GAMES AND ECONOMIC BEHAVIOR **15,** 203–227 (1996)
ARTICLE NO. 0066

# Equilibrium Selection in *n*-Person Coordination Games

Youngse Kim*

*Department of Economics, Queen Mary and Westfield College, University of London, London E1 4NS, England*

September 1, 1993

This paper investigates several approaches to equilibrium selection and the relationships between them. The class of games we study are *n*-person generalized coordination games with multiple Pareto rankable strict Nash equilibria. The main result is that all selection criteria select the same outcome (namely the risk dominant equilibrium) in two-person games, and that most equivalences break for games with more than two players. All criteria select the Pareto efficient equilibrium in voting games, of which pure coordination games are special cases. *Journal of Economic Literature* Classification Numbers: C70, C72, D82.


## 1. INTRODUCTION

Multiple equilibria are common in economic models. As an example, consider the stag hunt game. There are $n$ identical players who must choose simultaneously between two actions, $H$ and $L$. The safe strategy $L$ yields a fixed payoff $x \in (0, 1)$. The strategy $H$ yields the higher payoff of 1 if at least $\kappa$ players choose the same action $H$, but it yields nothing otherwise. For example, the median rule takes $\kappa$ to be the smallest integer larger than $n/2$, while the minimum rule takes $\kappa = n$. These games have two strict Nash equilibria in pure strategies, namely, **H** (all players choose $H$) and **L** (all choose $L$). Which equilibrium should be selected has provoked much debate.

One might argue that the Pareto-dominant equilibrium **H** is the focal point, but recent experimental results by van Huyck *et al*. (1990, 1991) show that

* This is the revised version of Chapter 2 in my dissertation at UCLA. Part of the research was done while I was at the University of Cambridge. I thank David K. Levine for his advice and encouragement. Comments from Costas Azariadis, Drew Fudenberg, Frank Hahn, Alan Kirman, Joe Ostroy, John Riley, Hamid Sabourian, and Bill Zame proved extremely beneficial. Suggestions by an associate editor and a referee improved the quality of the exposition. Financial support from the NSF and the UCLA Academic Senate is gratefully acknowledged. The usual disclaimer applies.



0899-8256/96 $18.00

subjects frequently fail to coordinate on the Pareto-dominant equilibrium. One can also question whether **H** is always the intuitively most appealing solution. For instance, action $L$ is less risky especially when $x$ and $\kappa$ are large. Harsanyi and Selten's (1988; HS henceforth) notion of *risk dominance* captures this idea in two-person games. They claim that in these games, **L** is risk dominant if and only if $x > \frac{1}{2}$. Another approach is Carlsson and van Damme's (1993a; CvD henceforth) global perturbations approach. This derives a selection rule by perturbing the original game with uncertainty about the players' information structure and embedding it in a game of incomplete information. In two-by-two games, CvD show that the selection based on global perturbations coincides with HS's risk dominance.

Another strand of literature addressed the selection problem by explicitly incorporating dynamic and evolutionary processes. Matsui and Matsuyama (1995; MM henceforth) study perfect foresight deterministic dynamics in which players discount the future and the opportunity to revise their strategies arrives randomly. MM show that, if players are sufficiently patient, or if each player can revise his strategy almost at will, a version of dynamic stability leads to the risk dominant equilibrium. The best response dynamics, which are obtained in the limit as players become myopic, do not readily distinguish between the two strict Nash equilibria since both equilibria are asymptotically stable. Young (1993) and Kandori, Mailath, and Rob (1993; KMR henceforth) have resolved this indeterminacy problem by introducing small random mutations at the individual player level, thus making the dynamic process stochastic. Foster and Young (1990; FY henceforth) and Fudenberg and Harris (1992) directly analyze stochastic replicator dynamics in which the process by which relative payoffs are translated into strategy adjustments is subject to continual perturbations. Noise causes play to shift perpetually from the neighborhood of one equilibrium to another. Long run equilibria are defined to be the states which appear with nonvanishing probability in the limit as the amount of noise vanishes. Each of these evolutionary models yields its cleanest prediction, namely the risk-dominant equilibrium, in $2 \times 2$ coordination games.[1]

This paper explores games with more than two players. Specifically, we focus on $n$-person binary action coordination games with two strict Pareto-ranked Nash

[1] There are several dynamic models in which the Pareto-dominant equilibrium is selected. Aumann and Sorin (1989) consider reputation effects in the repeated play of two-player games of common interests (i.e., games where there is a payoff vector that strongly Pareto dominates all other feasible payoffs). They show that when the possible types are all pure strategies with bounded recall then reputation effects pick out the Pareto-dominant outcome. Matsui (1991) considers a large population randomly matched to play a game of common interest with cheap talk. He shows that a unique cyclically stable set exists and contains only Pareto-dominant outcome. These works are excluded from our analysis, since we are concerned with the situation in which a large population *anonymously* play coordination games *without communication*.

equilibria. Generalization to multiperson games is also motivated by recent experimental results of van Huyck *et al*. (1990, 1991), which suggest that group size is important in determining the long run coordination outcome. The purpose of this paper is to study several approaches to equilibrium selection, to characterize fully the selection rules, and to expose the relationship among them. In particular, we study the following five models: three models of dynamic/evolutionary processes (MM, KMR, and FY) and two most salient selection models (HS and CvD).

The main result is that all selection criteria select the same outcome in two-person games and that predictions differ from each other in the games with more than two players. We provide geometric interpretations to clarify why the criteria are equivalent for two-person games but not for more general games. The idea behind our results can be understood as follows: in KMR or Young, the long run equilibrium depends only on the relative sizes of the strict equilibria's basins of attraction and not on the speed of adjustment in each basin. On the other hand, evaluation of some weighted integrals of the payoff difference function is central to characterizing the dynamic outcomes in MM, FY, and Fudenberg and Harris. All selection criteria coincide when the payoff difference function, or relative fitness, is linear in the state variable, which is the case only with a two-person game. In particular, the selected outcome in a two-person game coincides with the risk-dominant equilibrium. There is no guarantee of equivalence otherwise. As a counterexample, we later characterize the selection criteria of different approaches, in the stag hunt game described in the opening paragraph.

The rest of the paper is organized as follows. Section 2 formally defines the game of interest. Section 3 characterizes equilibrium selection criteria applying the MM, KMR, and FY dynamics. Section 4 investigates the most salient static selection criteria, namely HS's risk dominance and CvD's global perturbation. Section 5 compares these criteria, provides geometric interpretations, and proposes concrete answers to stag hunt games. Section 6 shows that all dynamic selection criteria studied in this paper support Pareto-dominance in voting games, of which pure coordination games are special cases. The final section concludes with some comments.

## 2. THE GAME

We consider a symmetric coordination game $G(n, \Pi)$, where $n$ is the number of players and $\Pi$ is the payoff matrix. Each player has binary choices available, denoted by high (H) and low (L). Consider a situation in which $(k-1)$ opponents choose H with the remaining $(n-k)$ opponents choosing L. Let $\pi_k^{\mathrm{H}}$ and $\pi_{n-k+1}^{\mathrm{L}}$, where $1 \leq k \leq n$, denote the payoff for a player taking H and L, respectively, where subscripts denote the *total* numbers of players choosing the strategy in superscripts. The class of games being studied is described by the space of payoff matrices as

$$\Omega \equiv \{\Pi \in \Re^{2n} \mid \pi_{k+1}^{\mathrm{H}} > \pi_k^{\mathrm{H}}, \text{ and } \pi_{k+1}^{\mathrm{L}} \geq \pi_k^{\mathrm{L}}, \forall k; \\ \pi_n^{\mathrm{H}} > \pi_1^{\mathrm{L}}, \pi_n^{\mathrm{L}} > \pi_1^{\mathrm{H}}; \pi_n^{\mathrm{H}} > \pi_n^{\mathrm{L}}\}, \tag{1}$$

where $\Re^{2n}$ is the $2n$-dimensional Euclidean space. The first set of inequalities in Eq. (1) imply that a player taking a particular action is no worse off when the number of opponents taking the same action increases. The next two inequalities require that all players playing a common action constitutes a strict Nash equilibrium. The last inequality means that the equilibrium when all players play H, denoted by **H**, is better than the one when all players play L, denoted by **L**. The following preliminary result is straightforward.

LEMMA 1. *If* $\Pi \in \Omega$ *then the only pure strategy equilibria of* $G(n, \Pi)$ *are the two strict Nash equilibria, viz.* **H** *and* **L**.

All the proofs are set out in the Appendix.

