Learning Processes, Mixed Equilibria and Dynamical Systems Arising from Repeated Games

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October 31, 1997

Keywords: Chain Recurrence, Dynamical Systems, Equilibrium Selection, Fictitious Play, Game Theory, Incomplete Information, Learning, Markov Process, Nash Equilibrium

Abstract

Fudenberg and Kreps (1993) consider adaptive learning processes, in the spirit of fictitious play, for infinitely repeated games of incomplete information having randomly perturbed payoffs. They proved the convergence of the adaptive process for 2×2 games with a unique completely mixed Nash equilibrium. Kaniovski and Young (1995) proved the convergence of the process for generic 2×2 games subjected to small perturbations. We extend their result to 2×2 games with several equilibria— possibly infinitely many, and not necessarily completely mixed. For a broad class of such games we prove convergence of the adaptive process; stable and unstable equilibria are characterized.

For certain 3-player, 2-strategy games we show that almost surely the adaptive process does not converge. We analyze coordination and anticoordination games.

The mathematics is based on a general result in stochastic approximation theory. Long term outcomes are shown to cluster at an attractor-free set for the dynamics of a vector field F canonically associated to an infinitely repeated μ-player game with randomized payoffs, subject to the long-run adaptive strategy of fictitious play.

The phase portrait of F can in some cases be explicitly described in sufficient detail to yield information on convergence of the learning process, and on stability and location of equilibria.

*M. Hirsch was partially supported by a grant from the National Science Foundation. Financial support from NATO (Grant CRG 950857) is gratefully acknowledged by both authors.

Contents

0 Introduction

Game theory, like any discipline which attempts to provide systematic tools for describing and analyzing real life situations, faces the problem of the confrontation of its idealized theoretical objects with their real-life counterparts. In this respect classical game theory suffers from several deficiencies:

(i) It is based on the unrealistic assumption that players are perfectly rational and possess full knowledge of the structure of the game, including the strategy spaces and the payoffs as well as the information and the rationality of the other players.

(ii) Nash's equilibrium concept, essential to the theory, is based on the players' ability to play *mixed* strategies. This raises the question of the justification and interpretation of mixed strategies.

(iii) A fundamental problem is that of *multiplicity of equilibria.* Even if the rationalistic justification of the Nash's equilibrium concept is satisfying when a game admits a unique equilibrium, it becomes highly problematic when there are several equilibria. Deductive equilibrium selection theories— such as Harsanyi and Selten's tracing procedure (1988)— argue that some equilibria are more reasonable than others. These theories, however, require very strong assumptions on the rationality of players, and there is no general and convincing argument in explaining how players can determine *which* equilibrium should be played.

One way of addressing theoretically some of the questions raised by (i) and especially (ii) is to introduce *perturbations* of the game in the sense of Harsanyi. In a perturbed game where each player has a small amount of private information represented by a privately observed (small) random perturbation of his own payoffs, all equilibria are essentially pure. Moreover, each regular equilibrium of the original game can be approximated by the beliefs resulting from the game of incomplete information (Harsanyi, 1973).

This approach provides a convincing justification and interpretation of mixed strategy equilibria. However it is not entirely satisfying with respect to (i), because the justification of equilibria in terms of rational behavior requires that the probability distributions of the payoffs be common knowledge.

An alternative approach to these problems is to replace the rationalistic explanations with an *adaptive* or *learning* interpretation of game-theoretic concepts such as equilibria and mixed strategies. Along these lines, Fudenberg and Kreps (1993) recently proposed an adaptive model for games with randomly disturbed payoffs, based on the method of *fictitious play* in which players— by playing the game over and over— adapt to their opponents' long term strategies and adapt their own responses over time.[1] At the start of each game, each player knows her own payoff matrix and the empirical frequencies of opponents' past actions, but no information about opponents' payoffs. Each player selects an action (pure strategy)

[1]There is an important literature on this subject. For an introduction and further reference the reader is refered to the recent book by Fudenberg and Levine (1996).

that maximizes her immediate expected payoff under the assumption that opponents will play the mixed strategy by the historical frequencies of past plays.

With regard to the points (i) and (ii) above, this kind of model enjoys some attractive features: There is no assumption of unrealistic common knowledge. Players have no any other information than their own payoff and the structure of the game. And players play only pure strategies, determined by their observation of the previous plays rather than by a complex reasoning process.

When adaptation according to fictitous play is added to Harsanyi's setup for a 2×2 game, it turns out that that empirical frequencies of action choices converge. Moreover, with sufficiently small noise in payoff matrices, *pure equilibria are almost surely selected over mixed equilibria* (Kaniovski & Young 1995; Hirsch & Benaïm 1994; Theorem 2.8 below). This is in surprising contrast to one of the fundamental conclusions of Harsanyi's theory, namely, that privately observed random perturbations of the payoffs should stabilize the mixed equilibria. This phenomenon illuminates the difference between Harsanyi's one-shot game situation, in which players are assumed to act on identical correct beliefs about their opponent strategies (regardless of the credibility of these beliefs), and the present context, in which players arrive at their beliefs— correct or not— as the result of the long-term learning adaptive process of fictitious play. We discuss this further in Section 2.

Stochastics and Dynamics of Fictitious Play Since payoffs vary stochastically in a Fudenberg-Kreps game, the infinitely repeated game gives rise to a stochastic process. The fundamental object of interest is the *state sequence* $\{x_k\}$, where x_k is a vector listing the empirical frequencies with which each player's pure strategies have been played in the first k games. The state sequence is a Markov process with countable state space, essentially a generalized urn process.

The stochastic process $\{x_k\}$ is caricatured by the dynamics of the deterministic *game vector field* F on the state space: $F(x)$ points in the direction of the conditional expected change in the state vector, given that the current state is x (see Equations (15), (16), (17)). Equilibria (zeroes) of F are Nash distribution equilibria of the unperturbed game. The mathematical relationship between state sequence and the game vector field is summarized in the crucial recurrence scheme

$$x_k + 1 - x_k = \frac{1}{k+1}[F(x_k) + Z_{k+1}],$$

where $\{Z_{k+1}\}$ is a random variable whose conditional expectation given x_k is zero. Such schemes are studied in Stochastic Approximation, whose results play a key role in analyzing fictitious play.

It has long been known that if the vector field F admits a unique, globally asymptotically stable equilibrium x_*, then almost surely $x_k \to x_*$; this is the classical Monro-Robbins theorem. Arthur *et al.* (1987) show that when F is a gradient vector field with finitely many equilibria, then almost surely $\{x_k\}$ converges; they also prove that $\{x_k\}$ cannot converge to a totally unstable equilbrium. A result of Pemantle (1990) implies that only asymptotically stable equilibria can have positive probability of being the limit of the state sequence.

Fudenberg and Kreps proved the convergence of the fictitious play adaptive process for those 2-player, 2-strategy games with *unique* Nash equilibria (possibly mixed). While their analysis provides quite an interesting interpretation of mixed equilibria as the result of learning in a situation of incomplete information, it doesn't address the important difficulties raised in (iii).

Kaniovski and Young (1995) and independently Benaïm & Hirsch (1994) showed that under generic conditions, when the payoffs are subject to sufficiently small independent random perturbations, the sequence of empirical frequency pairs converges almost surely to an equilibrium of the perturbed game which is close to an equilibrium of the unperturbed game. And this will be a pure equilibrium of the unperturbed game, if any exists. This result is given in Theorem 2.8 below under hypotheses slightly different from those of Kaniovski and Young (1995).

Previous results have been limited to proving convergence of the state sequence for certain situations, and to characterizing limiting equilibria. On the other hand, such games form an extremely small part of all games, and to concentrate on convergence is to ignore many interesting phenomena. Our point of view is that the entire limit set of the state sequence is of considerable interest. In this paper we use recent advances in dynamical systems and stochastic approximation to analzye the limit set for several classes of games, including some where convergence has zero probability.

As we will show, it is impossible to expect general convergence for noisy adaptive games having more than two players. In contrast to 2×2 games, even for games with a unique Nash distribution equilibrium there may be zero probability of convergence. Therefore an important question is to understand how qualitative features of the learning process, such as convergence or nonconvergence, can be deduced from structural features of the unperturbed game and the noise. We give several examples of this kind of analysis, using as our main tool the asymptotic dynamics of the game vector field.

The Main Results

In this paper we treat games with multiple equilibria, and games where oscillation and not convergence is the most likely behavior. The main tool is a recent theorem of Benaïm (1996), called here the *Limit Set Theorem*: Almost surely the limit set of $\{x_k\}$ has the important dynamical property of being *attractor-free* for the dynamics of F. In many cases this gives a great deal of information about the asymptotics of the game.

We prove that for 2×2 games having countably many Nash (distribution) equilibria, $\{x_k\}$ converges almost surely to a Nash equilibrium.

For any number of players, we show that every asymptotically stable equilibrium has positive probability of being selected (i.e., of being the limit of the game sequence). By a result of Pemantle (1990), an equilibrium which is not stable has no chance of being selected.

