Voting Rules and Threshold Phenomena

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Abstract

In this paper we extend two classic results concerning the majority rule to large classes of voting games. Condorcet studied an election between two candidates in which the voters' choices are random and independent and the probability of a voter choosing the first candidate is $p > 1/2$. Condorcet's "Jury Theorem" (Condorcet (1785), see Young (1988)) asserts that if the number of voters is sufficiently large, then the first candidate will be elected. We prove the assertion of Condorcet's Jury Theorem for arbitrary voting games in which the *Shapley-Shubik power index* of each voter is sufficiently small.

McGarvey (1953) proved that for every asymmetric relation R on a finite set of candidates there is a strict-preferences voter profile that has the relation R as its strict simple majority relation. We prove that McGarvey's theorem can be extended to arbitrary neutral monotone social welfare functions which can be described by a strong simple game G if the Shapley-Shubik power index of each individual is sufficiently small.

1 Introduction

In this paper we extend two basic results concerning the majority rule to general voting schemes. The first is Condorcet's Jury Theorem which asserts that for the majority rule, aggregation of information is asymptotically complete. The second is McGarvey's theorem, a far-reaching extension of Condorcet's paradox, which asserts that for every asymmetric relation R on a finite set of candidates there is a strict-preferences (linear orders, no ties) voter profile that has the relation R as its strict simple majority relation.

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The motivation for studying general voting schemes which aggregate individual preferences into the social decision is that simple majority is a very limited model for describing general forms of aggregation of preferences. More complex election rules involving two candidates or two alternatives are common in actual elections. Examples include the U.S. electoral system and the (idealized) Soviet multi-tier council system. In economic situations, even more than in actual elections, it can be difficult to determine the precise form of aggregation of individual preferences and it is therefore important to understand general phenomena which do not depend on a specific form of aggregation of preferences.

In Section 2, we consider an extension of Condorcet's Jury Theorem (hereon, CJT). It asserts that in an election between two candidates, say Alice and Bob, if every voter votes for Alice with probability $p > 1/2$ and for Bob with probability $1 - p$ and if these probabilities are independent, then as the number of voters tends to infinity the probability that Alice will be elected tends to one (see Young (1988)).

The reason usually given for the interest in CJT in economics and political science is that it can be interpreted as saying that even if agents receive very poor (independent) signals indicating which decision is correct, majority voting will nevertheless result in the correct decision being taken with a high probability if there are enough agents and each agent votes according to the signal he receives. Specifically, CJT deals with the following scenario: Every agent receives a single bit of information which is either 'Vote for Alice' or 'Vote for Bob' and these signals are independent. When Alice is the correct choice the probability of receiving the signal 'Vote for Alice' is $p > 1/2$. The voters vote precisely as the signal dictates and the decision is made according to the simple majority rule.

When we consider general political or economic situations the aggregation of agent's choices can be much more complicated than simple majority, the individual signal (or signals) may be more complicated than a single bit of information and the distribution of signals among agents may be much more general and, in particular, may violate independence. Furthermore, voters may vote strategically by taking into account the entire situation and not just their signal. We will deal with general forms of aggregation of preferences while leaving unchanged the assumptions on individual signals and the agent's method of voting. I expect that our findings as well as the technical tools we apply are also relevant in the case of more general signal distributions and strategic voting. Our results and methodology may also be relevant to other questions concerning the aggregation of information (even in cases where different agents have different goals) which is an important topic in theoretical economics (see e.g. Pesendorfer and Swinkels (1997)).

Suppose that the outcome of an election can be described in terms of a *strong simple game* G defined on the set of voters. A simple game defined on a set N of players (voters) is described by a function ν that assigns to every subset (coalition) S of players the value '1' or '0'. We assume that $\nu(\emptyset) = 0$ and $\nu(N) = 1$. A candidate is elected if the set S of voters which voted for him is a winning coalition in G, i.e., if $\nu(S) = 1$. A simple game G is *strong* if $\nu(S) + \nu(N \backslash S) = 1$ for every coalition S, i.e., if the complement of a winning coalition is a losing one. We will further assume that the game G is monotone i.e. the addition of an individual to a winning coalition does not change it into a losing one. The assertion of CJT does not extend to all strong monotonic simple games with a large number of players. Clearly, it fails for a dictatorship or when the outcome of the game is decided (in all cases or even only with high probability) by a small set of voters. We need to replace the assertion that the number of voters is large by the assertion that the "power" of each voter is small.

We rely on the Shapley-Shubik power index which assigns a real number between 0 and 1 to every player in a simple game. This index measures the power of the player in the game. A quick way to define the Shapley-Shubik power index is as follows: Suppose that there are n voters. We say that a voter i is *pivotal* with respect to a set S of voters if $\nu(S \cup \{i\}) = 1$ and $\nu(S \backslash \{i\}) = 0$. In other words, player i makes a difference. For a probability distribution $\mathbf{P}$ on all subsets of voters, the probability that a voter is pivotal is called the *influence* of the voter (with respect to $\mathbf{P}$). The Shapley-Shubik power index of a voter measures his influence under the following distribution: First, choose p uniformly between 0 and 1 and then let a player i belong to S with probability p (independently of other players.)

Theorem 1.1. *For every $p > 1/2$ and $\epsilon > 0$, there is $\delta = \delta(p, \epsilon) > 0$ such that the following assertion holds:*

For every election rule described by a monotone strong simple game G for an election between two candidates, Alice and Bob, if the Shapley-Shubik power index for each voter in G is at most δ and if each voter votes for Alice with probability p and for Bob with probability $1 - p$ and the votes are independent, then Alice will be elected with probability of at least $1 - \epsilon$.

The proof of Theorem 1.1 is given in Section 2 which also includes various examples of aggregation of preferences modeled on simple games. These models differ in nature from the simple majority model but do arise in real economic and political decision making. Of particular interest is aggregation based on multi-level hierarchical structures such as those in various types of organizations and in the Soviet multi-tier council system.

The proof of Theorem 1.1 uses recent results in probability theory and combinatorics concerning threshold phenomena. Threshold phenomena refer to situations in which the probability of an event changes rapidly as some underlying parameter varies within some interval. Denote the probability that Alice will be elected by $\mathbf{P}_p(G)$. If G is a monotone simple game then $\mathbf{P}_p(G)$ is a monotone function of p. We wish to analyze the interval $[p_1, p_2]$ in which $\mathbf{P}_{p_1}(G) = \epsilon$ and $\mathbf{P}_{p_2}(G) = 1 - \epsilon$. This interval is called the *threshold interval* for the game G. In the last two decades, conditions have been found which guarantee that the derivative of $\mathbf{P}_p(G)$ at a specific value of p is large. For the proof of our theorem we need to supplement these results with some observations concerning the behavior of the function at two different points in the threshold interval. (Some technical parts of the proof are presented in the appendix.) It is worth noting that for an election based on a strong simple game G, if every voter votes (independently) for Alice with probability $p > 1/2$, the probability that Alice will be elected (i.e., $\mathbf{P}_p(G)$) is maximal when G is a simple majority game. (Further details can be found in Section 2.)

The concepts of pivotal agents and influences are crucial to our analysis. The mathematical study of pivotal agents and influences is fundamental in the context of power indices in game theory, as well as in mechanism design and auction theory (Pesendorfer and Swinkels (1997)), other areas of theoretical economics, (Al-Najjar and Smorodinsky (2000)), reliability theory, statistical physics, probability theory and statistics, distributed computing (Ben-Or and Linial (1985,1990)) and complexity theory.

In Section 3, we consider extensions of a theorem by McGarvey (1953). Condorcet's "paradox" demonstrates that given three candidates A, B and C, the majority rule may result in the society preferring A to B , B to C and C to A. Arrow's theorem shows that under certain natural conditions, such "paradoxes" cannot be avoided under *any* non-dictatorial voting method.

McGarvey (1953) proved another far-reaching extension of Condorcet's paradox: For every asymmetric relation R on a finite set of candidates, there is a strict-preferences (linear orders, no ties) voter profile that has the relation R as its strict simple majority relation. McGarvey's theorem is an early and simple manifestation of the phenomenon that choice aggregated over many individuals may lead to arbitrary outcomes (or, in other words, will not have any testable implications). Another example of this phenomenon is the well-known result by Sonnenschein (1972) concerning demand functions.

A social welfare function is a map which associates an asymmetric relation on the alternatives to every profile of individual preferences. We require the condition of Independence of Irrelevant Alternative (IIA) which states that the social preference between two alternatives a and b is determined by the individual preferences between a and b. We also require the Pareto condition (P) that if every individual prefers a to b then so will the society. A social welfare function is called neutral if it is invariant under permutations of the alternatives. A neutral social welfare function that satisfies conditions (IIA) and (P) can be described by a strong simple game G defined on the set of individuals. Thus, given the order relations $R_1, R_2, \ldots, R_n$, which represent the individual preferences, the social preference relation R is defined by the following rule: For two alternatives a and b, aRb if the set of individuals for which aR_ib is a winning coalition in G. (Without neutrality, a social welfare function requires a simple game for every pair of alternatives; these games can be distinct and not strong.)

Theorem 1.2. *For every number m of alternatives, there is a real number $\delta = \delta(m) > 0$ such that if a neutral social welfare function described by a strong monotone simple game G in which the Shapley-Shubik power index of each individual is at most δ, then the social welfare function will lead to all asymmetric preference relations.*

While the statement of Theorem 1.2 does not involve any probability, its proof is based on an analysis of the social choice under some probabilistic assumptions on the individual preferences and on the extensions of Condorcet's Jury Theorem.

The extension of McGarvey's theorem is the starting point for this paper. My study of learnability and the testable implications of individual and collective choice (see Kalai (2002)) led me to the conjecture that under very general conditions, social choice leads to chaotic (highly non-learnable and non-testable) social preferences. Arrow's theorem asserts that for every non-dictatorial social welfare function and under very general conditions, if we observe that the society prefers alternative A over B and alternative B over C we cannot deduce the society's preferences between A and C. In other words, if we seek a mechanism which guarantees that A is preferred over C we must give one player all the power. Our theorem shows that if we require from a monotone and neutral social welfare function that it allow us to deduce *anything* from observing a sample of the society's preferences on some other society's preferences, then we must give one player a *substantial* amount of power (i.e. an amount of power which is bounded away from zero regardless of the size of the society). In section 3, we will present an application of Theorem 1.2 for social welfare functions with restricted individual domains.

