Buyer Coalition against Monopolistic Screening: On the Role of Asymmetric Information among Buyers*

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Abstract

The traditional theory of monopolistic screening tackles individual self-selection but has not paid enough attention to the possibility that buyers form a coalition to coordinate their purchases and to reallocate the goods. In this paper, we design the optimal sale mechanism which takes into account both individual and coalition incentive compatibility focusing on the role of asymmetric information among buyers.

We show that when buyer coalition is formed under asymmetric information, the monopolist can do as well as when there is no coalition. Although in the optimal sale mechanism marginal rates of substitution are not equalized across buyers (hence there exists room for arbitrage), they fail to realize the gains from arbitrage because of the transaction costs in coalition formation generated by asymmetric information.

Key Words: Monopolistic Screening, Coalition Incentive Compatibility, Asymmetric Information, Transaction Costs.

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1 Introduction

The theory of monopolistic screening[1] (second-degree price discrimination) studies monopolist's optimal pricing scheme when she has incomplete information about buyers' individual preferences.[2] According to the theory, the monopolist can maximize her profit by using a menu of packages which induces each type of buyer to select the package designed for him. While the theory tackles the self-selection issue at the individual level, it has not paid enough attention to the possibility that buyers might form a coalition to coordinate their purchases, possibly at the expenses of the seller. In other words, the theory is concerned with individual incentive compatibility but not with coalition incentive compatibility.

In reality, there exist rich evidences of buyers' joint actions. Bidders' collusive behavior in auctions is well acknowledged and auction literature has been devoting an increasing attention to the topic.[3] We observe a lot of cooperatives formed by buyers to purchase goods jointly.[4] In the case of information goods such as CDs, DVDs, softwares, consumers often share goods by illegally copying products among themselves.[5]

In this paper, we study the optimal sale mechanism which takes into account both individual and coalition incentive compatibility focusing on the role of asymmetric information among buyers (about each other's willingness to pay) at the coalition formation stage. In particular, we are interested in i) identifying the transaction costs in coalition formation that are generated by asymmetric information and ii) designing the optimal sale mechanism that exploits these transaction costs.

Consider for example the situation in which an upstream monopolist sells her goods to two downstream firms operating in separate markets. Given a menu of quantity-transfer pairs offered by the monopolist, the two downstream firms can employ two

[1] See, for instance, Maskin and Riley (1984) and Mussa and Rosen (1978) for an introduction.

[2] In what follows, we use 'she' to represent the monopolist or the seller and 'he' to represent a buyer or the third-party.

[3] For examples, see Caillaud and Jehiel (1998), Graham and Marshall (1987) and McAfee and McMillan (1992).

[4] See Heflebower, R. (1980). He describes three types of supply cooperatives. First, farmers form cooperatives to purchase feeds, fertilizers, petroleum products etc. Second, there also exist cooperatives run by urban businesses: for instance, baking companies form cooperatives to purchases materials and equipments cooperatively. Third, there are consumer cooperatives.

[5] See, for instance, Bakos et als (1999).

instruments to increase their joint payoffs. First, they can jointly decide which pair each buyer should choose. In our paper, this is modeled by manipulation of the reports which the buyers send into the sale mechanism. Second, after maximizing their pie through the manipulation of reports, they can reallocate among themselves the goods bought from the seller. We assume that they can use side-transfers to share the gains from manipulation of reports and/or reallocation of goods. As the main result, we show that when buyer coalition is formed under asymmetric information, the monopolist can do as well as when there is no coalition by fully exploiting the transaction costs in coalition formation. Although in the optimal sale mechanism the marginal rates of substitutions are not equalized across buyers with different types (hence there exists room for arbitrage), the buyers fail to realize any gain from arbitrage because of the incentive problem inside the coalition. We quantify the transaction costs due to asymmetric information and show that they are larger than the gains from arbitrage. We also show that the allocation obtained by the optimal sale mechanism which deters buyer coalition at no cost can be implemented through a menu of two-part tariffs.

In our model, the seller can produce any positive amount of homogeneous goods at a constant marginal cost.[6] She offers a sale mechanism to a finite number of buyers. For expositional simplicity, we focus on the two-type environment in most of the sections and show later on (in Section 6) that our main result extends to more general settings. In the two-type setting, a buyer has either high valuation (H-type) or low valuation (L-type) for the goods on sale. Types are independently and identically distributed and a buyer's type is his private information. In the optimal sale mechanism without buyer coalition, the quantity sold to a buyer depends solely on his report. It is a standard result that the quantity allocated to a type is determined by equalizing the marginal cost to the type's marginal surplus, evaluated with the virtual valuation. As is well known, L-type's virtual valuation is lower than his real valuation[7] and this results in a downward distortion in the quantity allocated to L-type compared to the first-best level.

The fact that the seller intentionally introduces a downward distortion in L-type's quantity creates room for buyer arbitrage since, at the optimal quantity profile, L-type has a higher marginal surplus than H-type. Consider the previous example of one

[6]This makes our setting different from an auction, in which the seller is tipically quantity-constrained.

[7]This is because the payment received by the monopolist from H-type is decreasing in the quantity sold to L-type.

upstream firm selling to two downstream firms. Suppose that the state of nature is such that one of the downstream firms has H-type while the other has L-type. Then, in the absence of transaction cost in coalition formation, they can successfully form a coalition to reallocate some quantity from H-type to L-type and increase their joint payoffs. Furthermore, this could alter buyers' incentive to report truthfully in the sale mechanism and eventually modify the seller's expected profit. In this paper, we focus on how asymmetric information affects buyers' ability to do arbitrage and how it affects the seller's profit.

Drawing on Laffont and Martimort (1997, 2000), we model coalition formation under asymmetric information by a side-contract offered to the buyers by a third-party who wishes to maximize the sum of the buyers' expected payoffs. The side-contract can specify manipulation of the reports made into the sale mechanism and/or reallocation of the goods obtained from the seller. The side-contract needs to satisfy incentive constraints as well as acceptance and budget balance constraints. The incentive constraints have to be satisfied since the third-party is not informed about the buyers' types. The acceptance constraints are defined with respect to the utilities that the buyers obtain when they play the sale mechanism non-cooperatively.

We first show that if the seller uses simple mechanisms in which the quantity allocated to a buyer and his payment do not depend on the other buyers' reports, buyer coalition strictly hurts the seller. However, we also show that if the seller judiciously designs the sale mechanism, buyer coalition cannot hurt the seller. More precisely, she can find sale mechanisms which deter any manipulation of reports and any reallocation of goods and yield the same profit as when there is no buyer coalition. We note that this result critically depends on the assumption that coalition forms under asymmetric information. In particular, the third party is not able to implement an efficient arbitrage between H-type and L-type because of the well-known tension between incentive and acceptance constraints in the side-contract. More precisely, since the rent that H-type can obtain by pretending to be L-type in the side mechanism is increasing in the quantity received by L-type, if the third-party reallocates some quantity from H-type to L-type, then H-type has a higher incentive to pretend to be L-type to the third-party. Hence, in order to induce him to truthfully report his type, the third-party has to concede him more rent.[8]

[8] The alternative of lowering L-type's payoff is not feasible since it would induce L-type to reject the side-contract.

This increase in the rent is defined as the transaction costs generated by asymmetric information. Since the transaction costs are larger than the gains from reallocating quantity from H-type to L-type, the efficient arbitrage cannot be realized.

There exists a small literature about consumer coalition which mostly addresses issues different from the one we consider in this paper. Alger (1999) is one exception: She studies the optimal menu of price-quantity pairs when consumers have access to multiple and/or joint purchases in a two-type setting. She finds that with multiple purchases only, the monopolist offers strict quantity discounts while, with joint purchases only, discounts are infeasible. However, her results are based on two specific assumptions. First, she supposes that consumer coalitions are formed under complete information among the consumers about each other's type. Therefore, there are no transaction costs in coalition formation in her setting. Furthermore, she restricts attention to homogeneous coalitions by assuming that only consumers with the same type can form coalitions. Second, she introduces a restriction on the set of mechanisms available to the seller by assuming that the quantity allocated to a consumer and his payment do not depend on the other consumers' choices. In contrast, in our model, coalition is formed under asymmetric information among buyers and consequently buyers with different types can form a coalition. Furthermore, we allow the seller to use complete contracts such that the quantity sold to a buyer and his payment can depend on the others' choices.

Innes and Sexton (1993, 1994) analyze the cases in which the monopolist is facing identical consumers who may form coalitions. They show that even though consumers' characteristics are homogeneous, the monopolist may price discriminate in order to deter the formation of coalitions, whereas price discrimination is unprofitable in the absence of the coalitions.

Using a third-party to model collusion under asymmetric information was first introduced in auction literature.[9] While that literature studies the optimal mechanism in a restricted set of mechanisms (they usually study the optimal reserve price in a first or second price auction), Laffont and Martimort (1997, 2000) use a more general approach in that they characterize the set of collusion-proof mechanisms and optimize in this set. In their papers, they do not consider quantity reallocation[10] and show that collusion has

[9] For examples, see Caillaud and Jehiel (1998), Graham and Marshall (1987) and McAfee and McMillan (1992).

[10] Reallocation is simply infeasible in their settings. In the first paper, the agents are regulated firms

no bite if the agents' types are independently distributed and if there is no restriction on the set of the principal's mechanisms. In contrast, collusion has bites in our model since we consider reallocation. We show that the optimal mechanism under no collusion can be implemented in a collusion-proof way if the seller fully exploits the transaction costs in coalition formation. Furthermore, we extend our result in two directions. First, while Laffont and Martmort do not study how to implement the outcome obtained from the optimal mechanism, we show that in our model, the outcome can be implemented by a menu of two-part tatiffs. Second, while Laffont and Martimort limit their analysis to the two-agent-two-type setting, we show in Section 6 that our main result extends to the n-buyer setting and to the three-type setting.

Our paper is to some extent related to the papers studying auctions with resale (Ausubel and Cramton (1999), Zheng (2001)). For instance, Ausubel and Cramton analyze the optimal auction under resale in a setting where buyers can engage in resale after receiving goods from the auctioneer and the resale is (assumed to be) always efficient. They prove that the seller maximizes his profit by allocating goods efficiently: any inefficient assignment would be corrected by ex post resale but the seller would fail to capture the gains from the resale. In contrast, in our setting, buyers sign a binding side-contract before each buyer chooses how much to buy. We show that when the seller judiciously designs her mechanism, the buyers fail to achieve efficient reallocation because of the transaction costs in coalition formation.[11]

The rest of the paper is organized as follows. Section 2 introduces the model; in particular, it describes the details of the coalition formation process. Section 3 reviews as a benchmark the well-known optimal sale mechanisms when no buyer coalition is possible and investigates whether those mechanisms leave any room for buyers' joint actions. Section 4 analyzes the third-party's problem of designing the side-contract and Section 5 finds the optimal sale mechanism in the presence of buyer coalition. Section 6 extends

producing complementary inputs. They have independently distributed types and there exists room for collusion since an exogenous restriction on the set of the principal's mechanisms is imposed. In the second paper, the agents are consumers of a public good. They have correlated types and therefore have incentives to collude since the principal will fully extract their rents if they behave non-cooperatively.

[11] Zheng (2001) allows resale in a setting of one-good auction in which buyers' values have different distributions. Any owner of the good is assumed to be able to choose a mechanism to sell it to others, taking into account that subsequent owners may wish in turn to resell the good. He proves the existence of an equilibrium which implements the same payoffs as when resale can be costlessly banned.

our analysis to the n-buyer setting and to the three-type setting. In Section 7, we show that our results are robust to relaxing assumptions about buyers' off-the-equilibrium-path beliefs and behavior in the coalition formation game. Concluding remarks are gathered in Section 8. Most of the proofs are left to Appendix.

2 The model

2.1 Preferences, information and mechanisms

A seller (for instance, an upstream monopolist) can produce any positive amount of homogeneous goods at a constant marginal cost $c > 0$[12] and sells the goods to $n \geq 2$ buyers (for instance, downstream firms operating in separate markets). We introduce the following informational assumptions: (i) the seller can observe only the amount of goods sold to each buyer and whether or not the buyer uses her goods but (ii) the seller cannot observe the actual quantity used by the buyer.[13] In what follows, for expositional simplicity, we focus on the two-buyer-two-type setting but we prove in Section 6 that our main result extends to the n-buyer setting and to the three-type setting.

Buyer i ($i = 1, 2$) obtains payoff $\theta^i u(q^i) - t^i$ from consuming quantity $q^i \geq 0$ of the goods and paying $t^i \in \mathsf{R}$ units of money to the seller. He privately observes his type $\theta^i \in \Theta \equiv \{\theta_L, \theta_H\}$, where $\Delta\theta \equiv \theta_H - \theta_L > 0$. The types θ^1 and θ^2 are identically and independently distributed with $\Pr\left\{\theta^i = \theta_L\right\} = \lambda \in (0, 1)$, $i = 1, 2$. The distribution of θ^1 and θ^2 is common knowledge. We suppose that u is twice differentiable, $u'(q) > 0 > u''(q)$ for any $q \geq 0$, $u(0) = 0$ and $(\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta)u'(0) > c > \lim_{q\to+\infty} \theta_H u'(q)$. The latter inequalities guarantee that each type will receive a positive and finite quantity in the optimal mechanism when buyer coalition is absent.[14] Each buyer's reservation utility is normalized to zero regardless of type.

The seller designs a sale mechanism to maximize her expected profit. A generic sale mechanism is denoted by M and, according to the revelation principle, we can restrict

[12] The assumption of constant marginal cost is only made to simplify the exposition. Our main result, Proposition 5 below, holds even if the marginal cost is increasing.

[13] These assumptions are similar to the assumptions adopted by Rey and Tirole (1986) to justify the use of two-part tariffs by an upstream monopolist.