These results lead to the following conclusion for fictitious play with randomly perturbed payoffs: Where there is more than one asymptotically stable equilibrium of the game vector field, it is certain that the empirical frequencies of players' pure strategies will converge to such an equilibrium. Unfortunately it does not appear to be possible to predict with certainty *which* equilibrium will be selected.

This seemingly unsatisfactory state of affairs is simply an ineluctable fact about fictitious play. Much as we might desire a unique equilibrium, nature does not behave that way except in very special cases. The limiting equilibrium in 2×2 games is a random variable; it is a challenging problem to analyze its distribution.

When there are more players or strategies, even very simple games can have complicated dynamics under randomized fictional play. We give an example of an n-player matching game whose sample paths have positive probability of clustering at all points of a limit cycle of the game vector field. For $n = 3$ this behavior has probability 1.

More generally, we investigate n-player, 2-strategy *coordination* and *anticoordination* games. In a coordination game, payoffs are such that it is to the advantage of each pair of players to make choose identical actions, while in an anticoordination came the opposite holds. Players do not have this knowledge, however. It turns out (under generic assumptions) that for 3 player coordination games, sample paths almost surely converge. For anticoordination games we give examples of parameter ranges where this holds, and others where convergence is not certain.

From a strictly mathematical point of view, our results are consequences of a theory of asymptotic pseudotrajectories and its applications to stochastic approximation processes, recently developed by Benaïm (1996), Benaïm and Hirsch (1995, 1996), relying heavily on methods developed in the literature on stochastic approximation (e.g., Kushner and Clark (1978), Arthur *et al.* (1987), Duflo (1990, 1996).) The results on coordination and anticoordination games use differentiable ergodic theory and the theory of monotone dynamical systems.

Results in this field depend in subtle ways on the nature of the noise in the payoff matrices. Some theorems need the noise to be sufficiently diffuse; some proofs require that the conditional distribution of payoffs to any player depends on her action choice alone. Sometimes abstract properties of noise densities, such as analyticity, are useful in drawing conclusions about the dynamics of the game vector field. As in all mathematical modeling, such assumptions may have no practical interpretation, and may in fact be unnecessary for validity of the theorems. In such cases the mathematical results should be viewed as insights into the nature of the real world situation being modeled.

Outline of Contents

The organization of the paper is as follows:

Section 1 briefly review FK-games.

Section 2 contains the general analysis of 2×2 FK-games and states the main convergence theorems, with applications to games whose payoffs are subjected to small privately observed random shocks. We discuss relations between our results and Harsanyi's purification on one hand, and the risk-dominance criterion on the other hand.

Section 3 contains the proof of our basic mathematical result, the Limit Set Theorem.

Section 4 contains applications of the Limit Set Theorem and further proofs.

Section 5 discusses some adaptive FK-games with more than two players
Section 6 contains the proof of Theorem 2.8.

Acknowledgments We are grateful to Emmanuelle Auriol and Paolo Ghirardato for helpful advice and discussion, and to Drew Fudenberg for valuable comments. Critiques by Douglas Gale and anonymous reviewers of earlier drafts have considerably improved the exposition.

1 Games with Randomly Perturbed Payoffs

In this section we introduce the adaptive model considered by Fudenberg and Kreps (1993, Section 7) in which players play pure strategies determined by privately observed noisy payoffs, in the spirit of Harsanyi's purification Theorem (1973).

The basic model is an infinitely repeated game played by $\mu \geq 1$ players labeled by $i \in \{1, \dots, \mu\}$ at times $k = 1, 2 \dots$ It is convenient to use superscript $-i$ to refer to the set of players $\neq i$. In a 2 player game, $-i$ refers to the player different from player i.

We assume that player i has a fixed finite set

$$A^i = \{1, 2, \dots, d_i\}$$

of (pure) strategies, called player i's *action space*. The cartesian product of the other players' action sets is denoted by A^{-i}; it has cardinality $d_{-i} = \Pi_{j \neq i} d_j$.

The set S^i of mixed strategies for player i is the unit simplex $\Delta^{d_i - 1} \subset \mathbf{R}^{d_i}$ of dimension $d_i - 1$:

$$S^i = \{z \in \mathbf{R}^{d_i} : \sum_{l=1}^{d_i} z_l = 1, \ z_l \geq 0\}.$$

We identify S^i with the set of probability measures on A^i. If $a^i \in A^i$ we let $\mathbf{a^i} \in \mathbf{S^i}$ denote the corresponding vertex (i.e, the lth component of $\mathbf{a^i}$ is 1 for $l = a^i$ and 0 otherwise).

We set $S^{-i} = \Pi_{j \neq i} S^j$. An element $r \in S^{-i}$ is a list $\{r^j\}_{j \neq i}$ of mixed strategies for player i's opponents.

The game's *state space* is the compact convex polyhedron

$$S = S^1 \times \cdots \times S^\mu \subset \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu}.$$

A state $s = (s^1, \dots, s^\mu) \in S$ of the game is a list of mixed strategies. We define

$$s^{-i} \in S^{-i}$$

by deleting s^i from S. The payoffs to player i are determined by her *payoff function*

$$V^i : A^i \times A^{-i} \to \mathbf{R}.$$

When she plays $l \in A^i$ and her opponents play $r \in A^{-i}$, her payoff is $V^i[l, r]$. We represent V^i by a matrix of shape $d_i \times d_{-i}$, also denoted by V^i. The payoff function V^i is extended to a function $V^i : S^i \times S^{-i} \to \mathbf{R}$ by multilinearity.

Nash Distribution Equilibria for a Randomly Perturbed Game

Before analyzing fictitious play, we review one-shot games in which payoffs are randomly perturbed; here we do not consider repeated plays or adaptation.

Let $\{V^i\}$ be the payoff matrices for a classical μ-player game. A corresponding *augmented* (or *perturbed*) game (Fudenberg and Kreps (1993)) is specified by random $d_i \times d_{-i}$ matrices

$$\mathbf{U}^i = V^i + \mathbf{E}^i,\ i = 1, \ldots, \mu; \tag{1}$$

where the $\mathbf{E}^i$ are matrix-valued random variables with zero means. There is no assumption here that $\mathbf{E}^i$ and $\mathbf{E}^j$ are independent for $i \neq j$.[2] When playing the game, each player knows only her own payoff matrix.

Properly speaking, the data (1) specify not a single game but a random game. We refer to it as an *FK-game*.

Let $r \in S^{-i}$ be a set of mixed strategies for player $-i$'s opponents. An action $l \in A^i$ is called a *best response* of player i to r if it maximizes the expected payoff to player i, assuming that she plays l and her opponents plays r. Thus

$$l = \mathrm{Argmax}_{m \in \{1,\ldots,d_i\}}\ \mathbf{U}^i[m, r].$$

To ensure uniqueness of the best response, we assume from now on that the random variables $\mathbf{U}^i$ satisfy the following *ad hoc* condition:

Hypothesis 1.1 *For e very mixed strategy $z \in S^{-i}$ and every $m, l \in A^i$ with $m \neq l$,*

$$\mathsf{P}\{\ \mathbf{U}^i[m, z] = \mathbf{U}^i[l, z]\ \} = 0.$$

For each $z \in S^{-i}$ the set of $d_i \times d_{-i}$ matrices U such that $U[m, z] = U[l, z]$ for some $l \neq m$ has zero Lebesgue measure. Therefore Hypothesis 1.1 is easily seen to be valid for a large class of random matrices.

The *best response map* of player i is the deterministic map

$$\beta^i : S^{-i} \to S^i$$

defined as follows: For $s^{-i} \in S^{-i}$ and $l \in \{1, \ldots, d_i\}$, let $\beta^i(s^{-i})_l$ denote the probability that action l is the best response of player i when the opponent uses the mixed strategy s^{-i}. Thus for each $l \in \{1, \ldots, d_i\}$:

$$\beta^i(s^{-i})_l \ = \ \mathsf{P}\{l = \mathrm{Argmax}_{m \in \{1,\ldots,d_i\}}\ \mathbf{U}^i[m, s^{-i}]\}. \tag{2}$$

Definition 1.2 The *Nash map* is the map

$$\nu : S \to S,$$
$$\nu(s^1, \ldots, s^\mu) = (\beta^1(s^{-1}), \ldots, \beta^\mu(s^{-\mu})). \tag{3}$$

To a joint mixed strategy s, the Nash map ν assigns the list comprising each player's best response to the opponents' strategies listed in s.

[2]Later we will assume such independence in order to compare our results with Harsanyi's theory of purification.

The Nash map will play a key role in the further analysis.

Definition 1.3 A *Nash distribution equilibrium* of the augmented game is a fixed point $s_* \in S$ of the Nash map.

Remark 1.4 If ν is continuous, the *existence* of Nash distribution equilibria follows from the Brouwer fixed point theorem. But the *calculation* of these equilibria by the players would require that the marginal probability laws of the random payoffs $\mathbf{U}^i$ are common knowledge. In this situation a Nash distribution equilibrium is the probability distribution of a Nash equilibrium for the augmented game (see Harsanyi (1973) or Fudenberg and Kreps (1993)). There is no implication here that the players have this knowledge. But we shall prove that in many cases, the adaptive process of fictitious play for infinitely repeated FK games causes empirical frequencies to converge to some Nash distribution equilibrium.

