left(\frac{\alpha - 2\varepsilon}{2\varepsilon}, \frac{1 - \alpha - 2\varepsilon}{2\varepsilon}\right)$$

for $x \in [x_f, x_l]$. Consequently for sufficiently small ε and all $s \geq s_\varepsilon$, we have

$$\frac{d}{dx} f_\beta^s(x) > 1 \qquad \text{for all } (x, f_\beta^s(x)) \in B_\varepsilon.$$

Similarly,

$$\frac{d}{dy} f_\alpha^s(y) > 1 \qquad \text{for all } (y, f_\alpha^s(y)) \in B_\varepsilon.$$

Suppose now that the curves of $y = f_\beta^s(x)$ and $x = f_\alpha^s(y)$ intersect in two distinct points (x_1, y_1), $(x_2, y_2) \in B_\varepsilon$. Then there is a point $(x^0, f_\beta^s(x^0)) \in B_\varepsilon$ such that $(d/dx) f_\beta^s(x^0) = (y_2 - y_1)/(x_2 - x_1)$ and a point $(f_\alpha^s(y^0), y^0) \in B_\varepsilon$ such that $(d/dy) f_\alpha^s(y^0) = (x_2 - x_1)/(y_2 - y_1)$. Since at least one of these is less than or equal to 1, we have arrived at a contradiction. This proves the uniqueness of the interior intersection point (x_β, y_α) for all

sufficiently large s and that $(x_\beta, y_\alpha) \to (\beta, \alpha)$ as $s \to \infty$. Thus Lemma 3 is proved.

From this it follows that *if* fictitious play with sampling converges, then it converges to a solution of (13), which is either a Nash equilibrium of G or close to one. The proof of convergence is given in the Appendix, where we also show that for case 1 games the limit is almost surely *not* contained in every sufficiently small neighborhood of the unstable Nash equilibrium. At this point let us also note that, for case 1 games, *both* of the pure equilibria are attained in the limit with positive probability if the initial state is rich; i.e., if in the initial state, strategies 1 and 2 are played with positive probability by both players.

4. MISTAKES AND OTHER SOURCES OF NOISE

In this section we relax the assumption that agents always choose best replies by supposing that they sometimes make "mistakes," that is, they do things for unexplained reasons. To make this idea concrete, assume that Row samples from the previous actions by Column and chooses a best reply with probability $1 - 2\varepsilon$, but with probability 2ε she chooses 1 or 2 at random. For notational convenience we shall assume that Row selects 1 and 2 with equal probability, though all that really matters is that she chooses both 1 and 2 with fixed positive probabilities whenever she randomizes. The column player acts similarly.

In place of the expressions in Lemma 1 we obtain the following:
Case 1.

$$P\{\xi^t(y) = 1\} = (1 - 2\varepsilon)[f_\alpha^s(y) + \delta_\alpha^s(b^t, y)] + \varepsilon,$$

$$P\{\psi^t(x) = 1\} = (1 - 2\varepsilon)[f_\beta^s(x) + \delta_\beta^s(a^t, x)] + \varepsilon;$$

Case 2.

$$P\{\xi^t(y) = 1\} = (1 - 2\varepsilon)[f_\alpha^s(y) + \delta_\alpha^s(b^t, y)] + \varepsilon,$$

$$P\{\psi^t(x) = 1\} = (1 - 2\varepsilon)[f_{1-\beta}^s(1 - x) + \delta_{1-\beta}^s(a^t, 1 - x)] + \varepsilon.$$

Consider case 1. Let $f_\alpha^{s,\varepsilon}(y) = (1 - 2\varepsilon) f_\alpha^s(y) + \varepsilon$, and $f_\beta^{s,\varepsilon}(x) = (1 - 2\varepsilon) f_\beta^s(x) + \varepsilon$. An argument similar to the preceding shows that if the stochastic process defined by (4) converges, then it converges to a solution of the system

$$x = f_\alpha^{s,\varepsilon}(y), \qquad y = f_\beta^{s,\varepsilon}(x).$$

If $\varepsilon < \alpha, \beta < 1 - \varepsilon$ the situation is similar to the previous one.

We remark that these inequalities have two interpretations. On the one hand, they allow us to conclude that the limit points of the process are close to the Nash equilibria of a *given game*, provided the rate of mistakes in the populations is sufficiently small. On the other hand, they say that if the rate of mistakes in a *given population* is ε, then the limit points are close to the Nash equilibria provided the game is such that α and β satisfy the given inequalities. Thus, this is a condition relating the characteristics of the nonhomogeneous pools of players and the characteristics of the game that they are playing.

For a fixed ε, the curves $f_\alpha^{s,\varepsilon}(\cdot)$ and $f_\beta^{s,\varepsilon}(\cdot)$ become more and more S-shaped as s increases. Hence the curves $x = f_\alpha^{s,\varepsilon}(y)$ and $y = f_\beta^{s,\varepsilon}(x)$ cross at exactly three points, which are the only possible limits of the stochastic process. In case 1, the interior crossing point is not stable, and the process converges to it with probability zero. The two stable crossing points are ε-close to the two pure Nash equilibria (when s is large), and hence the process converges almost surely to an ε-neighborhood of the pure Nash equilibria.

In case 2, if $\varepsilon < \alpha, \beta < 1 - \varepsilon$, there is a unique crossing point for a fixed ε and all sufficiently large s. The process converges to a neighborhood of the mixed Nash equilibrium of the game, and the neighborhood shrinks to (β, α) as s increases.

It should be remarked that nothing essential in this argument changes if the populations are heterogeneous with respect to the probability that they make mistakes, so long as all these probabilities are small.

5. Perturbed Best Reply Dynamics

We now formulate a general model that captures the preceding examples as special cases. Let G be a nondegenerate 2×2 game with mixed Nash equilibrium (β, α). We say that the coevolving process given by equations (4) is a *perturbed best-reply dynamic* of G if it is driven on average by slightly perturbed step functions plus terms that go to zero at least as fast as $t^{-\kappa}$ for some $\kappa > 0$. More precisely, suppose that for $\varepsilon > 0$ and $\delta \in [0, \frac{1}{2})$ there is a set of functions $h_\alpha^{\varepsilon,\delta}(\cdot)$, $g_\gamma^{\varepsilon,\delta}(\cdot)$, $\nu_\alpha^\varepsilon(b^t, \cdot)$, $\mu_\gamma^\varepsilon(a^t, \cdot)$ such that for some choice of the parameters, the Bernoulli random variables $\xi^t(\cdot)$ and $\psi^t(\cdot)$ can be represented in the following form:

Case 1.

$$P\{\xi^t(y) = 1\} = h_\alpha^{\varepsilon,\delta}(y) + \nu_\alpha^\varepsilon(b^t, y),$$

$$P\{\psi^t(x) = 1\} = g_\beta^{\varepsilon,\delta}(x) + \mu_\beta^\varepsilon(a^t, x);$$