[14] Our results below, however, extend to the case in which the seller refuses to serve L-type, which occurs if $(\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta)u'(0) \leq c$.

our attention to direct revelation mechanisms:

$$M=\left\{q^i(\widehat{\theta}^1,\widehat{\theta}^2),t^i(\widehat{\theta}^1,\widehat{\theta}^2);\ i=1,2\right\},$$

where $\widehat{\theta}^i\in\{\theta_L,\theta_H\}$ is buyer i's report, $q^i(\cdot)$ is the quantity he receives and $t^i(\cdot)$ is his payment to the seller. Since buyers are ex ante identical, without loss of generality we focus on symmetric mechanisms in which the quantity sold to a buyer and his payment depend only on the reports and not on his identity. Then, we can introduce the following notation, which simplifies the exposition: For quantities,

$$\begin{aligned} q_{HH} &= q^1(\theta_H,\theta_H)=q^2(\theta_H,\theta_H),\ q_{HL}=q^1(\theta_H,\theta_L)=q^2(\theta_L,\theta_H),\\ q_{LH} &= q^1(\theta_L,\theta_H)=q^2(\theta_H,\theta_L),\ q_{LL}=q^1(\theta_L,\theta_L)=q^2(\theta_L,\theta_L). \end{aligned}$$

$(t_{HH},t_{HL},t_{LH},t_{LL})\in\mathsf{R}^4$ are similarly defined. Let $q\equiv(q_{HH},q_{HL},q_{LH},q_{LL})$ denote the vector of quantities and $t\equiv(t_{HH},t_{HL},t_{LH},t_{LL})$ denote the vector of transfers.

2.2 Buyer coalition

Drawing on Laffont and Martimort (1997, 2000), we model buyers' coalition formation by a side-contract, denoted by S, offered by a benevolent third-party.[15] The third party designs S in order to maximize the sum of buyers' expected payoffs subject to incentive compatibility (since he does not know the types), participation and budget balance constraints. The participation constraints are written with respect to the utility that each type of buyer obtains when M is played non-cooperatively. Precisely, the game of seller's mechanism offer cum buyer coalition formation has the following timing.

Stage 1. Nature draws buyers' types (θ^1,θ^2); buyer i privately observes θ^i, $i=1,2$.

[15] The method of introducing a third party to model coalition formation may appear unrealistic, while it may seem natural to consider some bargaining models to describe coalition formation. However, we would like to point out an important property of the coalition formation model we analyze: the revelation principle implies that given a specific bargaining game G, any allocation achieved by a Bayesian equilibrium E of G can be obtained by a side-contract offered by the third party. Since we let the third party maximize the sum of the buyers' expected payoffs, we are describing the upper bound of what the coalition may achieve under asymmetric information. Furthermore, since we show later on that collusion does not hurt the seller, the property implies that specifying any particular bargaining game between the buyers would not change the main message of our paper as long as asymmetric information between them remains.

Stage 2. The seller proposes a sale mechanism M.

Stage 3. Each buyer simultaneously accepts or rejects M. If at least one buyer refuses M, then each buyer earns the reservation utility and the following stages do not occur.[16]

Stage 4. If both buyers accept to play M, then the third party proposes them a direct side-contract S in order to jointly manipulate their reports into M and to reallocate between themselves the goods bought from the seller.[17]

Stage 5. Each buyer simultaneously accepts or rejects S.

Stage 6. If at least one buyer refuses S, then mechanism M is played non-cooperatively. In this case, reports are directly made in M and stages 7 and 9 below do not occur. If instead S has been accepted by both buyers, then reports are made into S.

Stage 7. As a function of the reports in S, the third party enforces the manipulation of reports into M.

Stage 8. Quantities and transfers specified in M are enforced.

Stage 9. Quantity reallocation and side-transfers specified in S (if any) takes place in the buyer coalition.

Formally, a side-contract takes the following form:

$$\{\phi(\widetilde{\theta}^1,\widetilde{\theta}^2), x^i(\widetilde{\theta}^1,\widetilde{\theta}^2), y^i(\widetilde{\theta}^1,\widetilde{\theta}^2);\ i=1,2\},$$

where $\widetilde{\theta}^i \in \{\theta_L,\theta_H\}$ is buyer i's report to the third-party. Let $\phi(\cdot)$ represent the reports manipulation function, which maps any pair of reports made to the third-party $(\widetilde{\theta}^1,\widetilde{\theta}^2) \in \Theta^2$ into a pair of reports $(\widehat{\theta}^1,\widehat{\theta}^2) \in \Theta^2$ sent to the seller. We assume that ϕ can specify stochastic manipulations, as this convexifies the third-party's feasible set. After the buyers bought goods from the seller, the third-party can reallocate them within the coalition. Let $x^i(\cdot)$ represent the quantity of goods that buyer i receives from the third-

[16] We may also assume that if one buyer (say, buyer 1) vetoes M, then the seller can serve buyer 2 by offering a mechanism which is different from M. Our results below are robust to this modification as long as if the seller can prohibit buyer 2 from reselling to buyer 1 part of the goods he bought from the seller. Since we assume that the seller can observe whether or not a buyer uses her goods, the seller can deter any resale to a buyer who refused her offer.

[17] To be rigorous, the Revelation Principle applies to the third-party's design of S but does not apply to the seller's design of M. Thus, we should allow the seller to propose non-direct sale mechanisms. Nevertheless, as Proposition 3 in Laffont and Martimort (2000) establishes, any perfect Bayesian equilibrium outcome arising from a non-direct sale mechanism can be obtained as a perfect Bayesian equilibrium outcome induced by a direct sale mechanism.

party and $y^i(\cdot)$ the monetary transfer from him to the third-party. We impose the following ex post budget balance constraints: for the reallocation of goods,

$$\sum_{i=1}^{2} x^i(\theta^1, \theta^2) = 0, \text{ for any } (\theta^1, \theta^2) \in \Theta^2;$$

for the side transfers,

$$\sum_{i=1}^{2} y^i(\theta^1, \theta^2) = 0, \text{ for any } (\theta^1, \theta^2) \in \Theta^2.$$

After the third party proposed a side-contract S, a two-stage game is played by buyers: in its first stage (stage 5) each buyer accepts or rejects S; in the second stage (stage 6) the buyers report types either into M or into S depending on their decisions at the first stage. In what follows, we use the term "coalition formation game" to refer to the game which starts with the third-party's proposal of S (at stage 4). We are interested in (collusive continuation) equilibria of the coalition formation game in which both buyers accept S; thus, no learning about types occurs along the equilibrium path.[18] In sections 3-6 we make the following assumption:

$$\left\{ \begin{array}{c} \textsf{Assumption WCP}^{19}\text{: Given an incentive compatible } M \text{, if buyer } i \text{ vetoes } S \\ \text{(which is an off-the-equilibrium-path event), then buyer } j \neq i \text{ still has} \\ \text{prior beliefs about } \theta^i \text{ and the truthful equilibrium is played in } M. \end{array} \right.$$

By definition, truthtelling is an equilibrium in M under prior beliefs if and only if M is incentive compatible. Let $V(\theta^i)$ denote the expected payoff of buyer i as a function of his type in the truthful equilibrium in M.[20] Then, $V(\theta^i)$ is the reservation utility for type θ^i at the time of deciding whether to accept S or not: it is the payoff that the third-party should guarantee in order to induce a buyer with type $\theta^i \in \{\theta_L, \theta_H\}$ to accept S. In Section 7, we show that our results are robust to relaxing our initial assumptions about buyers' off-the-equilibrium-path beliefs and behavior.

[18] Notice, however, that there also exists an equilibrium in which both buyers refuse any side mechanism: if buyer i is vetoing any side mechanism, then rejecting is a best reply for buyer j.

[20] $V(\theta^i)$ does not depend on the identity i of the buyer since M is symmetric.

3 Do the optimal mechanisms without buyer coalition exhibit room for joint actions?

In this section, we first analyze the optimal mechanisms in the absence of buyer coalition and then examine whether in such mechanisms there exists any room for buyers' joint actions.

3.1 The optimal mechanisms without buyer coalition

In this subsection, we characterize the profit maximizing mechanisms when there is no buyer coalition. The seller's expected profit with mechanism $M = \{q, t\}$ is

$$\Pi = 2\lambda^2(t_{LL} - cq_{LL}) + 2\lambda(1-\lambda)(t_{HL} + t_{LH} - cq_{HL} - cq_{LH}) + 2(1-\lambda)^2(t_{HH} - cq_{HH})$$

M should satisfy the following Bayesian incentive compatibility constraints: for H-type,

$$\begin{aligned} (BIC_H) \quad & \lambda[\theta_H u(q_{HL}) - t_{HL}] + (1-\lambda)[\theta_H u(q_{HH}) - t_{HH}] \\ & \geq \lambda[\theta_H u(q_{LL}) - t_{LL}] + (1-\lambda)[\theta_H u(q_{LH}) - t_{LH}]; \end{aligned} \tag{1}$$

for L-type,

$$\begin{aligned} (BIC_L) \quad & \lambda[\theta_L u(q_{LL}) - t_{LL}] + (1-\lambda)[\theta_L u(q_{LH}) - t_{LH}] \\ & \geq \lambda[\theta_L u(q_{HL}) - t_{HL}] + (1-\lambda)[\theta_L u(q_{HH}) - t_{HH}]. \end{aligned} \tag{2}$$

M should also satisfy the following individual rationality constraints: for H-type,

$$(BIR_H) \quad \lambda[\theta_H u(q_{HL}) - t_{HL}] + (1-\lambda)[\theta_H u(q_{HH}) - t_{HH}] \geq 0; \tag{3}$$

for L-type,

$$(BIR_L) \quad \lambda[\theta_L u(q_{LL}) - t_{LL}] + (1-\lambda)[\theta_L u(q_{LH}) - t_{LH}] \geq 0. \tag{4}$$

The seller designs M to maximize Π subject to (1) to (4). We characterize the optimal mechanisms in the next proposition:

Proposition 1 *The optimal mechanisms in the absence of buyer coalition are characterized as follows.*

a) The optimal quantity schedule $q^* = (q^*_{HH}, q^*_{HL}, q^*_{LH}, q^*_{LL})$ is given by:

i) $q^*_{HH} = q^*_{HL} = q^*_H$, where $\theta_H u'(q^*_H) = c$;

ii) $q^*_{LH} = q^*_{LL} = q^*_L$, where $(\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta)u'(q^*_L) = c$.

b) Transfers are such that constraints (BIC_H) and (BIR_L) are binding.

Proof. The proof is standard and therefore it is omitted. ■

We first note that, in Proposition 1, q^*_H and q^*_L are equal to the optimal quantities allocated to H-type and L-type, respectively, when the seller faces only one buyer. In the one-buyer case, it is well known that L-type's virtual valuation is given by $\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta$ since an increase in the quantity received by L-type reduces through (BIC_H) the payment that the seller obtains from H-type. This makes her introduce a downward distortion in the quantity allocated to L-type with respect to the first-best level. Precisely, the seller determines q_L by equalizing the marginal cost to L-type's marginal utility evaluated with the virtual valuation. Proposition 1 states that, in the optimal mechanisms for the two-buyer case, the quantity obtained by a buyer is uniquely determined by his report regardless of the other buyer's report and is equal to the quantity he would receive in the one-buyer setting.

Inspecting (1) to (4) and Π shows that the transfer scheme t matters only to determine the values of $\bar{t}_L \equiv \lambda t_{LL} + (1-\lambda)t_{LH}$ and of $\bar{t}_H \equiv \lambda t_{HL} + (1-\lambda)t_{HH}$. Therefore, the seller has two degrees of freedom in choosing transfers which are expected payoff equivalent for her and for buyers. In particular, transfers can be designed in such a way that each buyer's payment is independent of the other buyer's report. Precisely, by setting $t_{LL} = t_{LH}$ and $t_{HL} = t_{HH}$, we obtain the optimal transfers in the one-buyer setting: $t^d_{HH} = t^d_{HL} = t^d_H \equiv \theta_H u(q^*_H) - (\Delta\theta)u(q^*_L)$ and $t^d_{LH} = t^d_{LL} = t^d_L \equiv \theta_L u(q^*_L)$. In what follows, we use M^d to denote the optimal mechanism in which the seller proposes the quantities and the transfers that she would offer in the one-buyer case: $M^d \equiv \{q^*, t^d\}$ where $t^d \equiv (t^d_{HH}, t^d_{HL}, t^d_{LH}, t^d_{LL})$. In M^d, the payoff of each buyer is fully determined by his report only and, as a consequence, truthtelling is a dominant strategy. Basically, in the absence of buyer coalition, the seller can maximize her profit by dealing with each buyer separately. It is easy to see that the outcome achieved by M^d can be implemented by a menu of two-part tariffs where the two-part tariff designed for L-type has a kink.[21]

[21] The two-part tariff for H-type takes the following form: $A_H + pq$ where $A_H = t^d_H - cq^*_H$ and $p = c$. The two-part tariff for L-type needs a kink at the point $q = q^*_L$ in order to prevent H-type from buying

3.2 Room for buyers' joint actions

In this subsection, we investigate whether the mechanisms characterized by Proposition 1 exhibit any room for buyers' joint actions. We say that room for joint actions exists if buyers can realize some gain by coordinating their actions in the absence of transaction costs in coalition formation. Therefore, this section identifies profitable joint actions in the absence of transaction costs and later on we verify whether these actions can be implemented in the presence of asymmetric information between the buyers after formally introducing the model of coalition formation under asymmetric information in Section 4.

We distinguish two kinds of joint actions: manipulation of reports and quantity reallocation. First, the buyers can coordinate their reports into M. Second, after buying some goods from the seller, they can reallocate them within the coalition.

Reports manipulation only First, it turns out that when the seller proposes M^d, in the absence of quantity reallocation, the buyers cannot generate any gain by coordinating their reports. This occurs because, as we noted before, in M^d, a buyer's payoff is independent of the other buyer's report. Therefore, there exists no joint manipulation of reports which is profitable.[22]

Quantity reallocation only Second, suppose that buyers can reallocate the goods bought from the seller but cannot jointly manipulate their reports. Then, it is manifest that when the buyers have the same types, there is no room for quantity reallocation since the seller allocates the same quantity to each of them: either q_{HH} (if $\theta^1 = \theta^2 = \theta_H$) or q_{LL} (if $\theta^1 = \theta^2 = \theta_L$). However, when one buyer has H-type and the other has L-type, the latter's marginal utility from consumption is strictly larger than the former's one since we have $\theta_H u'(q_H^*) = (\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta)u'(q_L^*) = c$. Therefore, they have an incentive to reallocate some quantity from H-type to L-type. We note that this incentive for

more than q_L^* in case he reports θ_L. This gives the seller some discretion in choosing the marginal price. For instance, she can use $A_L + pq$ such that $A_L = t_L^d - cq_L^*$, $p = c$ for $q \leq q_L^*$ and $p = \theta_H u'(q_L^*)$ for $q > q_L^*$.