Let $\mathcal{P}(A)$ denote the set of probability measures on A, which is naturally identified with the simplex $\Delta^{d-1} \subset \mathbf{R}^d$ where $d = \sum_{i=1}^{\mu} d_i$. Closely related to the Nash map for two players is the joint best response map.

Definition 1.5 The *joint best response map* is the map

$$\hat{\nu} : S \to \mathcal{P}(A) = \Delta^{d-1}, \tag{4}$$

defined for $a = (a^1 \dots a^\mu) \in A$ by

$$\hat{\nu}(s)_a = \mathsf{P}\{a^i = \mathrm{Argmax}_{j \in \{1,\dots,d_i\}}\ \mathbf{U}^i[j, s^{-i}]\ : i = 1, \dots \mu\}. \tag{5}$$

The number $\hat{\nu}(s)_a$ gives the probability that $a \in A$ constitutes the joint best responses to the mixed strategy profile $s \in S$.

Notice that when the matrices $\{\mathbf{E^i}\}$ are independent, then

$$\hat{\nu}(s)_a = \Pi_{i=1}^{\mu} \beta^i(s^{-i})_{a^i}. \tag{6}$$

The Game Vector Field for Adaptive FK-Games

We now consider fictitious play, with payoffs subject to random perturbations. Before each round of play, each player knows her own payoff matrix, but not the opponents'. But each player keeps track of the opponents' empirical frequencies of actions, and chooses her next action deterministically to optimize her expected payoff conditioned on the opponents playing the mixed strategy given by their current empirical frequencies of actions. This adaptive behavior is known as *fictitious play.*

We denote the set $\{0, 1, 2, \dots\}$ of natural numbers by $\mathbf{N}$ and the set of positive natural numbers by $\mathbf{N}_+$. The *Euclidean norm* of a vector $z \in \mathbf{R}^n$ is $||z|| = \sqrt{\sum_{j=1}^m x_j^2}$.

We consider the state space

$$S = \Delta^{d_1 - 1} \times \cdots \times \Delta^{d_\mu - 1} \subset \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu}$$

to be a submanifold (with corners) in $\mathbf{R}^d, d = d_1 + \cdots + d_\mu$. Its tangent space at every point is identified with the linear subspace

$$TS = \{(y^1, \ldots, y^\mu) \in \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu} \ : \ \sum_{j=1}^{d_i} y^i_j = 0, \quad i = 1, \ldots, \mu.\} \tag{7}$$

A *vector field* on S is a map from S to TS.

The action of player i in game $k \in \mathbf{N}_+$ is denoted by $a^i_k \in A^i$, and $\mathbf{a}^i_k$ denotes the corresponding vertex of the simplex $S^i = \Delta^{d_i - 1}$. We may consider $\mathbf{a}^i_k$ either as a pure strategy in the simplex of all strategies, or as a Dirac measure in the simplex of measures on the finite action space A^i.

The μ-tuple

$$a_k = (a^1_k, \ldots, a^\mu_k) \in A^1 \times \cdots \times A^\mu = A$$

is the *action profile* at time k. Corresponding to a_k is the *action vertex*

$$\mathbf{a}_k = (\mathbf{a}^1_k, \ldots, \mathbf{a}^\mu_k) \in S,$$

which is an extreme point of the convex polyhedron S. Thus a sequence of games produces the sequence $\{a_k\}$ of action profiles, and the equivalent sequence of $\{\mathbf{a}_k\}$ of vertices of S.

The *empirical frequency vector* $x^i_k \in S^i$ of player i after the first $k \geq 1$ games is the vector

$$x^i_k = \frac{1}{k} \sum_{j=1}^{k} \mathbf{a}^i_j \tag{8}$$

The mth component $(x^i_k)_m$ of x^i_k is the proportion of times in the first k games that player i has played action $m \in A^i$.

The *state of the game at time* $k \geq 1$ is the vector $x_k \in S$ listing the players' empirical frequencies at time k:

$$x_k = (x^1_k, \ldots, x^\mu_k) \in S^1 \times \cdots \times S^\mu,$$

or equivalently

$$x_{k+1} = \frac{k}{k+1} x_k + \frac{1}{k+1} \mathbf{a}_{k+1}. \tag{9}$$

We call $\{x_k\}_{k \in \mathbf{N}}$ the *state sequence* of the infinitely repeated game.

The *empirical joint frequency* tensor at time k is the $d_1 \times \cdots \times d_\mu$ tensor $C_k : A \to \mathbf{R}$ given by

$$C_k = \frac{1}{k} \sum_{j=1}^{k} \mathbf{a}^1_j \otimes \cdots \otimes \mathbf{a}^\mu_j; \tag{10}$$

where $y^1 \otimes \cdots \otimes y^\mu$ is defined by $y^1 \otimes \cdots \otimes y^\mu [l_1, \ldots, l_\mu] = \Pi^\mu_{i=1} y^i_{l_i}$. Thus $C_k[a]$ is the proportion of time in $\{1, \ldots, k\}$ that action profile $a \in A$ has been played.

An μ-player *adaptive FK-game* is a sequence of independent, identically distributed (IID) FK-games. It is specified by data

$$\{A^i, \{\mathbf{U}^i_k\}_{k \in \mathbf{N}_+}; \, k = 1, 2 \ldots\}$$

where A^i is player i's action set, and $\mathbf{U}^i_k$ is a random $d_i \times d_{-i}$ matrix of the form

$$\mathbf{U}^i_k = V^i + \mathbf{E}^i_k, \quad k = 1, 2, \ldots; \tag{11}$$

such that V_i is a fixed deterministic matrix, and $\{\mathbf{E}^i_k\}_{k \in \mathbf{N}_+}$ is a sequence of IID random matrices having mean zero. (We do not assume $\mathbf{E}^i_k$ and $\mathbf{E}^j_k$ are independent for $i \neq j$.)

Let $k \in \mathbf{N}$. After the first k games (if $k \geq 1$), the players play the augmented game defined by the matrices $\mathbf{U}^i_{k+1}$. They use the following adaptive procedure, *fictitious play*, for determining their next actions a^i_{k+1}. Player i knows her own payoff matrix $\mathbf{U}^i_{k+1}$ for round $k+1$ and her opponent's empirical frequency vector x^{-i}_k. She assumes the opponents will use the mixed strategy $x^{-i}_k \in S^{-i}$, and she computes and plays her best response action $a^i_{k+1} \in A^i$ to x^{-i}_k. Thus

$$a^i_{k+1} = \mathrm{Argmax}_{m \in \{1,\ldots,d_i\}} \, \mathbf{U}^i_{k+1}[m, x^{-i}_k].$$

Therefore *the state sequence $\{x_k\}$ is a nonstationary discrete-time Markov process, with values in the compact, convex set S.*

Using the best response maps β^i (see Equation (2)) and the Nash map ν (Equation (3)) we obtain the formulas:

$$\mathsf{P}\{a^i_{k+1} = l | x_k = x\} = (\beta^i(x^{-i}))_l, \tag{12}$$

$$\mathsf{E}(\mathbf{a}_{k+1} | x_k) = \nu(x^k). \tag{13}$$

From (9) and (13) we derive:

$$\mathsf{E}(x_{k+1} - x_k \mid x_k) = \frac{1}{k+1}(-x_k + \nu(x^k)). \tag{14}$$

We will describe the long term behavior of this Markov process in terms of the dynamics of the *game vector field* $F : S \to TS$ (see Equation (7)) on the state space, defined as:

$$\begin{aligned} F : S &\to TS \subset \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu}, \\ F(x) &= -x + \nu(x). \end{aligned} \tag{15}$$

$F(x)$ measures the discrepancy of $x \in S$ from being a Nash distribution equilibrium (Definition 1.3). Equations (13) and (14) give further meaning to F:

$$F(x) = \mathsf{E}(\mathbf{a}_{k+1} - x_k \, | x_k = x) \tag{16}$$

$$= (k+1) \, \mathsf{E}(x_{k+1} - x_k \mid x_k = x). \tag{17}$$

In other words: If the state at time k is x, then $F(x)$ is $k+1$ times the expected change in the state.

Our analysis of the fictitious play process will rely heavily on a close connection between the asymptotic behavior of sample paths of such a stochastic process $\{x_k\}$ and the deterministic dynamical system

$$\frac{dx}{dt} = F(x). \tag{18}$$

We call this the *game differential equation*. In a similar way, we will analyze the empirical joint frequencies in terms of the dynamics of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= F(x), \\ \frac{dC}{dt} &= -C + \hat{\nu}(x). \end{aligned}$$

In the next section we give some interesting consequences of the general theory which can be easily stated without the full mathematical formalism of Section 3.