Case 2.

$$P\{\xi^t(y) = 1\} = h_\alpha^{\varepsilon,\delta}(y) + \nu_\alpha^\varepsilon(b^t, y),$$

$$P\{\psi^t(x) = 1\} = g_{1-\beta}^{\varepsilon,\delta}(1-x) + \mu_{1-\beta}^\varepsilon(a^t, 1-x).$$

We assume that:

  (i)   for all small enough ε the functions $h_\alpha^{\varepsilon,\delta}(\cdot)$ and $g_\beta^{\varepsilon,\delta}(\cdot)$ are increasing, continuously differentiable, and convex–concave;

  (ii)

$$\lim_{\varepsilon\to 0} h_\alpha^{\varepsilon,\delta}(y) = \begin{cases} \delta & \text{if } y \in [0, \alpha), \\ 1 - \delta & \text{if } y \in (\alpha, 1], \end{cases} \qquad \lim_{\varepsilon\to 0} g_\gamma^{\varepsilon,\delta}(x) = \begin{cases} \delta & \text{if } x \in [0, \gamma), \\ 1 - \delta & \text{if } x \in (\gamma, 1], \end{cases}$$

where $\delta < \min(\alpha, \beta) \le \max(\alpha, \beta) < 1 - \delta$;

  (iii) $\sup_{y\in[0,1]} \nu_\alpha^\varepsilon(t, y) \le c'_{\alpha,\varepsilon} t^{-\kappa}$ and $\sup_{x\in[0,1]} \mu_\gamma^\varepsilon(t, x) \le c''_{\gamma,\varepsilon} t^{-\kappa}$ for some $\kappa > 0$, where $\gamma = \beta, 1 - \beta$.

THEOREM 2. *Let G be a nondegenerate* 2×2 *game. Every perturbed best reply dynamic converges almost surely provided ε is sufficiently small. The limit (X^*, Y^*) takes exactly two values for case 1 games, one close to (δ, δ), the other close to $(1 - \delta, 1 - \delta)$, with probabilities that depend on the initial state. For case 2 games the limit is deterministic, and lies close to the mixed equilibrium (β, α). In both cases, for every $\sigma > 0$ there is $\varepsilon_\sigma > 0$ such that whenever $\varepsilon < \varepsilon_\sigma$, (X^*, Y^*) lies within σ of a stable Nash equilibrium.*

The proof is given in the Appendix. We remark that, in case 1, for ε small enough, the curves $x = h_\alpha^{\varepsilon,\delta}(y)$ and $y = g_\beta^{\varepsilon,\delta}(x)$ cross at exactly three points: (x_0, y_0), (x_β, y_α) and (x_1, y_1). One of them, (x_β, y_α), approaches the mixed equilibrium as ε goes to 0. This point is not stable, so the process converges to it with zero probability. The other two lie close to (or coincide with) (δ, δ) and $(1 - \delta, 1 - \delta)$. It can be shown (though we shall not show it here) that the process (4) *converges to each of them with positive probability* as long as $P\{\zeta^t(y) = 1\} \in (0, 1)$ and $P\{\psi^t(x) = 1\} \in (0, 1)$ for all $y \in (0, 1)$, $x \in (0, 1)$, and $t \ge 1$. Similarly, in case 2 if ε is small enough there is a unique crossing point, which approaches the unique equilibrium of the game as ε goes to zero. The process converges to the crossing point with probability one. Similar conclusions hold if the limiting step functions in (ii) have different δ, say δ_1 and δ_2.

6. PERTURBED PAYOFFS

In this section we shall show how slight perturbations in the players' payoffs can be treated within this framework. This is the case considered by Fudenberg and Kreps (1993) and by Benaïm and Hirsch (1994). Assume that in each period t the payoffs α_{ij}^t, β_{ij}^t are random variables. Specifically, let us assume that

$$\alpha_{ij}^t = \alpha_{ij} + e_{ij}^t(\varepsilon), \qquad \beta_{ij}^t = \beta_{ij} + \tilde{e}_{ij}^t(\varepsilon),$$

where α_{ij}, β_{ij} are the mean payoffs and $e_{ij}^t(\varepsilon)$, $\tilde{e}_{ij}^t(\varepsilon)$ are independent random variables with mean zero and distribution functions $F_\varepsilon(\cdot)$ and $R_\varepsilon(\cdot)$. We want the errors to be "small" as $\varepsilon \to 0$; the most straightforward way to ensure this is to assume that ε are the standard deviations of the corresponding distributions. For simplicity of exposition we shall assume that $F_\varepsilon(\cdot)$ and $R_\varepsilon(\cdot)$ are normal, though it will be apparent that the analysis holds much more generally. We shall also assume for convenience that players have full information about past actions by the other side. Sampling, mistakes, and other perturbations can be included without substantially changing the conclusions, though at significant notational cost.

Our task is to show that this process satisfies the conditions of a perturbed best-reply dynamic. Assume that we have a case 1 game, and let us focus on the row player. Suppose that, at some period t, the proportion of column players choosing strategy 1 is y. Then the row player chooses 1 with probability

$$
\begin{aligned}
h_\alpha^\varepsilon(y) &= P\{\alpha_{12}^t - \alpha_{21}^t < (\alpha_{11}^t - \alpha_{12}^t - \alpha_{21}^t + \alpha_{22}^t)y\} \\
&= P\{(1 - y)e_{22}^t(\varepsilon) - (1 - y)e_{12}^t(\varepsilon) + ye_{21}^t(\varepsilon) - ye_{11}^t(\varepsilon) \\
&\quad < (\alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22})y - (\alpha_{22} - \alpha_{12})\}.
\end{aligned}
$$

Let $a = \alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22}$, so that $\alpha = (\alpha_{22} - \alpha_{12})a^{-1}$. Let us also recall that $a > 0$. We have

$$h_\alpha^\varepsilon(y) = P\left\{ N(0, 1) < \frac{a(y - \alpha)}{\varepsilon \sqrt{2[(1 - y)^2 + y^2]}} \right\},$$

where $N(0, 1)$ is a normal random variable with zero mean and variance 1. In particular,

$$N(0, 1) = \frac{(1 - y)e_{22}^t(\varepsilon) - (1 - y)e_{12}^t(\varepsilon) + ye_{12}^t(\varepsilon) - ye_{11}^t(\varepsilon)}{\varepsilon\sqrt{2[(1 - y)^2 + y^2]}}.$$

Clearly,

$$\lim_{\varepsilon \to 0} h_\alpha^\varepsilon(y) = 0 \text{ if } y < \alpha \qquad \text{and} \qquad \lim_{\varepsilon \to 0} h_\alpha^\varepsilon(y) = 1 \text{ if } y > \alpha.$$

It remains to be shown that, for all sufficiently small ε, $h_\alpha^\varepsilon(\cdot)$ is increasing and convex–concave.