[22] Laffont and Martimort (1997) also show that in their setting, where the agents' types are independently distributed, there exists a dominant-strategy optimal mechanism which eliminates any gain from joint manipulation of reports.

reallocation originates from the fact that the seller introduces a downward distortion in the quantity consumed by L-type in order to extract more rent from H-type. In contrast, if the seller knew θ^1 and θ^2, there would be no room for quantity reallocation since the first-best quantity schedule (q_H^{FB}, q_L^{FB}) would be implemented, which is characterized by $\theta_H u'(q_H^{FB}) = \theta_L u'(q_L^{FB}) = c$.

Manipulation of reports and quantity reallocation Last, consider the case in which buyers can jointly manipulate their reports and reallocate the goods. Now there exists room for joint manipulations of reports in some mechanisms which are optimal if reallocation is not feasible. For instance, when the seller proposes M^d, we have seen that buyers will report truthfully in the absence of reallocation. However, if reallocation is possible, then the coalition formed by two H-types has an incentive to report (θ_H, θ_L) to the seller and to reallocate the goods since the following inequality holds:

$$2\theta_H u(q_H^*) - 2t_H^d < 2\theta_H u(\frac{q_H^* + q_L^*}{2}) - t_H^d - t_L^d.$$

Moreover, it can be easily seen that the coalition formed by one H-type and one L-type also has an incentive to report (θ_L, θ_L) to the seller.

We show in the next section that the same manipulations can be implemented by a suitable side mechanism even though coalition formation takes place under asymmetric information. This suggests that the seller may wish to use a more sophisticated mechanism than M^d when buyers can form a coalition.

4 Coalition formation under asymmetric information

From now on, we assume that coalition formation occurs under asymmetric information and study the game of seller's mechanism offer cum coalition formation by allowing for both joint manipulation of reports and reallocation of goods. In particular, in this section, we analyze the third-party's design problem of S and characterize the constraints which buyer coalition imposes on the seller's design problem. In order to do this, we need to introduce some definitions.

Definition 1 *A side-contract $S^* = \{\phi^*(\cdot), x^{i*}(\cdot), y^{i*}(\cdot)\}$ is coalition-interim-efficient with respect to an incentive compatible mechanism M providing the reservation utilities $\{V(\theta_L), V(\theta_H)\}$ if and only if it solves the following program:*

$$\max_{\phi(\cdot),x^i(\cdot),y^i(\cdot)} \sum_{(\theta^1,\theta^2)\in\Theta^2} p(\theta^1,\theta^2) \sum_{i=1}^{2} \left[\theta^i u(q^i(\phi(\theta^1,\theta^2)) + x^i(\theta^1,\theta^2)) - t^i(\phi(\theta^1,\theta^2))\right]$$

$$\begin{array}{rcl} & & \textit{subject to} \\ U^i(\theta^i) & = & \sum_{\theta^j\in\Theta} p(\theta^j)[\theta^i u(q^i(\phi(\theta^i,\theta^j)) + x^i(\theta^i,\theta^j)) + y^i(\theta^i,\theta^j) - t^i(\phi(\theta^i,\theta^j))], \\ & & \textit{for any } \theta^i \in \Theta; \\ (BIC^S)\ U^i(\theta^i) & \geq & \sum_{\theta^j\in\Theta} p(\theta^j)[\theta^i u(q^i(\phi(\widetilde{\theta}^i,\theta^j) + x^i(\widetilde{\theta}^i,\theta^j) + y^i(\widetilde{\theta}^i,\theta^j) - t^i(\phi(\widetilde{\theta}^i,\theta^j))], \\ & & \textit{for any } (\theta^i,\widetilde{\theta}^i) \in \Theta^2; \\ (BIR^S)\ U^i(\theta^i) & \geq & V(\theta^i),\ \textit{for any } \theta^i \in \Theta; \\ (BB & : & x)\ x^1(\theta^1,\theta^2) + x^2(\theta^1,\theta^2) = 0,\ \textit{for any } (\theta^1,\theta^2) \in \Theta^2; \\ (BB & : & y)\ y^1(\theta^1,\theta^2) + y^2(\theta^1,\theta^2) = 0,\ \textit{for any } (\theta^1,\theta^2) \in \Theta^2. \end{array}$$

In words, a side-contract is coalition-interim-efficient with respect to M, if it maximizes the sum of the buyers' expected utilities subject to incentive, acceptance and budget balance constraints. Let us define the null side-contract, denoted by S^0, as $S^0 \equiv \{\phi(\cdot) = Id(\cdot), x^1(\cdot) = x^2(\cdot) = 0, y^1(\cdot) = y^2(\cdot) = 0\}$. In words, in the null-side contract, no manipulation of reports, no reallocation of quantity and no side-transfer occurs. When the third-party proposes S^0, a mechanism M is not affected by buyer coalition. The next definition refers to this class of mechanisms.

Definition 2 *An incentive compatible mechanism M is weakly collusion-proof if S^0 is coalition-interim-efficient with respect to M.*

The next proposition shows that M^d is not weakly collusion-proof: Even though the coalition forms under asymmetric information, the third party can find a side-contract which increases the buyers' expected payoff above the ones obtained by playing M^d truthtfully.

Proposition 2 *Suppose that the seller offers M^d. Then there exists a side-contract S^d such that*
(a) it satisfies (BIC^S), (BIR^S), $(BB : x)$ and $(BB : y)$;
(b) it manipulates buyers' reports into M^d such that they report (θ_H, θ_L) if $\theta^1 = \theta^2 = \theta_H$ and (θ_L, θ_L) if $\theta^1 \neq \theta^2$ and it reallocates the quantities after the manipulations of reports.;
(c) With respect to the case of no coalition, $M^d \circ S^d$ strictly increases each type of buyer's payoff while it strictly reduces the seller's profit.

Proof. See Appendix. ■

According to Proposition 2, when the seller offers the simple mechanism M^d, the buyer coalition can realize strict gains from suitably manipulating reports and reallocating goods. In this case, the buyer coalition strictly reduces the seller's profit because of the two following reasons: (i) in the states of nature in which the manipulations of reports occur, the quantity sold to the buyers is strictly reduced with respect to truthtelling and therefore the surplus which is generated by the trade is reduced; (ii) each type of buyer obtains a higher payoff than under truthtelling.

This proposition establishes that a "simple" mechanism like M^d does not allow the seller to obtain the same profit as under no coalition formation. Therefore, we need to ask whether better sale mechanisms than M^d exist. The following proposition is particularly important since it tells that in order to answer the previous question, we can restrict our attention to the set of weakly collusion-proof mechanisms.

Proposition 3 (weakly collusion-proofness principle) *There is no loss of generality in restricting the seller to offer weakly collusion-proof mechanisms in order to characterize the outcome of any perfect Bayesian equilibrium of the game of seller's mechanism offer cum coalition formation.*

Proof. The proof is omitted since it is a straightforward adaptation of the proof for Proposition 3 in Laffont and Martimort (2000). ■

The idea behind Proposition 3 is the following: since the third-party has no informational or instrumental advantage over the seller and is subject to the incentive, acceptance and budget balance constraints, any outcome that can be implemented by allowing coalitions to manipulate reports and/or reallocate goods can be mimicked by the seller in a collusion-proof way without any loss. The weakly collusion-proofness principle simplifies our analysis, since what can be achieved by the seller is contained in the set of weakly collusion-proof mechanisms.

In the next proposition, we characterize the set of weakly collusion-proof mechanisms.[23] Before stating the proposition it is useful to define the following variables θ_L^ε, $q_H^\varepsilon(x)$ and $q_L^\varepsilon(x)$, in which $\varepsilon \in [0,1)$ and $x > 0$:

$$\theta_L^\varepsilon \equiv \theta_L - \frac{1-\lambda}{\lambda}\varepsilon\Delta\theta,$$
$$q_H^\varepsilon(x) \equiv \arg\max_{z\in[0,x]} \theta_H u(z) + \theta_L^\varepsilon u(x-z) \text{ and } q_L^\varepsilon(x) \equiv x - q_H^\varepsilon(x) \tag{5}$$

We note that $q_H^\varepsilon(x)$ is uniquely defined since $\theta_H u(z) + \theta_L^\varepsilon u(x-z)$ is a strictly concave function of z. In particular, $(q_H^\varepsilon(x), q_L^\varepsilon(x))$ is the efficient allocation of a total quantity $x > 0$ between a buyer with valuation θ_H and a buyer with valuation θ_L^ε. Finally, we notice that if $q_H^\varepsilon(x) < x$, the first order condition needs to be satisfied with equality:

$$\theta_H u'[q_H^\varepsilon(x)] = \theta_L^\varepsilon u[q_L^\varepsilon(x)] \tag{6}$$

Proposition 4 *An incentive compatible sale mechanism $M = \{q, t\}$ is weakly collusion-proof if and only if there exists $\varepsilon \in [0,1)$ such that*
(a) the following coalition incentive constraints are satisfied: for HH coalition,

$$\begin{aligned} &2\theta_H u(q_{HH}) - 2t_{HH} \\ &\geq 2\theta_H u\left(\frac{q^1(\widehat{\theta}^1,\widehat{\theta}^2) + q^2(\widehat{\theta}^1,\widehat{\theta}^2)}{2}\right) - t^1(\widehat{\theta}^1,\widehat{\theta}^2) - t^2(\widehat{\theta}^1,\widehat{\theta}^2), \forall(\widehat{\theta}^1,\widehat{\theta}^2) \in \Theta^2; \end{aligned} \tag{7}$$

for HL coalition,

$$\begin{aligned} &\theta_H u(q_H^\varepsilon(q_{HL}+q_{LH})) + \theta_L^\varepsilon u(q_L^\varepsilon(q_{HL}+q_{LH})) - t_{HL} - t_{LH} \\ &\geq \theta_H u(q_H^\varepsilon(q^1(\widehat{\theta}^1,\widehat{\theta}^2) + q^2(\widehat{\theta}^1,\widehat{\theta}^2))) + \theta_L^\varepsilon u(q_L^\varepsilon(q^1(\widehat{\theta}^1,\widehat{\theta}^2) + q^2(\widehat{\theta}^1,\widehat{\theta}^2))) \\ &\quad - t^1(\widehat{\theta}^1,\widehat{\theta}^2) - t^2(\widehat{\theta}^1,\widehat{\theta}^2), \forall(\widehat{\theta}^1,\widehat{\theta}^2) \in \Theta^2; \end{aligned} \tag{8}$$

for LL coalition,

$$\begin{aligned} &2\theta_L^\varepsilon u(q_{LL}) - 2t_{LL} \\ &\geq 2\theta_L^\varepsilon u\left(\frac{q^1(\widehat{\theta}^1,\widehat{\theta}^2) + q^2(\widehat{\theta}^1,\widehat{\theta}^2)}{2}\right) - t^1(\widehat{\theta}^1,\widehat{\theta}^2) - t^2(\widehat{\theta}^1,\widehat{\theta}^2), \forall(\widehat{\theta}^1,\widehat{\theta}^2) \in \Theta^2, \end{aligned} \tag{9}$$

[23] We focus on the subset of mechanisms where L-type's Bayesian individual incentive constraint is not binding. We prove in Section 5 that the seller is not going to offer a mechanism M such that L-type's incentive constraint binds in the side-contract which is optimal with respect to M.

(b) the following no arbitrage constraint is satisfied

$$q_{HL} = q_H^{\varepsilon}(q_{HL} + q_{LH}), \tag{10}$$

(c) if $\varepsilon > 0$, then H-type's incentive constraint in the side mechanism is binding.

Proof. See Appendix. ■

When the coalition incentive constraints (7) to (9) are satisfied, the third-party has no incentive to manipulate the buyers' reports into M. In such a case, no room for reallocation exists if $\theta^1 = \theta^2$, since M allocates the same quantity to each buyer. Furthermore, if the no-arbitrage constraint (10) is satisfied, HL coalition has no incentive to reallocate the goods that are bought from the seller after making truthful reports. We note that in each coalition incentive constraint, both the left and right hand sides take into account the reallocation of the goods. When both agents report the same types to the third party, each buyer receives half of the total quantity available in the coalition: see (7) and (9). If the reports are different (i.e., $\tilde{\theta}^1 \neq \tilde{\theta}^2$), the total quantity is allocated according to (5): see (8).

In (8) to (10), $\varepsilon \in [0, 1)$ appears. Roughly speaking, ε is the Lagrange multiplier of H-type's incentive constraint in the third-party's design problem of S and it can be positive when that constraint is binding.[24] The seller has some flexibility in choosing ε because S^0 satisfies the necessary and sufficient conditions for optimality in the third-party's problem for any $\varepsilon \in [0, 1)$. In the presence of complete information within the coalition, the side mechanism does not need to satisfy any individual incentive constraint. Then the coalition incentive and the no-arbitrage constraints under complete information are obtained from (7) to (10) by taking ε equal to 0. Basically, in that setting, there are no transaction costs in coalition formation and the third party simply maximizes the sum of buyers payoffs for each profile of types by suitably manipulating reports or/and by efficiently reallocating the goods. In other words, whatever gains from joint actions - if there is any - are realized by buyers.

When the coalition is formed under asymmetric information, the side-contact has to satisfy not only the participation constraints but also the incentive constraints. Since the third-party has to guarantee each type of buyer the utility that he can obtain by

[24] Precisely, $\varepsilon = \frac{\delta}{a+\delta}$ where δ is the Lagrange multiplier of H-type's incentive constraint and a is a strictly positive constant.

non-cooperatively playing M, a tension between (BIR_L^S) and (BIC_H^S) may arise, and it may be costly to satisfy H-type's incentive constraint; ε measures how costly it is. In other words, ε captures the effect of asymmetric information on the third-party's decision making. The coalition incentive constraints under asymmetric information differ with respect to the constraints under complete information because L-type's valuation θ_L is replaced with the virtual valuation θ_L^ε, which is smaller than the real valuation for $\varepsilon > 0$. This is so because, as the quantity allocated to L-type (by the third party) increases, it is more difficult to satisfy H-type's incentive constraint in the side mechanism. When $\varepsilon = 1$, L-type's virtual valuation in the third-party's program is given by $\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta$, which is equal to L-type's virtual valuation in the seller's program under no buyer coalition studied in Subsection 3.1. The seller has some flexibility in choosing ε since S^0 is optimal for the third party if it satisfies the necessary and sufficient conditions for optimality in the third party's problem for at least one ε in $[0, 1)$.