## 3. DYNAMIC SELECTIONS

We deal with three types of dynamic processes, namely MM, KMR, and FY in turn, to calculate equilibrium selection rules in the game $G(n, \Pi)$. To that end, we illustrate the general features and definitions common to all three dynamic processes. Time runs from $t = 0$ to $\infty$. The game $G(n, \Pi)$ is played repeatedly in a society with $N$ identical players. The population size $N$ may be finite or infinite, and $N$ is divisible by $n$ if finite. At every point in time, each player is matched to form a group with the other $(n - 1)$ players, who are randomly drawn from the population playing the game anonymously. There is inertia in the sense that not every player is able to change his strategy at will. Given the chance to switch actions, players choose a best response with respect to some suitably defined objective function. Because of anonymity, they engage in this optimization without taking into account strategic considerations such as reputation, punishment, and forward induction.

Let $y_t$ denote the fraction of the players that are committed to action H at time $t$, where the state space is $Y \subseteq [0, 1]$. Given the state $y$, let $\Pi^{\mathrm{H}}(y)$ and $\Pi^{\mathrm{L}}(y)$ denote the value of playing action H and L, respectively. We derive the *payoff difference function,* $\Phi_N(y)$, using the following definitions of coefficient functions: if $N$ is finite,[2]

$$\gamma_k(y \mid N) = \begin{cases} \dfrac{\Pi_{i=1}^{n-1}(1 - y - \frac{i}{N})}{\Pi_{i=1}^{n-1}(1 - \frac{i}{N})}, & k = 1, \\ \dfrac{\Pi_{i=1}^{k-1}(y - \frac{i-1}{N})\Pi_{i=1}^{n-k}(1 - y - \frac{1}{N})}{\Pi_{i=1}^{n-1}(1 - \frac{i}{N})}, & 2 \le k \le n-1, \\ \dfrac{\Pi_{i=1}^{n-1}(y - \frac{i-1}{N})}{\Pi_{i=1}^{n-1}(1 - \frac{i}{N})}, & k = n. \end{cases}$$

If $N$ is infinite,

$$\gamma_k(y \mid \infty) = y^{k-1}(1-y)^{n-k} = \lim_{N \to \infty} \gamma_k(y \mid N).$$

Then the payoff difference function is expressed as

$$\begin{aligned} \Phi_N(y) &\equiv \Pi^{\mathrm{H}}(y) - \Pi^{\mathrm{L}}(y) \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \gamma_k(y \mid N)\phi_k, \end{aligned} \tag{2}$$

where $\phi_k \equiv \pi_k^{\mathrm{H}} - \pi_{n-k+1}^{\mathrm{L}}$ is increasing in $k$. We suppress the subscript $N$ or $\infty$ in $\Phi$ whenever there is no confusion.

### 3.1. *Matsui and Matsuyama*

We begin with the MM dynamics. Time is continuous. The population is a continuum, i.e., $N = \infty$, so the state space is $Y = [0, 1]$. If we make $N$ finite, we still obtain a similar result. The key assumption is that not every player can switch actions at will. More specifically, we assume that the opportunity to switch actions arrives randomly, following a Poisson process with parameter $\lambda$, the mean arrival rate. It is further assumed that this process is independent across the players and that there is no aggregate uncertainty. Due to the costly adjustment assumption, the social behavior pattern $y_t$ changes continuously over time with its rate of change belonging to $[-\lambda y_t, \lambda(1 - y_t)]$. Furthermore, any

[2] Let $z$ denote the number of players choosing action H. To avoid unnecessary complications, we may assume the case where $n \le z < N - n$. The formula for $\gamma_k(y \mid N)$ is derived simply by changing variable $y = z/N$, from

$$\Phi(z) = \sum_{k=1}^{n} \frac{\binom{z}{k-1}\binom{N-z-1}{n-k}}{\binom{N-1}{n-1}} \phi_k.$$

feasible path necessarily satisfies $y_0 e^{-\lambda t} \le y_t \le 1 - (1 - y_0)e^{-\lambda t}$, where the initial condition $y_0$ is given exogenously.

When the opportunity to switch arrives, players choose the action which results in the higher expected discounted payoffs, recognizing the future path of $y$ as well as their own inability to switch actions continuously. Given the opportunity, players commit to play $H$ if $V_t > 0$, $L$ if $V_t < 0$ and are indifferent if $V_t = 0$, where

$$V_t \equiv (\lambda + r) \int_0^\infty \Phi(y_{t+s}) e^{-(\lambda+r)s}\, ds \tag{3}$$

with $r > 0$ being the discount rate. We define $\rho \equiv r/\lambda$ to be the effective discount rate or the degree of friction. Therefore, $\{y_t\}_{t \in [0,\infty)}$ is an equilibrium path from $y_0$ if its right-hand derivative exists and satisfies

$$\dot{y}_t^+ = \begin{cases} \lambda(1 - y_t) & \text{if } V_t > 0, \\ [-\lambda y_t, \lambda(1 - y_t)] & \text{if } V_t = 0, \\ -\lambda y_t & \text{if } V_t < 0, \end{cases}$$

for any $t$. This states that all players currently playing action H (respectively L) switch, if given the chance, to L (resp. H), when $V_t <$ (resp. $>$) 0.

MM specify the stability concept as follows. A state $y$ is called *accessible* from $y'$, if an equilibrium path from $y'$ that reaches or converges to $y$ exists. It is called *globally accessible* if it is accessible from any $y' \in [0, 1]$. A state $y$ is called *absorbing*[3] if a neighborhood $U$ of $y$ exists such that any equilibrium path from $U$ converges to $y$. It is *fragile* if not absorbing.[4]

To state the properties of the state $y = 0$ and $y = 1$, let us define the partition $(\Omega_0(n, \rho), \Omega_1(n, \rho), \Omega_{01}(n, \rho))$ of the set $\Omega$. For this purpose, let $\alpha$ denote an $n$-dimensional vector whose $k$th element is $\alpha_k$, $k = 1, 2, \ldots, n$, and the vector $\beta$ is similarly defined. Also, let "$\cdot$" denote the inner product of two vectors. For example, $\alpha \cdot \Pi^\zeta = \sum_{k=1}^n \alpha_k \pi_k^\zeta$. We derive Lemma 2, using the definitions of sets,

$$\Omega_0(n, \rho) = \{\Pi \in \Omega \mid \alpha(n, \rho) \cdot \Pi^{\mathrm{H}} \le \beta(n, \rho) \cdot \Pi^{\mathrm{L}}\}, \tag{4}$$

$$\Omega_1(n, \rho) = \{\Pi \in \Omega \mid \beta(n, \rho) \cdot \Pi^{\mathrm{H}} \ge \alpha(n, \rho) \cdot \Pi^{\mathrm{L}}\}, \tag{5}$$

$$\Omega_{01}(n, \rho) = \Omega \backslash (\Omega_0(n, \rho) \bigcup \Omega_1(n, \rho)), \tag{6}$$

[3] Although this is the same concept as *asymptotically stable* according to standard terminology in dynamical systems, we simply use *absorbing* due to the presence of multiple paths. It should be noted that this has nothing to do with Markov processes.

[4] MM remarked that the definition does not rule out the possibility that a state may be both fragile and globally accessible, or that a state may be uniquely absorbing but not globally accessible. However, these situations do not occur in this model.

where

$$\alpha_k(n, \rho) \equiv \frac{1+\rho}{n} \prod_{j=k}^{n} \left( \frac{j}{j+\rho} \right), \quad \beta_k(n, \rho) \equiv \alpha_{n-k+1}(n, \rho). \tag{7}$$

Lemma 4 in the Appendix provides the properties of the coefficient vector $\alpha(n, \rho)$ and $\beta(n, \rho)$. We suppress $(n, \rho)$ in defining the partitioned set whenever there is no confusion.

LEMMA 2. *The state $y$ is globally accessible iff $\Pi \in \Omega_y$ for either $y = 0$ or $y = 1$; both $y = 1$ and $y = 0$ are absorbing iff $\Pi \in \Omega_{01}$. Moreover, if an absorbing state, $y$, is globally accessible, then it is a unique absorbing state in* $[0, 1]$ *and any other state must be fragile.*

Lemma 2 states that, for a given $\rho$, if the payoff matrix lies in the region $\Omega_0(n, \rho)$, then $y = 0$ is absorbing. It also implies that there are either one or two absorbing states and that a state is uniquely absorbing if and only if it is globally accessible. In summary, for any initial behavior patterns, there is an equilibrium path that converges to the state of everyone choosing L, and, if a sufficiently large fraction of population choose L initially, any equilibrium path converges to that state. Similarly, for a given $\rho$, if the payoff matrix is in the region $\Omega_1(n, \rho)$, then $y = 1$ is absorbing. If the payoff matrix lies in the region $\Omega_{01}(n, \rho)$, on the other hand, both states are absorbing. Proposition 1(a) states that, as friction vanishes, one state becomes fragile and the other becomes globally accessible. The regions $\Omega_0$ and $\Omega_1$ shrink as friction grows and, in the limit as friction goes to infinity, disappear. Proposition 1(b) states that, in the presence of large friction, both states become absorbing.