We record for reference the obvious but useful fact alluded to above:

Proposition 1.6 *The zeroes of the game vector field F are the Nash distribution equilibria.*

2 Asymptotic Behavior of Adaptive 2×2 FK-games

In this section we describe the asymptotic behavior of the state sequence for 2×2 adaptive 2×2 FK-games having arbitrarily many Nash distribution equilibria. Proofs are postponed to sections 3 and 4.

Convergence of Empirical Frequencies

Let $(A^1, \{\mathbf{U}^1_k\}, A^2, \{\mathbf{U}^2_k\})$ be an adaptive 2-player FK-game in which each player has two pure strategies; the 2×2 random matrices $\mathbf{U^i_k}$ are as in (11). Here each action set A^i, $i = 1, 2$ has cardinality 2, and S^i is a 1-dimensional simplex. We identify S^i with the closed unit interval $I = [0, 1]$ by the map $(s, 1 - s) \mapsto s$. In this way a game state in the original *simplicial coordinates*

$$((x^1, 1 - x^1), (x^2, 1 - x^2))$$

is given the *interval coordinates*

$$(x^1, x^2) \in I \times I.$$

In addition to Hypothesis 1.1 we also assume:

Hypothesis 2.1 *The Nash map $\nu : I \times I \to I \times I$ is Lipschitz continuous.*

This is a technical assumption which is crucial to our analysis, as it validates the standard theorems of existence, uniqueness and continuity of solutions to differential equations. In some cases, as for example in classical fictitious play for games with fixed payoff matrices, this hypothesis is not satisfied. However, as soon as the game is subject to some kind of random perturbation this assumption is very likely to be satisfied. This is the case with the adaptive FK-games of Section 1 under the mild restrictions of Hypothesis 1.1. Hypothesis 2.1 is valid for a large class of random matrices $\mathbf{E}^i$.

Given a state sequence $\{x_k\}_{k \in \mathbf{N}}$ in $I \times I$ resulting from infinitely repeated fictitious play— thus a sample path of a stochastic process satisfying (14)— we say that a point

$x_* \in I \times I$ is a *limit point* of $\{x_k\}_{k\in\mathbf{N}}$ if $\lim_{i\to\infty} x_{k_i} = x_*$ for some sequence $k_i \to \infty$. The *limit set* of $\{x_k\}_{k\in\mathbf{N}}$ is the set of all these limit points.[3]

The following theorem is one of the main results of the paper. It extends Proposition 8.1 of Fudenberg and Kreps (1993) to arbitrary 2×2 games. Recall that $\hat{\nu}$ is the joint best response map defined in Equation (5).

Let $\mathcal{E} \subset I \times I$ denote the set of Nash distribution equilibria.

Theorem 2.2 *Consider a 2×2, adaptive FK-game satisfying Hypotheses 1.1 and 2.1. Let $\{x_k\}$ denote the sequence of empirical frequency vectors, and $\{C_k\}$ the sequence of empirical joint frequency matrices. Then:*

(a) *With probability one, the limit set $L \subset I \times I$ of $\{x_k\}$ is a point or a compact arc in $\mathcal{E}$, such an arc being simultaneously the graph of a strictly increasing or decreasing function and the graph of the inverse of such a function.*

(b) *If $\mathcal{E}$ is finite or countably infinite then almost surely $\{x_k\}$ converges to a Nash distribution equilibrium.*

(c) *Let x_* be a Nash distribution equilibrium such that $\mathsf{P}\{x_k \to x_*\} > 0$. Then*

$$\mathsf{P}\{C_k \to \hat{\nu}(x_*) \mid x_k \to x_*\} = 1.$$

This result shows that when there are only finitely or countably many Nash distribution equilibria, players behave in the long run as though they have computed the equilibria of the game.[4]

It is a generic condition for a vector field to have finite equilibrium set, that is, it holds for a dense open set of C^1 fields on any compact manifold. In this sense, (b) and (c) imply that for most games of the type considered, sequences of sample paths and joint frequency matrices converge almost surely. The following corollary of (a) gives another condition for convergence:

Corollary 2.3 *In addition to the assumptions of 2.2, assume that the Nash map $\nu : [0,1] \times [0,1] \to \mathbf{R}^2$ is real analytic (e.g., noise matrices have real analytic distributions), and that at least one of the following conditions holds:*

(i) $(0,0)$ $(1,1)$ *are not both fixed points of ν;*

(ii) $(0,1)$ $(1,0)$ *are not both fixed points of ν.*

Then $\mathcal{E}$ is finite. Therefore sample paths, and joint frequency matrices, converge almost surely.

---

[3] The state limit set is actually not a set in the usual sense, but rather a set-valued random variable, that is, a function assigning a set to each element of a probability space. By exension of the usual probabilistic convention we call it a set, analogous to way that numerical random variables are treated as numbers.

[4] Players who know the probability laws of the stochastic matrices $\mathbf{U^1}$ and $\mathbf{U^2}$, and who read Remark 1.4, can do that.

Theorem 2.2(c) and Equation (6) imply:

Corollary 2.4 *Let $x_* \in I \times I$ be as in Theorem 2.2(c). Suppose that for each k, the two payoff perturbation matrices $\mathbf{E}_k^1$, $\mathbf{E}_k^2$ are independent random variables. Then almost surely the sequence of empirical joint frequency matrices C_n converges to the 2×2 matrix $[{x_*}^i {x_*}^j]$.*

The proof of Theorem 2.2 makes use of the material presented in Sections 3 and 4. Part (a) is a consequence of the Limit Set Theorem 3.3; part (b) derives from Lemma 4.4, and (c) from Theorem 4.1. Details are given in Section 4.

Remark 2.5 Our assumptions on the noise matrices in Theorem 2.2 and 2.4 are considerably less restrictive than those of Fudenberg and Kreps (1993).

Equilibrium Selection and Path Dependence

In this section we sharpen our analysis of 2×2 adaptive FK-games, focussing attention on the problem of equilibrium selection: Given that the empirical frequencies converge almost surely, can we predict which Nash distribution equilibria are likely to be selected? Theorem 2.6 states that under mild restrictions, only asymptotically stable equilibria of the game vector field can be limits of the state sequence.

In interval coordinates the Nash map is represented as

$$\nu : [0,1] \times [0,1] \to [0,1] \times [0,1],$$

$$\nu(x^1, x^2) = (\, b^1(x^2),\, b^2(x^1)\,); \tag{19}$$

the game vector field $F(x) = x - \nu(x)$ is then

$$F : [0,1] \times [0,1] \to \mathbf{R}^2,$$

$$F(x^1, x^2) = (-x^1 + b^1(x^2),\, -x^2 + b^2(x^1)\,).$$

Recall that a Nash distribution equilibrium $x_* \in I \times I$ is an equilibrium of F, i.e., a solution to $F(x) = 0$. Thus it is represented by a solution to the equation

$$x_*^1 = b^1(x_*^2),\; x_*^2 = b^2(x_*^1).$$

We call x_* *simple* if F (and thus also ν) is C^1 (continuously differentiable) in some neighborhood of x_*, and the Jacobian matrix $DF(x_*)$ is nonsingular. As the determinant of $DF(x_*)$ is $1 - {b^1}'(x_*^2){b^2}'(x_*^1)$, we see that x_* is simple if and only if the derivatives ${b^i}'$ at x_* verify

$${b^1}'(x_*^2){b^2}'(x_*^1) \neq 1.$$

We say that x_* is *linearly stable* if

$${b^1}'(x_*^2){b^2}'(x_*^1) < 1 \tag{20}$$

and *linearly unstable* if

$$b^{1'}(x_*^2)b^{2'}(x_*^1) > 1. \tag{21}$$

The reason for these names is given after Theorem 2.6.

The following result characterizes entirely the qualitative behavior of adaptive 2×2 FK-games under the generic assumption that Nash distribution equilibria of the extended game are simple:

Theorem 2.6 *Consider an adaptive 2×2 FK-game satisfying Hypotheses 1.1 and 2.1. Assume that all Nash distribution equilibria of the extended game are simple. Then:*

(i) *The sequence $\{x_k\}$ of empirical frequency vectors converges with probability one to a Nash distribution equilibrium.*

(ii) *Let x_* be a linearly unstable equilibrium such that the Nash map is C^2 in a neighborhood of x_*, and the joint best response matrix $\hat{\nu}(x_*)$ has strictly positive entries (see Equation (4)). Then*

$$\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} = 0.$$

(iii) *Suppose that each state $x \in I \times I$, the joint best response matrix $\hat{\nu}(x)$ has strictly positive entries. Then at every linearly stable equilibrium x_* we have:*

$$\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} > 0.$$

Part (i) of this theorem follows directly Theorem 2.2(b) since it is easy to prove that simple equilibria are isolated. Parts (ii) and (iii) follow from the more general results (Theorems 4.5 and 4.7) given in Section 4.

The assumption in (ii) that b^1 and b^2 are C^2 is technical and perhaps unnecessary. However, the assumption that $\hat{\nu}(x_*)$ has strictly positive entries is fundamental. It means every action profile has a positive probability of being chosen in response to any pair of mixed strategies.