Both our theorems require that the Shapley-Shubik power index for each voter be small. There are some cases in which this condition is automatically satisfied. In one such case the individuals are classified by "type", such that two individuals of the same type are indistinguishable, and there are many individuals of each type. Another such case arises when there is sufficient symmetry between the individuals to guarantee that every two individuals have the same power. Formally, "sufficient symmetry" means that there is a *transitive* group of permutations Γ on the alternatives such that the social welfare function is invariant under permutations in Γ.[1] In various political or economic situations it can be evident that individual agents have very small power even if it may be difficult to compute precisely the power indices of agents and even to describe the precise form of aggregation.

The quantitative estimates we obtain for our two general theorems are weak and probably not tight. For concrete examples, sharper threshold behavior can often be proved and a consequently sharper form of Theorem 1.2 can be deduced. We point out that Theorems 1.1 and 1.2 hold when we use the (non-normalized) Banzhaf power index rather than the Shapley-Shubik power index. (The Banzhaf power index of a voter is his influence with respect to the uniform probability distribution on all subsets of players.)

An interesting consequence of Theorem 1.1 and of the proof of Theorem 1.2 is that in some situations and for large societies, election outcomes hardly depend on the specific mechanism for aggregation and are primarily determined by the probability distribution of individual votes.

In Section 4, we examine to what extent our extension of CJT can be extended further to the cases of general distribution of signals and strategic voting. We point out that the relation between a sharp threshold and the asymptotically complete aggregation of information extends (though less obviously) to the case of strategic voting when we allow the voting rule to be biased towards one of the alternatives (as do Feddersen and Pesendorfer (1996, 1997 and 1998)). Understanding the situation for general distributions (namely, when the assumption of probability independence between individual signals is dropped) involves profound conceptual and technical problems concerning the aggregation of information. When the distribution of signals is not independent the notion of an agent's "influence" (namely, the probability of his being pivotal) can be extended in two different ways.

[1] A group Γ on the set of voters is transitive if for every two voters a and b, there is an element $g \in \Gamma$ such that $g(a) = b$.

We define the *effect* of a player (or, more precisely, the effect of knowing the player's vote) as the probability that Alice was elected conditioned on the player voting for Alice minus the probability that Alice was elected conditioned on the player voting for Bob. (The effect is a normalized form of the correlation between the election's outcome and the player's vote.) When the individual signals are independent, the effect is equal to the influence. We propose a general conjecture asserting that Theorem 1.2 further extends to large classes of distributions (FKG-distributions) under the condition that individual effects are small.

2 Threshold phenomena, influences and Condorcet's Jury Theorem

2.1 The mathematics behind the proof of Theorem 1.1

Consider a simple game $G =< N, \nu >$ with a set $N = \{1, 2, \ldots, n\}$ of players, where ν is a function from subsets of N to $\{0, 1\}$. A subset S of N is called a winning coalition if $\nu(S) = 1$; otherwise it is a losing coalition. We will assume that $\nu(\emptyset) = 0$ and $\nu(N) = 1$. We will also assume that G is monotone which means that if $R \subset T$ and $\nu(R) = 1$ then $\nu(T) = 1$. The game G is *strong* if $\nu(N \backslash S) = 1 - \nu(S)$ for every S.

The simple game G describes the way in which the individual preferences between our two candidates, Alice and Bob, aggregate. Now we will discuss how the voters are going to vote.

We suppose that the ith voter receives a signal s_i where $s_i = 1$ with probability $p > 1/2$, $s_i = 0$ with probability $1 - p$ and the signals are independent. $s_i = 1$ means "vote for Alice" and we assume that voters act according to their signals. Therefore, the set S of voters who vote for Alice is given by $S = \{i : s_i = 1\}$. S is a random set of players such that for each player i, $i \in S$ with probability p, $i \notin S$ with probability $1-p$ and the events "$i \in S$" are independent for $i = 1, 2, \ldots, n$. For a specific set $S \subset N$, the probability that the set of Alice's voters is precisely S is denoted by $\mathbf{P}_p(S)$ and is equal to $p^{|S|}(1-p)^{n-|S|}$. Denote by $\mathbf{P}_p(G)$ the probability that the random set S of Alice's voters is a winning coalition, i.e. the probability that Alice wins the election:

$$\mathbf{P}_p(G) = \sum\{\mathbf{P}_p(S) : \nu(S) = 1\}.$$

The proofs of the following two simple lemmas are presented in the appendix.

Lemma 2.1. *If G is a strong simple game, then:*

$$\mathbf{P}_p(G) = 1 - \mathbf{P}_{1-p}(G). \tag{2.1}$$

Lemma 2.2. *If G is a monotone simple game, then the function $\mathbf{P}_p(G)$ is a strictly monotone continuous function of p in the interval $[0,1]$.*

Let ϵ, $0 < \epsilon < 1/2$, be a real number. Since $\mathbf{P}_p(G)$ is a strictly monotone and continuous function of p, there is a unique value of p denoted by p_1 such that $\mathbf{P}_{p_1}(G) = \epsilon$. There is also a unique value of p denoted by p_2 such that $\mathbf{P}_{p_2}(G) = 1 - \epsilon$. The interval $[p_1, p_2]$ is called a *threshold interval* and its length $p_2 - p_1$ is denoted by $T_\epsilon(G)$. The value p_c, at which $\mathbf{P}_{p_c}(G) = 1/2$, is called the *critical probability* for G. It follows from relation (2.1) that if G is a strong simple game, then $p_c = 1/2$.

Let $\phi_k(G)$ denote the Shapley-Shubik power index for the k-th individual in G. Define $\bar{\phi}(G) = \max(\phi_1(G), \phi_2(G), \ldots, \phi_n(G))$. The main results of this section assert that if the power of every individual is small, then the threshold interval must also be. The following result is equivalent to Theorem 1.1:

Theorem 2.3. *For every $\epsilon, \delta > 0$ there exists $\gamma > 0$ such that for every monotone strong simple game G if $\bar{\phi}(G) \le \gamma$, then $T_\epsilon(G) \le \delta$.*

The proof extends to the following more general result.

Theorem 2.4. *For every $a, \epsilon, \delta > 0$ there exists $\gamma > 0$ such that for every monotone simple game G if $\bar{\phi}(G) \le \gamma$ and $a \le p_c(G) \le 1-a$ then $T_\epsilon(G) \le \delta$.*

We prove a similar result for the Banzhaf power index. Let $\bar{\beta}(G)$ be the maximum Banzhaf power index over all players in G.

Theorem 2.5. *For every $\epsilon, \delta > 0$ there exists $\gamma > 0$ such that for every monotone strong simple game G if $\bar{\beta}(G) \le \gamma$, then $T_\epsilon(G) \le \delta$.*

I will now present the mathematical concepts and results required for proving Theorem 2.3 and leave detailed proofs to the appendix. The *influence* of the k-th player on G, denoted by $I_k^p(G)$, is the probability that the player is *pivotal*, i.e., the probability that for a random coalition S (according to the probability distribution $\mathbf{P}_p$) which does not contain k, S is a losing coalition and $S \cup \{k\}$ is a winning one. The influence of a player is a normalized version of the correlation between his vote and the election's outcome. The total influence $I^p(G)$ equals $\sum I_k^p(G)$. Define

$$\phi_k(G) = \int_0^1 I_k^p(G)dp.$$

$\phi_k(G)$ is the Shapley-Shubik power index of player k in G. This integral representation of the Shapley value is due to Owen (1989). (Owen's representation of the Shapley-Shubik power index coincides with the description given in the introduction but is different from Shapley's original axiomatic definition.) The Banzhaf index of the k-th player equals $I_k^{1/2}(G)$. (We will omit the superscript p for $p = 1/2$.)

We require the following fundamental result:

Proposition 2.6. Russo's lemma (see Grimmett (1989))

$$\frac{d\mathbf{P}_p(G)}{dp} = I^p(G). \tag{2.2}$$

Russo's lemma implies that if the total influence in the threshold interval is large, then the threshold interval will be small. The lemma was discovered independently by several authors (before and after Russo's paper). The rather simple proof is presented in the appendix.

Next we require a result that shows that for a specific value of p, if all individual influences $I_k^p(G)$ are small, then their sum $I^p(G)$ is large.

Theorem 2.7 (Russo-Talagrand). *For some constant $C > 0$, if $I_k^p(G) \leq \delta$ for every $k = 1, 2, \ldots, n$, then:*

$$I^p(G) \geq C \log(1/\delta) \cdot \mathbf{P}_p(G)(1 - \mathbf{P}_p(G)). \tag{2.3}$$

We will not present the proof in this paper (though we will explain the basic setting in the appendix). Russo's original result (Russo (1982)) gave a weaker lower bound on $I^p(G)$ and his proof was elementary, quite short and a bit mysterious. Talagrand's proof relies on harmonic analysis techniques introduced in Kahn, Kalai and Linial (1988) (who essentially proved the case of $p = 1/2$). Kahn, Kalai and Linial (1988) is a good source for the basic mathematical tools. For a direct application of these harmonic analysis tools to the probabilistic study of social choice, see Kalai (2002).

Note that this result is already sufficient to deal with the cases considered in the introduction. Define a simple game to be *weakly anonymous* if the game is invariant under a transitive group of permutations on the players. (In this case, every two voters are equal; but not every two pairs of voters are necessarily equal.) An example would be an electoral voting system like that in the US where all states have the same number of voters and electors. Simple majority is the only anonymous strong simple game but the family of weakly anonymous strong simple games is very rich. For a weakly-anonymous game, $I_k^p(G)$ must be equal for every k. Therefore, from the Russo-Talagrand theorem, we have:

$$I^p(G) \geq C\log(n)\mathbf{P}_p(G)(1-\mathbf{P}_p(G)). \tag{2.4}$$

From this inequality it follows immediately that when ϵ is fixed, the slope of $\mathbf{P}_p(G)$ as a function of p is at least some constant times $\log n$ for every point p in the threshold interval and therefore the threshold interval is of length $O(1/\log n)$. With a slightly more delicate computation it can be deduced that

$$T_\epsilon(G) \leq C\log n\log(1/\epsilon) \tag{2.5}$$

(see Friedgut and Kalai (1996)). The bounds cannot be improved in this case (apart from the value of the constant C). Finding general conditions which guarantee that the length of the threshold interval will be as small as $n^{-\beta}$ for $\beta > 0$ appears to be important but beyond reach at this time. A similar argument applies to a different case mentioned in the Introduction in which voters are "replicated" sufficiently many times.