PROPOSITION 1. (a) *If $\Pi \in \Omega$ satisfies $\sum_{k=1}^{n} w_k \pi_k^{\mathrm{H}} > \sum_{k=1}^{n} w_k \pi_k^{\mathrm{L}}$, where the weights are defined by*

$$w_k = \frac{1}{n}, \quad k = 1, 2, \ldots, n,$$

*there exists $\epsilon > 0$ such that $y = 1$ is uniquely absorbing and globally accessible for any $\rho \in (0, \epsilon)$. If the inequality is reversed, the same statement holds with $y = 0$. In the nongeneric case of equality, both $y = 0$ and $y = 1$ are absorbing for any $\rho > 0$.*

(b) *For any $\Pi \in \Omega$, there exists $\eta > 0$ such that both $y = 0$ and $y = 1$ are absorbing and no state is globally accessible for any $\rho > \eta$.*

Recall that the smaller (larger) size of $\rho$ implies more (less) patience and/or a shorter (longer) duration of an action commitment.[5] The smaller the degree of

[5] MM pointed out the following feature of the dynamics. That $r \to 0$ implies that players are more

friction, $\rho$, gets, the more the long run equilibrium tends to rely on the payoff matrix specification and the less on the initial position of strategic uncertainty, and vice versa.[6] On the other extreme case of $\rho$ approaching infinity, called the best response dynamics, both states may obtain in the long run and exactly which one would come out depends solely upon what the initial state was. In fact, the dynamic paths would be close to those studied in Gilboa and Matsui (1991).

### 3.2. *Kandori, Mailath, and Rob*

Next we study the KMR dynamic with $n \geq 2$ players matching. Time is discrete, but Kandori (1991) verifies that the results derived by KMR are robust when extended to a continuous time formulation. Population size $N$ is finite, and the state space is $Y = \{0, 1/N, \ldots, 1 - 1/N, 1\}$. Within period $t$, there are a large number of random matches among the players so that each player's average payoff in that period is equal to the expected payoff.

Consider the MM dynamics in which players become myopic. Together with the inertia assumption, this implies that, given the chance to move, each player adopts a best response against the current strategy configuration of the society as a whole. In other words, players commit to action H if $\Phi(y_t) > 0$, and to action L if $\Phi(y_t) < 0$. A deterministic Darwinian dynamic $y_{t+1} = f(y_t)$ is then defined by

$$\text{sign}(f(y) - y) = \text{sign}(\Phi(y)) \quad \text{for } 0 < y < 1.$$

Since the game $G(n, \Pi)$ has two strict Nash equilibria, the Darwinian dynamic possesses multiple steady states and that the asymptotic behavior of the system depends on the initial condition $y_0$. Indeterminacy is resolved if we perturb the system with a constant flow of mutations. For a fixed $\varepsilon$, we may define a stochastic dynamic by composing the deterministic dynamic $f$ with a random mutation under which each player's strategy at time $t + 1$ is altered to the other action with probability $\varepsilon$. The stochastic model is described by a Markov process. Since $\varepsilon$ is strictly positive, the transition matrix is irreducible and, hence, the Markov

concerned about the future. That $\lambda \to \infty$ might have two opposite effects: players are less concerned about the future whilst the current strategy distribution becomes less important. Nevertheless, a strictly positive $r$ guarantees that the second effect always dominates the first one. Therefore, the smaller $\rho$ gets, the more players worry about the future.

[6] We have assumed that the speed of adjustment, represented by Poisson arrival parameter $\lambda$, is identical over the whole population. This does not seem a severe restriction since we have studied symmetric games. Nevertheless, we can in principle incorporate asymmetric speed of adjustment into the game of interest $G(n, \Pi)$ by assuming that each population $i$ has a Poisson arrival rate $\lambda_i$, $i = 1, 2, \ldots, n$. A fair amount of numerical simulations indicate that the equilibrium criterion depends on these numbers. However, a strong result can be obtained in any—symmetric or not—$2 \times 2$ games, which states: "If and only if action H is risk dominant with respect to L (in the sense of larger Nash product), then **H** is uniquely absorbing and globally accessible for sufficiently small $\rho_i$ for $i = 1, 2$, with $\rho_1/\rho_2 = \delta$ fixed." A proof is available upon request.

process has the unique steady state distribution which indicates the proportion of time that the system spends on each state in $Y$. A strategy configuration $y \in Y$ is defined as a *long run equilibrium* (LRE) if, as $\varepsilon \to 0$, the limit distribution assigns positive probability on $y$. KMR show that the LRE corresponds to the risk dominant equilibrium in two-by-two games.

The following proposition states the selection criterion for $n$-person generalized coordination game $G(n, \Pi)$. Since the game $G(n, \Pi)$ has two strict equilibria, and the relative sizes of basins of attractions determine the LRE, it matters whether or not the payoff difference function $\Phi(y)$ cuts the horizontal axis at a point less than a half. The following proposition generalizes existing results on two-person games (KMR's Theorems 3 and 4 and Young's Theorem 3) to multiperson games.

PROPOSITION 2. *For a given game $G(n, \Pi)$ satisfying $\sum_{k=1}^{n} w_k \pi_k^{\mathrm{H}} > \sum_{k=1}^{n} w_k \pi_k^{\mathrm{L}}$ where the weights are defined by*

$$w_k \equiv \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1} \quad \textit{for } k = 1, \ldots, n,$$

*there exists an* $\mathbf{N}$ *such that the unique LRE is* $y = 1$ *for any* $N > \mathbf{N}$. *If the inequality is reversed, the LRE becomes* $y = 0$. *In the nongeneric case of equality, the LRE can be either* $y = 1$ *and* $y = 0$, *with the limit distribution placing probability half on each.*

### 3.3. *Foster and Young*

The last dynamic we study is Foster and Young (1990), which is acknowledged to be the first to consider a stochastic differential equation model of evolutionary dynamics. Time is continuous, and the population size $N = \infty$. Given the state $y_t$, the current rate of increase for H is $\Pi^{\mathrm{H}}(y)$, while the average rate of increase of the whole population is $y\Pi^{\mathrm{H}}(y) + (1 - y)\Pi^{\mathrm{L}}(y)$, where $\Pi^{\mathrm{H}}$ and $\Pi^{\mathrm{L}}$ are the value of playing action H and L, respectively. The relative rate of increase in the fraction of H is given by the deterministic replicator equation

$$\begin{aligned} \frac{dy_t/dt}{y_t} &= \Pi^{\mathrm{H}}(y_t) - [y_t\Pi^{\mathrm{H}}(y_t) + (1 - y_t)\Pi^{\mathrm{L}}(y_t)] \\ &= (1 - y_t)\Phi(y_t), \end{aligned} \tag{8}$$

where the payoff difference function $\Phi$ is defined in Eq. (2). Equation (8) can be written as

$$dy_t = y_t(1 - y_t)\Phi(y_t)dt. \tag{9}$$

This system has two asymptotically stable states, namely $y = 0$ and $y = 1$, and exactly which one is obtained depends completely on the initial state $y_0$. FY resolve this indeterminacy problem by perturbing the deterministic system with continual and nonnegligible shocks. We then obtain the following stochastic differential equation

$$dy_t = y_t(1 - y_t)\Phi(y_t)dt + \sigma dW_t, \tag{10}$$

where $W_t$ is a Gaussian noise with zero mean and unit variance. To keep the state $y$ always positive, the state space must be $Y = [\Delta, 1 - \Delta]$ for some small $\Delta > 0$.[7]

Our goal is to study the asymptotic behavior of Eq. (10) as $\sigma$ converges to zero. The state $y \in Y$ is called a *stochastically stable equilibrium* (SSE in short) if, as $\sigma \to 0$, the limiting density assigns positive probability to every small neighborhood of $y$. Theorem 2 of FY shows that computation of the SSE can be done by finding the minimum of a suitably defined potential function. The potential function $U(y)$ can be explicitly computed from the formula Eq. (5) of FY as

$$U(y) = -\int_0^y x(1 - x)\Phi(x)\, dx. \tag{11}$$

Combining all the arguments implies that the problem is to find $y \in [0, 1]$ minimizing $U(y)$. The following proposition provides the selection criterion according to the SSE notion.