The intuition behind assertions (ii) and (iii) is part of the general philosophy (which will become precise mathematics in Section 3), according to which the long term behavior of the state sequence $\{x_k\}$ is closely related to the dynamics of the game differential equation (18). Indeed, as already noticed (Proposition 1.6), the Nash distribution equilibrium x_* is also an equilibrium for the dynamics of the game vector field

$$F = (F^1, F^2) : [0,1] \times [0,1] \to \mathbf{R}^2,$$

$$\begin{aligned} F^1(x^1, x^2) &= -x^1 + b^1(x^2), \\ F^2(x^1, x^2) &= -x^2 + b^2(x^1). \end{aligned}$$

An elementary computation shows that the Jacobian matrix $DF(x_*^1, x_*^2)$ has eigenvalues $-1 \pm \sqrt{b^{1'}(x_*^2)\, b^{2'}(x_*^1)}$. Our definitions of linear stability and instability of x_* correspond to the synonymous properties of $x_* = (x_*^1, x_*^2)$ as an equilibrium of F: Linear stability implies that the forward trajectories of all initial states sufficiently near x_* converge uniformly to x_*; this dynamic condition is called *asymptotic stability* of x_*. Linear instability implies that exactly two forward trajectories other than x_* converge to x_*. We point out in Theorem 4.7 that the condition of linear stability in assertion (iii) of Theorem 2.6 can be replaced by asymptotic stability.

If we view the sequence $\{x_k\}$ as a noisy numerical integration of F (see Equation (31)), it seems natural that linear instability should imply that a typical sequence $\{x_k\}$ of game states does not converge toward any unstable equilibrium of F; Theorem 4.7(ii) validates this intuition under somewhat restrictive hypotheses.

Independent Small Perturbations of Payoffs

Let ? (0) denote a classical 2-player, 2-strategy game in normal form, specified by the pair of 2×2 payoff matrices V^1 and V^2. We call such a game *generic* if the four numbers

$$M^i = V^i_{11} - V^i_{12}, \tag{22}$$
$$N^i = V^i_{12} - V^i_{22} \tag{23}$$

are nonzero for $i = 1, 2$. The set of all generic 2×2 games is identified, *via* the components of the payoff matrices, with an open subset of $\mathbf{R}^8$ whose complement has Lebesgue measure zero.

We consider two levels of extensions of the game ? (0). First, for each $\varepsilon > 0$ we consider an augmented game ? (ε) determined by random payoff matrices $\mathbf{U}^i = V^i + \varepsilon \mathbf{E}^i$, $i = 1, 2$. Second, we consider the infinitely repeated FK-game specified by the random matrices $\mathbf{U}_i$. That is, for each i let $\{\mathbf{E}^i_k\}_{k \in \mathbf{N}}$ be an IID sequence of random matrices with the same distribution as $\mathbf{E}^i$, and take player i's payoff matrix at round k to be

$$\mathbf{U}^i_k = V^i + \varepsilon \mathbf{E}^i_k.$$

Each player uses fictitious play to select deterministically the next action a^i_k in her action set $A^i = \{1, 2\}$.

We now make the following assumptions:

Hypothesis 2.7

(i) *The payoff matrix for player i is*

$$\mathbf{U}^i = V^i + \varepsilon \mathbf{E}^i,\ i = 1, 2$$

where $\mathbf{E}^i$ is a random matrix having the following special form:

$$\mathbf{E}^i = \begin{bmatrix} \eta^i_1 & \eta^i_1 \\ \eta^i_2 & \eta^i_2 \end{bmatrix},$$

whose columns $\eta^1 = (\eta^1_1, \eta^1_2)^\top$ and $\eta^2 = (\eta^2_1, \eta^2_2)^\top$ are independent random vectors with zero mean.

(ii) *Each random variable* $\eta_2^i - \eta_1^i$ *admits a strictly positive continuous density function* $f^i : \mathbf{R} \to \mathbf{R}_+$ *such that* $\lim_{|t|\to\infty} tf^i(t) = 0$.

Assumption (i) means that the perturbations to player i's payoffs are independent of the other player's strategies. It is chosen here for the sake of simplicity, but is not really essential and could easily be weakened. Assumption (ii), a technical one needed for our proof, is satisfied by many density functions. Remark 2.9 discusses what can be proved for densities that are merely integrable.

Corresponding to the first two levels of games there are two levels of Nash equilibria:

- Mixed or pure Nash equilibria for $?(0)$.
- Nash distribution equilibria for the augmented game $?(\varepsilon)$, for a given value of ε (see Definition 1.3).

It is straightforward to show that $?(0)$ has at most two pure and one mixed Nash equilibrium. More precisely: using interval coordinates (subsection 2), denote by $(p, q) \in [0,1] \times [0,1]$ the pair of mixed strategies where player 1 plays action 1 with probability p and player 2 plays action 1 with probability q. Then a computation shows that (p, q) is a Nash equilibrium provided $M^1 q + N^1(1-q) \geq 0$ and $M^2 p + N^2(1-p) \geq 0$ (see Equations (22) and (23)). In interval coordinates we have the following easily verified characterization of equilibria:

- $(1,1) \in [0,1] \times [0,1]$ is a pure Nash equilibrium if $M^1, M^2 > 0$.
- $(0,0)$ is a pure Nash equilibrium if $N^1, N^2 < 0$.
- $(1,0)$ is a pure Nash equilibrium if $N^2 > 0 > M^2$.
- $(0,1)$ is a pure Nash equilibrium if $N^1 > 0 > M^2$.
- (p, q) is a mixed Nash equilibrium if

$$
\begin{aligned}
0 < p &= \frac{N^2}{N^2 - M^2} < 1, \\
0 < q &= \frac{N^1}{N^1 - M^1} < 1.
\end{aligned}
$$

A direct computation shows that in interval coordinates the Nash map (see Equation (19))

$$\nu_\varepsilon : [0,1] \times [0,1] \to [0,1] \times [0,1]$$

for the augmented game $?(\varepsilon)$ is given as follows. Set

$$H^i(s) = \int_{-\infty}^{s} f^i(t)\,dt.$$

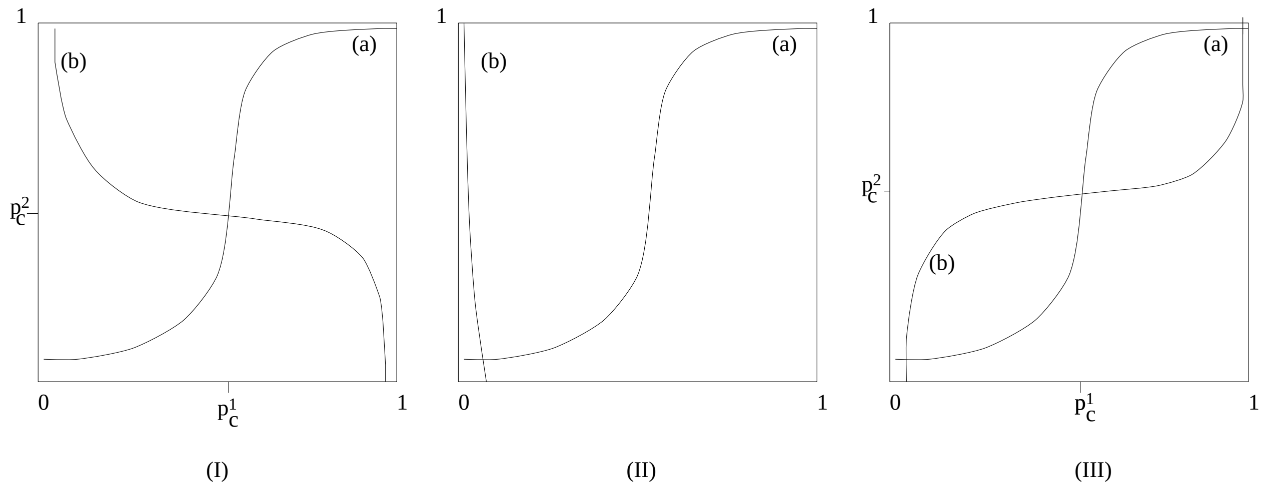

Figure 1: Three possibilities for $\Gamma(\varepsilon)$ and ε. (I): $M^1 > 0 > M^2$ and $N^1 < 0 < N^2$. The unperturbed game admits one mixed Nash equilibrium. (II): $M^1 > 0 > N^1$ and $M^2 < N^2 < 0$. The unperturbed game admits one pure Nash equilibrium, at $(0,0)$. (III): $M^i > 0 > N^i$, $i = 1,2$. The unperturbed game admits one mixed Nash equilibrium, and pure Nash equilibria at $(0,0)$ and $(1,1)$.