In the proof of Theorems 2.3 and 2.5 presented in the appendix, we require some additional observations that will enable us to relate the influences for one value of p to the influences for the entire threshold interval.

2.2 Multi-tier council democracy

Consider the following example from Ben-Or and Linial (1986,1990) whose origins going back to von-Neumann (1956). The society is divided into three parts, each of which is again divided into three parts, and so on and so forth until every part is a single voter. (For simplicity assume that the number of individuals is a power of three.) The election begins with voting according to majority rule at the level of single voters followed by a majority vote at the next aggregated level and continuing recursively to the top level.

This is a weakly-anonymous strong simple game but it is very different from simple majority.

---------Insert Figure 1 here--------------------

The threshold behavior in this example can be computed directly. (The proof is presented in the appendix.)

Proposition 2.8. $T_\epsilon(G) \leq C\log(1/\epsilon)/n^{0.379\cdots}$.

(The exponent is $1 - \log 2/\log 3$.)

Remark: This hierarchical voting method has an obvious resemblance to the multi-tier system of councils ("soviets" in Russian). Lenin (and others) advocated this system during the 1917 Russian revolution. Friedgut (1979) is a good source for the early writings of Marx, Lenin and others and for an analysis of the Soviet election systems in the 70's. Lenin's concept of centralized democracy is based on a hierarchical method of voting and was implemented in the Soviet Union and its satellites, for party institutions, national bodies and labor unions. (For national bodies, the method was changed in 1936.) For the party institutions there could be as many as seven layers. For example, party members of the local organizations, say, the department of mathematics in Budapest, elected representatives to the science faculty party meeting who in turn elected representatives to the university meeting. They elected representatives to the meeting of the 5th district of Budapest who in turn elected representatives to the Budapest meeting. This continued with the election of the party congress, the Central Committee and finally the Politburo.

This recursive model can be regarded as an idealized version of centralized democracy. In the next example we consider, the power of voters increases as we move up the hierarchy. This would appear to be closer to how the system was actually implemented.

2.3 Aggregation of information in an hierarchical organization

We will describe now a hierarchical method of aggregation of preferences where people higher up in the hierarchy have more power. This example also serves another purpose. Our extension of CJT allows for generalized methods of aggregation of preferences but leaves unchanged the assumption that the individual signals are independent. The example describes here shows that general methods for aggregating preferences and general distributions for signals can be at times interchangeable.

Consider a company with an hierarchical structure. Each first level manager has K employees under him, and each second level manager is responsible for K first level managers, and so on... decisions are made by the head of the company who is at the top of the hierarchy.

The company is facing an important decision which involves choosing one of two alternatives A and B. In order to make the example as concrete as possible assume that the firm is a software company and that the decision is whether Alice or Bob should write the core of the new generation operating system. For such a crucial decision, everyone in the firm forms an opinion.

Suppose that the ith employee receives a signal s_i indicating which is the better alternative. The signals are not independent and are based on the following aggregation of individual assessments. To start with, every employee of the company receives an independent signal s'_i. $s'_i = 1$ with probability p and $s'_i = 0$ with probability $1 - p$. The vector of independent signals $(s'_1, s'_2, \ldots, s'_n)$ determines the actual signals $(s_1, s_2, \ldots, s_n)$ as follows: Starting from the bottom up every manager forms his opinion according to his signal except for the case in which all the workers directly under him favor the opposite alternative. In such a case he changes his mind. Formally, for the simple workers i, $s_i = s'_i$ while for managers i, $s_i = s'_i$ if for some of the employees j directly below him, $s'_i = s_j$ and $s_i = 1 - s'_i$ otherwise.

We can regard the situation as a dictatorship with a rather complicated structure of distribution of signals (given by the s_i's). We can also consider it to be a rather complicated strong simple game (rather similar in nature to the multi-tier council democracy) with independent individual signals (the original s'_i's).

`-------------Insert Figure 2 here --------------`

In this game, if $K \geq 2$ is fixed then people higher up the hierarchy will have more power. For example, for $K = 2$, the Banzhaf power index for the boss is $1/2$ while the Banzhaf power index for a manager r steps below the boss is $1/2^{r+1}$.

Proposition 2.9. *For $K = 2$, asymptotic complete aggregation of information will take place as the number of agents grows. This is not the case for $K = 3$.*

Remark: Larger values of K and weaker conditions for the managers to change their minds are perhaps more realistic. In such cases, the question of when there will be asymptotically complete aggregation of information may be subtle. For example, when $K = 10$ and when the opinions of seven employees are sufficient to change their manager's mind, then asymptotic complete aggregation of information will take place (as the total number of employees grows), however, if it takes eight opinions to change the manager's opinion then it will not take place. For mathematical models of aggregation of information (from the leaves to the root) on tree-like structures (and the mathematically similar problem of noisy information from the root to the leaves), see Peres (1999).

2.4 Two further examples

The following two examples are useful in seeing more precisely how far our extension of CJT can be extended when there are influential players.

In the first example there are n voters, one of whom is more distinguished than the others (think of her as the Supreme Court Chief Justice). If the gap between the number of votes for Alice and Bob is at least $E = E(n)$, then the election is decided by majority. If the gap is smaller than E, then the outcome is determined by the preference of the distinguished voter. The analysis depends on the value of E. Note that the Shapley-Shubik power index of the distinguished voter is $o(1)$ as long as $E = o(n)$. It follows from Theorem 1.1 (and can easily be derived directly) that when $E = o(n)$, the assertion of CJT holds. Note that if $\lim_{n\to\infty} E(n)/\sqrt{n} = \infty$, for example if $E(n) = n^{2/3}$, then the Banzhaf power index for the distinguished voter tends to one as n tends to infinity.

When $E(n)$ is proportional to n, then the CJT does not hold. When $E(n) = [\alpha \cdot n]$ and $0 < \alpha < 1$, then the Shapley-Shubik power index for the distinguished voter tends to β where $\beta = \beta(\alpha)$ and $0 < \beta < 1$. In this case, the example demonstrates that the assertion of CJT does not follow from the assumption that no small set of voters has decisive power as measured by their combined Shapley-Shubik power index.

In our second example, there are n voters, three of whom are more distinguished than the others. If these three are in agreement, they determine the outcome of the election. Otherwise, the election is decided by simple majority. In this case CJT does not hold and indeed the Shapley-Shubik power index for each of the three distinguished voters is bounded away from zero. Furthermore, the combined Shapley-Shubik power for these players is bounded away from one and therefore no small set of voters has decisive power as measured by their combined Shapley-Shubik power index.

Note also that in this example $I(G)$ tends to infinity as does n. This demonstrates that $I(G)$ can be arbitrarily large and CJT may still not hold. In the appendix, we will give conditions for CJT to hold even when some of the voters do have a large amount of power. We will also state a theorem which shows that the monotonicity assumption on the simple game can be relaxed.

2.5 Related phenomena: The superiority of simple majority and sensitivity to noise

We mentioned in the Introduction that the study of pivotal agents and influences (under different names) is central in a number of areas in mathematics, physics, statistics and economics. I will mention here four additional basic phenomena concerning pivotal agents while leaving some technical explanations to the appendix. A phenomenon that goes back to Banzhaf's original work (and perhaps earlier) and which is relevant to our discussion is the following:

- Simple majority maximizes total influence.

Remark: Here, when the number n of players is even we will regard as simple majority game any strong simple game such that every coalition with more than $n/2$ players is a winning coalition.

By Russo's lemma it follows that:

Proposition 2.10. *Let G be a monotone strong simple game G with n players and let M be a simple majority game with n players. Then, for every $p > 1/2$, $\mathbf{P}_p(M) \geq \mathbf{P}_p(G)$. Equality holds if and only if G is a simple majority game (in the slightly more general sense mentioned above).*

Thus, the threshold interval for every strong simple game of n players is at least as large as the threshold interval in a simple majority game. In our context we conclude that simple majority is superior in terms of aggregation of information. (In contrast, Maug and Yilmaz (2002) describe a situation in which splitting the voters into two groups yields better aggregation of information (for strategic behavior) than simple majority. In their situation the population is divided into two groups with different interests.)

Al-Najjar and Smorodinsky (2000) considered a general framework of influence relative to a mechanism and proved a tight upper bound on the average influence. They showed that every mechanism (in their sense) can be replaced by another based on simple majority such that influence increases. Their work was influenced by Mailath and Postlewaite (1990) who considered applications to the problem of public goods. Fudenberg, Levine and Pesendorfer (1998) proved a similar result (although it is harder to see the connection from their formulation) and provided game-theoretic applications. Chayes, Chayes, Fisher and Spencer (1986) showed that the upper bound of $n^{1/2}$ on the average number of pivotal agents extends to various probability distributions which exist in physics.

Another basic property of influences is the following:

- For a fixed value of p, there cannot be more than a few agents with a large amount of influence.

Al-Najjar and Smorodinsky (2000) consider this phenomenon (in a more general context) and derive some economic consequences.

In our context, this property follows from the inequality:

$$\sum (I_k^p(G))^2 \leq 4/p(1-p) \tag{2.6}$$

which is a simple consequence of the basic (harmonic analysis) setting in Kahn, Kalai and Linial (1988) (for $p = 1/2$) and Talagrand (1994) (for general p).

A third basic phenomenon is the following:

- The sum of influences cannot be overly small.

For a strong simple game G,

$$I(G) \geq 1. \tag{2.7}$$

($I(G) = 1$ if and only if G is a dictatorship). This inequality has its origins in the works of Whitney and Loomis, Harper, Bernstein, Hart and others and has great importance in many mathematical contexts. A good reference for the basic result is Hart (1977) who gives for arbitrary simple games, the precise lower bound for $I(G)$ in terms of the number of winning coalitions.

The fourth and last phenomenon is the following:

- Simple and weighted majorities are stable in the presence of noise.

Motivated by mathematical physics, Benjamini, Kalai and Schramm (1999) studied the sensitivity of an election's outcome to low levels of noise in the signals (or, if you wish, to small errors in the counting of votes). Their assumption is that there is a probability $\epsilon > 0$ for a mistake in counting a vote and these probabilities are independent. Simple majority tends to be quite stable in the presence of noise. Two-level majority like the US electoral system is less stable and multi-tier council democracy is quite sensitive to noise.