PROPOSITION 3. *If* $\Pi \in \Omega$ *satisfies* $\sum_{k=1}^n w_k \pi_k^H > \sum_{k=1}^n w_k \pi_k^L$, *where the weights are defined by*

$$w_k \equiv \frac{6k(n - k + 1)}{n(n + 1)(n + 2)} \quad \text{for } k = 1, \ldots, n, \tag{12}$$

*then* $y = 1$ *is the unique SSE. If the inequality is reversed,* $y = 0$ *is the unique SSE. In the nongeneric case of equality, the SSE can be either* $y = 0$ *or* $y = 1$, *with the limit distribution placing probability half on each.*

## 4. STATIC SELECTIONS

### 4.1. *Global Perturbation*

The global perturbation approach of Carlsson and van Damme (1993a) is based on a perturbation of the players' payoff information in 2 × 2 games. The game to be played is determined by a random draw from some subclass of all 2 × 2 games. Each player observes the selected game with some noise and then chooses one of his two available actions. If the initial subclass of

[7] Fudenberg and Harris (1992) avoid this boundary problem by adding the stochastic noise to Eq. (8) instead of Eq. (9).

games is large enough and contains games with different equilibrium structures, iterative elimination of dominated strategies in the incomplete information game yields a surprising result. When the $2 \times 2$ game actually selected by Nature is a coordination game, iterated dominance forces the players to coordinate on the risk dominant equilibrium, if the amount of noise in the players' observations is sufficiently small. Carlsson and van Damme (1993b) consider a class of $n$-person binary choice games that are described in the Introduction of this paper. They analyze the global game in which the value of $x$ is observed with some noise and show that the derived selection rule differs from the HS's risk dominance. We apply CvD's idea to $n$-person generalized coordination game $G(n, \Pi)$ defined in Eq. (1). Notice that the $n$-person stag-hunt game studied by CvD is a special class of our game.

Nature draws $\theta$, which determines the payoff matrix. Each player $i$ receives a private signal $\theta_i$ that provides an unbiased estimate of $\theta$ with some noise. After observing their own signals, players then choose either H or L. Payoffs are determined by the true game and the players' choices. Let $\Theta$ be a one-dimensional random variable and let $\{E_i\}_{i=1}^n$ be an $n$ tuple of i.i.d. random variables, each having zero mean. The $E_i$ is independent of $\Theta$, with a continuous density and a support within $[-1, 1]$. For $\varepsilon > 0$, write

$$\Theta_i^\varepsilon = \Theta + \varepsilon E_i. \tag{13}$$

That $\varepsilon = 0$ implies that the true payoff realization $\theta$ is common knowledge. We are interested in what happens when $\varepsilon$ is arbitrarily small, namely under almost common knowledge.

Let $P(\theta)$ denote a $2n$-dimensional payoff vector of a perturbed game, i.e., $P(\theta) \equiv (p_1^H(\theta), \ldots, p_n^H(\theta); p_1^L(\theta), \ldots, p_n^L(\theta))$. We confine our attention to the perturbations that satisfy the following two conditions.

*Assumption* 1. (a) For each $k$, the function $p_k^{\mathrm{H}}$ (resp. $p_{n-k}^{\mathrm{L}}$) is continuous, monotonically increasing (resp. decreasing) in $\theta$, and unbounded above and below; (b) the original unperturbed game $G(n, \Pi)$ obtains with $\theta = 0$, i.e., $P(0) = \Pi$.

Let us define $\bar{\theta} \equiv \min\{\theta \mid \underline{p}^{\mathrm{H}}(\theta) \geq \bar{p}^L(\theta)\}$ and $\underline{\theta} \equiv \max\{\theta \mid \underline{p}^L(\theta) \geq \bar{p}^H(\theta)\}$, where $\underline{p}^a(\theta) = \min\{p_k^a(\theta) \mid 1 \leq k \leq n\}$ and $\bar{p}^a(\theta) = \max\{p_k^a(\theta) \mid 1 \leq k \leq n\}$ for $a \in \{\mathrm{H}, \mathrm{L}\}$. Assumption 1(a) above guarantees that $\underline{\theta}$ and $\bar{\theta}$ exist, and that $-\infty < \underline{\theta} < 0 < \bar{\theta} < +\infty$. Clearly, if $\theta$ is greater than $\bar{\theta}$ and the value of $\theta$ is common knowledge among all players, strategy H is strictly dominant in a game with payoff matrix $P(\theta)$. Similarly, if $\theta < \underline{\theta}$, strategy L is strictly dominant. The next assumption guarantees that the possibility of each strategy being strictly dominant is real.

*Assumption* 2. The $\Theta$ is uniformly distributed over an interval which contains $[\underline{\theta}, \bar{\theta}]$.

The realized value $\theta$ is almost common knowledge if $\varepsilon$ is positive but tiny. Lack of common knowledge, together with A2, suggests applying an iterative elimination of strictly dominated strategies. The next lemma shows that the Bayesian Nash equilibrium has the cutoff property and that the game considered here is indeed dominance solvable.

LEMMA 3. *If A1 and A2 hold, then the equilibrium is characterized by cut-off $\theta_{GP}$ such that player i optimally chooses H (resp. L) iff $\theta_i > (\text{resp.} <)$ $\theta_{GP}$. Furthermore, $\theta_{GP}$ is the unique root of the equation $(1/n)\sum_k p_k^H(\theta) = (1/n)\sum_k p_k^L(\theta)$.*

Recall from A1(b) that the perturbed game corresponds to the original unperturbed game $G(n, \Pi)$ when $\theta = 0$. We are interested in what happens at $\theta = 0$ when the true payoff is almost common knowledge. Recall from Eq. (13) that $|\theta_i| < \varepsilon$ if $\theta = 0$. So if $\theta_{GP} <$ (resp. $>$) 0, then $\theta_i > \theta_{GP}$ for all $i$ when $\theta = 0$ and $\varepsilon$ is sufficiently small; hence all players should optimally play H (resp. L) by Lemma 3. So we say that the equilibrium **H** in the unperturbed game is *robust with respect to global perturbation* if $\theta_{GP} < 0$, and that **L** is robust if $\theta_{GP} > 0$. Now the main result of this section follows.

PROPOSITION 4. *If $\Pi \in \Omega$ satisfies $\sum_{k=1}^n w_k \pi_k^H > \sum_{k=1}^n w_k \pi_k^L$, where the weights are defined by*

$$w_k = \frac{1}{n}, \quad k = 1, \ldots, n,$$

*then **H** is robust with respect to global perturbation. If the inequality is reversed, then **L** is robust.*

## 4.2. *Risk Dominance*

We now turn to Harsanyi and Selten's (1988) notion of risk dominance. The definition of risk dominance is based on a hypothetical process of expectation formation starting from an initial situation, where it is common knowledge that either the equilibrium **H** or **L** must be the solution without knowing which one is the solution. Roughly speaking, the coordinated equilibrium **H** risk-dominates the other coordinated equilibrium **L** if the net gain from coordination with H is relatively larger than that with L. Net gain again is defined as the payoff from successful coordination minus the loss incurred when all the opponents *collectively* choose the other action. This implies that risk dominance measures "risk" from taking a particular action in a too extreme manner. Thus, the relevant threshold value is $-\phi_1/(\phi_n - \phi_1)$, where $\phi_k = \pi_k^H - \pi_{n-k+1}^L$. Recall that the payoff difference function, $\Phi$ defined in Eq. (2), is monotone increasing and satisfies $\Phi(0) < 0 < \Phi(1)$. Hence, the equilibrium **H** risk-dominates **L** if and only if

$$\Phi\left(\frac{\phi_n}{\phi_n - \phi_1}\right) > 0. \tag{14}$$

We can easily check that Eq. (14) becomes linear in payoff matrix if $n = 2$, but it is nonlinear if $n$ exceeds two. CvD (1993b) calculate the risk dominant equilibrium in $n$-person stag-hunt game, which is a special class of the game studied in this paper.