Let

$$q(y) = a \frac{y - \alpha}{\sqrt{2[(1 - y)^2 + y^2]}}.$$

Then

$$\frac{d}{dy} h_\alpha^\varepsilon(\cdot) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left[-\frac{q^2(\cdot)}{2\varepsilon^2}\right] q'(\cdot).$$

A straightforward calculation shows that $q'(y) > 0$ for all $y \in [0, 1]$; hence $h_\alpha^\varepsilon(\cdot)$ is increasing. To demonstrate that $h_\alpha^\varepsilon(\cdot)$ is convex–concave we need to show that $(d^2/dy^2) h_\alpha^\varepsilon(\cdot)$ is first positive then negative. Now

$$\frac{d^2}{dy^2} h_\alpha^\varepsilon(\cdot) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left[-\frac{q^2(\cdot)}{2\varepsilon^2}\right] \{q''(\cdot) - \frac{1}{\varepsilon^2} q(\cdot)[q'(\cdot)]^2\}.$$

Let us examine the behavior of

$$Q_\varepsilon(\cdot) = q''(\cdot) - \frac{1}{\varepsilon^2} q(\cdot)[q'(\cdot)]^2.$$

Since $q(y)(y - \alpha) > 0$ for $y \neq \alpha$ and $q''(\cdot)$ is bounded on $[0, 1]$, for all small enough ε the function $Q_\varepsilon(\cdot)$ changes sign over $[0, 1]$ from plus to minus. Hence there is at least one point y_ε where $Q_\varepsilon(y_\varepsilon) = 0$. Moreover, this point must belong to a neighborhood of α, which shrinks to zero as $\varepsilon \to 0$. Let us show that the root is unique.

If there were more than one such root, there would exist y_ε' (in the same neighborhood) such that $Q_\varepsilon'(y_\varepsilon') = 0$. But

$$Q_\varepsilon'(\cdot) = q'''(\cdot) - \frac{1}{\varepsilon^2} [q'(\cdot)]^3 - \frac{2}{\varepsilon^2} q(\cdot)q'(\cdot)q''(\cdot).$$

Since $q(y_\varepsilon') \to 0$ as $\varepsilon \to 0$, the sign of this derivative coincides with the sign of $-[q'(\cdot)]^3$, which is negative. Consequently $Q_\varepsilon'(\cdot)$ cannot have zeros in a sufficiently small neighborhood of α, and hence y_ε is unique.

Thus we have established that there is a unique point y_ε such that

$Q_\varepsilon(y) > 0$ for $y \in [0, y_\varepsilon)$ and $Q_\varepsilon(y) < 0$ for $y \in (y_\varepsilon, 1]$. This implies that $h_\alpha^\varepsilon(\cdot)$ is convex–concave.

We have proved that $h_\alpha^\varepsilon(\cdot)$ has all the properties of a perturbed best-reply dynamic for the row player in a case 1 game. The other cases are verified similarly.

Note that

$$h_\alpha^\varepsilon(0) = P\left\{ N(0, 1) < -\frac{a\alpha}{\varepsilon\sqrt{2}} \right\} > 0,$$

$$h_\alpha^\varepsilon(1) = P\left\{ N(0, 1) < \frac{a(1 - \alpha)}{\varepsilon\sqrt{2}} \right\} < 1.$$

Similarly $g_\beta^\varepsilon(0) > 0$ and $g_\beta^\varepsilon(1) < 1$, where $g_\beta^\varepsilon(x)$ stands for the probability that Column chooses strategy 1 given that the proportion of row players choosing 1 is x. The inequalities imply that in case 1 there are two limits (x_0, y_0) and (x_1, y_1) for the random process generated by the perturbed best-reply dynamic, which *do not coincide* with $(0, 0)$ and $(1, 1)$, though they approach these values as $\varepsilon \to 0$.

From this and Theorem 2 it follows that, *when the variance (standard deviation) of perturbations is sufficiently small, fictitious play with normally perturbed payoffs converges almost surely to a neighborhood of a stable Nash equilibrium of the game and the size of the neighborhood goes to zero with the variance.*

A similar argument holds for a wide class of perturbations. For example, we could consider bounded perturbations that are uniformly distributed over some finite interval. In this case, ε is proportional to the length of the interval. The corresponding functions $h_\alpha^\varepsilon(\cdot)$ and $g_\beta^\varepsilon(\cdot)$ attain the values 0 and 1 for *all* sufficiently small ε, and hence (for case 1 games) the dynamic converges to the pure equilibria *exactly*.

7. RATE OF CONVERGENCE

We conclude by studying the rate of convergence of perturbed best-reply dynamics. As we have already mentioned, (4) can be thought of as a two-dimensional stochastic approximation procedure. The central limit theorem for stochastic approximation (see Fabian, 1968; Nevelson and Hasminskii, 1976) shows that when the process has a unique limit, the deviation from the limit is approximately $\vec{N}(\vec{0}, t^{-1}K)$ as $t \to \infty$, where $\vec{N}(\vec{0}, t^{-1}K)$ stands for a normal distribution with mean zero and variance matrix $t^{-1}K$. For case 2 games, where the limit is a singleton, we can apply this result directly

to our processes and conclude that they converge at the rate $1/\sqrt{t}$, which is the typical convergence rate for statistical estimation processes. In case 1, there are two points that are attainable with positive probability in the limit. In this situation we need to use conditional limit theorems (see Arthur *et al.*, 1987, 1988). These results imply the following.

THEOREM 3. *Let G be a nondegenerate 2 × 2 game. Every perturbed best-reply dynamic converges to its limit at rate at least $1/\sqrt{t}$ as $t \to \infty$. More precisely, there exists $\sigma > 0$ such that for all $\varepsilon \le \sigma$ and $\vec{z} \in R^2$*

$$\lim_{t \to \infty} P\{\sqrt{t}[(X^t, Y^t) - (x, y)] < \vec{z}, \lim_{m \to \infty} (X^m, Y^m) = (x, y)\}$$
$$= P\{\vec{N}(\vec{0}, K(x, y)) < \vec{z}\} P\{\lim_{m \to \infty} (X^m, Y^m) = (x, y)\}.$$

Here (x, y) can be any of the points to which the process converges with positive probability, and $\vec{N}(\vec{0}, K(x, y))$ is a two-dimensional normal random vector with mean zero and variance matrix $K(x, y)$ satisfying the equation

$$L(x, y)K(x, y) + K(x, y)L^T(x, y) = -B(x, y),$$

where $L(x, y) = J(x, y) + (1/2)I$, the sign ^{T} designates transposition, I stands for the identity matrix in R^2, and

$$B(x, y) = \begin{pmatrix} x(1 - x) & 0 \\ 0 & y(1 - y) \end{pmatrix}.$$

Case 1.

$$J(x, y) = \begin{pmatrix} -1 & \dfrac{d}{dy} h_\alpha^{\varepsilon, \delta}(y) \\ \dfrac{d}{dx} g_\beta^{\varepsilon, \delta}(x) & -1 \end{pmatrix},$$

Case 2.

$$J(x, y) = \begin{pmatrix} -1 & \dfrac{d}{dy} h_\alpha^{\varepsilon, \delta}(y) \\ \dfrac{-d}{dx} g_{1-\beta}^{\varepsilon, \delta}(1 - x) & -1 \end{pmatrix},$$

This theorem is proved in the Appendix.