3 The chaotic nature of social preferences

3.1 Social welfare functions

We consider a social welfare function which, given a profile of n order relations R_i, $i = 1, 2, \ldots, n$ on m alternatives, yields an asymmetric relation R

for the society. Thus, $R = F(R_1, R_2, \ldots R_n)$ where F is the social welfare function. aR_ib states that the i-th individual prefers alternative a over alternative b. aRb indicates that the society prefers alternative a over alternative b. The social preferences are not assumed to be transitive.

The condition of independence from irrelevant alternatives (IIA), states that for every two alternatives a and b the individual preferences between a and b determine the social preference between a and b. Formally, the set $\{i : aR_ib\}$ determines whether aRb. The social preference between a and b can thus be described by a strong simple game $G_{a,b}$ as follows: Let S be the set of individuals which prefer alternative a over alternative b (i.e., $S = \{i : aR_ib\}$). S is a winning coalition for the game $G_{a,b}$ if aRb.

The Pareto condition is another standard assumption, which asserts that if all individuals prefer alternative a over b then so will the society. This means that for every two alternatives a and b in the game $G_{a,b}$, the set of all voters is a winning condition and the empty set of voters is a losing one.

We will also assume that the social welfare function is monotone which means that if an individual who prefers an alternative a over alternative b changes his preferences, this will not result in the opposite change in the society's preferences.

Finally, we assume that the social welfare function is *neutral*, namely that it is invariant to permutations of the *alternatives*. Assuming neutrality is equivalent to the assertion that all simple games $G_{a,b}$ are strong and identical. Therefore, a neutral social welfare function can be described in terms of a simple game. (Our main result and its proof extend to the case in which the games $G_{a,b}$ are strong simple games although they may be distinct.)

A convenient way to think about the function is as a rule for elections between two candidates. There is a pool of several candidates and every individual has an order relation on all candidates. We wish to understand the possible outcomes of two-candidate elections between pairs of candidates within the pool.

3.2 An outline of the proof of Theorem 1.2

Let me first explain informally why our extension of CJT combined with McGarvey's theorem itself leads to an extension of McGarvey's theorem to arbitrary neutral social welfare functions in which each voter has a small amount of power. Suppose we wish to realize an asymmetric relation R on m alternatives. We start with a profile of preferences on M individuals such that for every two alternatives a and b if aRb then the majority of individuals prefer a to b. According to McGarvey's theorem such a profile exists. Let R_i be the preference relation of the ith individual, $i = 1, 2, \dots, M$. Next, consider a much larger society with n individuals and for every individual choose a preference relation which agrees with R_i with probability $1/M$. Now suppose that aRb. The proportion of individuals which prefer a over b in the small society is at least $1/2 + 1/M$. Therefore, the probability that an individual in the large society will prefer a to b is at least $1/2 + 1/M$. It follows from the extended version of CJT that for every $\epsilon > 0$ when n is sufficiently large the society will prefer a over b with a probability of at least $1 - \epsilon$. The probability that for *every* two alternatives a and b we will have that aRb is at least $1 - \epsilon \cdot \binom{m}{2}$, and this quantity is positive if ϵ is small enough. The detailed proof is given in the appendix.

3.3 Some quantitative estimates

Consider a profile $\mathcal{P}$ consisting of order relations of k individuals $R_1, R_2, \dots R_k$ on a set X of m alternatives. For two alternatives $a, b \in X$ let $p(a, b)$ be the proportion of individuals that prefer alternative a over b. Suppose that $p(a, b) \neq 1/2$ for every a and b. Let $q(a, b) = p(a, b) - 1/2$. Let R be a relation on X defined by $aR_\nu b$ if $p_\nu(a, b) > 1/2$. Define

$$t = \min\{|q(a, b)| : a, b \in X\}.$$

We say that the profile $\mathcal{P}$ realizes the asymmetric relation R with *quality* t. The quantitative estimates derived from our proof of Theorem 1.2 depend on the quality required to represent an asymmetric relation.

McGarvey showed that every asymmetric relation R can be realized by $m(m-1)$ voters. McGarvey's proof relies on the following simple observation: Combining two voters with order relations $1 <_1 2 <_1 3 <_1 \cdots <_1 m$ and $m <_2 m-1 <_2 <_2 \cdots <_2 3 <_2 1 <_2 1$ we reach a situation in which these two voters combined prefer alternative '2' to alternative '1' but are indifferent between every other pair. For every pair of alternatives a and b, if aRb we can define order relations for two voters according to which both prefer a to b but have the opposite preferences on every other pair of alternatives. Combining such pairs of voters for every two alternatives a and b such that aRb implies that R can be realized by $m(m-1)$ voters. Stearns (1959) found a way to reduce the number of voters to m and noticed that a simple counting argument implies that at least $m/\log m$ voters are needed. Erdős and Moser (1964) were able to reduce the number of voters required to realize every asymmetric relation to $O(m/\log m)$.

The quality t of a distribution based on McGarvey's proof is $1/m^2$ and can be improved to $1/m$ and $\log m/m$ using the subsequent results by Stearns and Erdős-Moser. Alon (2002) recently showed that every asymmetric relation can be realized with quality proportional to $1/\sqrt{m}$ and that this is the best result possible.

If we base our analysis on Alon's result then for a neutral weakly anonymous social welfare function on m alternatives, it follows that the number of voters required to realize all asymmetric relations is $2^{O(\sqrt{m})}$. It is quite possible that a number of voters which is only polynomial in m suffices but I cannot prove it. I do not even have an example in which m voters are not sufficient. As for Theorem 1.2, I conjecture that m can be taken as a polynomial of $1/\delta$. In specific cases for which we have better upper bounds on the threshold interval, we can deduce that a smaller number of voters suffices. For example, From the threshold behavior of the multi-tier system which we proved in Proposition 2.8, we can deduce that for this case m^3 agents suffice.

3.4 Social choice with restricted individual preferences

There has been intensive study of social choice when individual preferences are restricted to a subset T of all order relations (see, for example Kalai and Muller(1977)). One famous example is single-picked order relations. The following Corollary to the proof of Theorem 1.2 is in the spirit of results by Maskin (1995) and Dasgupta and Maskin (1997). Maskin's theorem asserts that if the number n of voters is odd and if for some restriction T of all order relations on m alternatives some anonymous election rule always leads to transitive social preferences, then the majority rule will also always lead to transitive social preferences. (A social welfare function is anonymous if it is invariant under all permutations of the voters.) The following Corollary to the proof of Theorem 1.2 is in the same spirit.

Corollary 3.1. *There exists $\gamma = \gamma(m) > 0$ with the following property: Let T be a set of linear orders on m alternatives. If the majority rule yields a non-transitive social preferences for a profile restricted to T, then so does every monotone neutral social welfare function based on a simple game G for which the Shapley-Shubik power index of every individual is at most γ.*

This result is a weak form of a conjecture by Maskin (1995) which asserts that in his theorem we can replace the condition that the social welfare function be anonymous by the condition that it not be dictatorial and every voter has *some* influence (in other words, there are no "dummies").

3.5 More general forms for aggregation of individual preferences

A far-reaching extension of McGarvey's theorem by Saari (1989) asserts that the plurality method for large societies gives rise to *all* choice functions. For every choice function c defined on subsets A of a finite set X of candidates, Saari described voter profiles such that for every set A of candidates, the candidate who is ranked first among the elements of A for the largest number of voters is precisely $c(A)$.

We have considered social welfare functions which associate an asymmetric social preference to the individual preference relations. The plurality rule is an example of a more general notion of social welfare functions which, based on the individual preferences, define a choice function on every subset of alternatives.

Consider the following setting: We have an election rule which allows for an arbitrary number of candidates. Given the individual preferences on a pool X of candidates, we wish to ascertain for every subset A of candidates, who will win an election among the candidates in A. (We assume that people vote naively.)

Saari's conclusion regarding the plurality rule does not hold if we choose for A the candidate who ranks first according to the Borda rule, another well-known voting method. The Borda rule involves each individual ranking the candidates of A by the numbers $1, 2, \ldots, |A|$ (the candidate ranked '1' is the least favorable) and choosing the candidate for which the sum of the individual ranks is maximal. In fact, the class of choice functions that arises from the Borda rule is quite restricted. It is well-known (and perhaps was already known to Borda himself) that the choices for the Borda rule cannot be prescribed in an arbitrary manner. For example, if $x = c(A)$ then x cannot lose in a two-candidate election against any other element of A (see, Saari (1995)).

What accounts for the difference in results between the Borda rule and the plurality rule? According to the plurality rule, the society's choice for a set A depends only on the individual choices for A. According to the Borda rule, the society's choice for a set A also depends on the individual preferences for the elements of A.

We say that a social welfare function (in the generalized form considered here) satisfies the "Irrelevance of Rejected Alternatives" (IRA) condition if the choice of the society depends only on the choices of the individuals. IRA is a natural assumption to make in political and economic situations in which the society's choice between several alternatives depends only on the individual choices and cannot take into account the individual preferences among (*relevant*) alternatives that were not chosen. The results of this paper can be extended to prove that, under fairly general conditions and when the society is large, if (IRA) is assumed then the class of choice functions that arises includes all choice functions. Extensions in this direction can be found in Kalai (2001).

4 Influence without independence and strategic voting.

4.1 The nature and role of independence in CJT

To make the discussion more concrete, suppose again that an institute has to decide whether to hire Alice or Bob. The files of the two candidates provide the data on which each individual in the institute bases his opinion. The opinion of individual i is what we referred to as his signal s_i. It consists of a single bit of information: if $s_i = 1$ then the ith individual favors Alice; if $s_i = 0$ he favors Bob. The probability p that an individual i will form an opinion in favor of Alice depends on the strength of the evidence in her favor. Given p we have assumed that the signals s_i are identical and independent.

The assumption that the signals are identical does not appear to be a severe restriction. A simple observation worth mentioning is that monotonicity implies that if the probability of Alice being elected is at least $1-\epsilon$ when every voter votes (independently) for Alice with probability p, then the same conclusion holds when voter i votes (independently) for Alice with probability p_i and $p_i \geq p$ for every i. For CJT it is sufficient to require that the average value of $p_1, \ldots, p_n$ be at least p (see Grofman, Owen and Feld (1983)). It would be interesting to extend our result in this manner as well.