## 5. DISCUSSION

We summarize the dynamic selection criteria characterized in previous sections. Recall that $\Pi$ is payoff matrix, $\Phi$ is the payoff difference function defined in Eq. (2) and $\phi_k \equiv \pi_k^H - \pi_{n-k+1}^L$. In the game $G(n, \Pi)$, each model in parenthesis selects the equilibrium **H** if and only if $\Pi \in \Omega$ satisfies the following condition:

$$\text{[MM]} \quad \sum_{k=1}^{n} \frac{1}{n}[\pi_k^H - \pi_k^L] > 0;$$

$$\text{[KMR]} \quad \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} [\pi_k^H - \pi_k^L] > 0;$$

$$\text{[FY]} \quad \sum_{k=1}^{n} \frac{6k(n-k+1)}{n(n+1)(n+2)}[\pi_k^H - \pi_k^L] > 0.$$

We offer brief comments on how selection mechanisms differ. In KMR and FY, the constant flow of nonnegligible noises play a crucial role in selecting among strict equilibria. This allows the dynamical process to always restart. Hence, the resulting stochastic process is ergodic, which in turn implies that each state is eventually visited with probability one. What matters is how often the different states are visited over a long time period. Both papers show that the dynamic process assigns virtually all the probability to the risk-dominant equilibrium in two-by-two games. Exactly which equilibrium is selected depends crucially on details of noise distribution. In FY, it is the drift term depending on the payoff difference in Brownian motion that leads to the present result. In KMR, it is the state-independent rate of mutation that makes only sizes of basins of attraction relevant.[8] MM investigate equilibrium selection in two-by-two games, using an explicit adjustment process. They impose perfect foresight, and there is no mutation. The perfect-foresight restriction turns out to be sufficient to elicit a unique equilibrium, which is risk-dominant in two-by-two games.

[8] It has been recognized that the selection equilibrium is sensitive to details of noise process. Bergin and Lipman (1994) verify the following result: in KMR and Young, if one allows mutation rates to vary with the state of the system, then it is always possible to introduce mutation process in such a way that, given any strict Nash equilibrium, the unique invariant distribution with mutations converges to that equilibrium as mutation vanishes. Vaughan (1994) shows that the FY-type stochastic differential equation approach may lead to different selections if the drift term is assumed to be state dependent.

Now we elaborate on what makes three dynamic selections equivalent for two-by-two games but not for more general games. Recall that the state $y \in Y$ denotes the population fraction choosing action H. Also recall that the payoff difference function, $\Phi(y)$, is strictly increasing in $y \in [0, 1]$, and that $\Phi(0) = \pi_1^{\mathrm{H}} - \pi_n^{\mathrm{L}} < 0 < \pi_n^{\mathrm{H}} - \pi_1^{\mathrm{L}} = \Phi(1)$. This immediately implies that a unique cutoff exists such that $\Phi(y) = 0$. Both MM and FY dynamics use weighted integrals of payoff difference along a potential path. To be more specific, in MM, Eq. (15) in the Appendix implies that there is a path from $y = 0$ (everyone chooses action L) to $y = 1$ (everyone chooses H) as people become increasingly patient if and only if the area below and above $\Phi(y)$ is positive. In FY, the integrand is $y(1 - y)\Phi(y)$, which is a symmetric sign-preserving transformation of $\Phi(y)$. Hence, the system Eq. (10) stays almost surely in the neighborhood of $y = 1$ if and only if the area below and above $y(1 - y)\Phi(y)$ is positive. On the other hand, in KMR, the threshold for different basins of attraction matters. More specifically, a unique long run equilibrium is $y = 1$ (everyone chooses H) if and only if $\Phi(y)$ cuts the horizontal axis at a point less than a half. If $\Phi(y)$ is linear in $y$, then the condition that the integral value of $\Phi(y)$ is positive is equivalent to the condition that $\Phi(y)$ cuts the horizontal axis at a point less than a half. If $\Phi$ is not linear, there is no guarantee of equivalence. But if the underlying game is two-person game, $\Phi$ is linear, but not if the game involves more than two players.

Now we summarize the static selection criteria:

$$[\mathrm{CvD}] \quad \sum_{k=1}^{n} \frac{1}{n}[\pi_k^{\mathrm{H}} - \pi_k^{\mathrm{L}}] > 0;$$

$$[\mathrm{HS}] \quad \sum_{k=1}^{n} \binom{n-1}{k-1} \mu^{k-1}(1-\mu)^{n-k}[\pi_k^{\mathrm{H}} - \pi_k^{\mathrm{L}}] > 0,$$

where

$$\mu = \frac{\pi_n^{\mathrm{L}} - \pi_1^{\mathrm{H}}}{(\pi_n^{\mathrm{H}} - \pi_1^{\mathrm{L}}) + (\pi_n^{\mathrm{L}} - \pi_1^{\mathrm{H}})}.$$

It is easy to check algebraically that, if $n = 2$, all five selection criteria are reduced to

$$\pi_2^{\mathrm{H}} - \pi_1^{\mathrm{L}} > \pi_2^{\mathrm{L}} - \pi_1^{\mathrm{H}}.$$

This inequality is the well-known condition that the equilibrium **H** is risk-dominant in two-by-two games. It is also immediate to show that, if $n \geq 3$, all equivalences break, except that between the MM selection and the CvD selection. As a counterexample, let us take the stag-hunt game described in the Introduction, where $n = 3$ and $\kappa = 3$ (i.e., a three-person game under minimum rule). Applying the above formula to this game at hand, we can show that each approach selects the equilibrium **L** if and only if the payoff from action L, $x$, is larger than $\frac{1}{3}$ [MM] and [CvD], $\frac{1}{4}$ [KMR], $\frac{3}{10}$ [FY], and $(\sqrt{5}-1)/2$ [HS], respectively. Arguments thus far yield the following main result.

PROPOSITION 5. *Consider the game $G(n, \Pi)$. If $n = 2$, then all five approaches select the risk-dominant equilibrium. If $n \geq 3$, the equivalences break.*

One may remark that the MM dynamic and the CvD global game approach generate the same selection criteria. We suspect that this equivalence also breaks if the uniformity assumption A2 is relaxed. Notice from Eq. (17) in the Appendix that the weight $1/n$ in the CvD formula emerges owing to the following reason: the probability that a certain number of opponents receive signals larger than my signal is independent of the exact location of my signal. But this property holds only when the distribution of true parameters is uniform.

This paper may also have substantial implications with regard to recent experimental results by van Huyck *et al.* (1990, 1991). The experiments are as follows. Each treatment lasts for 10 stages. At the end of each treatment, subjects are paid the sum of their payoffs in the games they play. In each of the games, each player chooses among seven effort levels. In each stage, each player's payoff is determined by his own effort and a simple summary statistic. This statistic is either the minimum or median of group effort choices. The parameter values are given for the normal forms to be of coordination games with seven strict Pareto-ranked symmetric Nash equilibria. A large group consists of 14 to 16 players. One interesting result was that, in large group minimum treatments, subjects initially chose widely dispersed efforts and then rapidly approached the Pareto worst equilibrium. We claim that our results can capture this aspect. To this end, consider the stag hunt game in which $n = 15$ and $\kappa = 15$. Applying the selection formula to this game, then numerical calculations show that each approach selects the Pareto-dominant equilibrium **H** if and only if $x$ is less than 0.0667 [MM and CvD], 0.0001 [KMR], 0.0221 [FY], and 0.134 [HS], respectively. This implies that, unless $x$ is extremely small, subjects' choices converge to the Pareto inferior Nash equilibrium.

## 6. MORE GENERAL GAMES

The class of games we look at is admittedly restrictive, since only binary choices are allowed. Extension to a class of games with more than two actions would be not only complicated, but the equilibrium selection would be often impossible due to the typical intransitivity among strict Nash equilibria. We study an interesting class of generalized pure coordination or simply "voting games," in which intransitivity does not arise. We define a voting game $G(n, m; \Pi^{\kappa})$, where $n$ is the number of players, $m$ is the number of choices, and the voting rule $\kappa$ will be defined below. The payoff to the player action $s = 1, 2, \ldots, m$ is described as

$$\pi^s(\sharp(1), \ldots, \sharp(m)) = \begin{cases} a_s & \text{if } \sharp(s) \geq \kappa \\ 0 & \text{otherwise,} \end{cases}$$

where $\sharp(s)$ denotes the total number of players choosing action $s$, and $\kappa$ may be $2, \ldots, n$. Moreover, all coordinated equilibria are ordered, that is, $0 < a_s \leq a_{s'}$, $\forall s < s'$. The game $G(n, m; \Pi^\kappa)$ possesses $m$ pure strategy Pareto rankable Nash equilibria, where everyone chooses action $s = 1, 2, \ldots, m$. It requires that both the voting rule (represented by $\kappa$) and the security (normalized to zero) be identical over all choices.[9]

Now we have the following.

PROPOSITION 6. *All three dynamic criteria, namely MM, KMR, and FY, select the Pareto efficient Nash equilibrium in any* $G(n, m; \Pi^\kappa)$.