Let us consider some implications of this result. For case 1 games, if a limit coincides with a *pure* equilibrium, then Theorem 3 gives a zero variance matrix for the limiting normal distribution. This indicates that the actual convergence rate is faster than $1/\sqrt{t}$. In fact, if the payoff perturbations are *bounded*, and these are the only perturbations, then it can be shown (we shall not do so here) that convergence occurs nearly as fast as $1/t$, that is, faster than $t^{\tau-1}$ for *every* small $\tau > 0$. A similar result holds if the perturbations arise solely from sampling.

For case 2 games, by contrast, under both types of perturbations convergence occurs only at rate $1/\sqrt{t}$, and the limit lies only in a neighborhood of the mixed Nash equilibrium. The same holds for case 1 games if agents make *systematic* random errors. In other words, sampling variability and payoff perturbations lead to a substantially faster rate of learning when the stable equilibria are pure as opposed to mixed. When the players make small systematic errors, the rates are the same in both cases. We suspect that similar results hold for a wide class of games, but for the moment this remains a matter of conjecture.

APPENDIX

This Appendix gives proofs of Theorems 2 and 3.

THEOREM 2. *Let G be a nondegenerate* 2×2 *game. Every perturbed best-reply dynamic converges almost surely. The limit* (X^*, Y^*) *takes exactly two values for case 1 games, one close to* (δ, δ), *the other close to* $(1 - \delta, 1 - \delta)$, *with probabilities that depend on the initial state. For case 2 games the limit is deterministic and lies close to the mixed equilibrium* (β, α). *That is, for every* $\sigma > 0$ *there is* $\varepsilon_\sigma > 0$ *such that whenever* $\varepsilon < \varepsilon_\sigma$, (X^*, Y^*) *lies within* σ *of a stable Nash equilibrium.*

Proof. Consider the system of ordinary differential equations

$$Case\ 1. \qquad \dot{x} = h_\alpha^{\varepsilon,\delta}(y) - x, \qquad \dot{y} = g_\beta^{\varepsilon,\delta}(x) - y,$$

$$Case\ 2. \qquad \dot{x} = h_\alpha^{\varepsilon,\delta}(y) - x, \qquad \dot{y} = g_{1-\beta}^{\varepsilon,\delta}(1 - x) - y. \tag{a1}$$

The functions involved in (a1) are continuously differentiable on $\mathcal{K} = [0, 1] \times [0, 1]$. Hence for every $\vec{x} \in \mathcal{K}$ there is a unique solution $\vec{y}(\vec{x}, t)$, $t \geq 0$, of (a1) such that $\vec{y}(\vec{x}, 0) = \vec{x}$.

Since in case 1,

$$\frac{\partial}{\partial x}[h_\alpha^{\varepsilon,\delta}(y) - x] + \frac{\partial}{\partial y}[g_\beta^{\varepsilon,\delta}(x) - y] = -2,$$

and in case 2,

$$\frac{\partial}{\partial x}[h_\alpha^{\varepsilon,\delta}(y) - x] + \frac{\partial}{\partial y}[g_{1-\beta}^{\varepsilon,\delta}(1 - x) - y] = -2,$$

Bendixson's criterion (see Hahn, 1967, p. 66) says that (a1) does not have cycles or phase polygons belonging to $\mathfrak{K}$. From the theorem of Bendixson (Hahn, 1967, p. 66), we conclude that every half trajectory of (a1) belonging to $\mathfrak{K}$ either is identically equal to some singular point of these equations or approaches one of them. In what follows next we shall deal only with half trajectories of (a1) belonging to $\mathfrak{K}$.

Let us first consider case 1. Arguing as in the proof of Lemma 3, we conclude that there are three singular points, (x_0, y_0), (x_β, y_α), and (x_1, y_1) such that

$$(x_0, y_0) \rightarrow (\delta, \delta), (x_\beta, y_\alpha) \rightarrow (\beta, \alpha), \text{ and } (x_1, y_1) \rightarrow (1 - \delta, 1 - \delta), \quad \text{(a2)}$$

as $\varepsilon \rightarrow 0$. The argument also shows that

$$\frac{d}{dy}h_\alpha^{\varepsilon,\delta}(y_\alpha)\frac{d}{dx}g_\beta^{\varepsilon,\delta}(x_\beta) > 1 \quad \text{(a3)}$$

for all sufficiently small ε.

Introduce the Jacobian of (a1)

$$J(x, y) = \begin{pmatrix} -1 & \dfrac{d}{dy}h_\alpha^{\varepsilon,\delta}(y) \\ \dfrac{d}{dx}g_\beta^{\varepsilon,\delta}(x) & -1 \end{pmatrix},$$

Its eigenvalues are

$$-1 \pm \sqrt{\frac{d}{dy}h_\alpha^{\varepsilon,\delta}(y)\frac{d}{dx}g_\beta^{\varepsilon,\delta}(x)}.$$

Due to (a3) one of the eigenvalues of $J(x_\beta, y_\alpha)$ is positive, so the matrix is not stable. Using standard results in stochastic approximation on the nonattainability of unstable points (see Arthur et al., 1988, Lemma 2, Pemantle, 1990, Theorem 1) we conclude that

$$P\{(X^t, Y^t) \rightarrow (x_\beta, y_\alpha)\} = 0 \quad \text{(a4)}$$

for every initial state of the system.

Next we show that the matrices $J(x_0, y_0)$ and $J(x_1, y_1)$ are stable. To

prove this, note that $h_\alpha^{\varepsilon,\delta}(\cdot)$ is increasing over $[0, 1]$, is convex–concave, and converges to the step function at α. Hence its derivative (which is monotone since the function is convex–concave) converges uniformly to 0 on any closed set that does not contain α. The same holds for the derivative of $g_\beta^{\varepsilon,\delta}(\cdot)$ on any closed set which does not contain β. Consequently $J(x_i, y_i) \to -I$ as $\varepsilon \to 0$ for $i = 0, 1$. Hence the matrices are stable for all sufficiently small ε.