The virtual valuation θ_L^ε affects the coalition incentive constrains through two channels. First, given a quantity consumed by L-type, the third-party evaluates his surplus with θ_L^ε instead of θ_L. Second, given a total quantity available to a coalition, the value of θ_L^ε affects the third-party's decision to reallocate the goods. As we said above, when buyers report (θ_H, θ_H) or (θ_L, θ_L) in S, the third party gives each agent the half of the total available quantity regardless of the value of ε. However, if buyers report (θ_H, θ_L) in S, then the third-party reallocates the goods by equalizing H-type's marginal surplus to L-type's marginal surplus and the latter is evaluated with θ_L^ε. The larger is $\varepsilon > 0$, the smaller is the quantity obtained by L-type in HL coalition. Finally, we observe that the no-arbitrage constraint (10) requires H-type's marginal surplus to be equal to L-type's virtual marginal surplus.

One might argue that the seller might ask the buyers for the information that they may have learned during the course of coalition formation. However, since we show that even though the seller is restricted to use grand-mechanisms which only depend on the buyers' types, she can deter buyer coalition at no cost, we do not need to consider more general grand-mechanisms.

5 The optimal weakly collusion-proof mechanism

In this section, we analyze the optimal weakly collusion-proof mechanism. Observe that when the third party proposes S^0, (i) the Bayesian incentive constraints (BIC^S) in the side mechanism are equal to (BIC_H) and (BIC_L) introduced in subsection 3.1; (ii) the acceptance constraints (BIR^S) in the side mechanism are automatically satisfied with equality. Hence, in the presence of buyer coalition, the seller's maximization program - denoted by (P) - is defined as follows: she designs M and selects ε in order to maximize her expected profit Π subject to Bayesian individual incentive, Bayesian participation, coalition incentive and no arbitrage constraints. However, the following lemma shows that we can substantially reduce the number of constraints to take into account in program (P). Before stating the lemma, it is useful to define the following reduced program (RP):

$$\max_{\{q,t,\varepsilon\}} \quad \Pi \text{ subject to (1)-(4) and (10)}$$

Lemma 1 *If the inequalities*

$$2q_{HH} \geq q_{HL} + q_{LH} \geq 2q_{LL}, \tag{11}$$

hold at the solution to Program (RP), then the solution of Program (RP) is equivalent to that of Program (P) in terms of the expected payoff for the seller and for each type of buyer.

Proof. See Appendix. ■

Lemma 1 basically says that (7)-(9) can be satisfied at no cost. In other words, (10) is the only relevant constraint which buyer coalition introduces into the seller's program of Subsection 3.1. We now provide a simple intuition of why the coalition incentive constraints do not reduce the seller's profit.

We start by considering transfers which satisfy $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ - written below - with equality.

$$\begin{array}{llcl} (CIC_{HH,HL}) & 2\theta_H u(q_{HH}) - 2t_{HH} & \geq & 2\theta_H u(\frac{q_{HL}+q_{LH}}{2}) - t_{HL} - t_{LH} \\ (CIC_{HL,LL}) & \theta_H u(q_{HL}) + \theta_L^{\varepsilon} u(q_{LH}) - t_{HL} - t_{LH} & \geq & \theta_H u(q_H^{\varepsilon}(2q_{LL})) + \theta_L^{\epsilon} u(q_L^{\varepsilon}(2q_{LL})) - 2t_{LL} \end{array}$$

Then we can prove that all the other coalition incentive constraints are automatically satisfied if (11) holds.[25] Let us regard each coalition as a consolidated agent and let $V_k^{\epsilon}(x)$ denote the total surplus that a coalition having k number of buyers with H-type derives from consuming a total quantity $x > 0$; $k \in \{0, 1, 2\}$ is viewed as the "type" of the coalition. Then, the following single crossing condition holds: $\frac{\partial V_2^{\epsilon}(x)}{\partial x} > \frac{\partial V_1^{\epsilon}(x)}{\partial x} > \frac{\partial V_0^{\epsilon}(x)}{\partial x}$ for any $x > 0$ and for any $\varepsilon \in [0, 1)$: the marginal surplus from consumption is strictly increasing in k. Now we can apply a standard result from the theory of monopolistic screening [see Maskin and Riley (1984)] to conclude that when local downward coalition incentive constraints bind - in our case, $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ -, all the coalition incentive constraints are satisfied if the quantity profile for coalitions is monotone - i.e., if (11) holds- and if the single crossing condition holds.

Satisfying $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ with equality requires the seller to use two degrees of freedom from the transfer schedule t, say t_{HH} and t_{LL}. The two remaining degrees of freedom t_{LH} and t_{HL} are sufficient to deal with (1)-(4) without reducing profit since, as we noted in Subsection 3.1, transfers appear in (1)-(4) and Π only through the values of $\bar{t}_L \equiv \lambda t_{LL} + (1 - \lambda)t_{LH}$ and of $\bar{t}_H \equiv \lambda t_{HL} + (1 - \lambda)t_{HH}$.[26]

Lemma 1 is consistent with the findings in Laffont and Martimort (1997, 2000). In these papers, if the agents have independent types, all the coalition incentive constraints can be satisfied without reducing the principal's payoff, unless there exist exogenous restrictions on the mechanisms which are available to the principal.[27] However, reallocation between agents is not considered in their settings simply because it is infeasible.

Since there is one more constraint in (RP) than in the seller's program without buyer coalition, the seller cannot earn more profit in the presence of buyer coalition than in its absence. However, the next proposition states that actually the profit level is the same in the two cases. More precisely, it guarantees the existence of a transfer schedule which, paired with the quantity profile q^* of Proposition 1, enables the seller to achieve the profit that she obtains in the absence of buyer coalition.

Proposition 5 *There exists a transfer scheme t^* such that*

[25] Actually, (11) is also necessary for the existence of transfers satisfying (7)-(9).

[26] Formally, in the proof of Lemma 1 in Appendix, we show that any value of $\bar{t}_H$ and $\bar{t}_L$ can be attained by suitably choosing t_{HL} and t_{LH} as when the vector $t \in \mathsf{R}^4$ is unconstrained.

[27] Furthermore, to obtain that result, the principal does not need to exploit the transaction costs created by asymmetric information at the coalition formation stage.

(a) $M^* \equiv \{q^*, t^*\}$ *is an optimal mechanism in the absence of buyer coalition;*
(b) M^* *is also weakly collusion-proof.*

Proof. In the solution to (RP), both (BIC_H) and (BIR_L) bind, as in the optimal mechanism with no buyer coalition (the usual arguments can be applied). Hence, the seller obtains the same expected profit in the two cases by setting $q = q^*$ if (10) is satisfied at q^*; indeed, both (BIC_L) and (BIR_H) are automatically satisfied. The profile q^* is such that $\theta_H u'(q^*_{HL}) = (\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta)u'(q^*_{LH}) = c$, while condition (10) is satisfied by q^* if there exists $\varepsilon \in [0, 1)$ such that $\theta_H u'(q^*_{HL}) = \theta_L^{\varepsilon} u'(q^*_{LH})$. Since we are interested in the Sup of the seller's profit, we allow ε to take the value equal to one.[28] Given that q^* satisfies (10) for $\varepsilon = 1$, we conclude that in (RP) the seller earns as much profit as when buyer coalition is absent. Finally, we observe that the payoff equivalence between (RP) and (P) applies because (11) holds at q^*. In particular, given q^* and $\varepsilon = 1$, we can find a (unique) transfer profile t^* such that (BIC_H), (BIR_L), $(CIC_{HH,HL})$ and $(CIC_{HH,HL})$ bind[29]; (11) guarantees that all the other coalition incentive constraints are satisfied. ■

Proposition 5 implies that all the constraints generated by collusion-proofness can be satisfied at no cost. Hence, the seller can implement the same quantity profile q^* as in the absence of buyer coalition and earn the same profit. Under asymmetric information, the possibility to form a coalition does not help the buyers to increase their payoffs. Even though the third party aims at maximizing the buyers' payoffs and there clearly exists room to increase these payoffs by reallocating the goods within HL coalition, there exists no side mechanism implementing a desirable reallocation when the seller proposes M^*.[30]

To give a clear intuition of why the third-party fails to efficiently reallocate the goods, in the next proposition we quantify both the gains from reallocation and the transaction costs created by asymmetric information and then we show that the latter is strictly larger than the former.

[28] Admittedly, $\varepsilon = 1$ is not feasible, according to Proposition 4. However, $\varepsilon = 1$ can be arbitrarily closely approximated by feasible values of ε.

[29] See Appendix for the complete description of the transfers.

[30] We note that there exist infinitely many transfer schemes - let $\widehat{t}$ denote one - such that $(q^*, \widehat{t})$ is optimal under no coalition formation and moreover *strictly* satisfies all the CIC. We considered t^* above in order to fix the ideas but there is no compelling reason to prefer t^* to any transfer scheme which strictly satisfies all CIC.

Proposition 6 *Suppose that the seller offers $M^* = \{q^*, t^*\}$ and that the third-party does not manipulate reports but reallocates quantity $\Delta q \in (0, q_H^*]$ from H-type to L-type in HL coalition. Then*
(a) the expected gains from the reallocation is given by:

$$G \equiv 2\lambda(1-\lambda)\left\{\theta_L\left[u(q_L^*+\Delta q)-u(q_L^*)\right]-\theta_H\left[u(q_H^*)-u(q_H^*-\Delta q)\right]\right\},$$

(b) the transaction costs created by asymmetric information is given by:

$$TC \equiv 2(1-\lambda)^2(\Delta\theta)\left[u(q_L^*+\Delta q)-u(q_L^*)\right], \tag{12}$$

(c) we have $TC - G > 0$ for any $\Delta q \in (0, q_H^*]$.

Proof. Since it is straightforward to compute the gains from reallocation, we focus on the computation of the transaction costs. We first note that since the rent that H-type can obtain by pretending to be L-type is increasing in the quantity received by L-type, the reallocation increases H-type's incentive to report L-type in the side mechanism. Suppose now that buyer 2 reports his type truthfully in S and compute the payoff that H-type of buyer 1 obtains by pretending to be L-type to the third-party. Then, the H-type's expected surplus from consumption is given by $\theta_H\left[(1-\lambda)u(q_L^*+\Delta q)+\lambda u(q_L^*)\right]$ and his expected payment is equal, from the binding L-type's participation constraint, to $\theta_L\left[(1-\lambda)u(q_L^*+\Delta q)+\lambda u(q_L^*)\right]$. Hence, in order to implement the reallocation, the third-party has to give an H-type a rent equal to $(\Delta\theta)\left[(1-\lambda)u(q_L^*+\Delta q)+\lambda u(q_L^*)\right]$, which is larger than $(\Delta\theta)u(q_L^*)$, an H-type's rent in the absence of reallocation. This increase in H-type's rent represent the transaction costs in coalition formation created by asymmetric information. From the ex ante point of view, the transaction costs are given by (12). Since $TC - G$ is a strictly convex function of Δq, the following inequality holds:

$$TC - G > 2\lambda(1-\lambda)\left[\theta_H u'(q_H^*) - (\theta_L - \frac{1-\lambda}{\lambda}\Delta\theta)u'(q_L^*)\right](\Delta q) = 0 \text{ for any } \Delta q \in (0, q_H^*] \quad \blacksquare$$

We show below that the seller can implement the outcome achieved by the optimal collusion-proof mechanism through a menu of two-part tariffs: so she does not need to use a direct mechanism. Before that, however, we study the features of the transfers t^*

that satisfy (BIC_H), (BIR_L), $(CIC_{HH,HL})$ and $(CIC_{HH,HL})$ with equality when $q = q^*$ and $\varepsilon = 1$. This will be helpful in understanding the payment schemes used in the menu. We run a thought experiment in two stages. First, we investigate how the transfers look like when the coalition can manipulate reports but cannot reallocate goods. Second, we examine how introducing reallocation modifies the transfers.

In the absence of reallocation, $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ with $q = q^*$ and $\varepsilon = 1$ are written as follows:

$$\begin{array}{rrcl} (CIC_{HH,HL}) & 2\theta_H u(q_H^*) - 2t_{HH} & \geq & \theta_H[u(q_H^*) + u(q_L^*)] - t_{HL} - t_{LH} \\ (CIC_{HL,LL}) & \theta_H u(q_H^*) + \theta_L^1 u(q_L^*) - t_{HL} - t_{LH} & \geq & (\theta_H + \theta_L^1)u(q_L^*) - 2t_{LL} \end{array}$$

The transfer schedule which satisfies (BIC_H), (BIR_L), $(CIC_{HH,HL})$ and $(CIC_{HH,HL})$ with equality is given by: $t_{HL}^d = t_{HH}^d = \theta_H u(q_H^*) - (\Delta\theta)u(q_L^*)$ and $t_{LL}^d = t_{LH}^d = \theta_L u(q_L^*)$. In fact, these are exactly the transfers specified in mechanism M^d.[31]

When reallocation is feasible, coalition becomes more powerful because the buyers can increase their joint payoffs from misreporting by suitably reallocating the goods.[32] The transfers are now such that

$$t_{LH}^* < \theta_L u(q_L^*) < t_{LL}^* \text{ and } t_{HH}^* < \theta_H u(q_H^*) - (\Delta\theta)u(q_L^*) < t_{HL}^* \tag{13}$$

This means that upon reporting a type, each buyer faces a lottery which determines his payment as a function of the report of the other buyer. In particular, facing an L-type is always bad news because then the payment is higher than when facing an H-type. The intuition can be given as follows. As reallocation increases the gross payoff that HL coalition obtains after manipulating its reports to LL, a t_{LL} larger than t_{LL}^d is needed to make such a manipulation less attractive. However, since (BIR_L) is binding, an increase in t_{LL} must be accompanied with a decrease in t_{LH}; thus $t_{LH}^* < \theta_L u(q_L^*) = t_{LL}^d = t_{LH}^d < t_{LL}^*$. A similar argument applies to $(CIC_{HH,HL})$: $t_{HL} > t_{HL}^d$ relaxes that constraint and this implies, since (BIC_H) binds, a smaller t_{HH}.

We now study the implementation through a menu of two-part tariffs. Suppose that the seller offers two tariffs, tariff $T_H = \{(A_{HH}, p_{HH}), (A_{HL}, p_{HL})\}$ and tariff $T_L =$

[31] In M^d, it is unprofitable to manipulate reports because of strategic independence: the payoff a buyer obtains does not depend on the report of the other buyer.