Then:

$$\nu_\varepsilon(x^1, x^2) = (\, b_\varepsilon^1(x^2), b_\varepsilon^2(x^1)\,)$$

where

$$b_\varepsilon^1(x^2) = H^1\left(\frac{(M^1 - N^1)x^2 + N^1}{\varepsilon}\right),$$

$$b_\varepsilon^2(x^1) = H^2\left(\frac{(M^2 - N^2)x^1 + N^2}{\varepsilon}\right).$$

The Nash distribution equilibria, fixed points of the Nash map, are the solutions (x^1, x^2) of the system

$$x^1 = b_\varepsilon^1(x^2), \;\; x^2 = b_\varepsilon^2(x^1). \tag{24}$$

They are equivalently determined by taking x^1 to be a fixed point of the composite mapping

$$b_\varepsilon^1 \circ b_\varepsilon^2 : [0,1] \to [0,1]$$

and setting $x^2 = b_\varepsilon^1(x^1)$.

Figure 1 illustrates some generic situations of ? (ε) for a small value of ε. The curves $p_2 = b_\varepsilon^2(p_1)$ and $p_1 = b_\varepsilon^1(p_2)$ are labeled (a) and (b) respectively. Their intersection points are the Nash distribution equilibria.

Let x_* be a Nash equilibrium for the unperturbed game ? (0), with interval cooordinates $(x_*^1, x_*^2) \in I \times I$, and simplicial coordinates

$$(\alpha_*^1, \alpha_*^2) = ((x_*^1, 1 - x_*^1), (x_*^2, 1 - x_*^2)) \in \Delta^1 \times \Delta^2 \subset \mathbf{R}^2 \times \mathbf{R}^2.$$

The expected payoff $P^i(x_*)$ to player i, given that both players play these mixed strategies, is given by the formula

$$P^i(x_*) = V^i_{11}x^1_*x^2_* + V^i_{12}x^1_*(1-x^2_*) + V^i_{21}(1-x^1_*)x^2_* + V^i_{22}(1-x^1_*)(1-x^2_*) \tag{25}$$

in interval coordinates, and by

$$P^i(\alpha_*) = \sum_{m,l=1,2} V^i_{ml}(\alpha^1_*)_m(\alpha^2_*)_l). \tag{26}$$

in simplicial coordinates.

Consider now the further extension of the extended game to the stochastic sequence of infinitely repeated FK-games driven by fictitious play, with noise parameter ε. The *cumulative average payoff* to player i at time k is defined, using simplicial coordinates, to be

$$P^i_k(\varepsilon) = \frac{1}{k}\sum_{j=1}^{k} \mathbf{U}^i_j[a^i_j, a^{-i}_j].$$

If $P^i_k(\varepsilon)$ converges as $k \to \infty$, the limit can be regarded as the long run average payoff to player i in the infinitely repeated game. More generally, the closed interval

$$[\liminf_{k\to\infty} P^i_k(\varepsilon),\ \limsup_{k\to\infty} P^i_k(\varepsilon)]$$

gives the *essential range* of player i's long run average payoffs, for fixed noise parameter $\varepsilon > 0$.

Theorem 2.8 *Let ? (0) be a generic 2×2 game, and assume Hypothesis 2.7 for the one-parameter family of adaptive FK-games $\{?(\varepsilon)\}$. Then:*

(i) *If $\varepsilon > 0$ is fixed at a sufficiently small value, the sequence of empirical frequencies of ? (ε) converges almost surely to a Nash distribution equilibrium $x_*(\varepsilon)$ of ? (ε).*

(ii) *As ε goes to zero, $x_*(\varepsilon)$ converges to a Nash equilibrium $x_*(0)$ of ? (0).*

(iii) *The essential range of long run average payoffs to each player i reduces to the expected payoff at $x_*(0)$ for ? (0), as $\varepsilon \to 0$:*

$$\lim_{\varepsilon\to 0}\liminf_{k\to\infty} P^i_k(\varepsilon) = \lim_{\varepsilon\to 0}\limsup_{k\to\infty} P^i_k(\varepsilon) = P^i(x_*(0)).$$

(iv) *If ? (0) admits a pure Nash equilibrium, then $x_*(0)$ in (ii) is necessarily a pure Nash equilibrium. If there are two pure Nash equilibria (I) and (II) for ? (0), each has a positive probability to be selected as $x_*(0)$ in* **(ii)**. *That is:*

$$\begin{aligned} 0 &< \mathsf{P}\{x_*(0) = (I)\} \\ &= 1 - \mathsf{P}\{x_*(0) = (II)\} \\ &< 1. \end{aligned}$$

The proof of this theorem is based on Theorems 2.2 and 2.6. As it is rather long and technical, although elementary, we have postponed it to Section 6.

Remark 2.9 We do not know if the full strength of Hypothesis 2.7 is needed for Theorem 2.8. If we weaken part (ii) of the hypothesis by assuming only that the densities f^i are integrable, then the first part of the proof goes through to show: *If ε is sufficiently small, then ? (ε) has a linearly stable Nash distribution equilibrium arbitrarily near any given pure Nash equilibrium of ? (0).*

Remark 2.10 Results similar to parts (i) and (ii) of Theorem 2.8 were obtained by Kaniovski and Young (1995) under different assumptions on the noise matrices $\mathbf{E}^i$: Instead of Hypothesis 2.7, they assume the 8 matrix entries to be independent and normally distributed with variance ϵ. Part (iii) appears to be new. Part (iv) may be well known to urn theorists.

Equilibrium Selection, Stability and Harsanyi's Purification

At this stage it is interesting to compare our results with Harsanyi's justification of mixed equilibria. For this purpose let ? (0) be a 2×2 game which is generic (see Section 2) and admits three Nash equilibria (I), (II) and (III), with (I) and (II) pure and (III) mixed.

As predicted by Harsanyi's theory (1973), all three equilibria of ? (0) can be approximated by distribution equilibria of ? (ε) as ε goes to zero. In particular, the mixed equilibrium (III) is the limit of a Nash distribution equilibrium $(III)_\varepsilon$. For the unperturbed game, (III) is *unstable* in the sense that a player can make small deviations from this equilibrium without loss.

An important conclusion of Harsanyi's theory is that the randomness introduced in the game *stabilises* this equilibrium, in the following sense. Suppose (in the notation of Hypothesis 2.7(i)) that the matrices V^1, V^2, the parameter ε and the probability distribution of $\mathbf{E^1}$ and $\mathbf{E^2}$ are common knowledge. Assume that player $-i$ plays the mixed strategy given by the Nash equilibrium whose distribution is $(III)_\varepsilon$. Then, almost surely, at each round of the game, the best response of player i is *uniquely determined*, and player i cannot deviate from it without penalty.

An alternative interpretation, more closely related to our context, is that only one player computes and plays this equilibrium, while the other player simply follows the behavior rule of fictitious play. It can be shown in this case that the empirical frequencies of both players will converge with probability one to $(III)_\varepsilon$.

On the other hand, suppose players know neither opponents' payoff matrices, nor the distributions of their own matrices, and let both players adapt their strategies by fictitious play— the adaptive FK-game. Then according to Theorem 2.8(iv), almost surely the sequence of empirical frequencies will *not* converge to $(III)_\varepsilon$; in this context $(III)_\varepsilon$ is unstable. Hence we see that there is a considerable difference between stability of a Nash equilibrium in Harsanyi's context and in ours.

The key point is that these two notions of stability correspond to different scenarios, based on very different degrees of knowledge. For Harsanyi it is implicit and *essential* that each player knows the mixed strategy played by his opponent and chooses his own accordingly. (Perhaps an outside mediator has informed the players of the strategy profile whose distribution is $(III)_\varepsilon$.) In the repeated FK game, in contrast, neither player acts on any *a priori* knowledge of the opponent's strategies. Instead, each player predicts—generally in error!— that the opponent plays the mixed strategy based on the opponent's past empirical frequencies of actions, and the player then chooses his own pure strategy accordingly.

Theorem 2.2 means that these predictions become self-fulfilling prophecies as players gradually obtain knowledge of each other's play and empirical frequencies of actions converge; and Theorem 2.8 shows that mixed equilibria of the unperturbed game are unstable.

Payoff Dominance, Risk-Dominance and Path Dependence

Suppose that ?(0) is the symmetric coordination game given by the (constant) matrices

$$V_1 = V_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{27}$$

with $a > c, d > b$. As a normalization we also suppose $a \geq b$. This game admits three (necessarily symmetric) Nash equilibria (p_*, p_*), which we denote by (I), (II) and (III) as follows:

$$\begin{array}{ll} (I) & p(I)_* = 1, \\ (II) & p(II)_* = 0, \\ (III) & p(III)_* = \Theta/(1+\Theta), \end{array}$$

where $\Theta = \dfrac{d-b}{a-c}$. Equilibria (I) and (II) are pure and (III) is mixed. The payoffs at (I), (II), (III) are respectively

$$a,\ d,\ \text{and}\ \frac{a\Theta^2 + (c+b)\Theta + d}{(1+\Theta)^2}.$$

From these expressions it is easily seen that (I) Pareto dominates (II) and (III). Thus, if the players could coordinate they would certainly choose to coordinate on (I). However in absence of coordination, the riskiness of (I) relative to (II) is relevant and can lead the players to choose other strategies.