Now we can prove convergence with probability one of the vector (X^t, Y^t) as $t \to \infty$. Rewrite (4) in the vector form

$$\vec{X}^{t+1} = \vec{X}^t + (1/a^{t+1})[\vec{R}(\vec{X}^t) + \vec{Q}(t, \vec{X}^t) + \vec{\zeta}(t, \vec{X}^t)], \tag{a5}$$

where for every $\vec{x} = (x, y)$ we have

$$\vec{R}(\vec{x}) = (h_\alpha^{\varepsilon,\delta}(y) - x, g_\beta^{\varepsilon,\delta}(x) - y), \qquad \vec{\zeta}(t, \vec{x}) = (\Xi^t(y), \frac{a^{t+1}}{b^{t+1}} \Psi^t(x)),$$

$$\Psi^t(x) = \psi^t(x) - E\psi^t(x), \qquad \Xi^t(y) = \xi^t(y) - E\xi^t(y),$$

$$\vec{Q}(t, \vec{x}) = (\nu_\alpha^\varepsilon(b^t, y), \quad \frac{a^{t+1}}{b^{t+1}} \mu_\beta^\varepsilon(a^t, x) + \frac{a^1 - b^1}{b^{t+1}} [g_\beta^{\varepsilon,\delta}(x) - y]).$$

Note that $\vec{R}(\cdot)$ and $\vec{Q}(t, \cdot)$ are deterministic, whereas $\vec{\zeta}(t, \cdot)$ is a random vector. The martingale convergence theorem implies that

$$\sum_{i=1}^\infty (1/a^{i+1})\vec{\zeta}(i, \vec{X}^i)$$

exists with probability one. Designate by Ω_0 the joint event that the series limit exists and that $\vec{X}^t$ does not converge to (x_β, y_α) as $t \to \infty$. Then owing to (a4), $P\{\Omega_0\} = 1$. Fix an elementary outcome ω from Ω_0. Then the realization of the stochastic sequence (a5) takes the form

$$\vec{x}^{t+1} = \vec{x}^t + (1/a^{t+1})\vec{R}(\vec{x}^t) + \vec{\sigma}^t, \tag{a6}$$

where $\vec{x}^t$ and $\vec{\sigma}^t$ stand for the realizations of $\vec{X}^t$ and $(1/a^{t+1})[\vec{Q}(t, \vec{X}^t) + \vec{\zeta}(t, \vec{X}^t)]$ and

$$\lim_{t \to \infty} \left\| \sum_{i=t}^\infty \vec{\sigma}^i \right\| = 0. \tag{a7}$$

What remains to be shown is that $\{\vec{x}^i\}$ converges to either (x_0, y_0) or (x_1, y_1).

The set $\mathcal{L}$ of all limit points of $\{\vec{x}^i\}$ is a closed, connected set, $\mathcal{L} \subseteq \mathcal{H}$. We have to show that it is a singleton set consisting of either (x_0, y_0) or (x_1, y_1).

First, let us establish that $\mathcal{L}$ is an invariant set for (a1). The series $\sum_{i \geq 1} 1/a^{i+1}$ diverges; hence for all integers $n \geq 2$ and all real $t \geq 0$ we can define

$$\vec{x}^n(t) = \vec{x}^i, \qquad \text{where } \sum_{j=n}^{i} (1/a^{j+1}) \leq t < \sum_{j=n}^{i+1} (1/a^{j+1}).$$

We include the possibility that $i = n - 1$, in which case the left-hand sum is vacuous and takes the value zero. For every convergent subsequence $\{\vec{x}^{n_p}\}$, one can show (see, for example, Benveniste *et al.*, 1990, pp. 230, 231) that for every fixed $T > 0$

$$\lim_{p \to \infty} \sup_{t \in [0,T]} \|\vec{x}^{n_p}(t) - \vec{y}(\vec{x}, t)\| = 0, \tag{a8}$$

where $\vec{x}^{n_p} \to \vec{x}$ as $p \to \infty$. For each $\vec{x} \in \mathcal{L}$ there is such a subsequence $\vec{x}^{n_p} \to \vec{x}$. Denote by $n_p(t)$ the number i such that $\vec{x}^{n_p}(t) = \vec{x}^i$. Hence by (a8)

$$\|\vec{y}(\vec{x}, t) - \vec{x}^{n_p(t)}\| \to 0 \qquad \text{as } p \to \infty, \tag{a9}$$

for an arbitrary but fixed $t > 0$. Since $\mathcal{L}$ is a closed set, by (a9), $\vec{x} \in \mathcal{L}$ implies that $\vec{y}(\vec{x}, t) \in \mathcal{L}$. Hence $\vec{y}(\vec{x}, t) \in \mathcal{L}$ for all $t > 0$. The same is true for all $t < 0$. Thus we have shown that if $\vec{x} \in \mathcal{L}$, then $\vec{y}(\vec{x}, t) \in \mathcal{L}$ for $-\infty < t < \infty$. This means that $\mathcal{L}$ is an invariant set for (a1).

Every invariant set is made up of whole trajectories (see Hahn, 1967, p. 60). Since $\mathcal{L}$ is compact, we can use the theory of Bendixson to analyze the half trajectories that begin in $\mathcal{L}$. Namely, each of them either approaches a singular point or coincides with it. Hence, if $\vec{x} \in \mathcal{L}$, then $\vec{y}(\vec{x}, t)$ converges as $t \to \infty$ to one of (x_0, y_0), (x_β, y_α), or (x_1, y_1). Since $\mathcal{L}$ is a closed set, by (a8) we conclude that at least one of these points belongs to $\mathcal{L}$.

There are two situations to consider: first, (x_0, y_0) or (x_1, y_1) or both are in $\mathcal{L}$; second, neither is in $\mathcal{L}$. In the latter case, $\mathcal{L}$ must contain (x_β, y_α). We shall show that in the first situation $\mathcal{L}$ is a singleton set, while the latter situation is impossible.

Consider the first situation. Without loss of generality we can assume that $(x_0, y_0) \in \mathcal{L}$. If $\mathcal{L} = \{(x_0, y_0)\}$, then there is nothing to prove. Let us

assume that $\mathscr{L}$ is not a singleton set and let us show that this assumption leads to a contradiction.

Since $J(x_0, y_0) \to -I$ as $\varepsilon \to 0$, by Taylor's expansion we conclude that there exists τ^0 such that $\|\vec{x} - (x_0, y_0)\| \le \tau^0$ implies

$$\langle \vec{R}(\vec{x}), \vec{x} - (x_0, y_0) \rangle \le -\tfrac{1}{2} \|\vec{x} - (x_0, y_0)\|^2 \tag{a10}$$

whenever ε is sufficiently small.