The proof in the Appendix is lengthy, but the idea is intuitive. The previous sections suggest that Pareto efficiency is guaranteed when the number of actions is two; i.e., $m = 2$. With three or more actions, we apply the selection criterion in a pairwise way. The only case that we have to worry about is lack of transitivity, but this cannot occur in the class of games considered. The proposition implies that players eventually learn to play the efficient outcome in voting games. This observation is consistent with van Huyck *et al.*'s (1990) experimental results with pure coordination games, showing that actual subjects move swiftly to the Pareto best equilibrium effort level, regardless of group size.

## 7. CONCLUSION

We have generalized results on equilibrium selection in the direction of group size. However, the assumption of binary strategies is obviously restrictive. Effort is needed to generalize in encompassing multiactions. Pairwise comparisons may be a natural criterion, but we have to restrict the class of games, in order to preserve transitivity. As is shown in Section 6, a generalized pure coordination or voting game preserves such transitivity. On the other hand, it is easy to construct a game in which transitivity does not hold. Young (1993) analyzes a two-person three-action game where pairwise risk dominance fails but, nevertheless, a unique long run equilibrium exists. This fact suggests modification or refinement of risk-dominance. Ellison (1994) characterizes KMR-style long run equilibria in two-person multiaction games. More importantly, he shows that Morris, Rob, and Shin's (1995) refinement of risk-dominance, called $\frac{1}{2}$-dominance, is a sufficient condition for an equilibrium to be the unique long run equilibrium.

[9] We can easily construct counterexamples demonstrating the fact that both identical rule and equal security are necessary and sufficient to guarantee the Pareto efficiency.

A more important research agenda will be to clarify the general relationship between the nature of the underlying dynamics and selected static equilibrium. Recent papers, such as Binmore, Samuelson, and Vaughan (1995) and Börgers and Sarin (1993), attempt to address such an issue. Binmore *et al.* emphasize the importance of the order in which certain limits are taken and of the time span over which one desires to study the behavior of the selection model. Börgers and Sarin show that, in the continuous time limit, a version of a stochastic aspiration-based learning model coincides with the deterministic, continuous time replicator dynamics. We will have to await further research in these directions for answers.

## APPENDIX

*Proof of Lemma* 1. Suppose, to the contrary, that the pure strategy profile of exactly $k$ players choosing H and $(n-k)$ players choosing L is a Nash equilibrium. Then both $\pi^{\mathrm{L}}_{n-k} \geq \pi^{\mathrm{H}}_{k+1}$ and $\pi^{\mathrm{H}}_{k} \geq \pi^{\mathrm{L}}_{n-k+1}$ hold for such $k$. Adding the two inequalities yields

$$-(\pi^{\mathrm{L}}_{n-k+1} - \pi^{\mathrm{L}}_{n-k}) \geq \pi^{\mathrm{H}}_{k+1} - \pi^{\mathrm{H}}_{k}$$

which contradicts the definition of the set $\Omega$. ■

*Characterization of the vector $\alpha$ and $\beta$.* Equations (4) to (6) define the sets $\Omega_0$, $\Omega_1$, and $\Omega_{01}$, where the element of the coefficient vectors $\alpha$ and $\beta$ is

$$\alpha_k(n,\rho) \equiv \frac{1+\rho}{n}\prod_{j=k}^{n}\left(\frac{j}{j+\rho}\right), \qquad \beta_k(n,\rho) \equiv \alpha_{n-k+1}(n,\rho).$$

The following lemma characterizes the properties of the coefficient vectors.

LEMMA 4. *For any n given,* (a) $\sum_{k=1}^{n}\alpha_k = \sum_{k=1}^{n}\beta_k = 1, \forall\rho$; (b) $\alpha_{k+1} > \alpha_k$ *and* $\beta_{k+1} < \beta_k$, $\forall k$, $\rho \in (0,\infty)$; (c) $\lim_{\rho\to 0}\alpha_k = \lim_{\rho\to 0}\beta_k = 1/n$, $\forall k$; (d) $\lim_{\rho\to\infty}\alpha = (0,\ldots,0,1)$; *and* $\lim_{\rho\to\infty}\beta = (1,0,\ldots,0)$.

*Proof.* (a) Via mathematical induction. Checking the case of $n=2$ is trivial. Supposed that it holds for $n-1$, i.e., $\sum_{k=1}^{n-1}\prod_{j=k}^{n-1}(j/(j+\rho)) = (n-1)/(1+\rho)$, then for $n$

$$\begin{aligned}\sum_{k=1}^{n}\alpha_k &= \frac{1+\rho}{n}\sum_{k=1}^{n}\prod_{j=k}^{n}\left(\frac{j}{j+\rho}\right)\\ &= \frac{1+\rho}{n}\left[\frac{n}{n+\rho} + \frac{n}{n+\rho}\sum_{k=1}^{n-1}\prod_{j=k}^{n-1}\left(\frac{j}{j+\rho}\right)\right]\\ &= \frac{1+\rho}{n}\frac{n}{n+\rho}\left[1+\frac{n-1}{1+\rho}\right] = 1.\end{aligned}$$

The fact that $\sum_{k=1}^{n} \beta_k = 1$ is trivial since the elements of the vector $\beta$ are just a rearrangement of those of $\alpha$. To check (b), (c), and (d) is straightforward. ■

*Proof of Lemma* 2. First of all, notice that $\Phi(0) = \pi_1^H - \pi_n^L < 0 < \Phi(1) = \pi_n^H - \pi_1^L$ and that $\Phi$ is strictly increasing, since

$$\Phi'(y) = (n-1)\sum_{k=0}^{n-2} \binom{n-2}{k} y^k (1-y)^{n-k-2}[\phi_{k+2} - \phi_{k+1}] > 0$$

by the definition of the $\phi$ function and the non-decreasing property of the $\pi_k$ sequences.

The outcome **H** can be upset when players have an incentive to deviate for a feasible path from $y = 1$. Because of the monotonicity of $\Phi$, the incentive to deviate is the strongest if all players are anticipated to switch from H to L in the future, i.e., $y_t = e^{-\lambda t}$. Hence, the condition for $y = 1$ being fragile is

$$V_0 = (\lambda + r)\int_0^{\infty} \Phi(e^{-\lambda s})e^{-(\lambda+r)s}\, ds \leq 0,$$

which would be by the change-of-variable technique

$$(1+\rho)\int_0^1 \Phi(y)y^{\rho}\, dy \leq 0. \tag{15}$$

Using Eq. (2), the definition and properties of the beta and gamma functions[10] and some algebraic manipulation, Eq. (15) becomes

$$\begin{aligned} 0 \;\geq\; & (1+\rho)\sum_{k=1}^{n} \binom{n-1}{k-1} \phi_k \int_0^1 y^{k+\rho-1}(1-y)^{n-k}\, dy \\ =\; & (1+\rho)\sum_{k=1}^{n} \binom{n-1}{k-1} \phi_k \frac{\Gamma(k+\rho)\Gamma(n-k+1)}{\Gamma(n+\rho+1)} \\ =\; & \sum_{k=1}^{n} \alpha_k \phi_k, \end{aligned}$$

or, equivalently,

$$\sum_{k=1}^{n} \alpha_k \pi_k^{\mathrm{H}} \leq \sum_{k=1}^{n} \alpha_k \pi_{n-k+1}^{\mathrm{L}} = \sum_{k=1}^{n} \beta_k \pi_k^{\mathrm{L}}, \tag{16}$$

which corresponds to the condition defining the $\Omega_0$ set.

[10] Refer to any text on mathematical statistics.

We claim: $y = 0$ is globally accessible if and only if $\Pi \in \Omega_0$. To prove the "if" part, it suffices to show that, if Eq. (16) holds, i.e., $\Pi \in \Omega_0$, a feasible path from $y = 1$ to $y = 0$, $y_t = e^{-\lambda t}$, satisfies the equilibrium condition, i.e., $V_t \le 0$ $\forall t$ along the path. This can be checked as follows:

$$\begin{aligned} V_t &= (\lambda + r)\int_0^\infty \Phi(y_{t+s})e^{-(\lambda+r)s}\,ds \\ &\le (\lambda + r)\int_0^\infty \Phi(e^{-\lambda s})e^{-(\lambda+r)s}\,ds \le 0 \quad \forall t. \end{aligned}$$

To prove the "only if" part, it suffices to demonstrate that, if $\Pi \in \Omega\backslash\Omega_0$, the equilibrium path is unique and converges to $y = 1$ for $y_0$ sufficiently close to 1. Reminding that any feasible path from $y_0$ satisfies $y_t \ge y_0 e^{-\lambda t}$, we get

$$V_0 \ge (\lambda + r)\int_0^\infty \Phi(y_0 e^{-\lambda s})e^{-(\lambda+r)s}\,ds.$$

Since the right-hand side is strictly positive at $y_0 = 1$ and continuous in $y_0$, it is still positive for $y_0$ sufficiently close to 1.