The key assumption in CJT, as it is in our generalization as well, is that the signals s_i are distributed independently. It is important to note that in our setting when we consider the individual votes over many decisions, the observed individual votes are not independent. Our assumption is that the only dependence between the votes is via the data. This form of "conditional independence" is less damaging than, for example, assuming that the voters vote identically and independently according to a fixed distribution (say, equal probability for each candidate). The assumption that the decision makers base their decision on some evidence and that the individual interpretations of the evidence are independent from one another is reasonable in various economic and political situations (see, for example, the discussion in Ladha (1992)). Nevertheless, to assume that voters are voting independently is quite unrealistic in many circumstances. In fact, the independence assumption seems especially unreasonable in "real" elections and is perhaps more justified for mundane day-to-day forms of aggregation of individual preferences. It can also be argued that when the voting rule is complex, the assumption of the independence of signals is less realistic and the lack of independence is more damaging than in cases of simple majority. (This last statement, due to a referee, is far from obvious and warrants further study.)

Without the assumption of independence, CJT (for simple majority) clearly does not hold. It is no longer the case that when each individual votes for Alice with probability $p > 1/2$, Alice will win with high probability.

One extreme example is the case in which all voters vote with probability p for Alice but they all vote the same way. Alice will be elected with probability p regardless of the number of voters when the election is based on simple majority and for every other simple game. CJT for simple majority does not hold in situations with high positive correlations between all voters or between voters in a small number of sectors.

The following is a further example: Let $p = 1/2 + \epsilon/2$. Consider the following distribution on individual signals: First choose a number t in the interval $[\epsilon, 1]$ uniformly and then independently choose each voter's signal to be '1' with probability t and '0' with probability $1 - t$. The probability for each individual signal to be '1' is p. The outcome of the elections, even with many players, will favor the election of Alice with probability p.

Alternatively, we can take t to be $1/2 + \epsilon$ with probability $1/2$ and $1/2 - \epsilon/2$ with probability $1/2$ and proceed as before.

One way to think about these last two examples is that we have an asymptotic complete aggregation of information although the information itself is not sufficient to determine the election's outcome.

4.2 Influences and effects for general distributions

I expect that the mathematical methodology which enables us to move from simple majority to general voting games is similar to that needed for dealing with signals which are not independent and that the phenomenon we describe based on the assumption of independence can be extended to more general contexts.

Note that the notion of influence extends to arbitrary distributions in two different ways:

Let $\mathbf{P}$ be an arbitrary distribution on signals for the voters, namely on 0-1 vectors of length n. Suppose that the voters vote according to their signals.

Define the *influence* of the k-th player as the probability that the kth player is pivotal. Denote by $I_k^{\mathbf{P}}(G)$ the influence of the kth player for the simple game G w.r.t. the distribution $\mathbf{P}$.

Define the *effect* of the kth player on a strong simple game G as the difference between the probability that S is a winning coalition conditioned on $k \in S$ minus the probability that S is a winning coalition conditioned on $k \notin S$. Denote by $e_k^{\mathbf{P}}(G)$ the effect of the kth player for the simple game G w.r.t. the distribution $\mathbf{P}$. (The effect is a normalized form of the correlation between the individual's vote and the election's outcome.) More precisely, $e_k^{\mathbf{P}}(G)$ is the effect of knowing the kth player's vote on the election outcome. The effect is undefined if the probability that the kth player votes for Alice is 1 or 0.

When $\mathbf{P}$ represents a product probability measure, namely when the individual signals are independent, then the effect and the influence coincide.

Remarks and examples:

1. The effect of an agent's vote on the election's outcome is a measure we can estimate or "get a feel for" in real life situations. In real life elections the influence of each voter is small while the effect of each voter is large (namely, it is bounded away from zero regardless of the number of voters).

2. For general distributions, the effect of an agent can be negative. This will be the case for a voter who always votes for the candidate who is the underdog in the election polls.

3. A dummy (a voter k such that $\nu(S \cup \{k\}) = \nu(S)$ for every S) has no influence but may have a large effect. This will be the case if he always votes for the candidate who is expected to win according to election polls. An observer on a committee without the right to vote but who is likely to convince the committee of his opinion also has a large effect. Note that in the first case we do not attribute to that player real "influence" (in the (non-technical) English sense of the word) while in the second case we would consider him "influential".

4. In the three examples of the previous paragraph, in which CJT breaks down for the simple majority rule, the effects of individuals were large. In the first example, in which all voters voted identically, the effect of each voter is 1 while the influence is 0. In the second and third examples the effect of each voter is bounded away from zero as a function of ϵ.

5. In Section 2.3, we considered the aggregation of information within a firm. There we saw a very similar situation being described as a dictatorship with a complex distribution of signals or as a complex strong simple game with a distribution of independent signals.

Consider the description as a dictatorship in which the distribution of votes is complex. The influence of each employee, except the head of the company, is, of course, 0. The influence of the head of the company is 1. His effect is also 1 although the effect of the other employees is still positive. We might ask ourselves whether the other employees have any real "influence" (in the daily non-technical sense of the word). In the scenario we have described, it appears that the employees do have some real influence. However, the same distribution could result in a different scenario in which each employee has no "influence" and the head of the firm makes up his own mind. His choice then influences the views of the employees below him.

6. The uncertainty in interpreting effects as real "influences" is genuine. On the other hand, it can be argued that the effects of agents is a reasonable measure of an agent's "satisfaction" with the social decision process and his ability to identify with the choice of the group. The distinction between influence and effect may be relevant to the "voting paradox" which has been extensively discussed in the political science and philosophy literature.

4.3 Asymptotically complete aggregation of information when individual effects are small

The following general form of Theorem 1.1 proposes that for a large class of probability distributions that describe voters' behavior, if the individual votes are biased towards Alice and individual effects are small, then Alice will win the election with high probability.

A natural condition to impose on the distribution $\mathbf{P}$ is the FKG condition (see Liggett (1985)). One definition of FKG measure on $\{0,1\}^n$ goes as follows: A distribution $\mathbf{P}$ on $\{0,1\}^n$ (or on $\mathbb{R}^n$) is called an FKG-distribution if for every $x, y \in \{0,1\}^n$

$$\mathbf{P}(x)\mathbf{P}(y) \leq \mathbf{P}(\max(x,y))\mathbf{P}(\min(x,y)).$$

Here, for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ $\max(x,y) = (\max(x_1,y_1), \ldots, \max(x_n,y_n))$ and $\min(x,y) = (\min(x_1,y_1), \ldots, \min(x_n,y_n))$. The FKG property is a profound notion of non-negative correlations between agents' signals. It implies (but stronger than) the following condition (known as *non-negative association*): For all increasing real functions f and g, it holds that $E[fg] \geq E[f]E[g]$. This is equivalent to the condition that for all increasing events A and B it holds that $P[AB] \geq P[A]P[B]$. Under the FKG property if the simple game is monotone, all effects are non-negative. This form of non-negative correlation is a plausible assumption to make in the context of collective choice.

Remark: The FKG condition is defined in a similar way for $\mathbb{R}^n$ rather than $\{0,1\}^n$. A special case of FKG is the notion of MLRP (monotone likelihood ratio property) which is heavily used in theoretical economics.

We propose the following far-reaching generalization of Theorem 1.1.

Conjecture 4.1. *For every $p > 1/2, \epsilon > 0$ there is $\delta = \delta(p, \epsilon) > 0$ such that for every strong simple game G if $\mathbf{P}$ is an FKG-distribution on individual votes such that the effect of every agent w.r.t. $\mathbf{P}$ is at most δ and the probability of each voter voting for Alice is at least p then Alice will be elected with a probability of at least $1 - \epsilon$.*

When we consider simple or weighted majority we can assume an arbitrary distribution of voters.

Theorem 4.2 (Kalai and Mossel (2003)). *1. For every $p > 1/2, \epsilon > 0$, there is $\delta = \delta(p, \epsilon) > 0$ such that for an election determined by a weighted majority strong simple game if $\mathbf{P}$ is a distribution on the individual votes such that the effect of every agent w.r.t. $\mathbf{P}$ is at most δ and the probability for each voter to vote for Alice is at least p, then Alice will be elected with a probability of at least $1 - \epsilon$.*

2. If the outcome of the elections are determined by a strong simple game which is not a weighted majority game (e.g., the US electoral method) then there is a probability distribution $\mathbf{P}$ for the individual signals such that the probability of every voter to vote for Alice is larger than 1/2 and Bob will be elected with probability 1. (The effect of every voter is therefore 0.)

My interpretation of Theorem 4.2 and Conjecture 4.1 is that the diminishing effects of individual voters and biased marginal probabilities simultaneously guarantee two things: asymptotically complete aggregation of information and the assertion that the information alone is sufficient to decide the election.

4.4 Strategic voting

Another key assumption of CJT, as well as our generalization, is that each agent votes according to his signal. There have recently been some interesting papers on the case in which voters vote strategically based on their signal. We return now to the case in which Alice is the better candidate and each voter receives a signal telling him to vote for Alice with probability $p > 0.5$ where these probabilities are independent. However, we now assume that the voters vote strategically. We therefore need to specify what each voter wishes to optimize and to assume that every voter wishes to minimize the probability of a mistake. This is a simplified version of the assumption in Feddersen and Pesendorfer (1998) (see also Feddersen and Pesendorfer (1996,1997)) who considered juries and naturally gave much larger weight to an innocent person being convicted than to a guilty one being acquitted. Note that all agents have a common goal. (This is another important assumption which drew considerable criticism even in the original context of Condorcet Jury theorem.) Overall, it appears that asymptotic complete aggregation of information becomes more likely when agents vote strategically rather than naively.

Conjecture 4.3. *For every strong simple game G when the agents wish to minimize the probability for a mistake naive voting is a Nash-equilibrium point.*

An interesting result of strategic voting is that asymptotic complete aggregation of information occurs even when the voting rule is biased towards one outcome. Feddersen and Pesendorfer (1998) proved that an asymptotic complete aggregation of information will take place when a proportion of α jurors are needed to convict for every $0 < \alpha < 1$. However, when all jurors are required for conviction, no matter how large is the jury, the probability for a mistake is bounded away from zero.

It is interesting to note that the relation between a "sharp threshold" and "asymptotically complete aggregation of information", which was obvious in our previous setting, extends to the case of strategic voting, though it is less obvious.