Fix $\tau > 0$ such that $\tau < \tfrac{1}{2} \min(\tau^0, \max_{\vec{x} \in \mathscr{L}} \|\vec{x} - (x_0, y_0)\|)$. Let U_σ define the open σ-neighborhood of (x_0, y_0). Since there are limit points inside of U_τ and outside of $U_{2\tau}$, there exists a sequence $\{k_p\}$ of instants when $\{\vec{x}^n\}$ is inside of U_τ, that is, $\|\vec{x}^{k_p} - (x_0, y_0)\| < \tau$, and also there exists a sequence $\{q_p\}$ of instants when $\{\vec{x}^n\}$ is outside of $U_{2\tau}$, that is, $\|\vec{x}^{q_p} - (x_0, y_0)\| \ge 2\tau$. Let us form new sequences $\{k_p'\}$ and $\{q_p'\}$ in the following way:

$$k_1' = k_1, \qquad q_p' = \min\{q_i : q_i > k_p'\}, \qquad k_{p+1}' = \min\{k_i : k_i > q_p'\}, p \ge 1.$$

Then $k_p' < q_p' < k_{p+1}'$ for all $p \ge 1$. To simplify notation, we can assume that the original sequences have this property, that is,

$$k_p < q_p < k_{p+1}, \qquad \vec{x}^{k_p} \in U_\tau, \qquad \vec{x}^{q_p} \notin U_{2\tau}, p \ge 1.$$

Now we define two new sequences $\{m_p\}$ and $\{r_p\}$. One, $\{m_p\}$, is formed by the last instants from $[k_p, q_p]$ when $\{\vec{x}^n\}$ is still in U_τ, while the other, $\{r_p\}$, is formed by the first instants from $[m_p, q_p]$ when $\{\vec{x}^n\}$ is outside of $U_{2\tau}$. That is,

$$m_p = \max\{n : k_p \le n \le q_p \text{ and } \vec{x}^n \in U_\tau\},$$

$$r_p = \min\{n : m_p \le n \le q_p \text{ and } \vec{x}^n \notin U_{2\tau}\}.$$

Since between k_p and q_p the sequence $\{\vec{x}^n\}$ moves from inside of U_τ to outside of $U_{2\tau}$, such moments always exist. The value $M = \max_{\vec{x} \in \mathscr{K}} \|\vec{R}(\vec{x})\|$ is finite; hence by (a6) and (a7)

$$\lim_{n \to \infty} \|\vec{x}^{n+1} - \vec{x}^n\| = 0. \tag{a11}$$

This implies that for all sufficiently large n the sequence $\{\vec{x}^n\}$ needs several steps to get from inside of U_τ to outside of $U_{2\tau}$. Hence $r_p > m_p$ for all sufficiently large p. Consequently, dropping some finite number of initial

terms in these sequences, we can assume that this inequality holds for all $p \geq 1$.

Since m_p is the last moment from $[k_p, q_p]$ when $\{\vec{x}^n\}$ belongs to U_τ, we have $\|\vec{x}^{m_p} - (x_0, y_0)\| < \tau$ and $\|\vec{x}^{m_p+1} - (x_0, y_0)\| \geq \tau$. By (a11) these inequalities imply that

$$\|\vec{x}^{m_p} - (x_0, y_0)\| \to \tau \qquad \text{as } p \to \infty.$$

Hence, selecting a subsequence $\{p_l\}$ if necessary, we can assume that there is $\vec{x}(\tau)$ such that

$$\vec{x}^{m_{p_l}} \to \vec{x}(\tau) \qquad \text{as } l \to \infty,$$

where $\|\vec{x}(\tau) - (x_0, y_0)\| = \tau$.

From now on we shall identify $\{m_{p_l}\}$ and $\{r_{p_l}\}$ with $\{m_p\}$ and $\{r_p\}$ to simplify notation. The properties we need are summarized in the relations

$$\lim_{p \to \infty} \vec{x}^{m_p} = \vec{x}(\tau), \qquad \|\vec{x}(\tau) - (x_0, y_0)\| = \tau, \qquad m_p < r_p, \qquad p \geq 1. \quad \text{(a12)}$$

Fix a small positive $\tau' < \tau$. For all sufficiently large p, there is a subsequence $\{l_p\}$ such that

$$l_p = \max\{n : m_p \leq n \leq r_p \text{ and } \|\vec{x}^n - \vec{x}^{m_p}\| \leq \tau'\}.$$

Since $\tau' < \tau$ and $\{\vec{x}^n\}$ between m_p and r_p moves from inside of U_τ to outside of $U_{2\tau}$, we have, for all sufficiently large p,

$$m_p < l_p < r_p,$$

Hence, dropping some finite number of initial terms in these sequences, we can assume that the above inequalities hold for all $p \geq 1$.

From (a6)

$$\vec{x}^{l_p} - \vec{x}^{m_p} = \sum_{i=m_p}^{l_p-1} (1/a^{i+1}) \vec{R}(\vec{x}^i) + \vec{\sum}^p, \qquad \text{(a13)}$$

where

$$\vec{\sum}^p = \sum_{i=m_p}^{l_p-1} \vec{\sigma}^i.$$

Letting L be a Lipschitz constant for $\vec{R}(\cdot)$ and

$$T_p = \sum_{i=m_p}^{l_p-1} (1/a^{i+1}),$$

one obtains from (a13)

$$\|\vec{x}^{l_p} - \vec{x}^{m_p}\| \le [\|\vec{R}(\vec{x}^{m_p})\| + L\tau']T_p + \left\|\vec{\Sigma}^p\right\|$$

and

$$\|\vec{x}^{l_p} - \vec{x}^{m_p}\| \ge [\|\vec{R}(\vec{x}^{m_p})\| - L\tau']T_p + \left\|\vec{\Sigma}^p\right\|.$$

By (a11) and the definition of $\{l_p\}$ we have $\|\vec{x}^{l_p} - \vec{x}^{m_p}\| \to \tau'$. Furthermore (a12) implies that $\|\vec{R}(\vec{x}^{m_p})\| \to \|\vec{R}(\vec{x}(\tau))\|$, where $\|\vec{R}(\vec{x}(\tau))\| > 0$. Let $R = \|\vec{R}(\vec{x}(\tau))\|$. From the above inequalities and (a7) it follows that

$$(R - L\tau')T_p \le \tau'[1 + o_p(1)] \le (R + L\tau')T_p,$$

where $o_p(1) \to 0$ as $p \to \infty$. Thus we may choose positive constants c and C and a $\tau^* > 0$ such that for all $\tau' < \tau^*$ and all sufficiently large p

$$c\tau' \le T_p \le C\tau'. \tag{a14}$$

From (a13) we get

$$\|\vec{x}^{l_p} - (x_0, y_0)\|^2 = \left\|\vec{x}^{m_p} - (x_0, y_0) + \sum_{i=m_p}^{l_p-1} (1/a^{i+1})\vec{R}(\vec{x}^i) + \vec{\Sigma}^p\right\|^2$$

$$= \|\vec{x}^{m_p} - (x_0, y_0)\|^2 + 2T_p(\vec{x}^{m_p} - (x_0, y_0), \vec{R}(\vec{x}^{m_p}))$$

$$+ 2\left\langle \vec{x}^{m_p} - (x_0, y_0), \sum_{i=m_p}^{l_p-1} (1/a^{i+1})[\vec{R}(\vec{x}^i) - \vec{R}(\vec{x}^{m_p})]\right\rangle \tag{a15}$$