[32] Indeed, both in $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$, the first term in the right hand side increases: $2\theta_H u(\frac{q_H^* + q_L^*}{2}) > \theta_H[u(q_H^*) + u(q_L^*)]$ and $\theta_H u(q_H^1(2q_L^*)) + \theta_L^1 u(q_L^1(2q_L^*)) > (\theta_H + \theta_L^1)u(q_L^*)$.

$\{(A_{LH}, p_{LH}), (A_{LL}, p_{LL})\}$ where, for instance, A_{HL} and p_{HL} represent the fixed fee and the marginal price that the buyer who chooses tariff T_H has to pay when the other buyer chooses tariff T_L. Consider now the following tariffs $\{T_H^*, T_L^*\}$:

$$A_{jk} = t_{jk}^* - cq_j^*, \text{ for } j,k \in \{H,L\} \text{ and } p_{HH} = c,$$

$$p_{jk} = c \text{ for } q \leq q_j^* \text{ and } p_{jk} = \theta_H u'(q_L^*) \text{ for } q > q_j^* \text{ for } jk \in \{HL, LH, LL\}.$$

The next proposition shows that the outcome achieved by M^* can be implemented through $\{T_H^*, T_L^*\}$.

Proposition 7 *Suppose that the seller offers $\{T_H^*, T_L^*\}$. Then, regardless of whether or not the buyers can form a coalition,*

(a) each buyer accepts the offer,

(b) j-type of buyer, with $j \in \{H,L\}$, chooses the tariff T_j^* and buys quantity q_j^*.

Proof. The proof is long and hence omitted.[33] In the proof, we first redefine the third-party's program taking into account the fact that he can now choose the total quantity to buy given a choice of tariffs and then study the program as in the proof of Proposition 4. ■

We mentioned in Subsection 3.1 that when coalition formation is impossible, the optimal outcome can be implemented through a menu of two-part tariffs in which the tariff designed for L-type has a kink. The above proposition states that a more complicated menu of two-part tariffs can be used to implement the optimal outcome when the buyers can form a coalition. Now the suitable menu of two-part tariff is such that (i) the fixed fee paid by a buyer depends on the two-part tariff chosen by the other buyer; (ii) the tariff a buyer faces has a kink unless both buyers choose the tariff designed for H-type. The kink is necessary because of the downward coalition incentive constraints $(CIC_{HH,HL})$, $(CIC_{HH,LL})$ and $(CIC_{HL,LL})$. Consider $(CIC_{HH,HL})$, for instance, and assume that there is no kink in T_H^*. Then, when both buyers have H-type, they have an incentive to coordinate their purchases such that only one buyer chooses T_H^*, buys more than q_H^* and shares it with the other buyer who chooses T_L^*.[34] This deviation is

[33] The proof can be received upon request from the authors.

[34] Likewise, if there were no kink in T_L^*, then the buyer who pretended to be L-type may buy more than q_L^* and then share with the other buyer.

prevented by the increase in the marginal price - the kink - from c to $\theta_H u'(q_L^*)$. The fixed fee paid by a buyer needs to be dependent on the other buyer's choice of two-part tariff since the optimal transfer scheme requires this sort of randomness.

6 Extensions

In the previous sections, we considered the two-buyer-two-type setting in order to keep the exposition and the intuition for the results as simple as possible. In this section, we show that our main result (Proposition 5) can be extended to the n-buyer-two-type setting and the two-buyer-three-type setting.

6.1 The case of n buyers

We here show that our main result holds when the seller faces $n > 2$ buyers if we assume that the only feasible coalition is the grand coalition, the one including all the buyers. More precisely, we suppose that if at least one buyer rejects the side mechanism, then M is played non-cooperatively (and with prior beliefs): coalitions of size smaller than n are not going to arise. This assumption can be justified when any attempt to organize a coalition - after the grand coalition was rejected - is sufficiently time consuming such that it is impossible for the third party to design a new side mechanism which is tailored for the buyers who accepted the original side mechanism. Clearly, this assumption is not needed if $n = 2$ but it makes the model quite tractable when $n > 2$.

Without loss of generality, we restrict our attention to symmetric sale mechanisms, which are now introduced. Let q_{Lk} $(k = 0, 1, ..., n-1)$ denote the quantity allocated to each L-type by the seller when the profile of reports $\hat{\theta} \equiv (\hat{\theta}^1, ..., \hat{\theta}^n) \in \Theta^n$ includes exactly k number of buyers with H-type. The variables q_{Hk}, t_{Hk} and t_{Lk} are defined similarly. Let $q_n \equiv (q_{L0}, ..., q_{Ln-1}, q_{H1}, ..., q_{Hn})$ and $t_n \equiv (t_{L0}, ..., t_{Ln-1}, t_{H1}, ..., t_{Hn})$ so that a sale mechanism is given by $M_n = \{q_n, t_n\}$. Any optimal mechanism $\{q_n^*, t_n\}$ without buyer coalition is such that $q_{Lk}^* = q_L^*$ and $q_{Hk}^* = q_H^*$ for any k and the expected payment of type L and H is equal to $\theta_L u(q_L^*)$ and $\theta_H u(q_H^*) - \Delta\theta u(q_L^*)$, respectively.

Proposition 3, the weakly collusion-proofness principle, applies to this setting. Here we generalize Proposition 4 by describing the conditions under which an incentive compatible mechanism M_n is weakly collusion-proof. In order to do that, we need to investigate how goods are reallocated by the third party in a coalition containing $k \in \{1, ..., n-1\}$ number of buyers with H-type (in what follows, such a coalition will be referred to as a "$k-$coalition") when $x > 0$ is the total quantity available; the cases $k = 0$ and $k = n$ are obvious. In any $k-$coalition, the third-party allocates the same quantity to each buyer of the same type since $u'' < 0$. Precisely, if quantity z is allocated to each H-type, then each L-type receives $\frac{x-kz}{n-k}$ and the quantity received by H-type $q^{\varepsilon}_{Hk}(x)$ is defined as

$$q^{\varepsilon}_{Hk}(x) \equiv \arg \max_{z \in [0, \frac{x}{k}]} k\theta_H u(z) + (n-k)\theta^{\varepsilon}_L u(\frac{x-kz}{n-k})$$

Hence, the no-reallocation condition for a $k-$coalition (if $q_{Lk} > 0$) is:

$$\theta_H u'(q_{Hk}) = \theta^{\varepsilon}_L u'(q_{Lk}) \tag{14}$$

If (14) is satisfied, a $k-$coalition which reports truthfully in M_n has no incentive to alter the allocation determined by the seller. Notice that

$$V^{\varepsilon}_k(x) \equiv \max_{z \in [0, \frac{x}{k}]} k\theta_H u(z) + (n-k)\theta^{\varepsilon}_L u(\frac{x-kz}{n-k})$$

is the gross payoff for a $k-$coalition when it has the total quantity x. Moreover, we can define $q^{\varepsilon}_{Hn}(x) = q^{\varepsilon}_{L0}(x) = \frac{x}{n}$ and we have, as a consequence, $V^{\varepsilon}_n(x) = n\theta_H u(\frac{x}{n})$ and $V^{\varepsilon}_0(x) = n\theta^{\varepsilon}_L u(\frac{x}{n})$. Again, we may regard each coalition as a consolidated agent and interpret V^{ε}_k as the surplus function for a coalition with type k. For a $k-$coalition, manipulating its reports is equivalent to reporting a number $k\prime \neq k$ of buyers with H-type. The next proposition summarizes the coalition incentive constraints and the no-arbitrage constraint.

Proposition 8 *An incentive compatible sale mechanism M_n is weakly collusion-proof if and only if there exists $\varepsilon \in [0, 1)$ such that*
(a) the following coalition incentive constraints are satisfied:

$$\begin{aligned} & V^{\varepsilon}_k[kq_{Hk} + (n-k)q_{Lk}] - kt_{Hk} - (n-k)t_{Lk} \\ \geq\ & V^{\varepsilon}_k[k'q_{Hk'} + (n-k')q_{Lk'}] - k't_{Hk'} - (n-k')t_{Lk'} \text{ for any } (k, k') \in \{0, 1, ..., n\}^2 \end{aligned}$$

(b) the no-arbitrage condition (14) holds for $k = 1, ..., n-1$.
(c) if $\varepsilon > 0$, then H-type's incentive constraint in the side mechanism is binding.

The next proposition establishes that the buyer coalition does not create any loss to the seller, as in the case of $n = 2$.

Proposition 9 *Given the quantity schedule q_n^*, there exists transfers t_n^* such that $M_n^* \equiv \{q_n^*, t_n^*\}$ is optimal under no buyer coalition and is also weakly collusion-proof.*

Proof. See Appendix. ■

6.2 The case of three types

Mechanism design problems under collusion are viewed qualitatively more complicated when there are more than two types than when there are only two types. For instance, Laffont and Martimort (1997, 2000) limit their analysis to the two-type setting since it is hard to determine the binding CIC constraints when there are more than two types. However, we can show that in our model the main result – Proposition 5 – extends to the three-type setting. The main difficulty in performing such an extension comes from the fact that the single-crossing condition for coalitions holds only partially since no order can be made between coalitions HL and MM. Nevertheless, we are able to prove Proposition 12.

We assume $n = 2$ for simplicity. Buyer i privately observes his type $\theta^i \in \Theta \equiv \{\theta_L, \theta_M, \theta_H\}$, where $\Delta_H \equiv \theta_H - \theta_M > 0$ and $\Delta_M \equiv \theta_M - \theta_L > 0$. The types θ^1 and θ^2 are identically and independently distributed with $p_L \equiv \Pr\left\{\theta^i = \theta_L\right\}$, $p_M \equiv \Pr\left\{\theta^i = \theta_M\right\}$ and $p_H \equiv \Pr\left\{\theta^i = \theta_H\right\}$. The distribution of θ^1 and θ^2 is common knowledge. In the absence of buyer coalition, the virtual valuations of type M and L are given by:

$$\theta_M^v \equiv \theta_M - \frac{p_H}{p_M}\Delta_H \qquad \theta_L^v \equiv \theta_L - \frac{p_H + p_M}{p_L}\Delta_M$$

Clearly, $\theta_H > \max\{\theta_M^v, \theta_L^v\}$ but the order between θ_M^v and θ_L^v depends on the parameters of the problem. If $\theta_M^v \geq \theta_L^v$, then virtual valuations are said to be monotone. If $\theta_M^v < \theta_L^v$, then let $\bar{\theta}_{ML}^v \equiv \frac{p_L\theta_L^v + p_M\theta_M^v}{p_L + p_M}$. In any case, we assume that $\min\{\theta_M^v u'(0), \theta_L^v u'(0)\} > c > \lim_{q \to +\infty} \theta_H u'(q)$, so that each type gets a positive and bounded quantity in case of no coalition.

As in section 2.1, we can restrict our attention to direct revelation mechanisms:

$$M = \left\{q^i(\widehat{\theta}^1, \widehat{\theta}^2), t^i(\widehat{\theta}^1, \widehat{\theta}^2);\ i = 1, 2\right\}.$$

We here focus on symmetric mechanisms and introduce the following notation:

$$q_{jk} \equiv q^1(\theta_j, \theta_k) = q^2(\theta_k, \theta_j), \qquad t_{jk} \equiv t^1(\theta_j, \theta_k) = t^2(\theta_k, \theta_j), \qquad j, k = L, M, H$$

Therefore, $M \equiv \{q, t\}$ where $q \equiv \{q_{jk}\}_{j,k=L,M,H}$ and $t \equiv \{t_{jk}\}_{j,k=L,M,H}$. Let $\bar{t}_j \equiv p_L t_{jL} + p_M t_{jM} + p_H t_{jH}$ and $\bar{u}_j \equiv p_L u(q_{jL}) + p_M u(q_{jM}) + p_H u(q_{jH})$ with $j = L, M, H$. Then, the expected profit is given by:

$$\begin{aligned} \Pi \;=\; & 2(p_L \bar{t}_L + p_M \bar{t}_M + p_H \bar{t}_M) - 2c[p_L^2 q_{LL} + p_L p_M (q_{LM} + q_{ML}) + p_L p_H (q_{HL} + q_{LH})] \\ & -2c[p_M^2 q_{MM} + p_M p_H (q_{MH} + q_{HM}) + p_H^2 q_{HH}] \end{aligned}$$

The Bayesian incentive compatibility and participation constraints are given by:

$$\begin{aligned} (BIC) \quad \theta_j \bar{u}_j - \bar{t}_j \;&\geq\; \theta_j \bar{u}_{j'} - \bar{t}_{j'}, \quad j, j' = L, M, H \\ (BIR) \quad \theta_j \bar{u}_j - \bar{t}_j \;&\geq\; 0, \quad j = L, M, H \end{aligned}$$

An optimal mechanism solves the problem $\max_{\{q,t\}} \Pi$ s.t. (BIC) and (BIR). The next proposition characterizes the optimal mechanisms in the absence of buyer coalition.

Proposition 10 *The optimal mechanisms in the absence of buyer coalition are characterized by*

a) The optimal quantity schedule $q^* = \left\{q^*_{jk}\right\}_{j,k=L,M,H}$ *is monotone* $q^*_H > q^*_M \geq q^*_L$ *and given by:*

i) $q^*_{Hj} = q^*_H$ *for* $j = L, M, H$, *where* $\theta_H u'(q^*_H) = c$;

ii) If $\theta^v_M \geq \theta^v_L$, *then* $q^*_{Mj} = q^*_M$ *and* $q^*_{Lj} = q^*_L$ *for* $j = L, M, H$, *where* $\theta^v_M u'(q^*_M) = c$ *and* $\theta^v_L u'(q^*_L) = c$.