Consider monotone simple games with arbitrary critical probabilities: We say that a sequence of simple games G_n satisfies the sharp threshold property if for every $\epsilon > 0$, the length of the threshold interval satisfies

$$T_\epsilon(G_n) = o(\min(p_c(G_n), (1 - p_c(G_n))),$$

as n tends to infinity. When we examine the argument of Feddersen and Pesendorfer we realize that the reason for asymptotically complete information aggregation is precisely the sharp threshold property which holds for the majority rule and fails for the anonymous rule.

Proposition 4.4. *A sequence of games G_n display sharp threshold behavior if and only if there are symmetric strategies which lead to asymptotically complete aggregation of information.*

I further conjecture that if G_n displays sharp threshold behavior, there are (not necessarily symmetric) Nash-equilibrium strategies which lead to asymptotically complete aggregation of information and that, for weakly-anonymous games, these Nash-equilibrium strategies can be taken to be symmetric. Proving this conjecture as well as Conjecture 4.3 seems to require further understanding of influences.

We will mention two examples in which modeling based on a simple game is appropriate and for which our discussion appears to be relevant.

The HU tenure game

Returning to Alice and Bob, Alice was hired and she is now up for tenure. For Alice to receive tenure at the Hebrew University she must go through a procedure which involves four committees: The departmental committee has to vote in favor of tenure; then a committee next to the dean of the faculty has to approve it; next a nomination committee has to grant tenure; and finally if the university rector or president objects to the decision they can appeal to a higher university committee. Each of the first three committees requires a 2/3 majority for a positive decision and the appeal committee requires a 2/3 majority to reverse the earlier decision. With some simplification this process can be modeled as a simple game. Thus, we assume that tenure is decided in a somewhat complex way by the opinions of 40 or so people on the various committees.[2] The rationale for such a complex tenure procedure is certainly to avoid mistakes, especially that of awarding tenure to candidates which do not deserve it and to a lesser extent not to reject candidates who indeed deserve it.

In my opinion, modeling this procedure as a simple game, assuming that agents vote strategically, that all voters have the same goal and the independence of the individual signals is quite reasonable. (I am aware of the arguments against each of these assumptions.) Voting naively might lead to catastrophic outcomes so I assume that agents do indeed strategically: However, this does not mean that voters explicitly use randomization for mixed strategies but rather that they are more open to the positive interpretation of evidence and to arguments in favor of candidates. The correct strategy is not deduced from complex computations but rather through adjustments in strategy over time in order to achieve the common goal and the conventional wisdom that one should sometimes vote in favor of a candidate that one would have rejected were he the only judge.

[2] Note that modeling this procedure as a simple game is appropriate even if people in the higher committees take into consideration (in a monotone way) the (numeric) outcomes in the lower committees. Of course, other aspects of the sequential nature of this process are neglected in the simple game model.

A vaccination game

Finally, the following is an example of a different nature in which our analysis is relevant: Consider a population with a set N of n individuals that is threatened by a contagious disease. For a subset S of the population, let $\nu(S) = 1$ if when the individuals in S are immunized, then the whole population is protected against the disease and $\nu(S) = 0$ otherwise. Let p_c be the critical probability for this monotone simple game and say $p_c = 0.6$. In such a scenario, it might be hard to identify the precise winning coalitions but it is very realistic to assume that no individual will have a great deal of "power". It follows from Theorem 2.4 that if every individual decides randomly and independently with probability $p > 0.6$ to take the vaccination, then the society will be protected with a high probability and if $p < 0.6$ then, with a high probability, the society will not be protected.

We can add another ingredient to this story. Suppose further that each individual has to decide whether to take the vaccination where the payoffs are as follows: 0 if he does not take the vaccination and the population is protected anyway, -1 if he does take the vaccination and -50 if he does not take the vaccination and the population is not protected. (Note: this is not a common-goal game.) I conjecture that there exists a Nash equilibrium point at which the ith individual chooses to take the vaccination with probability p_i where the p_i's are very close to 0.6. A proof appears to require a deeper understanding of influences.

5 Concluding remarks

Condorcet's Jury Theorem provided an appealing philosophical justification for democracy, which was a new idea at the time (with some old roots). It does not appear to yield any verifiable predictions. The only way we can estimate the value of p is by counting the votes but once we do so the assertion of CJT becomes tautological. If we try to verify the assumptions of CJT we will certainly identify patterns of voting which violate the condition of independence. In short, the assumptions are false and the conclusions cannot be tested. Our extension of CJT suggests something that we can in fact test. When the power of voters is small, the choice of election method does not matter that much. A referee pointed out that this "prediction" fails in the case of elections for the US Senate which can be regarded as a choice between two alternatives: a Democrat-controlled senate and Republican-controlled senate. He proposed the explanation that voters in small states tend to vote differently than voters in large states. This is an insightful remark and places this work on more realistic ground. Condorcet's Jury Theorem and our generalization are about "cleaning" the "noise" from the signals to reveal the consequences of the underlying information.[3] For this purpose, games in which agents have little power yield qualitatively similar behavior to that under simple majority rule. There is much still to be learned regarding quantitative aspects. I consider the link, even if only a partial one, between Condorcet's Jury Theorem and the power (or effects) of agents to be an important one. I expect that this link will have a role to play in other areas of theoretical economics which involve the aggregation of information.

Condorcet's Jury theorem and of our extension constitute an example of a threshold phenomenon in which the probability of an event changes rapidly as some underlying parameter varies within some interval. Identifying threshold phenomena in other economic situations and especially in the context of general equilibrium theory, is of particular interest.

The extension of CJT relies on known mathematical ideas and results (Russo (1984), Ben Or and Linial (1985,1990) Kahn, Kalai and Linial (1988), Talagrand (1994) and Friedgut and Kalai (1996)) with one crucial addition. These mathematical results and methods are related to early works on the indices of power in game theory as well as to more recent works in economics on pivotal agents (such as Al-Najjar and Smorodinsky (2000)) and may find further applications.

Our extension of McGarvey's theorem is similar in spirit to Arrow's impossibility theorem. Arrow's theorem asserts that the only way we can force social preferences to be transitive (or rational) is by giving all the power to one individual. We show that the only way we can force *any restriction* on social preferences is by giving one individual a substantial amount of power (i.e., an amount of power which is bounded away from zero independently of the size of the society). Here as well, stronger quantitative versions of the theorem are desirable.

A large number of agents, none of whom is too powerful, is an assumption which lies at the heart of much of economic theory: Complete aggregation of information as many aspects of perfect equilibrium theory can be re-

[3]For the US senate election, even if there was a clear answer to the question: "is a Democrat-controlled senate superior?" and every voter would receive a signal with this information, the information is not sufficient to determine the election's outcomes since the choice depends also on the question: "who will better represent the state of Texas?". Therefore, we cannot expect a very high correlation for Senate elections between the popular vote and the outcomes. We can expect a much higher correlation between the popular vote and the outcomes in presidential elections.

garded as pleasant features of this scenario, while chaotic social preferences as chaotic demand functions can be seen as unpleasant features. The simple link between the "pleasant" and "unpleasant" sides which we observed in our extension of McGarvey's theorem may also apply in other economic situations.

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6 Appendix

6.1 Russo's lemma and other basic facts

1. If G is a strong game then $\mathbf{P}_p(G) = 1 - \mathbf{P}_{1-p}(G)$.

Proof: Let $\bar{S} = N \backslash S$. Note that for every subset S of N, $\mathbf{P}_p(S) = \mathbf{P}_{1-p}(\bar{S})$ (both are equal to $p^{|S|}(1-p)^{n-|S|}$). Since G is strong, $\nu(S) = 1$ if and only if $\nu(\bar{S}) = 0$, therefore

$$\mathbf{P}_p(G) + \mathbf{P}_{1-p}(S) = \sum_{S \subset N, \nu(S)=1} p^{|S|}(1-p)^{n-|S|} + \sum_{S \subset N, \nu(S)=1} (1-p)^{|S|} p^{n-|S|} =$$

$$\sum_{S \subset N, \nu(S)=1} p^{|S|}(1-p)^{n-|S|} + \sum_{S \subset N, \nu(\bar{S})=0} p^{|\bar{S}|}(1-p)^{n-|\bar{S}|} =$$

$$= \sum_{R \subset N} p^{|R|}(1-p)^{n-|R|} \ = \ 1.$$

□

2. If G is a monotone simple game then the function $\mathbf{P}_p(G)$ is a strictly monotone continuous function of p in the interval $[0,1]$.

Proof: Let $0 < p < q < 1$. Consider a random subset S of N according to the probability measure $\mathbf{P}_p$. Let R be a random subset of N according to the probability measure $\mathbf{P}_r$ where $r = (q-p)/(1-p)$. Consider $T = S \cup R$. The probability that a player i belongs to T is $1-(1-p)(1-r) = q$ and the events $i \in T$ are independent. $\mathbf{P}_p(G)$ is the expected value of $\nu(R)$ and $\mathbf{P}_q(G)$ is the expected value of $\nu(T)$. By monotonicity, $\nu(T) \geq \nu(R)$ and therefore $\mathbf{P}_q(G) \geq \mathbf{P}_p(G)$. Since there is a positive probability that $R = \emptyset$ and $T = N$, we conclude that $\mathbf{P}_q(G) > \mathbf{P}_p(G)$. (The monotonicity of $\mathbf{P}_p(G)$ also follows from Russo's lemma, but the direct argument used here will serve us again in the proof of Theorem 1.1.) □

3. A proof of Russo's lemma:

Let

$$\mathbf{P}^{p_1,p_2,\dots p_n}(S) = \prod\{p_i : i \in S\} \cdot \prod\{(1-p_i) : i \notin S\}.$$

Let $I_k^{p_1,p_2,\dots p_n}(G)$ be the probability that the kth player is pivotal according to the probability measure $\mathbf{P}^{p_1,p_2,\dots p_n}(S)$. Note that $\mathbf{P}^{p_1,p_2,\dots p_n}(S)$ is a linear function of p_i and $\partial\mathbf{P}^{p_1,p_2,\dots p_n}(S)/\partial p_i = I_k^{p_1,p_2,\dots p_n}(G)$. By the chain rule, $d\mathbf{P}_p(G)/dp = d\mathbf{P}_{p,p,p\dots,p}/dp$ is equal to

$$\sum_{k=1}^{n} \partial\mathbf{P}_{p_1,p_2,\dots,p_n}/\partial p_k = \sum_{k=1}^{n} I_k^{p_1,p_2,\dots p_n}(G),$$

at the point $(p_1,p_2,\dots,p_n) = (p,p,\dots,p)$. Therefore, $d\mathbf{P}_p(G)/dp = \sum_{k=1}^{n} I_k^p(G) = I^p(G)$. □

6.2 Proof of Theorem 1.1

We require the following result by Friedgut (1998) that asserts (in our terminology) that a simple game with a small influence (w.r.t. $\mathbf{P}_p$) is determined with high probability (w.r.t. $\mathbf{P}_p$) by a small set of players.