$$+ \left\|\sum_{i=m_p}^{l_p-1} (1/a^{i+1})\vec{R}(\vec{x}^i)\right\|^2 + o_p(1),$$

where we included in $o_p(1)$ all terms containing $\vec{\Sigma}^p$. Since all terms containing space variables are bounded and, by (a7), $\|\vec{\Sigma}^p\| \to 0$, it follows that $o_p(1) \to 0$ as $p \to \infty$. By (a10) and (a14),

$$2T_p\langle \vec{x}^{m_p} - (x_0, y_0), \vec{R}(\vec{x}^{m_p})\rangle \leq - T_p\|\vec{x}^{m_p} - (x_0, y_0)\|^2$$

$$\leq -c\tau'\|\vec{x}^{m_p} - (x_0, y_0)\|^2. \tag{a16}$$

Also, since $\|\vec{x}^i - \vec{x}^{m_p}\| \leq \tau'$ for $m_p < i \leq l_p$, the Lipschitz property and (a14) imply that

$$2\left|\left\langle \vec{x}^{m_p} - (x_0, y_0), \sum_{i=m_p}^{l_p-1} (1/a^{i+1})[\vec{R}(\vec{x}^i) - \vec{R}(\vec{x}^{m_p})]\right\rangle\right|$$

$$\leq 2\|\vec{x}^{m_p} - (x_0, y_0)\|L\tau' T_p \leq 2\|\vec{x}^{m_p} - (x_0, y_0)\|LC(\tau')^2. \tag{a17}$$

Finally, since $\|\vec{x}^{l_p} - \vec{x}^{m_p}\| \leq \tau'$, it follows from (a13) that

$$\left\|\sum_{i=m_p}^{l_p-1} (1/a^{i+1})\vec{R}(\vec{x}^i)\right\|^2 \leq (\tau')^2 + o'_p(1), \tag{a18}$$

where $o'_p(1) \to 0$ as $p \to \infty$. Putting together (a15)–(a18), we get

$$\|\vec{x}^{l_p} - (x_0, y_0)\|^2 \leq \|\vec{x}^{m_p} - (x_0, y_0)\|^2(1 - c\tau')$$
$$+ 2\|\vec{x}^{m_p} - (x_0, y_0)\|LC(\tau')^2 + (\tau')^2 + o_p(1) + o'_p(1).$$

Taking into account (a12) and passing to the limit as $p \to \infty$, we obtain

$$\limsup_{p \to \infty} \|\vec{x}^{l_p} - (x_0, y_0)\|^2 \leq \tau^2(1 - c\tau') + (2\tau LC + 1)(\tau')^2.$$

If τ' is chosen small enough that $\tau^2(1 - c\tau') + (2LC + 1)(\tau')^2 < \tau^2$, then

$$\limsup_{p \to \infty} \|\vec{x}^{l_p} - (x_0, y_0)\| < \tau.$$

However, $m_p < l_p < r_p$ and $\vec{x}^n$, $m_p < n < r_p$, lie outside of U_τ. Hence $\vec{x}^{l_p}$ must be outside of U_τ, that is,

$$\|\vec{x}^{l_p} - (x_0, y_0)\| \geq \tau, \qquad p \geq 1,$$

which is a contradiction. This contradiction shows that $\mathcal{L}$ cannot be a nonsingleton set that contains (x_0, y_0) or (x_1, y_1) or both.

We conclude that either $\mathcal{L} = \{(x_0, y_0)\}$, or $\mathcal{L} = \{(x_1 \ y_1)\}$, or $\mathcal{L}$ contains neither (x_0, y_0) nor (x_1, y_1) and does contain $(x_\beta; x_\alpha)$. Let us show that this latter situation is impossible.

First, let us show that for every $\vec{x} \in \mathcal{L}$

$$\vec{y}(\vec{x}, t) \to (x_\beta, y_\alpha) \qquad \text{as } t \to \infty. \tag{a19}$$

Assume to the contrary that (15) does not hold. Since for every $\vec{z} \in \mathcal{L}, \vec{y}(\vec{x}, t)$ must converge to a singular point as $t \to \infty$, we have

$$\vec{y}(\vec{z}, t) \to (x_0, y_0) \qquad \text{or} \qquad \vec{y}(\vec{z}, t) \to (x_1 \ y_1) \text{ as } t \to \infty.$$

Since $\mathcal{L}$ is a closed invariant set, the latter implies that (x_0, y_0) or (x_1, y_1) belongs to $\mathcal{L}$, which is contrary to our current hypothesis.

Let γ be a whole trajectory and $\gamma \in \mathcal{L}$, $\gamma \neq (x_\beta, y_\alpha)$. Set γ_+ and γ_- for any half trajectories constituting γ. By (a19), γ_+ approaches (x_β, y_α) at $t \to \infty$; so does γ_- as $t \to -\infty$. (By the theory of Bendixson, each half-trajectory must approach a singular point, but due to the argument above this point cannot be (x_0, y_0) or (x_1, y_1).) Then $\gamma \cup (x_\beta, y_\alpha)$ is a phase 1-gon (see Hahn, 1967, p. 66). This is not possible, since, by the theory of Bendixson, (a1) does not have phase polygons. Consequently, the trajectories constituting $\mathcal{L}$ must coincide with (x_β, y_α), which implies that $\mathcal{L} = \{(x_\beta, y_\alpha)\}$. Thus in the second situation $\mathcal{L}$ is always a singleton set consisting of (x_β, y_α). This is not possible by (a4) and the construction of Ω_0. Consequently the second situation is not possible at all. This completes the proof for the second situation and case 1 as a whole.

In case 2, we find the solutions of the system

$$h_\alpha^{\varepsilon, \delta}(y) = x \qquad \text{and} \qquad g_{1-\beta}^{\varepsilon, \delta}(1 - x) = y.$$

An argument similar to the proof of Lemma 3 shows that it has a unique solution (x_β', y_α') and as $\varepsilon \to 0$

$$(x_\beta', y_\alpha') \to (\beta, \alpha), \qquad \text{and} \qquad \begin{aligned} &\frac{d}{dy} h_\alpha^{\varepsilon, \delta}(y_\alpha') \to \infty, \\ &\frac{d}{dx} g_{1-\beta}^{\varepsilon, \delta}(1 - x_\beta') \to \infty. \end{aligned} \tag{a20}$$

The Jacobian of this system is

$$
J(x, y) = \begin{pmatrix} -1 & \dfrac{d}{dy} h_\alpha^{\varepsilon,\delta}(y) \\[3ex] -\dfrac{d}{dx} g_{1-\beta}^{\varepsilon,\delta}(1-x) & -1 \end{pmatrix}
$$

and its eigenvalues are

$$
-1 \pm \sqrt{-\frac{d}{dy} h_\alpha^{\varepsilon,\delta}(y) \frac{d}{dx} g_{1-\beta}^{\varepsilon,\delta}(1-x)}.
$$

From (a20) we see that for sufficiently small ε the eigenvalues of $J(x_\alpha', y_\beta')$ have a negative real part. Consequently the matrix is stable. Using an argument similar to the one given above,[3] it follows that, with probability one, (X', Y') converges to (x_β', y_α'). This concludes the proof of Theorem 2.