We also claim that $y = 1$ is absorbing if and only if $\Pi \in \Omega\backslash\Omega_0$. To prove the "only if" part is exactly the same as to prove the "if" part of the statement that $y = 0$ is globally accessible iff $\Pi \in \Omega_0$. Similarly, to prove the "if" part is exactly the same as to prove the "only if" part of the statement that $y = 0$ is globally accessible iff $\Pi \in \Omega_0$.

Similarly, the condition for $y = 0$ being fragile combined with the change of variable technique will be

$$\begin{aligned} V_0 &= (\lambda + r)\int_0^\infty \Phi(1 - e^{-\lambda s})e^{-(\lambda+r)s}\,ds \\ &= (1 + \rho)\int_0^1 \Phi(y)(1 - y)^\rho\,dy \ge 0. \end{aligned}$$

Again by the definition of $\Phi$ function, the properties of gamma and beta functions and some algebraic manipulation, we have

$$\begin{aligned} 0 &\le \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_k \frac{\Gamma(k)\Gamma(n-k+\rho)}{\Gamma(n+1+\rho)} \\ &= \sum_{k=1}^{n} \beta_k \phi_k, \end{aligned}$$

or, equivalently,

$$\sum_{k=1}^{n} \beta_k \pi_k^{\mathrm{H}} \le \sum_{k=1}^{n} \beta_k \pi_{n-k+1}^{\mathrm{L}} = \sum_{k=1}^{n} \alpha_k \pi_k^{\mathrm{L}},$$

which is the condition defining $\Omega_1$. A symmetric argument as before shows that $y = 1$ is globally accessible if and only if $\Pi \in \Omega_1$ and that $y = 0$ is absorbing if and only if $\Pi \in \Omega \setminus \Omega_1$.

Combining all the facts shown yields the desired result. ■

*Proof of Proposition* 1. Part (a) is clear from Lemma 2(b) and (c). As $\rho \to \infty$, Lemma 2(d), together with Eq. (1) implies that both $\Omega_0$ and $\Omega_1$ converge to the empty set, while $\Omega_{01}$ converges to the whole set $\Omega$. ■

To prove Proposition 2, the following two lemmas are helpful.

LEMMA 5. *For N sufficiently large,* $\Phi_N(y) = 0$ *has the unique root in* $[0, 1]$.

*Proof.* Differentiate $\Phi_N$ defined in Eq. (2) with respect to $y$, expand the resulting equation, and rearrange terms; then we have

$$\Phi'(y) \geq \frac{(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k} y^k(1-y)^{n-k-2}(\phi_{k+2}-\phi_{k+1}) + C/N}{\prod_{i=1}^{n-1}(1-i/N)}$$

Here the constant $C$ is obtained from the exact expansion by replacing $y$ and $1/N$ with 0's (resp. 1's) if the coefficient $\phi_k$ is positive (resp. negative). Note that the first term of the numerator and the denominator are strictly positive, regardless of $N$. For any $\epsilon > 0$, the second term $C/N > -\epsilon$ for $N$ sufficiently large. Hence, $\Phi(y)$ is increasing in $y$ for $N$ large enough. It is trivial to show that $\Phi(0) < 0 < \Phi(1)$. Combining these facts yields the desired result. ■

LEMMA 6. *For N sufficiently large and any Darwinian deterministic dynamic, the limit distribution for* $G(n, \Pi)$ *puts probability one on* 1 *if* $y^* < \frac{1}{2}$, *or probability one on* 0 *when the inequality is reversed.*

*Proof.* The same as that of KMR's Theorem 3; thus it is omitted. ■

*Proof of Proposition* 2. In principle, we can calculate the unique root $y^*$ as a function of $n$, $\Pi$, and $N$, and then see what happens to the equation $y^*(N) = \frac{1}{2}$ as $N$ becomes large. But this procedure is rather complicated. The trick is to plug $y = 1/2$ directly into the equation $\Phi(y) = 0$, and then see what happens in the limit as $N \to \infty$. Since it is easy to check

$$\lim_{N\to\infty} \gamma_k\left(\frac{1}{2} \mid N\right) = \left(\frac{1}{2}\right)^{n-1},$$

we are done. ■

*Proof of Proposition* 3. The problem is $\min_{y\in[0,1]} U(y)$, where the potential function $U$ is defined in Eq. (11). We claim that this is equivalent to the following problem: to choose $y = 0$ if $U(1) > 0$, and choose $y = 1$ if $U(1) < 0$. Notice that $-U(y)$ is the value of integral of the function $x(1-x)\Phi(x)$ over $[0, y]$. Since $\Phi$ is strictly increasing and $\Phi(0) < 0 < \Phi(1)$, it is clear that $-U(1) > -U(y)$ for any $y \in [0, 1)$. Hence, if $-U(0) > 0$, the maximum $-U(1)$ obtains at $y = 1$. On the other hand, if $-U(0) < 0$, the maximum $-U(0) = 0$ obtains at $y = 0$. But,

$$\begin{aligned} U(1) &= \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_k \int_0^1 x^k (1-x)^{n-k+1}\, dx \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_k \frac{\Gamma(k+1)\Gamma(n-k+2)}{\Gamma(n+3)} \\ &= \sum_{k=1}^{n} \frac{k(n-k+1)}{n(n+1)(n+2)} \phi_k. \end{aligned}$$

Let insert $\phi_k \equiv \pi_k^H - \pi_{n-k+1}^L$ into the above expression and multiply both sides by six in order to make the weights sum to one. ∎

*Proof of Proposition* 4. Notice that the existence and uniqueness of such $\theta_{GP}$ are guaranteed by Assumption 1(a) and 1(c). As was suggested, we maintain the assumption that no player will choose strictly dominated strategies. Player $i$ will certainly choose H if $\theta_i > \bar{\theta}$: Since the expected value is $E(\Theta \mid \theta_i^\varepsilon = \theta_i) = \theta_i$, player $i$ knows that H is strictly dominant at each such observation.

Consider an observation $\theta_i$ of player $i$ slightly below $\bar{\theta}$, such be that $|\bar{\theta} - \theta_i| < 2\varepsilon$. Player $i$ knows that his opponent will play H if $\theta_j > \bar{\theta}$; hence, $i$'s payoff if he chooses H at $\theta_i$ is approximately

$$\sum_{k=1}^{n} \Pr(\theta_j > \theta_i \quad \text{for exactly } k-1 \text{ opponents } \mid \Theta_i^\varepsilon \approx \bar{\theta})\, p_k^{\mathrm{H}}(\bar{\theta}) \tag{17}$$

$$= \sum_{k=1}^{n} \Pr(E_j > E_i \quad \text{for exactly } k-1 \text{ opponent})\, p_k^{\mathrm{H}}(\bar{\theta}) \tag{18}$$

$$= \frac{1}{n} \sum_{k=1}^{n} p_k^{\mathrm{H}}(\bar{\theta}). \tag{19}$$

Assumption 2 allows us to conclude that the probability in the Eq. (17) is independent of $\theta_i$, at least as long as $\theta_i$ lies $\varepsilon$ inside the support of $\Theta$. This observation allows us to conclude that this probability must be equal to the a priori probability that $E_i$ is the $(k+1)$th smallest among the errors. Thus, Eq. (18) ensues, the probability in which is clearly the same for all players. This fact, combined with the assumption that the i.i.d. of $E_i$ has a continuous density, yields Eq. (19).

A similar reasoning shows that the expected payoff to action L is at most approximately $(1/n)\sum_{k=1}^{n} p_k^L(\bar{\theta})$, which is strictly lower than $(1/n)\sum_{k=1}^{n} p_k^H(\bar{\theta})$ calculated above by the monotonicity assumption 1(a). Hence, if $\theta_{GP} < \bar{\theta}$, there exists $\bar{\theta}^1$ such that H is strictly dominant for any $\theta_i > \bar{\theta}^1$ in the reduced game where player $j$ is constrained to play H when $\theta_j > \bar{\theta}$. In a similar way one can construct $\bar{\theta}^2 < \bar{\theta}^1$ and continuing inductively, we can find sequences $\bar{\theta}^m$ such that H is iteratively dominant for $\theta_i > \bar{\theta}^m$.