Theorem 6.1. *For every real numbers $z > 0, A > 1$ and $\gamma > 0$, there is $C = C(\gamma, A, z)$ such that if $z \leq p \leq 1-z$ the following assertion holds: For a monotone simple game $G =< N, \nu >$, if $I^p(G) \leq A$ then there exists* *a collection S of at most C players in N and a monotone simple game $H =< S, \nu_0 >$ such that*

$$\mathbf{P}_p\{T \subset N : \nu(T) \neq \nu_0(T \cap S)\} < \epsilon. \tag{6.1}$$

Proof of Theorems 2.3 and 2.5: Since by Russo's lemma $I^p(G) = d\mathbf{P}_p(G)/dp$ Theorem 2.7 and 6.1 give conditions for the derivative of $\mathbf{P}_p(G)$ to be large at a given point p, in order to prove that the threshold interval is small we need to move from local information (for specific values of p) to global information for the entire threshold interval.

Lemma 6.2. *Let G be a monotone simple game. Let $p < q \in [1/3, 2/3]$. Suppose that $\mathbf{P}_p(G) \geq a > 0$, $I^p(G) \leq A$ and that $\mathbf{P}_q(G) \leq b < 1$. Let S be the set of players guaranteed by Theorem 6.1. Then $\max\{I^q_k(G) : k \in S\} \geq U$ where $U > 0$ depends only on a, A and B.*

Proof: Let $\gamma = \min((1-b)/2, a/2)$. Theorem 6.1 guarantees the existence of a set S of at most $C(\gamma, A)$ players and a simple game $H =< S, \nu_0 >$ such that

$$\mathbf{P}\{T \subset N : \nu(T) \neq \nu_0(T \cap S)\} < \gamma.$$

Claim:

$$\mathbf{P}_p\{T : \nu(T \cup S) = 1\} \geq 1 - (2/a) \cdot \gamma.$$

Proof of the claim: Let $\mathbf{P}^0_p$ and $\mathbf{P}^1_p$ be the probability distributions induced from $\mathbf{P}_p$ on subsets of S and on subsets of $N \backslash S$, respectively. Note that whether $\nu(T \cap S) = 1$ depends only on $T \backslash S$ so we have to show that

$$\mathbf{P}^1_p\{T : T \cap S = \emptyset, \nu(T \cup S) = 1\} \geq 1 - (2/a) \cdot \gamma.$$

Now, $\mathbf{P}^0_p\{R \subset S : \nu_0(R) = 0\} \geq a/2$ and therefore if T is disjoint from S and $\nu(T \cap S) = 1$ then $\mathbf{P}^0_p\{R : R \subset S, \nu(T \cap R) \neq \nu_0(R)\} \geq a/2$. It follows that indeed

$$\mathbf{P}^1_p\{T : T \cap S = \emptyset, \nu(T \cup S) = 1\} \geq 1 - 2/a \cdot \gamma.$$

We return now to the proof of the Lemma. Consider the following operation: Start with a random subset R of players according to $\mathbf{P}_p$. For $j \notin R$ add j to R with probability $(q-p)/(1-p)$. Let R^* be the resulting set of players. The probability that $\nu(R^*) = 0$ is at least $1 - b$ and the probability that also $\nu(R^* \cup S) = 1$ is at least $1 - b - 2/a\gamma$. (since $\nu(R \cup S) = 1$ implies $\nu(R^* \cup S) = 1$). This means that when we draw a coalition R^* at random according to $\mathbf{P}_q$, then the probability that $\nu(R^*) = 0$

and $\nu(R^* \cup S) = 1$ is at least $1 - b - 2/a \cdot \gamma$. Now we can examine the effect of adding the players in S one by one. Since $q \in [1/3, 2/3]$ we deduce that $\max\{I_k^q(G) : k \in S\} \geq 3^{-C}(1 - b - 2/a \cdot \gamma)$, as required. $\square$

We return now to the proof of Theorems 2.3 and 2.5. We can assume that $\delta \leq 1/10$. Suppose that $I_\epsilon(G) > \delta$. By Russo's lemma (and the mean-value theorem) there exists p in $[1/3, 2/3]$ such that $I^p(G) \leq A$, where $A = 3/\delta$. Since G is a strong simple game we can assume that $p \leq 1/2$. By Theorem 6.1, there is a set S of players and a simple game $H =< S, \nu_0 >$ such that relation 6.1 holds.The cardinality of S is bounded by a function C of A and ϵ. By our lemma, for every $q \geq p$ in the threshold interval there is a player $k \in S$ such that $I_k^q(G) \geq U$, where U depends only on ϵ and A. Taking $q = 1/2$, we have $\bar{\beta}(G) \geq U$. This proves Theorem 2.5. Since for every $q \geq p$ in the threshold interval there is a player in S whose influence is at least U, we conclude that $\bar{\phi}(G)$ is larger than $1/2 \cdot \delta \cdot C^{-1} \cdot U$. $\square$

The proof of Theorem 2.4 is identical. Just replace the interval $[1/3, 2/3]$ by an appropriate interval around the critical probability of the game.

6.3 Allowing strong players and relaxing monotonicity

We will remark here on two possible extensions of Theorem 1.1. We have already shown that we cannot replace the assertion in Theorem 1.1 that "no player has a large power index" with the assertion that "No small group of players has almost all the power". However, by combining Theorem 6.1 with Russo's lemma our theorem can be extended to the case in which for *every* p there is no small set S of voters such that the combined influence of voters outside S is small.

When we consider general schemes of aggregation, monotonicity is a natural condition to demand but is not always realistic. The question of whether this condition can be weakened is therefore of interest. Indeed for the assertion of Theorem 1.1 a weaker condition suffices. Let p be fixed and let $I_k^p(A, +)$ be the probability that adding player k will change a losing coalition into a winning one and let $I_k^p(A, -)$ be the probability that it will change a winning coalition into a losing one. The proof of CJT applies if there is a constant $\theta > 1$ such that for every probability p in the threshold interval and every k, $I_k^p(A, +) \geq \theta I_k^p(A, -)$.

6.4 Analyzing the hierarchical voting methods

Proof of Proposition 2.8: Let z_r be the probability that Alice will be elected when the probability of every voter to vote for Alice is p and there are r levels of hierarchy. Recall that $n = 3^r$.

We obtain that $z_1 = p$ and

$$z_{r+1} = z_r^3 + 3z_r^2(1 - z_r).$$

Note that if we write $z_r = 1/2 + t$ then $z_{r+1} = 1/2 + 3/2t - 2t^3$. It follows immediately that if $p = 1/2 + (2/3)^r = 1/2 + n^{-0.379..}$ then $z_r \geq 3/4$. If we write $z_r = 1 - s$ then $z_{r+1} = 1 - 3s^2 + 3s^3$ and it again follows immediately that if $p \geq 3/4$ then $z_r > 1 - \epsilon$ when $r = \log(1/\epsilon)$. □

Proof of Proposition 2.9: Let z_r be the probability that alternative A is chosen for the case of a firm with r layers. For $r = 1$ we have a company with a single employee and thus $z_1 = p$. When the number of layers is $r + 1$ there are two possibilities for the CEO to decide on Alice: Either he supported Alice to begin with and at least one of the employees below him supported her or he supported Bob to begin with and all employees below him supported Alice. This yields:

$$z_{r+1} = p \cdot (1 - (1 - z_r)^K) \ + \ (1 - p) \cdot z_r^K \ .$$

For $K = 2$ we get for every x, $1/2 < x < 1$,

$$p \cdot (1 - (1 - x)^2) \ + \ (1 - p) \cdot x^2 \ > \ x \ .$$

It follows that the sequence z_r tends to 1 as r tends to infinity. For $K = 3$, if $p < 2/3$, a simple calculation shows that z_r tends to $3p - 1$. □

6.5 Additional phenomena

1. Simple majority maximizes influence.

   The influence $I_k^p(G)$ is equal to:

$$I_k^p(G) = \sum_{S \subset N} p^{|S|}(1 - p)^{n-|S|}(\nu(S \cup \{k\}) - \nu(S)).$$

   Summing over all k, $k = 1, 2, \ldots, n$ we obtain:

$$\sum_{k=1}^{n} I_k^p(G) = \sum_{k=1}^{n} \sum_{S \subset N} p^{|S|}(1 - p)^{n-|S|}(\nu(S \cup \{k\}) - \nu(S)) =$$

$$\sum_{k=1}^{n} \sum_{S \subset N} p^{|S|}(1 - p)^{n-|S|}\nu(S \cup \{k\}) - n \cdot \sum_{S \subset N} p^{|S|}(1 - p)^{n-|S|}\nu(S).$$

   By inspection of the total contribution of $\nu(S)$ we find that

$$\sum_{k=1}^{n} I_k^p(G) = \sum_{S\subset N} p^{|S|}(1-p)^{n-|S|}\nu(S)((1-p)/p\cdot|S| - (n-|S|)) =$$

$$= \sum_{S\subset N} p^{|S|}(1-p)^{n-|S|}\nu(S)((1/p\cdot|S| - n).$$

Suppose we want to maximize the sum of influences under the condition that G is a strong game (or even that G has precisely 2^{n-1} winning coalitions). We must let $\nu(G) = 1$ if $|S| > n/2$. (If n is even we can choose half the coalitions of size $n/2$ to be winning in an arbitrary way.)

2. The proof of relations (2.6) and (2.7):

Here we need to rely on the very basic harmonic analysis setting. For simplicity I will present the proof for $p = 1/2$. A simple game can be described by a *Boolean function*, namely $f(x_1, x_2, \ldots, x_n)$ where the variables x_k take the values 0 or 1 and the value of f itself is also 0 or 1. Every 0-1 vector $x = (x_1, x_2, \ldots, x_n)$ corresponds to a subset of players $S = \{k : x_k = 1\}$ and we let $f(x_1, x_2, \ldots, x_2) = \nu(S)$. Let Ω_n denote the set of 0-1 vectors $(x_1, \ldots, x_n)$. Let $L_2(\Omega_n)$ denote the space of real functions on Ω_n, endowed with the inner product $< f, g >= \sum 2^{-n} f(x_1, \ldots, x_n)g(x_1, \ldots, x_n)$. $L_2(\Omega_n)$ is a 2^n-dimensional vector space.