THEOREM 3. *Let G be a nondegenerate 2×2 game. Every perturbed best reply dynamic converges to its limit at rate at least $1/\sqrt{t}$ as $t \to \infty$. More precisely, there exists $\sigma > 0$ such that for all $\varepsilon \le \sigma$ and $\vec{z} \in R^2$*

$$
\lim_{n \to \infty} P\left\{ \sqrt{t}[(X', Y') - (x, y)] < \vec{z}, \lim_{m \to \infty} (X^m, Y^m) = (x, y) \right\}
$$

$$
= P\{\vec{N}(\vec{0}, K(x, y)) < \vec{z}\} P\left\{ \lim_{m \to \infty} (X^m, Y^m) = (x, y) \right\}.
$$

Here (x, y) can be any of the points to which the process converges with positive probability, and $\vec{N}(\vec{0}, K(x, y))$ is a two-dimensional normal random vector with mean zero and variance matrix $K(x, y)$ satisfying the equation

$$
L(x, y)K(x, y) + K(x, y)L^T(x, y) = -B(x, y),
$$

---

[3] Alternatively, if we set $X = x - x_\beta'$, $Y = y - y_\alpha'$, $a(X) = -1$, $b(Y) = [h_\alpha^{\varepsilon,\delta}(Y + y_\alpha') - x_\beta']Y^{-1}$, $d(Y) = -1$, $c(X) = [g_{1-\beta}^{\varepsilon,\delta}(1 - X - x_\beta') - y_\alpha']X^{-1}$, then, taking into account monotonicity of $h_\alpha^{\varepsilon,\delta}(\cdot)$ and $g_{1-\beta}^{\varepsilon,\delta}(\cdot)$, the global stability of the point (x_β', y_α') follows from a result of Krasovskii (see Hahn, 1967, p. 132). Hence we can apply in this case the standard stochastic approximation technique (see, Benveniste *et al.*, 1990).

where $L(x, y) = J(x, y) + (1/2)I$, the sign ^{T} designates transposition, I stands for the identity matrix in R^2, and

$$B(x, y) = \begin{pmatrix} x(1 - x) & 0 \\ 0 & y(1 - y) \end{pmatrix}.$$

Case 1.

$$J(x, y) = \begin{pmatrix} -1 & \dfrac{d}{dy} h_\alpha^{\varepsilon, \delta}(y) \\ \dfrac{d}{dx} g_\beta^{\varepsilon, \delta}(x) & -1 \end{pmatrix},$$

Case 2.

$$J(x, y) = \begin{pmatrix} -1 & \dfrac{d}{dy} h_\alpha^{\varepsilon, \delta}(y) \\ -\dfrac{d}{dx} g_{1-\beta}^{\varepsilon, \delta}(1 - x) & -1 \end{pmatrix}.$$

Proof. Consider our game dynamic given as a stochastic approximation procedure (a5). Since the noise term $\vec{\zeta}(t, \cdot)$ is bounded, the Lindeberg condition holds. (See, for example, condition 4 in Theorem 6.6.1 from Nevelson and Hasminskii, 1976, Theorem 6.6.1.)

Let us check that $E\vec{\zeta}(t, (x, y))\vec{\zeta}(t, (x, y))^T$ is close to $B(x, y)$. For case 1 games we have

$$E\vec{\zeta}(t, (x, y))\vec{\zeta}(t, (x, y))^T$$

$$= \begin{pmatrix} h_\alpha^{\varepsilon, \delta}(y)[1 - h_\alpha^{\varepsilon, \delta}(y)] & 0 \\ 0 & g_\beta^{\varepsilon, \delta}(x)[1 - g_\beta^{\varepsilon, \delta}(x)] \end{pmatrix} + o_t^{(1)}(1),$$

where $o_t^{(1)}(1)$ stands for a matrix whose elements converge uniformly to 0 as $t \to \infty$. Similarly for case 2 games

$$E\vec{\zeta}(t, (x, y))\vec{\zeta}(t, (x, y))^T$$

$$= \begin{pmatrix} h_\alpha^{\varepsilon, \delta}(y)[1 - h_\alpha^{\varepsilon, \delta}(y)] & 0 \\ 0 & g_{1-\beta}^{\varepsilon, \delta}(1 - x)[1 - g_{1-\beta}^{\varepsilon, \delta}(1 - x)] \end{pmatrix} + o_t^{(2)}(1).$$

But if (x, y) is a singular point of our dynamic, then
  Case 1.

$$h_\alpha^{\varepsilon,\delta}(y) = x, \qquad g_\beta^{\varepsilon,\delta}(x) = y,$$

Case 2.

$$h_\alpha^{\varepsilon,\delta}(y) = x, \qquad g_{1-\beta}^{\varepsilon,\delta}(1 - x) = y.$$

These formulae imply

$$h_\alpha^{\varepsilon,\delta}(y)[1 - h_\alpha^{\varepsilon,\delta}(y)] = x(1 - x), \qquad g_\beta^{\varepsilon,\delta}(x)[1 - g_\beta^{\varepsilon,\delta}(x)] = y(1 - y),$$

$$g_{1-\beta}^{\varepsilon,\delta}(1 - x)[1 - g_{1-\beta}^{\varepsilon,\delta}(1 - x)] = y(1 - y).$$

Hence,

$$E\vec{\zeta}(t, (x, y))\vec{\zeta}(t, (x, y))^T = B(x, y) + o_t(1).$$

To apply the limit theorem for stochastic approximation, we need to check the stability of the matrix $J(\cdot, \cdot) + 1/2I$ at the stable singular points (see, for example, Nevelson and Hasminskii, 1976, Theorem 6.6.1, condition 2). But we already showed in the proof of Theorem 2 that in case 1 for all small enough ε, $J(\cdot, \cdot)$ is close to $-I$. In case 2, $J(\cdot, \cdot)$ has two eigenvalues with the real part equal to -1. Thus in both cases for all small enough ε, $J(\cdot, \cdot) + 1/2I$ has eigenvalues with negative real parts at the stable singular points (i.e., the limits of our process) which means that the matrices are stable. Now, taking into account that Theorem 2 guarantees convergence of (X^t, Y^t) with probability one to the limit, Theorem 3 follows from Theorem 6.6.1 by Nevelson and Hasminskii (1976) in case 2 games and from Theorems 3 and 4 by Arthur *et al.* (1987) in case 1 games.

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