Harsanyi and Selten (1988) say that (I) *risk-dominates* (II) if (I) is associated with the largest product of deviation losses, that is if $a - c > d - b$. Similarly (II) risk-dominates (I) if $a - c < d - b$. In the latter situation there is conflict between payoff dominance and risk-dominance and it is not obvious that a Nash equilibrium will be played.

Consider now a parameterized family ?(ε) of adaptive 2×2 FK-games. Theorem 2.8 (iii) shows that players are led to coordinate on a Nash equilibrium whose distribution is close (for small noise) to a pure equilibrium of the unperturbed game, and *both* equilibria $(I)_\varepsilon$ and $(II)_\varepsilon$ have a positive probability to be selected. Therefore, while $(I)_\varepsilon$ is the more

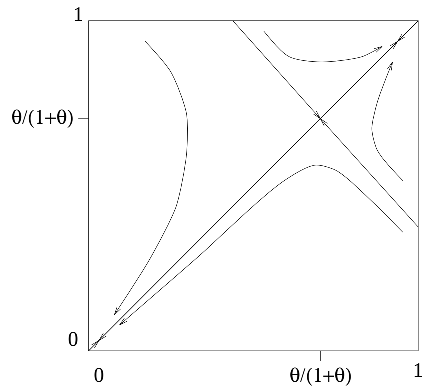

Figure 2: The phase portrait of a vector field corresponding to a small perturbation $\Gamma(\varepsilon)$ of a symmetric coordination game $\Gamma(0)$ with $\Theta > 1$ (i.e (II) risk-dominates (I)). The equilibrium with the larger basin of attraction is close to the risk-dominant equilibrium of the unperturbed game.

efficient equilibrium, it can happen that the state sequence converge toward the less efficient equilibrium $(II)_\varepsilon$.

The economic phenomenon captured by this last result is usually called *path dependence* and has been largely discussed in the literature on economic history (Arthur, (1989)).

Contrasting path selection is presented by models considered by Young (1993), Kandori *et al.* (1993) or Ellison (1993): noise and myopic responses by bounded rational players lead to selection of the risk-dominant equilibrium. In these models, players selected from a finite population are repeatedly matched. They adapt their strategies according to a deterministic rule based on the current strategy distribution of the population; but they also can always deviate from this rule and play any arbitrary strategy with a small probability controlled by a *mutation parameter* ε.

There are several differences between the process considered in this paper and the models studied by these authors. Without going into details, we point out the fundamental difference that our model is a nonstationary Markov chain, while theirs is stationary. Therefore, under natural assumption, their underlying process is ergodic, meaning that the current probability distribution converges toward an invariant measure $\lambda(\varepsilon)$ regardless of the initial conditions.[5] By using a characterisation of such invariant measures due to Freidlin and Wentzel (1984), the authors cited above show that as ε goes to zero, the invariant measure $\lambda(\varepsilon)$ tends to concentrate on the Dirac measure supported at the risk-dominant equilibrium.

In our process, each equilibrium $(I)_\varepsilon$ and $(II)_\varepsilon$ has a positive probability to be selected. This gives some *plasticity* to the process; in the beginning of the play the state sequence $\{x_k\}$ behaves roughly as an ergodic process in the sense that it has a nonnegligeable probability to visit abitrary small neighborhoods of both equilibria. However, since players take into account the entire past in adapting their strategies, the effect of a new information tends

[5] As pointed out by Ellison (1993), this property is meaningful only if the limiting measure is reached in a reasonable amount of time.

to vanishe in the long run, and the process eventually homes in to one of the two equilibria. One cannot predict which equilibrium will be selected since this depends on the initial state and the particular sequence of payoff perturbations.

We conjecture, however, that the risk-dominant equilibrium has a larger probability to be selected than the other equilibrium. The intuition behind this conjecture is that for the deterministic dynamical system (18), the risk-dominant equilibrium corresponds to the equilibrium of (18) having the larger basin of attraction (see Figure 2).

3 Continuous Time Dynamics Arising From Fictitious Play

This section introduces the mathematical basis for our analysis of adaptive FK-games, based on the dynamics of the game vector field (15):

$$\begin{aligned} F : S &\to TS, \\ F(x) &= -x + \nu(x) \end{aligned}$$

and the corresponding game differential equation on S:

$$\frac{dx}{dt} = F(x). \tag{28}$$

The Limit Set Theorem

Throughout the remainder of this section we assume:

Hypothesis 3.1 *The game vector field F is locally Lipschitz.*

Next we introduce tools enabling us to analyze game asymptotics in terms of the dynamics of F.

The *Limit Set Theorem* given below describes the state limit set $L\{x_k\}$ in terms of the dynamics of the game vector field F. The point of this result is that for various types of games, it leads to much information about the probable location and shape of the limit set L of sample paths of the repeated game. As we will show, for some games L must be (with probability one) a stable equilibrium; for others L must be contained in a rather small attractor approximating a point; for still others there is a positive probability that L is a limit cycle.

Denoting the dimension of S by

$$n = \sum_1^\mu (d_i - 1) = d - \mu,$$

we identify S with a compact convex subset of $\mathbf{R}^n$ having nonempty interior, and TS with $\mathbf{R}^n$; for convenience we assume the origin belongs to the interior of S. Under this identification, the game vector field is a Lipschitz map

$$F : S \to \mathbf{R}^n.$$

It is convenient to extend F to a Lipschitz map defined on all of $\mathbf{R}^n$. This is possible by standard theory; an explicit construction is to define $F(x)$, for x outside S, to be $F(\rho(x))$ where $\rho : \mathbf{R}^n \to S$ is the retraction along rays emanating from the origin. Notice that this makes $F : \mathbf{R}^n \to \mathbf{R}^n$ a bounded Lipschitz map.

It follows that F is *completely integrable*, meaning that its trajectories are defined for all values of t. Therefore F generates a *flow*

$$\Phi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n,$$

where for each $y \in \mathbf{R}^n$, the function $t \mapsto \Phi_t(y)$ is the solution to the initial value problem

$$\frac{dx}{dt} = F(x) \tag{29}$$

$$x(0) = y. \tag{30}$$

The parameterized curve $t \mapsto \Phi_t(y)$ is the *trajectory* of y; the image of this curve is the *orbit* of y.

For each fixed $t \in \mathbf{R}$, the map $y \mapsto \Phi_t y$ is a homeomorphism of $\mathbf{R}^n$. We view the flow as the collection of maps $\{\Phi_t : \mathbf{R}^n \to \mathbf{R}^n\}_{t \in \mathbf{R}}$, with Φ_0 denoting the identity map of $\mathbf{R}^n$. We have the composition law $\Phi_s \circ \Phi_t = \Phi_{s+t}$.

An *equilibrium* (or stationary point) p is a zero of F; this is equivalent, by uniqueness of solutions, to $\Phi_t(p) = p$ for all t. We call a point y *periodic* if $\Phi_T(y) = y$ for some $T > 0$. The *limit set* (more properly, the omega limit set) of y (and of its orbit and trajectory) is the set of points of the form $\lim_{k\to\infty} \Phi_{t_k}(y)$ for some sequence $t_k \to \infty$.

An *invariant set* for F is a set $Q \subset \mathbf{R}^n$ such that $\Phi_t(Q) = Q$ for all t. Equilibria and periodic orbits are invariant sets; more generally, limit sets of orbits are invariant.

For any invariant set Q we denote by $\Phi|Q$ the *restriction* of the flow Φ to Q, that is, the collection of maps $\Phi_t|Q : Q \to Q$ obtained by restricting each Φ_t to Q.

Let Q denote a compact invariant set. A subset K of Q is called an *attractor* for $\Phi|Q$ provided K is nonempty, compact and invariant, and there is an neighborhood $U \subset Q$ of K with the property that $\lim_{t\to\infty} \mathrm{dist}(\Phi_t x, K) = 0$ uniformly for $x \in U$. Here $\mathrm{dist}(a, K)$ means the distance from a to the nearest point of K. Speaking loosely, we say that an attractor captures the orbits of nearby points.

An asymptotically stable limit cycle or equilibrium is an example of an attractor. The whole space Q is, trivially, an attractor; any other attractor is a *proper* attractor.

The *basin* of an attractor K is the set of all points whose trajectories tend to K. If the basin of K is all of Q then K is a *global* attractor.

We call Q *attractor-free* if Q is a nonempty compact invariant set that contains no proper attractor. Many compact invariant sets are known to be attractor-free, such as periodic orbits and limit sets of trajectories. The closure of the union of any collection of attractor-free subsets of Q is attractor-free, as is the intersection of a nested collection. If the flow is ergodic for a Borel measure supported everywhere in Q, then Q is attractor-free. If $\Phi_t y \to p \neq y$ as $t \to \pm\infty$ then the closure of the orbit of y (called a homoclinic loop) is attractor-free.