For a subset S of N consider the function $u_S(x) = (-1)^{\sum\{x_i : i\in S\}}$. It is not difficult to check that the 2^n functions u_S for all subsets S form an orthonormal basis for the space of real functions on Ω_n.

For a function $f \in \Omega_n$, let $\hat{f}(S) =< f, u_S >$ ($\hat{f}(S)$ is called a Fourier-Walsh coefficient). Since the functions u_s form an orthogonal basis it follows that

$$\sum_{S\subset N} \hat{f}^2(S) = \sum_{x\in\Omega_n} 2^{-n} f^2(x).$$

(This relation is called Parseval's formula.) If f is a Boolean function, then $f^2(x)$ is either 0 or 1 and therefore $\sum_{x\in\Omega_n} 2^{-n} f^2(x)$ is simply the probability that $f = 1$ (with respect to the uniform distribution).

We are ready to verify relation (2.6). Suppose that f is a Boolean function which corresponds to a strong simple game G. $u_\emptyset$ is simply the constant-one function and therefore $\hat{f}(\emptyset) = \mathbf{P}_{1/2}(G) = 1/2$. Next, note that $u_{\{k\}}(x_1, x_2, \ldots, x_n) = 1$ if $x_k = 0$ and $u_{\{k\}}(x_1, x_2, \ldots, x_n) = -1$ if $x_k = 1$ and it follows that

$$\hat{f}(\{k\}) = I_k^{1/2}(G).$$

Parseval's formula implies that

$$\sum_{S \subset N} \widehat{f}^2(S) = 1/2,$$

and since $\widehat{f}^2(\emptyset) = 1/4$ we conclude that:

$$\sum I_k^2(f) = \sum_{k=1}^{n} \widehat{f}^2(\{k\}) \leq$$

$$\leq \sum_{S \subset N, S \neq \emptyset} \widehat{f}^2(S) = 1/4.$$

Let G be a strong simple game and f be the associated Boolean function. A basic (but quite easy to prove) relation between influences and Fourier coefficients from Kahn, Kalai and Linial (1988) is

$$I(G) = 4 \sum_{S \subset N} \widehat{f}^2(S)|S|. \tag{6.2}$$

Note that since $\sum_{S \subset N} \widehat{f}^2(S) = 1/2$ and $\widehat{f}(\emptyset) = 1/2$ we obtain that $\sum_{S \subset N, S \neq \emptyset} \widehat{f}^2(S) = 1/4$ and this implies relation (2.7) $I(G) \geq 1$. (Various proofs are known for this fundamental fact.) Relation (6.2) and its extension to $\mathbf{P}_p$ is the starting point for the proof of Theorem 2.7.

6.6 Proof of Theorem 1.2

In order to fill in the outline of the proof presented in Section 3 we will need a few notations. For a real number $\gamma > 0$, let $\mathcal{G}[\gamma]$ denote the class of strong simple games G such that $\bar{\phi}(G) \leq \gamma$. Let $\mathcal{F}[\gamma, m]$ denote the class of neutral social welfare functions on m alternatives which are based on a simple game G in $\mathcal{G}[\gamma]$.

Let $\mathcal{O}(m)$ denote the set of order relations on a fixed set X of m alternatives. Let ν be a probability distribution on $\mathcal{O}(m)$. Thus, $\nu = (\nu(\pi) : \pi \in \mathcal{O}(m))$, $0 \leq \nu(\pi) \leq 1$ and $\sum\{\nu(\pi) : \pi \in \mathcal{O}(m)\} = 1$. For two alternatives $a, b \in X$ let

$$p_\nu(a, b) = \sum\{\nu(\pi) : \pi \in \mathcal{O}(m),\ a >_\pi b\}.$$

(For an order relation π and two alternatives a and b we denote that a is preferred over b with respect to π by $a >_\pi b$.) For example, if ν is the uniform distribution, then $p_\nu(a, b) = 1/2$ for every two alternatives a and b. Let $q_\nu(a, b) = p_\nu(a, b) - 1/2$. Let R_ν be a relation on X defined by $aR_\nu b$ if

$p_\nu(a,b) > 1/2$. If $\min\{|q_\nu(a,b)| : a,b \in X\} = t$ we say that the distribution ν realizes the asymmetric relation R_ν with *quality* t.

Recall now the function $\delta(p,\epsilon)$ that appeared in Theorem 1.1, our extension of CJT. Consider a distribution ν on $\mathcal{O}(m)$ which realizes an asymmetric relation R with quality $t > 0$. Theorem 1.1 asserts that if a strong simple game G has the property that all individual powers are smaller than $\delta(1/2+t,\epsilon)$ then $\mathbf{P}_{1/2+t}(G) > 1-\epsilon$. Consider now a social welfare function in $\mathcal{F}[\delta(1/2+t,\epsilon),m]$ and a pair of alternatives a and b. If the individual preferences are independently drawn at random according to the distribution ν, then the probability that aR_ib is $p_\nu(a,b) \geq 1/2+t$. Therefore, the social choice between a and b coincides with R with probability of at least $\mathbf{P}_{1/2+t}(G) \geq 1-\epsilon$.

McGarvey's theorem implies that for every asymmetric relation R there is a distribution ν which realizes the asymmetric relation R. Indeed, given a set of orderings that yields the relation R by the majority rule, let ν be the uniform probability distribution on this set of orderings. The proof of McGarvey's theorem shows that m^2 voters are sufficient and therefore the quality t of the realization is at least $1/m^2$. Let

$$\delta(m) = \delta(1/2+t, 2\epsilon/(m(m-1)).$$

For every social welfare function in $\mathcal{F}[\gamma,m]$, if the individual preferences are drawn independently according to ν, then the probability that, for two specific alternatives a and b, the social preference between a and b coincides with R is at least $1-2\epsilon/m(m-1)$, and therefore the probability that the social preferences will coincide with R is at least $1-\epsilon$. □

6.7 Strategic voting

Proof of Proposition 4.4 (outline only): Suppose first that the sequence of simple games G_n displays the sharp threshold phenomena. Let $p(n)$ be the critical probability for G_n. Suppose that $p(n) \geq 1/2$ and consider the following strategy: If the signal is '1', vote '1'. If the signal is '0' vote '1' with probability $q(n) = 2(p(n)-1/2)$ and '0' with probability $1-q(n)$. A cursory inspection shows that for this strategy the superior candidate will be elected with probability tending to one as n tends to infinity. The proof in the other direction is similar.

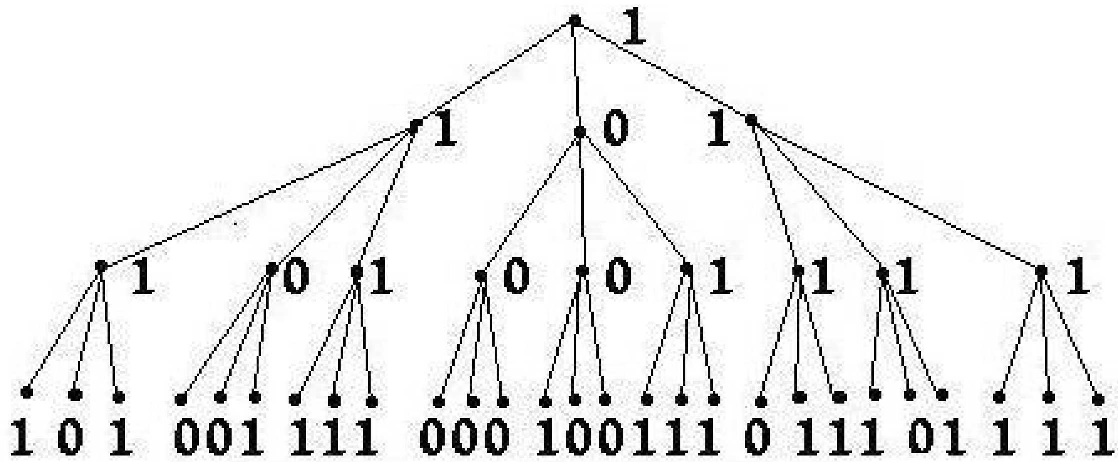

Fig. 1: Multi tier counsil system with four leyers. The signals of the 27 voters are shown as well as how the votes aggregate.

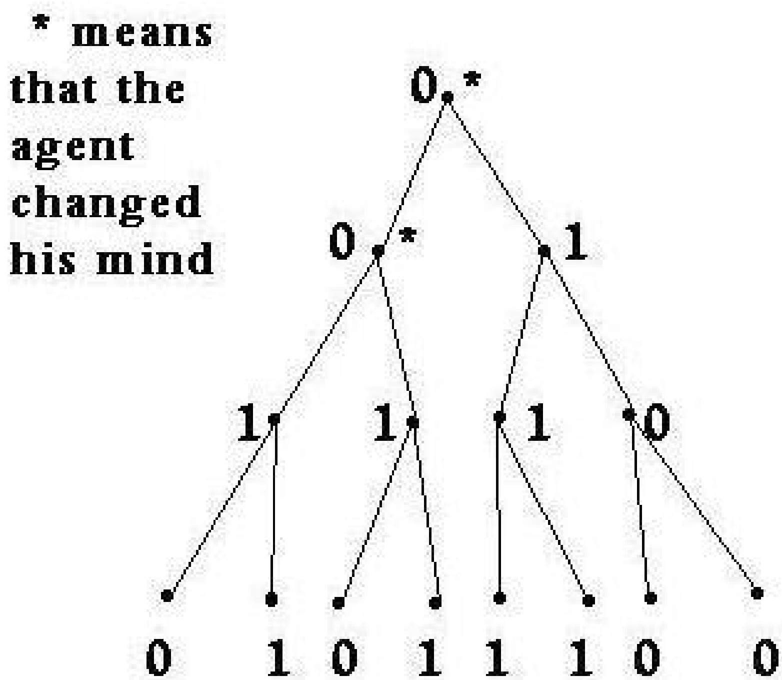

Fig. 2: Four level hierarchy, K=2. The original (independent) signals of the 15 employees are described.