On the other hand, starting from the maintained assumption that action L will be chosen when $\theta_i < \underline{\boldsymbol{\theta}}$, we inductively find a sequence $\underline{\boldsymbol{\theta}}^m$ such that L is iteratively dominant for $\theta_i < \underline{\boldsymbol{\theta}}^m$. By the definition of $\theta_{GP}$, it is obvious that $\bar{\theta}^m \downarrow \theta_{GP}$ and $\underline{\boldsymbol{\theta}}^m \uparrow \theta_{GP}$ as $m \to \infty$. ■

*Proof of Proposition* 6. (1) MM. All the proofs of Section 3.2 apply straightforwardly, so we omit them. After all, we are able to show that: if $\rho \in (0, \bar{\rho}]$ for some $\bar{\rho} > 0$, then the Pareto efficient outcome is uniquely absorbing and globally accessible.

(2) KMR. Let $z^s$ denote the number of players choosing strategy $s = 1, 2, \ldots, m$. Given the chance to move and the state $\mathbf{z} = (z^1, \ldots, z^m)$, the expected average payoff for the player who has been choosing action $s$ is calculated as

$$\begin{array}{ll} f_\kappa(z^s - 1)a_s & \text{if he chooses } s \text{ again} \\ f_\kappa(z^{s'})a_{s'} & \text{if he chooses } s' \neq s, \end{array} \tag{20}$$

where

$$f_\kappa(z) = \sum_{k=\kappa}^{n} \frac{\binom{z}{k-1}\binom{N-z-1}{n-k}}{\binom{N-1}{n-1}}. \tag{21}$$

and $z \in Z \equiv \{n-1, n, \ldots, N-n\}$. The next lemma is just a technical result but plays an important role in what follows.

LEMMA 7. *For any $\kappa$, the function $f_\kappa(z)$ is strictly increasing in $z \in Z$.*

*Proof.* We ignore the denominator of Eq. (21), since it is positive independently of $\kappa$ or $z$. If $\kappa = n$, it is straightforward to show that

$$f_n(z) - f_n(z-1) = \binom{z-1}{n-2}\binom{N-z-1}{0}.$$

If $\kappa = n - 1$, then

$$\begin{aligned} f_{n-1}(z) - f_{n-1}(z-1) &= \left[\binom{z}{n-2}\binom{N-z-1}{1} - \binom{z-1}{n-2}\binom{N-z}{1}\right] \\ &\quad - (f_n(z) - f_n(z-1)) \\ &= \binom{z}{n-2}\binom{N-z-1}{1} \\ &\quad - \left[\binom{z-1}{n-2}\binom{N-z}{1} - (f_n(z) - f_n(z-1))\right] \\ &= \binom{z-1}{n-3}\binom{N-z-1}{1}. \end{aligned}$$

Likewise, we can show

$$f_\kappa(z) - f_\kappa(z-1) = \binom{z-1}{\kappa-2}\binom{N-z-1}{n-\kappa},$$

which is positive for any $z \in Z$. Since $f_\kappa(z) > f_\kappa(z-1)$ for all $z \in Z$ and for any $\kappa$, we obtain the desired result. ■

LEMMA 8. *Any mixed strategy is unstable.*

*Proof.* Assume not; i.e., there exist $s, s' \in C(\mathbf{z})$ with $s < s'$, and both $s$ and $s'$ are best responses to $\mathbf{z}$. Then we get

$$f(z^{s'} - 1)a_{s'} \geq f(z^s)a_s > f(z^s - 1)a_s \geq f(z^{s'})a_{s'} > f(z^{s'} - 1)a_{s'}.$$

The strict inequalities follow from Lemma 7 and the weak inequalities follow from the presumed optimality of $s$ and $s'$ relative to $\mathbf{z}$. The contradiction establishes the desired result. ■

LEMMA 9. *The collection of limit sets is* $\{e^s\}_{s=1}^m$, *where* $e^s$ *is the state of all population choosing strategy* $s$.

*Proof.* The same logic as in Proposition 9(2) of Kandori and Rob (1995; KR henceforth) applies, so the proof is omitted. ■

*Proof of the KMR part.* The first task is to compute costs of transition $C_{s's}$ between limit sets, $e^s$ and $e^{s'}$. Assume the society is initially clustered at $e^{s'}$, then the minimum number of mutations, $x$, needed to switch it over into the basin of attraction of $e^s$ is determined by $f(x)a_s \geq f(N-1-x)a_{s'}$. This represents an immediate jump to escape the best response region of $s'$, and the triangular inequality argument of KR's Proposition 5 guarantees that no gradual escape is less costly than this immediate jump. Note that we mutate individuals taking $s'$ into $s$, because any other mutation will only raise the transition cost more. Thus, the cost of transition $C_{s's}$ is the minimum integer $x$ satisfying

$$f(x) \geq f(N-1-x)(a_{s'}/a_s). \tag{22}$$

It has a unique root, since Lemma 7 implies that the left-hand side of Eq. (22) is strictly increasing and so its right-hand side is strictly decreasing in $x$.

Since a pure coordination game $G(n, m; \Pi^\kappa)$ specifies $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m$, we can easily check that

$$C_{s'm} < C_{s's} \quad \forall s < m, \forall s' \neq s; \quad C_{m,m-1} < C_{s',m-1} \quad \forall s' < m-1.$$

Therefore, the first step of the optimum branching algorithm as in KR, pages 407–410, is to choose a minimum cost outgoing branch from each state, which results in the system of branches $(s \to m)$, $s = 1, 2, \ldots, m-1$, and $(m \to m-1)$. The longest branch among these is of length $C_{m,m-1}$. Therefore we drop it and are left with an $m$-tree. This completes the algorithm. ∎

(3) FY. Due to Young's Theorem 2 and FY's Theorem 2, it is essentially the same as case (2) above; thus the proof is omitted. ∎

## REFERENCES

Aumann, R. J., and Sorin, S. (1989). "Cooperation and Bounded Recall," *Games Econ. Behav.* **1**, 5–39.

Bergin, J., and Lipman, B. L. (1994). "Evolution with State-Dependent Mutations," *Econometrica*, forthcoming.

Binmore, K. G., Samuelson, L., and Vaughan, R. (1995). "Musical Chairs: Modelling Noisy Evolution," *Games Econom. Behav.* **11**, 1–35.

Börgers, T., and Sarin, R. (1993). "Learning Through Reinforcement and Replicator Dynamics." Mimeo, University College London.

Carlsson, H., and van Damme, E. (1993a). "Global Games and Equilibrium Selection," *Econometrica* **61**, 989–1018.

Carlsson, H., and van Damme, E. (1993b). "Equilibrium Selection in Stag Hunt Games," in *Frontiers of Game Theory* (K. G. Binmore, A. Kirman, and P. Tani, Eds.). Cambridge, MA: MIT Press.

Ellison, G. (1994). "Stochastic Evolution, Basins of Attraction, and Half-dominant Equilibria." Mimeo, Harvard University.

Foster, D., and Young, P. H. (1990). "Stochastic Evolutionary Game Dynamics," *Theoret. Population Biol.* **38**, 219–232.

Fudenberg, D., and Harris, C. (1992). "Evolutionary Dynamics with Aggregate Shocks," *J. Econ. Theory* **57**, 420–441.

Gilboa, I., and Matsui, A. (1991). "Social Stability and Equilibrium," *Econometrica* **59**, 859–868.

Harsanyi, J. C., and Selten, R. (1988). *A General Theory of Equilibrium Selection in Games*. Cambridge, MA: MIT Press.

Kandori, M. (1991). "Independent Mutations and Long Run Equilibria in Games: A Continuous Time Formulation." Mimeo, Princeton University.

Kandori, M., Mailath, G., and Rob, R. (1993). "Learning, Mutation and Long Equilibria in Games," *Econometrica* **61**, 27–56.

Kandori, M., and Rob, R. (1995). "Evolution of Equilibria in the Long Run: A General Theory and Applications," *J. Econ. Theory* **65**, 383–414.

Matsui, A. (1991). "Cheap-Talk and Cooperation in a Society," *J. Econ. Theory* **54**, 245–258.

Matsui, A., and Matsuyama, K. (1995). "An Approach to Equilibrium Selection," *J. Econ. Theory* **65**, 415–434.

Morris, S., Rob, R., and Shin, H. S. (1995). "p-Dominance and Belief Potential," *Econometrica* **63**, 145–158.

van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1990). "Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure," *Amer. Econ. Rev.* **80**, 234–248.

van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1991). "Strategic Uncertainty, Equilibrium Selection, and Coordination Failure in Average Opinion Games," *Quart. J. Econ.* **106**, 885–910.

Vaughan, R. (1994). "Evolutive Equilibrium Selection 1: Symmetric Two-player Binary-choice Games." Mimeo, University College London.

Young, P. H. (1993). "The Evolution of Conventions," *Econometrica* **61**, 57–84.