If instead $\theta^v_M < \theta^v_L$, *then* $q^*_{Mj} = q^*_M = q^*_{Lj} = q^*_L$ *for* $j = L, M, H$, *where* $\bar{\theta}^v_{ML} u'(q^*_L) = c$.

b) Transfers are such that constraints (BIC_{HM}), (BIC_{ML}) *and* (BIR_L) *bind.*

Proof. The proof is standard and therefore is omitted. ■

As in the two-type case, the weakly collusion proof principle holds. Before we state the characterization of weakly collusion proof mechanisms, it is useful to define i) the variables θ^ε_H, θ^ε_M and θ^ε_L; ii) the functions $q^\varepsilon_j(x; jk)$ and $q^\varepsilon_k(x; jk)$, $jk = HM, HL, ML$;

iii) the functions $V^{\varepsilon}_{jk}(x)$, $j,k = L,M,H$ as follows:

$$\theta^{\varepsilon}_{H} \equiv \theta_{H}, \quad \theta^{\varepsilon}_{M} \equiv \theta_{M} - \frac{p_H}{p_M}\Delta_H \varepsilon_{HM}, \quad \theta^{\varepsilon}_{L} \equiv \theta_{L} - \frac{p_H}{p_L}\Delta_M \varepsilon_{ML},$$

$$q^{\varepsilon}_{j}(x;jk) \equiv \arg\max_{z\in[0,x]} \theta^{\varepsilon}_{j} u(z) + \theta^{\varepsilon}_{k} u(x-z) \quad \text{and} \quad q^{\varepsilon}_{k}(x;jk) \equiv x - q^{\varepsilon}_{j}(x;jk)$$

$$V^{\varepsilon}_{jk}(x) \equiv \max_{z\in[0,x]} \theta^{\varepsilon}_{j} u(z) + \theta^{\varepsilon}_{k} u(x-z), \quad j,k = L,M,H$$

where $\varepsilon \equiv (\varepsilon_{HM}, \varepsilon_{ML}) \in [0,1) \times [0,+\infty)$ and $x > 0$.

The next proposition characterizes weakly collusion-proof mechanisms.

Proposition 11 *An incentive compatible sale mechanism M is weakly collusion-proof if and only if there exists $\varepsilon \in [0,1) \times [0,+\infty)$ such that*
(a) the coalition incentive constraints are satisfied

$$V^{\varepsilon}_{jk}(q_{jk} + q_{kj}) - t_{jk} - t_{kj} \geq V^{\varepsilon}_{jk}(q_{j'k'} + q_{k'j'}) - t_{j'k'} - t_{k'j'}, \qquad \text{for any } j,k,j',k' \tag{15}$$

(b) the no arbitrage constraints hold

$$q_{jk} = q^{\varepsilon}_{j}(q_{jk} + q_{kj}; jk), \qquad \text{for } jk = HM, HL, ML. \tag{16}$$

(c) if $\varepsilon_{HM} > 0$ (resp. $\varepsilon_{ML} > 0$), then (BIC^{S}_{HM}) [resp. (BIC^{S}_{ML})] binds.

Proof. The proof is long (but is very similar to the proof of proposition 4) and hence omitted.[35] ■

Finally, we can prove that the buyer coalition does not create any loss to the seller.

Proposition 12 *Given the quantity profile $q^* = \left\{q^*_{jk}\right\}_{j,k=L,M,H}$, there exists a transfer scheme $t^* = \left\{t^*_{jk}\right\}_{j,k=L,M,H}$ such that $M^* \equiv \{q^*, t^*\}$ is an optimal mechanism in the absence of buyer coalition and is also weakly collusion-proof.*

Proof. The proof goes along the same lines of the proof of proposition 5 (they are briefly sketched below), but is considerably longer, hence it is omitted.[36] ■

We now briefly sketch the proof of the above result. First, as in the two-type case, the principal can choose $\varepsilon^* \equiv (\varepsilon^*_{HM}, \varepsilon^*_{ML})$ such that the third-party has the same virtual

[35] The proof can be received upon request from the authors.
[36] The proof can be received upon request from the authors.

valuations that she has: $\theta_M^{\varepsilon^*} = \theta_M^v$ and $\theta_L^{\varepsilon^*} = \theta_L^v$. This implies in particular that conditional on that there is no manipulation of reports, the third-party will not reallocate goods: the no-arbitrage constraints are satisfied. Second, there remain some degrees of freedom in transfers in the optimal mechanisms under no coalition and the principal can use this freedom to satisfy all the coalition incentive constraints. We conjecture that our result will hold when there are more than three types as well.

7 Robustness

In the previous sections we have made a specific assumption about buyers' beliefs and behavior in case the side mechanism is vetoed – namely, assumption WCP: buyers are expected to play the truthtelling equilibrium of the sale mechanism (with prior beliefs). In this section, we show that our results are robust to eliminating this assumption.

We recall that, given a sale mechanism M, in the coalition formation game, first the third party proposes a side mechanism S, then each buyer simultaneously announces whether he accepts or refuses S and finally buyers report in S if S was unanimously accepted, or in M otherwise. Under assumption WCP, we established above that if $M = M^*$, then (i) the third party proposes $S = S^0$; (ii) both buyers accept S^0. In this section, we show that (i) and (ii) do hold even though we eliminate assumption WCP. Finally, we analyze the structure of the equilibria of the game which is played after both buyers accepted S^0. We focus here on the case of $n = 2$ and in Subsections 7.2 and 7.3 we make the following assumption[37]

$$\frac{u''(x)}{u'(x)} \text{ is weakly increasing in } x. \tag{17}$$

7.1 Is M^* more collusion-proof than weakly collusion-proof?

Mechanism M^* is weakly collusion-proof according to definition 2, which rests on assumption WCP. This assumption determines precisely the reservation utility for each type of buyer at the time of deciding whether to accept or reject the side mechanism. What if the third party expects the buyers to coordinate – following buyer 1's (say)

[37] When u is a Bernoulli utility function over money, this assumption is called "non-increasing absolute risk-aversion".

rejection of the side mechanism – on a non-truthful equilibrium of M^*, possibly under non-prior beliefs of buyer 2 about θ^1? Then, the acceptance constraints of buyer 1 in the side mechanism design problem may be altered and eventually the third party may be induced to select a non-null side mechanism. Nevertheless, the next proposition proves that if $M = M^*$, then there exists no side mechanism $S \neq S^0$ which might be accepted by both agents and increase the third party's payoff.

Proposition 13 *When M^* is proposed, even without assumption WCP, there exists no equilibrium in the coalition formation game in which the third party designs $S \neq S^0$ and both buyers accept S.*

Proof. See Appendix. ■

This proposition basically says that M^* is collusion-proof not only if the third party believes that truthtelling is played in case S is rejected. Actually, S^0 is proposed in *any* collusive continuation equilibrium of the coalition formation game.

7.2 Robustness to cheap talk

In Subsection 7.1 we established that the choice of S^0 by the third party is robust to the various equilibria of M^* he may expect the buyers to coordinate on if S^0 is vetoed. Here we prove that even without assumption WCP, buyers still have incentives to accept S^0. In principle, this result is not straightforward. Buyer 1 – for instance –, depending on what he expects to be played if S^0 is rejected, may try to increase his payoff by vetoing S^0. We below describe this issue and our answer in more detail.

As we mentioned above, a two stage game starts *after* S^0 is proposed by the third party. In the first stage, each buyer i makes a preplay announcement (veto or accept) which may signal some information about θ^i; in the second stage, buyers report in M^* or in S^0. In any case, however, in the second stage M^* is actually played since S^0 is null: the first stage is just a sort of cheap-talk stage in which a buyer may signal his type. We know that no type wishes to reject S^0 under assumption WCP, but what if buyer 1 expects that a non-truthful equilibrium of M^* will be played (possibly under non-prior beliefs of 2 about θ^1) in case he vetoes S^0? Here we study whether it is possible for some type of buyer 1 to veto S^0 – which is an out-of-equilibrium message – in order to manipulate beliefs of buyer 2 about θ^1 and then reach some better outcome for himself when playing M^* at the next stage. In other words, we ask whether beliefs of buyer 2 – following a deviation of 1 – exist such that buyer 1 (or just a type of buyer 1) gains from rejecting S^0. The answer to this question is negative and the following lemma provides a useful step.

Lemma 2 *Under (17), in M^**
(a) reporting L is strictly dominant for type L,
(b) type H strictly prefers reporting H to L if his opponent plays H and strictly prefers reporting L to H if his opponent plays L.

Proof. See Appendix. ■

By using Lemma 2(a), we can prove that buyer 1 cannot gain from trying to manipulate buyer 2's beliefs through the cheap-talk stage.

Proposition 14 *Under (17), there exists no belief of buyer* 2 *(following a deviation of 1) which supports an equilibrium of M^* in which one (or both) type of buyer* 1 *is better off with respect to truthtelling behavior.*

Proof. Inequalities (13) imply that buyer 1 (regardless of his type) has a chance to be better off with respect to the truthtelling equilibrium only if his opponent plays H more often than under truthtelling. However, this cannot occur in any equilibrium of M^* – regardless of buyer 2's beliefs about θ^1 - because reporting L is strictly dominant for type L of buyer 2. Hence, in any equilibrium of M^* the probability that 2 reports H is at most equal to the probability that 2 reports H under truthtelling. ■

We note that this proposition is stronger than Proposition 9 in Laffont and Martimort (2000). Indeed, their result refers to the notion of ratifiability [see Cramton and Palfrey (1995)], which allows buyer 2 to have only "reasonable" or "consistent" beliefs about θ^1. In contrast, we do not need any "sophisticated" argument in order to make our point: simply no beliefs of 2 support buyer 1's rejection of S^0.

7.3 Multiplicity of equilibria in M^*

Consider the game of coalition formation immediately after both buyers accepted S^0: At that point in time, buyers have to report in S^0. However, as we observed above, that is equivalent to playing non-cooperatively M^* with prior beliefs for both buyers, since each buyer i has prior beliefs about θ^j ($j \neq i$) after S^0 has been unanimously accepted. Although truthtelling is an equilibrium in M^*, there may exist other equilibria in M^* which buyers may coordinate on. The next proposition addresses this issue.

Proposition 15 *Under (17), there exists only one non-truthful equilibrium of M^* played with prior beliefs. In it, every buyer type reports L. For buyers, the latter equilibrium is strictly Pareto-dominated by truthtelling.*

Proof. See Appendix. ■

Since buyers strictly Pareto prefer truthtelling to the non-truthful equilibrium, coordination on the latter seems unlikely to occur. Hence, non-uniqueness in M^* does not appear to be a problem for the seller.

8 Concluding remarks

We found that simple sale mechanisms in which the quantity sold to a buyer and his payment depend solely on his own report create room for buyers' joint actions such that the buyers can realize strict gains at the seller's loss by coordinating their purchases and reallocating the goods. However, we showed that when the seller judiciously designs her mechanism(s) by exploiting the transaction costs in coalition formation, buyer coalition does not hurt her and, in particular, the buyers are unable to implement efficient arbitrage. We also showed that this outcome can be implemented through a menu of two-part tariffs.

Our result is derived in a complete contract setting in which there is no restriction on the set of contracts available to the seller. This setting corresponds to a situation in which the seller faces a small number of buyers and knows well each buyer's identity. This allows the seller to use state-contingent contracts in which a buyer's payment can depend on the other buyers' reports.

In contrast, when there are a large number of buyers (in particular, a mass of buyers), the seller would not have complete information about the identities of the potential buyers. This might impose some restrictions on the set of contracts available to the seller as in Alger (1999). It would be interesting to study the impact of asymmetric information on buyers' joint actions in this setting.

APPENDIX
Proof of Proposition 2

The side mechanism S^d mentioned in the statement of Proposition 2 is formally defined as follows.

Reports manipulations are: $\phi^d(HH) = HL$, $\phi^d(HL) = LL$, $\phi^d(LH) = LL$, $\phi(LL) = LL$.

Goods are reallocated as follows: $x^{1d}(HH) = -\frac{q_H^* - q_L^*}{2}$, $x^2(HH) = \frac{q_H^* - q_L^*}{2}$; $x^{1d}(HL) = \hat{x} > 0$, with $\hat{x}$ close to 0, $x^{2d}(HL) = -\hat{x}$; $x^{2d}(LH) = -x^{1d}(HL) = \hat{x}$; $x^{1d}(LL) = x^{2d}(LL) = 0$.

Side transfers are: $y^{1d}(HH) = \frac{t_H^d - t_L^d}{2}$, $y^{2d}(HH) = -\frac{t_H^d - t_L^d}{2}$; $y^{1d}(HL) = y^{2d}(LH) = \hat{y}$, $y^{2d}(HL) = y^{1d}(LH) = -\hat{y}$; $y^{1d}(LL) = y^{2d}(LL) = 0$, where $\hat{y} > 0$ is still to be defined.

In words, a coalition HH reports HL; then goods and transfers are equally shared between the buyers. A coalition HL or LH reports LL; then goods are slightly reallocated from L-type to H-type and H-type pays $\hat{y}$ to L- type.

We prove that there exists an $\hat{y} > 0$ is such that all the incentive and participation constraints in the side mechanism are satisfied (actually, they are slack). This establishes that S^d can be implemented and that the payoff of each buyer type is strictly larger than the one from playing M^d non-cooperatively.

Define $\hat{q}_H \equiv q_L^* + x^{1d}(HL)$ and $\hat{q}_L \equiv q_L^* - x^{1d}(HL)$. Constraint (BIC_H^S) is

$$\begin{aligned} & \lambda[\theta_H u(\hat{q}_H) - \theta_L u(q_L^*) - y] + (1-\lambda)[\theta_H u(\frac{q_L^* + q_H^*}{2}) - \theta_L u(q_L^*) - \frac{\theta_H}{2}(u(q_H^*) - u(q_L^*))] \\ \geq \;& \lambda(\Delta\theta) u(q_L^*) + (1-\lambda)[\theta_H u(\hat{q}_L) - \theta_L u(q_L^*) + y] \end{aligned} \tag{18}$$

Consider $\tilde{y} = \theta_H[u(q_L^*) - u(\hat{q}_L)]$. With $y = \tilde{y}$, (i) the right hand of (18) is exactly equal to $V(\theta_H)$; (ii) since $\hat{x}$ is close to 0, (18) is strictly satisfied, hence (BIR_H^S) holds; (iii) (BIR_L^S) holds. Now consider increasing y above $\tilde{y}$ until the point $\hat{y}$ at which (18) binds. At that point, (BIR_H^S) still holds because the right hand side of (18) increased above $V(\theta_H)$; clearly, also (BIR_L^S) still holds since $\check{y} > \tilde{y}$. In order to prove that (BIC_L^S) is satisfied, a standard argument can be used: just sum (BIC_L^S) and (BIC_H^S) (which binds) and obtain an inequality which is strictly satisfied because $\hat{q}_H > q_L^*$ and $\frac{q_L^* + q_H^*}{2} > \hat{q}_L$.

The side mechanism S^d may not be the optimal side mechanism against M^d. In particular, reallocation is not performed efficiently in HL coalition since otherwise we are not sure that it is possible to satisfy all the incentive and acceptance constraints.

However, in the optimal side mechanism against M^d (denote it by S^{Od}), the seller certainly loses with respect to the case of no coalition. Indeed, from Proposition 3, there exists a weakly collusion proof sale mechanism denoted by M'which achieves the profit she obtains with M^d and S^{Od} . However, in M', at least one of the two types must have strictly higher payoff with respect to the case of no coalition otherwise S^{Od} would not be the optimal side mechanism against M^d.

Proof of Proposition 4

We are interested in grand-mechanisms such that L-type's incentive constraint is not binding. The third-party maximizes the following objective,

$$\begin{gathered}(1-\lambda)^2[\theta_H u(q^1(\phi_{HH})+x^1_{HH})-t^1(\phi_{HH})+\theta_H u(q^2(\phi_{HH})+x^2_{HH})-t^2(\phi_{HH})]+\\ \lambda(1-\lambda)[\theta_L u(q^1(\phi_{LH})+x^1_{LH})-t^1(\phi_{LH})+\theta_H u(q^2(\phi_{LH})+x^2_{LH})-t^2(\phi_{LH})]+\\ \lambda(1-\lambda)[\theta_H u(q^1(\phi_{HL})+x^1_{HL})-t^1(\phi_{HL})+\theta_L u(q^2(\phi_{HL})+x^2_{HL})-t^2(\phi_{HL})]+\\ \lambda^2[\theta_L u(q^1(\phi_{LL})+x^1_{LL})-t^1(\phi_{LL})+\theta_L u(q^2(\phi_{LL})+x^2_{LL})-t^2(\phi_{LL})]\end{gathered}$$

subject to the following constraints.

- Budget balance constraints: for the quantity reallocation

$$\sum_{i=1}^{2} x^i(\theta^1,\theta^2)=0,\ \text{for any }(\theta^1,\theta^2)\in\Theta^2;$$

for the side transfers

$$\sum_{i=1}^{2} y^i(\theta^1,\theta^2)=0,\ \text{for any }(\theta^1,\theta^2)\in\Theta^2,$$

- H-type's Bayesian incentive constraint for buyer 1:

$$\begin{aligned}&\lambda[\theta_H u(q^1(\phi_{HL})+x^1_{HL})-t^1(\phi_{HL})-y^1_{HL}]+(1-\lambda)[\theta_H u(q^1(\phi_{HH})+x^1_{HH})-t^1(\phi_{HH})-y^1_{HH}]\\ \geq\ &\lambda[\theta_H u(q^1(\phi_{LL})+x^1_{LL})-t^1(\phi_{LL})-y^1_{LL}]+(1-\lambda)[\theta_H u(q^1(\phi_{LH})+x^1_{LH})-t^1(\phi_{LH})-y^1_{LH}],\end{aligned}$$

- H-type's Bayesian incentive constraint for buyer 2 :

$$\begin{aligned}&\lambda[\theta_H u(q^2(\phi_{LH})+x^2_{LH})-t^2(\phi_{LH})-y^2_{LH}]+(1-\lambda)[\theta_H u(q^2(\phi_{HH})+x^2_{HH})-t^2(\phi_{HH})-y^2_{HH}]\\ \geq\ &\lambda[\theta_H u(q^2(\phi_{LL})+x^2_{LL})-t^2(\phi_{LL})-y^2_{LL}]+(1-\lambda)[\theta_H u(q^2(\phi_{HL})+x^2_{HL})-t^2(\phi_{HL})-y^2_{HL}],\end{aligned}$$

- H-type's acceptance constraint for buyer 1:

$$\lambda[\theta_H u(q^1(\phi_{HL}) + x^1_{HL}) - t^1(\phi_{HL}) - y^1_{HL}] + (1-\lambda)[\theta_H u(q^1(\phi_{HH}) + x^1_{HH}) - t^1(\phi_{HH}) - y^1_{HH}]$$

$$\geq V(\theta_H)$$

- H-type's acceptance constraint for buyer 2:

$$\lambda[\theta_H u(q^2(\phi_{LH}) + x^2_{LH}) - t^2(\phi_{LH}) - y^2_{LH}] + (1-\lambda)[\theta_H u(q^2(\phi_{HH}) + x^2_{HH}) - t^2(\phi_{HH}) - y^2_{HH}]$$

$$\geq V(\theta_H)$$

- L-type's acceptance constraint for buyer 1:

$$\lambda[\theta_L u(q^1(\phi_{LL}) + x^1_{LL}) - t^1(\phi_{LL}) - y^1_{LL}] + (1-\lambda)[\theta_L u(q^1(\phi_{LH}) + x^1_{LH}) - t^1(\phi_{LH}) - y^1_{LH}] \geq V(\theta_L),$$

- L-type's acceptance constraint for buyer 2:

$$\lambda[\theta_L u(q^2(\phi_{LL}) + x^2_{LL}) - t^2(\phi_{LL}) - y^2_{LL}] + (1-\lambda)[\theta_L u(q^2(\phi_{HL}) + x^2_{HL}) - t^2(\phi_{HL}) - y^2_{HL}] \geq V(\theta_L),$$

We introduce the following multipliers:

- $\rho^x(\theta^1, \theta^2)$ for the budget-balance constraint for the quantity reallocation in state (θ^1, θ^2),
- $\rho^y(\theta^1, \theta^2)$ for the budget-balance constraint for the side-transfers in state (θ^1, θ^2),
- δ^i for the H-type's Bayesian incentive constraint concerning buyer i,
- v^i_L for the L-type's acceptance constraint concerning buyer i,
- v^i_H for the H-type's acceptance constraint concerning buyer i.

We define the Lagrangian as follows:

$$L = E(U_1 + U_2) + \sum_{i=1,2} \delta^i (BIC^S)_i(\theta_H) + \sum_{i=1,2} v^i_H (BIR^S)_i(\theta_H) + \sum_{i=1,2} v^i_L (BIR^S)_i(\theta_L)$$

$$+\sum_{\theta_1,\theta_2}\rho^x(\theta^1,\theta^2)(BB:x)(\theta^1,\theta^2)+\sum_{\theta_1,\theta_2}\rho^y(\theta^1,\theta^2)(BB:y)(\theta^1,\theta^2)$$

Step 1: Optimizing with respect to $y^i(\theta^1,\theta^2)$

After optimizing with respect to y^i_{HH}, we have:

$$\rho^y_{HH}-\delta^i(1-\lambda)-v^i_H(1-\lambda)=0,\ \text{for } i=1,2.$$

After optimizing with respect to y^1_{HL} and y^2_{HL} respectively, we have:

$$\begin{aligned}\rho^y_{HL}-\delta^1\lambda-v^1_H\lambda &= 0;\\ \rho^y_{HL}+\delta^2(1-\lambda)-v^2_L(1-\lambda) &= 0\end{aligned}$$

After optimizing with respect to y^1_{LH} and y^2_{LH} respectively, we have:

$$\begin{aligned}\rho^y_{LH}+\delta^1(1-\lambda)-v^1_L(1-\lambda) &= 0;\\ \rho^y_{LH}-\delta^2\lambda-v^2_H\lambda &= 0\end{aligned}$$

After optimizing with respect to y^i_{LL}, we have:

$$\rho^y_{LL}+\delta^i\lambda-v^i_L\lambda=0,\ \text{for } i=1,2.$$

From the above eight equations, we have:

$$\begin{aligned}\delta^1+v^1_H &= \delta^2+v^2_H,\quad \delta^1-v^1_L=\delta^2-v^2_L,\\ \lambda(\delta^j+v^j_H) &= (1-\lambda)\left(v^k_L-\delta^k\right),\ \text{for } j\neq k.\end{aligned}$$

In what follows, without loss of generality, we restrict our attention to symmetric multipliers:

$$\delta^1=\delta^2=\delta, v^1_H=v^2_H=v_H, v^1_L=v^2_L=v_L.$$

Step 2: Optimizing with respect to $x^i(\theta^1,\theta^2)$

After optimizing with respect to x^i_{HH}, we have:[38]

$$\rho^x_{HH}+\left[(1-\lambda)+\delta^i+v^i_H\right](1-\lambda)\theta_H u'(q^i(\phi_{HH})+x^i_{HH})=0,\ \text{for } i=1,2.$$

[38]In homogeneous coalitions, HH and LL, the reallocation cannot lead to corner solutions. In coalition HL, instead, this is conceivable but it is not going to occur when the seller designs the sale mechanism optimally. Hence, we only consider interior solutions for the reallocation problem.

Since $\delta^1+v_H^1 = \delta^2+v_H^2$, the above equation implies that $q^1(\phi_{HH})+x_{HH}^1 = q^2(\phi_{HH})+x_{HH}^2$. Since $x_{HH}^1 + x_{HH}^2 = 0$ from the budget balance constraint, we have $q^i(\phi_{HH}) + x_{HH}^i = \frac{q^1(\phi_{HH})+q^2(\phi_{HH})}{2}$.

After optimizing with respect to x_{HL}^1 and x_{HL}^2 respectively, we have:

$$\begin{aligned} \rho_{HL}^x + \left[(1-\lambda) + \delta^1 + v_H^1\right] \lambda\theta_H u'(q^1(\phi_{HL}) + x_{HL}^1) &= 0, \\ \rho_{HL}^x + \left[\lambda\theta_L - \delta^2\theta_H + v_L^2\theta_L\right] (1-\lambda)u'(q^2(\phi_{HL}) + x_{HL}^2) &= 0. \end{aligned}$$

By using the symmetry of the multipliers and $\lambda(\delta^j + v_H^j) = (1-\lambda)\left(v_L^k - \delta^k\right)$, we obtain from the two above equations:

$$\theta_H u'(q^1(\phi_{HL}) + x_{HL}^1) = \left(\theta_L - \frac{1-\lambda}{\lambda}(\Delta\theta)\varepsilon\right) u'(q^2(\phi_{HL}) + x_{HL}^2),$$

where $\varepsilon = \frac{\delta}{1-\lambda+\delta+v_H}$.

After optimizing with respect to x_{LH}^1 and x_{LH}^2 respectively, we have:

$$\begin{aligned} \rho_{LH}^x + \left[\lambda\theta_L - \delta^1\theta_H + v_L^1\theta_L\right] (1-\lambda)u'(q^1(\phi_{LH}) + x_{LH}^1) &= 0, \\ \rho_{LH}^x + \left[(1-\lambda) + \delta^2 + v_H^2\right] \lambda\theta_H u'(q^2(\phi_{LH}) + x_{LH}^2) &= 0. \end{aligned}$$

From the two above equations, and recalling that $\theta_L^\varepsilon = \theta_L - \frac{1-\lambda}{\lambda}(\Delta\theta)\varepsilon$ we obtain:

$$\theta_H u'(q^2(\phi_{LH}) + x_{LH}^2) = \theta_L^\varepsilon u'(q^1(\phi_{LH}) + x_{LH}^1).$$

After optimizing with respect to x_{LL}^i, we have:

$$\rho_{LL}^x + \left[\lambda\theta_L - \delta^i\theta_H + v_L^i\theta_L\right] \lambda u'(q^i(\phi_{LL}) + x_{LL}^i) = 0, \text{ for } i = 1, 2.$$

Since the multipliers are symmetric, the above equations imply that $q^1(\phi_{LL}) + x_{LL}^1 = q^2(\phi_{LL}) + x_{LL}^2$. Since $x_{LL}^1 + x_{LL}^2 = 0$ from the budget balance constraint, we have $q^i(\phi_{LL}) + x_{LL}^i = \frac{q^1(\phi_{LL})+q^2(\phi_{LL})}{2}$.

Step 3: Optimizing with respect to $\phi(\theta_1, \theta_2)$

• Optimizing with respect to ϕ_{HH} yields:

$$\phi_{HH}^* \in \arg\max_{\phi_{HH}} \left\{2\theta_H u(\frac{q^1(\phi_{HH}) + q^2(\phi_{HH})}{2}) - t^1(\phi_{HH}) - t^2(\phi_{HH})\right\}.$$

- Optimizing with respect to ϕ_{HL} yields:

$$\phi^*_{HL} \in \arg\max_{\phi_{HL}} \left\{\theta_H u(q^1(\phi_{HL}) + x^1_{HL}) + \theta^\varepsilon_L u(q^2(\phi_{HL}) + x^2_{HL}) - t^1(\phi_{HL}) - t^2(\phi_{HL})\right\},$$

where

$$\theta_H u'(q^1(\phi_{HL}) + x^1_{HL}) = \theta^\varepsilon_L u'(q^2(\phi_{HL}) + x^2_{HL}) \text{ holds.}$$

- Optimizing with respect to ϕ_{LH} yields:

$$\phi^*_{LH} \in \arg\max_{\phi_{LH}} \left\{\theta^\varepsilon_L u(q^1(\phi_{LH}) + x^1_{LH}) + \theta_H u(q^2(\phi_{LH}) + x^2_{LH}) - t^1(\phi_{LH}) - t^2(\phi_{LH})\right\},$$

where

$$\theta_H u'(q^2(\phi_{LH}) + x^2_{LH}) = \theta^\varepsilon_L u'(q^1(\phi_{LH}) + x^1_{LH}) \text{ holds.}$$

- Optimizing with respect to ϕ_{LL} yields:

$$\phi^*_{LL} \in \arg\max_{\phi_{LL}} \left\{2\theta^\varepsilon_L u(\frac{q^1(\phi_{LL}) + q^2(\phi_{LL})}{2}) - t^1(\phi_{LL}) - t^2(\phi_{LL})\right\}$$

Proof of Lemma 1

We start by proving the single crossing condition: $\frac{\partial V^\varepsilon_2(x)}{\partial x} > \frac{\partial V^\varepsilon_1(x)}{\partial x} > \frac{\partial V^\varepsilon_0(x)}{\partial x}$. Clearly, $V^\varepsilon_2(x) = 2\theta_H u(\frac{x}{2})$ and $V^\varepsilon_0(x) = 2\theta^\varepsilon_L u(\frac{x}{2})$, hence $\frac{\partial V^\varepsilon_2(x)}{\partial x} = \theta_H u'(\frac{x}{2})$ and $\frac{\partial V^\varepsilon_0(x)}{\partial x} = \theta^\varepsilon_L u(\frac{x}{2})$. For a coalition HL let us consider for simplicity interior allocations (but the proof is easily adapted to the non-interior case). Then $q^\varepsilon_H(x)$ is such that $\theta_H u'[q^\varepsilon_H(x)] = \theta^\varepsilon_L u'[q^\varepsilon_L(x)]$ and the envelope theorem implies $\frac{\partial V^\varepsilon_1(x)}{\partial x} = \theta_H u'[q^\varepsilon_H(x)] = \theta^\varepsilon_L u'[q^\varepsilon_L(x)]$. Since u' is strictly decreasing and $\theta_H > \theta^\varepsilon_L$, we have $q^\varepsilon_H(x) > \frac{x}{2} > q^\varepsilon_L(x)$; hence $\theta_H u'(\frac{x}{2}) > \theta_H u'[q^\varepsilon_H(x)] = \theta^\varepsilon_L u'[q^\varepsilon_L(x)] > \theta^\varepsilon_L u(\frac{x}{2})$. Then we can use a result from section 3 in Maskin and Riley (1984) to conclude that if (11) holds and $\text{CIC}_{HH,HL}$ and $\text{CIC}_{HL,LL}$ (the local downward coalition incentive constraints) bind, then all the other incentive coalition constraints are satisfied.

For any q satisfying (11), consider now a transfer profile such that $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind. In particular, let

$$t_{HH} = A_1 + \frac{t_{HL} + t_{LH}}{2} \text{ and } t_{LL} = A_2 + \frac{t_{HL} + t_{LH}}{2} \tag{19}$$

where A_1 and A_2 depend on q. Then we need to maximize Π with respect to $(q, t_{HL}, t_{LH}, \varepsilon)$ subject to (1)-(4) and (10). Conditions (19) do not actually represent any further constraints, since transfers appear in Π and in (1) to (4) only through the values of $\bar{t}_L$ and $\bar{t}_H$, which are equal to $\frac{1+\lambda}{2}t_{HL} + \frac{1-\lambda}{2}t_{LH} + (1-\lambda)A_1$ and $\frac{\lambda}{2}t_{HL} + \frac{2-\lambda}{2}t_{LH} + \lambda A_2$ in view of (19). Since these expressions are linearly independent, by suitably choosing t_{HL} and t_{LH} any value of the above expressions can be attained, as when (19) does not need to be satisfied.

The transfers which make $M^* = \{q^*, t^*\}$ weakly collusion proof

$$\begin{aligned}
t^*_{HL} &= \frac{(1+\lambda)\theta_L - (3-\lambda^2)\theta_H}{2}u(q^*_L) + \theta_H\frac{\lambda(3-\lambda)}{2}u(q^\bullet_H) + (1-\lambda)(2-\lambda)\theta_H u(\frac{q^*_H + q^*_L}{2}) \\
&\quad + \frac{\lambda(1-\lambda)}{2}V^1_1(2q^*_L)] \\
t^*_{LH} &= \frac{(\lambda+3)\theta_L + (2\lambda+\lambda^2-1)\theta_H}{2}u(q^*_L) + \theta_H\frac{\lambda(1-\lambda)}{2}u(q^\bullet_H) - \lambda(1-\lambda)\theta_H u(\frac{q^*_H + q^*_L}{2}) \\
&\quad - \frac{\lambda(1+\lambda)}{2}V^1_1(2q^*_L) \\
t^*_{HH} &= \frac{(\lambda+2)\theta_L - (1-\lambda)(2+\lambda)\theta_H}{2}u(q^*_L) + \theta_H\frac{2+2\lambda-\lambda^2}{2}u(q^\bullet_H) - \lambda(2-\lambda)\theta_H u(\frac{q^*_H + q^*_L}{2}) \\
&\quad - \frac{\lambda^2}{2}V^1_1(2q^*_L) \\
t^*_{LL} &= \frac{(\lambda^2+2\lambda-1)\theta^1_L}{2}u(q^*_L) - \theta_H\frac{(1-\lambda)^2}{2}u(q^\bullet_H) + (1-\lambda)^2\theta_H u(\frac{q^*_H + q^*_L}{2}) + \frac{1-\lambda^2}{2}V^1_1(2q^*_L)
\end{aligned}$$

Proof of Proposition 9

The proof of this proposition is very similar to the one provided for $n = 2$, hence it is only sketched. First, the single crossing condition holds: $\frac{\partial V^\varepsilon_{k+1}(x)}{\partial x} > \frac{\partial V^\varepsilon_k(x)}{\partial x}$ for $k = 0, 1..., n-1$. Hence, if

$$(k+1)q_{Hk+1} + (n-k-1)q_{Lk+1} \geq kq_{Hk} + (n-k)q_{Lk} \text{ for } k = 0, ..., n-1 \tag{20}$$

and the local downward coalition incentive constraints bind [the ones preventing a $(k+1)-$coalition from reporting k], then all the other coalition incentive constraints are satisfied. Moreover, at $q = q^*$ both (20) and (14) hold (with $\varepsilon = 1$) and there exist transfers satisfying with equality (BIC_H), (BIR_L) and local downward coalition incentive constraints.

Proof of Proposition 13

Assume that if buyer 1 (say) rejects the side mechanism then a non-truthful equilibrium is played in M^*, possibly because 2 has non-prior beliefs about θ^i. Let V_j^N denote the payoff of j-type ($j = L, H$) of buyer 1 in such equilibrium. Proposition 14 implies that $V_H^N \leq V(\theta_H)$ and $V_L^N \leq V(\theta_L)$.
If $V_L^N < V(\theta_L) = 0$, then the acceptance constraint of L-type of buyer 1 is not relaxed in the design of the side mechanism since otherwise L-type would have a negative payoff by accepting S and therefore, by anticipating this fact, he would have rejected the seller's offer of M^* at stage two.
If $V_H^N < V(\theta_H)$, then observe that H-type's acceptance constraint in the side-mechanism is slack when his reservation utility is $V(\theta_H)$ (in the sense that the Lagrange multiplier is zero), hence it is a fortiori slack when the reservation utility of H-type is below $V(\theta_H)$. The reason is that, when $V_H^N \leq V(\theta_H)$, constraint (BIC_H^S) determines the rent of H-type rather than (BIR_H^S).

Proof of Lemma 2

It is useful to write down the payoff matrices in M^* for L-type and H-type, respectively. For example, $\theta_L u(q_H^*) - t_{HL}^*$, the entry in the left table below corresponding to row H and column L, is the payoff to L-type if he claims H and his opponent reports L.

$L-type$	L	H
L	$\theta_L u(q_L^*) - t_{LL}^*$	$\theta_L u(q_L^*) - t_{LH}^*$
H	$\theta_L u(q_H^*) - t_{HL}^*$	$\theta_L u(q_H^*) - t_{HH}^*$

$H-type$	L	H
L	$\theta_H u(q_L^*) - t_{LL}^*$	$\theta_H u(q_L^*) - t_{LH}^*$
H	$\theta_H u(q_H^*) - t_{HL}^*$	$\theta_H u(q_H^*) - t_{HH}^*$

In order to prove lemma 2 we start by establishing the following inequality

$$t^*_{LH} - t^*_{LL} \geq t^*_{HH} - t^*_{HL} \tag{21}$$

Recalling that $(CIC_{HH,HL})$ and $(CIC_{HL,LL})$ bind in the transfer scheme t^*, we find

$$\begin{aligned} t^*_{HH} &= \theta_H u(q^*_H) - \theta_H u(\frac{q^*_H + q^*_L}{2}) + \frac{t^*_{LH} + t^*_{HL}}{2} \\ t^*_{LL} &= \frac{V^1(2q^*_L) - V^1(q^*_H + q^*_L)}{2} + \frac{t^*_{LH} + t^*_{HL}}{2} \end{aligned}$$

hence (21) is equivalent to

$$V^1(q^*_H + q^*_L) - V^1(2q^*_L) \geq 2\theta_H u(q^*_H) - 2\theta_H u(\frac{q^*_H + q^*_L}{2}) \tag{22}$$

Define $g(z) = V^1(q^*_H + q^*_L + z) - V^1(2q^*_L) - [2\theta_H u(q^*_H + \frac{z}{2}) - 2\theta_H u(\frac{q^*_H + q^*_L}{2})]$ and notice that $g(0) > 0$ is equivalent to (22) and $g(-q^*_H + q^*_L) = 0$. Since we prove in what follows that $g'(z) > 0$ for $z < 0$, we conclude that $g(0) > 0$.

Indeed,

$$g'(z) = \theta_H u'[q^1_H(q^*_H + q^*_L + z)] - \theta_H u'(q^*_H + \frac{z}{2})$$

and $g'(0) = \theta_H u[q^1_H(q^*_H + q^*_L)] - \theta_H u(q^*_H) = 0$. $g'(z) > 0$ for $z < 0$ if $q^1_H(q^*_H + q^*_L + z) < q^*_H + \frac{z}{2}$ for $z < 0$. That is true since we can show that the following inequality holds:

$$\frac{d[q^1_H(q^*_H + q^*_L + z)]}{dz} > \frac{1}{2}. \tag{23}$$

By applying the implicit function theorem to (6) we obtain

$$\frac{dq^1_H(x)}{dx} = \frac{\theta^1_L u''[x - q^1_H(x)]}{\theta_H u''[q^1_H(x)] + \theta^1_L u''[x - q^1_H(x)]} = \frac{1}{\frac{\theta_H u''[q^1_H(x)]}{\theta^1_L u''[q^1_L(x)]} + 1}$$

We need to prove that $1 \geq \frac{\theta_H u''[q^1_H(x)]}{\theta^1_L u''[q^1_L(x)]}$ and - in view of (6) - the right hand side is equal to $\frac{u'[q^1_L(x)]u''[q^1_H(x)]}{u'[q^1_H(x)]u''[q^1_L(x)]}$. The assumption that $\frac{u''}{u'}$ is increasing implies $\frac{u'[q^1_L(x)]u''[q^1_H(x)]}{u'[q^1_H(x)]u''[q^1_L(x)]} \leq 1$.

(a) Lemma 2(a) is clearly equivalent to

$$\theta_L u(q^*_L) - t^*_{LL} > \theta_L u(q^*_H) - t^*_{HL} \text{ and } \theta_L u(q^*_L) - t^*_{LH} > \theta_L u(q^*_H) - t^*_{HH} \tag{24}$$

We first show that the second inequality holds and then prove the first. The second inequality in (24) is equivalent to

$$[\theta_L - (1+\lambda)\theta_H]u(q_L^*) + [(2+\lambda)\theta_H - 2\theta_L]u(q_H^*) > 2\lambda\theta_H u(\frac{q_H^* + q_L^*}{2}) - \lambda V^1(2q_L^*)$$

The definition of V^1 and the strict concavity of u yield the following inequalities: $V^1(2q_L^*) > (\theta_H + \theta_L^1)u(q_L^*)$ and $u(\frac{q_H^*+q_L^*}{2}) < u(q_H^*) - \frac{q_H^*-q_L^*}{2}u'(q_H^*) = u(q_H^*) - \frac{q_H^*-q_L^*}{2}\frac{c}{\theta_H}$. Hence it is sufficient to prove that

$$[2\theta_L - (2-\lambda)\theta_H]u(q_L^*) + [(2+\lambda)\theta_H - 2\theta_L]u(q_H^*) \geq 2\lambda\theta_H[u(q_H^*) - \frac{q_H^* - q_L^*}{2}\frac{c}{\theta_H}],$$

which is equivalent to

$$[(2-\lambda)\theta_H - 2\theta_L][u(q_H^*) - u(q_L^*)] + \lambda c(q_H^* - q_L^*) \geq 0$$

If $(2-\lambda)\theta_H - 2\theta_L \geq 0$, then we are done. If instead $(2-\lambda)\theta_H - 2\theta_L < 0$, then we use the mean value theorem to write $u(q_H^*) - u(q_L^*) = u'(\xi)(q_H^* - q_L^*)$ for some $\xi \in (q_L^*, q_H^*)$, hence $u'(\xi) < u'(q_L^*) = \frac{c}{\theta_L^1}$. Finally, it is easy to verify that the following inequality holds

$$[(2-\lambda)\theta_H - 2\theta_L]\frac{c}{\theta_L^1}(q_H^* - q_L^*) + \lambda c(q_H^* - q_L^*) \geq 0$$

In order to prove the first inequality in (24), it is sufficient to observe that it is obtained by adding $t_{LH}^* - t_{LL}^*$ and $t_{HH}^* - t_{HL}^*$to the left and the right hand side of the second inequality in (24), respectively. Since the latter holds, (21) implies that the first inequality is satisfied as well.

(b) The payoff matrices written at the beginning of the proof reveal that lemma 2(b) is equivalent to

$$\theta_H u(q_L^*) - t_{LL}^* > \theta_H u(q_H^*) - t_{HL}^* \text{ and } \theta_H u(q_L^*) - t_{LH}^* < \theta_H u(q_H^*) - t_{HH}^* \qquad (25)$$

These inequalities are readily proved as follows. H-type has no dominant strategy since (BIC_H) binds: either he prefers H against H and L against L, or vice-versa.. However, we can rule out that H-type wishes to report L against H, because otherwise in view of (21) he would also prefer to play L against L and he would have a dominant strategy. From here (25) follows.

Proof of Proposition 15

In view of (24) and (25), M^* is a strategically non-trivial game only for the two H-types, 1_H and 2_H. The game between them is symmetric and we represent it through the payoff matrix below, which describes the payoff of 1_H (the row player) as a function of his report and the report of 2_H (the column player) while taking into account that 2_L is playing L for sure

$$\begin{array}{ccc} 1_H \backslash 2_H & L & H \\ L & \theta_H u(q_L^*) - t_{LL}^* & (\Delta\theta) u(q_L^*) \\ H & \theta_H u(q_H^*) - t_{HL}^* & (\Delta\theta) u(q_L^*) \end{array}$$

Here L weakly dominates H [see the first inequality in (25)], hence there exists a non-truthful equilibrium; in it, each buyer type reports L. In order to prove that buyers are Pareto worse off in the latter equilibrium than under truthtelling, it is sufficient to recall inequalities (13): they establish that each buyer is better off when his opponent reports H rather than L. Hence, the truthful equilibrium yields Pareto higher payoffs than the equilibrium in which everybody reports L.

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