The interplay between attractors and attractor-free sets is very useful in analyzing long-term dynamical behavior. In the following simple but useful result we consider the basic dynamical system to be $\Phi|Q$. It says that an attractor-free compact invariant set is contained in every attractor whose basin it meets:

Lemma 3.2 *Let Q and $\Lambda \subset Q$ be compact invariant sets for the flow Φ, and assume $\Phi|\Lambda$ is attractor-free. Then if Λ meets the basin of an attractor A for $\Phi|Q$, it follows that $\Lambda \subset A$.*

Proof Let $x \in \Lambda$ be a point in the basin of A. Since the trajectory of x has limit points in A, and Λ is closed, limit points lies in Λ. Thus $\Lambda \cap A$ is a nonempty compact invariant set, and it is an attractor for the flow in Λ. Being attractor-free, Λ therefore coincides with $\Lambda \cap A$. **QED**

The following theorem, the mathematical basis for our results, concerns the game vector field F of an adaptive FK-game. Recall that $L\{x_k\}$ denotes the state limit set.

Theorem 3.3 (Limit Set Theorem) *With probability one, the state limit set $L\{x_k\}$ has the following properties:*

(a) *$L\{x_k\}$ is an invariant set for the flow of the game vector field F.*

(b) *$L\{x_k\}$ is compact, connected and attractor-free.*

From Lemma 3.2 we obtain a useful corollary:

Corollary 3.4 *With probability one, the state limit set is contained in every attractor whose basin it meets. In particular it is contained in every global attractor.*

The proof of Theorem 3.3 is based on the following recursion relation

$$x_{k+1} - x_k = \frac{1}{k+1}[F(x_k) + Z_{k+1}], \tag{31}$$

where $\{Z_{k+1}\}_{k \in \mathbf{N}}$ is a sequence of random variables defined by (31), that is,

$$Z_{k+1} = (k+1)(x_{k+1} - x_k) - F(x_k).$$

Lemma 3.5 *The processes $\{x_k\}, \{Z_{k+1}\}$ in Equation (31) satisfy the following conditions:*

(i) *The vector field F is locally Lipschitz.*

(ii) *There exists $R > 0$ such that $||x_k|| < R$, $||Z_{k+1}|| < R$ for all $k = 0, 1, 2, \ldots$*

(iii)

$$\mathsf{E}(Z_{k+1} \,|x_k) = 0.$$

Proof Conditions (i) is Hypothesis 3.1, while (ii) and (iii) easily follow from equation (14). **QED**

Proof of Theorem 3.3. A recursion such as (31) is a particular form of a *stochastic approximation process*, and Theorem 3.3 follows from a general result[6] proved in (Benaïm, 1996) (see also (Benaïm and Hirsch (1996)), concerning the asymptotic behavior of stochastic approximation processes satisfying Lemma 3.5. **QED**

4 Applications of the Limit Set Theorem

Correlated Strategies and Average Payoffs

Theorem 3.3 give us valuable information on the asymptotic behavior of the empirical frequencies of actions played by *each* player. But there are other interesting questions, such as:

(a) What is the long term behavior of the joint empirical frequencies of action profiles?[7]

(b) Where do the payoffs of the infinitely repeated game tend to cluster?

To address such questions, we consider more generally an arbitrary function $H : A \to \mathbf{R}^m$. After round k, this function is evaluated on the current action profile $a_k \in A$; in this way we obtain a stochastic process $\{H(a_k)\}$. The empirical frequency of $\{H(a_k)\}$ is the vector

$$\langle H \rangle_k = \frac{1}{k} \sum_{j=1}^{k} H(a_j).$$

The limit set of the sequence $\{\langle H \rangle_k\}$ is denoted by $L[H]$.

The following result uses the machinery of the Limit Set Theorem to estimate the location of $L[H]$:

Theorem 4.1 *Let*

$$H : A \to \mathbf{R}^m$$

be such that the map $\overline{H} : S \to \mathbf{R}^m$ *is Lipschitz, where*

$$\overline{H}(x) = \mathsf{E}(H(a_{k+1})|x_k = x) = \sum_{a \in A} H(a)\hat{\nu}(x)_a$$

and $\nu(x)_a$ *is given by definition 1.5. Then the limit set of the sequence* $\{\langle H \rangle_k\}$ *is almost surely a compact connected subset of the closed convex hull of* $\overline{H}(L(\{x_k\}))$.

[6] This result is stated in terms of chain recurrent sets rather than attractor-free sets, but the two notions are equivalent by a theorem of Conley (1978).)

[7] Even though the players act independently, it can happen that the state of Nature correlates their strategies.

Before proving Theorem 4.1 we make some remarks, and apply the theorem to some specific functions H:

Remark 4.2

(a) Observe first of all that when the state sequence converges almost surely toward an equilibrium x_* (for example, as in Theorem 2.2(b)), then Theorem 4.1 implies that $\langle H\rangle_n$ converges to $\overline{H}(x_*)$.

(b) Consider the function

$$H : A \to \mathbf{R}^A,$$
$$(a^1, \ldots, a^\mu) \mapsto \mathbf{a}^1 \otimes \cdots \otimes \mathbf{a}^\mu.$$

In this case $\langle H\rangle_k$ is just the empirical joint frequency tensor C_k (see Equation (10)) and Theorem 4.1 implies that the limit set of the sequence $\{C_k\}$ is almost surely a compact connected set contained in the convex hull of $\hat{\nu}(L\{x_k\})$, where $\hat{\nu}$ is the joint best response map (Equation (5)). Therefore in case $\{x_k\}$ converges almost surely, we can conclude that $\{C_k\}$ converges to $\hat{\nu}(\lim_{k\to\infty} x_k)$.

(c) Now consider the function

$$H : A\times \to \mathbf{R},$$
$$a \mapsto \mathbf{U}^i[a^i, a^{-i}].$$

Then $\{\langle H\rangle_k\}$ is the sequence of cumulative average payoffs to player 1 at time k, and Theorem 4.1 characterizes its limit set.

Proof of Theorem 4.1 We define a new repeated game, the *cascaded game*, having one additional *silent* player. This $(\mu+1)th$ player takes no actions, or rather, always takes the same action. At time k the silent player's game state is $u_k = \langle H\rangle_k \in \mathbf{R}^m$. Thus the state sequence of the cascaded game is

$$\{(x_k, u_k) \in \mathbf{R}^n \times \mathbf{R}^m\}_{k\in\mathbf{N}}.$$

Let F denote the game vector field of the orginal game. Then the game vector field for the cascaded game, on $\mathbf{R}^n \times \mathbf{R}^m$, is easily worked out to give the tame differential equation

$$\frac{dx}{dt} = F(x), \quad \frac{du}{dt} = -u + \overline{H}(x). \tag{32}$$

Notice this system is in cascade form, that is, the evolution of $x(t)$ is independent of u.

Let $L \subset \mathbf{R}^n$ denote the limit set of $\{x_k\}$, and let $L' \subset L \times \mathbf{R}^m$ denote the limit set of the sequence $\{(x_k, u_k)\}$.

Since L is invariant under F, it is clear that $L \times \mathbf{R}^m$ is invariant under system (32).

Let Φ denote the flow of (32) and let $C \subset \mathbf{R}^m$ denote the closed convex hull of $\overline{H}(L)$. We claim *$L \times C$ is a global attractor for $\Phi|(L\times\mathbf{R}^m)$.*

Suppose this claim is true. Then $L' \subset L \times C$ by Lemma 3.2. This implies the conclusion of Theorem 4.1.

We now pass to the proof of the claim. Let $V : L \times \mathbf{R}^m \to \mathbf{R}_+$ be the map defined by $V(x, u) = \mathrm{dist}(u, C)$. By convexity of C, we have

$$\begin{aligned} \mathrm{dist}((1-t)u + t\overline{H}(x), C) &\leq (1-t)\mathrm{dist}(u, C) + t\mathrm{dist}(\overline{H}(x), C) \\ &= (1-t)\mathrm{dist}(u, C), \end{aligned}$$

where the last equality holds because $\overline{H}(x) \in C$. Thus

$$\frac{\mathrm{dist}(u + t(-u + \overline{H}(x)), C) - \mathrm{dist}(u, C)}{t} \leq -\mathrm{dist}(u, C).$$

Now the map $V(x, u)$ being convex and continuous because C is convex, so it admits a right partial derivative with respect to u. Therefore letting t goes to zero in the last inequality gives

$$\frac{d}{dt} V(\Phi_t(x, u)) \leq -V(\Phi_t(x, u)),$$

whence $V(\Phi_t(x, u)) \leq e^{-t} V((x, u))$ by a standard theorem in differential inequalities. This implies that $L \times C$ is a global attractor for $\Phi|(L \times \mathbf{R}^m)$. **QED**

Proof of Theorem 2.2

Let $\mathcal{E} \subset S$ denote the set of Nash distribution equilibria, which are the equilibria (zeroes) of F.

The proof is based on the following:

Lemma 4.3 *The flow Φ of the game vector field is area-decreasing.*

Proof Here $n = 2$. As usual we identify the state space with $I \times I$. Consider first the case where the case where F and hence the Nash map ν are C^1 (continuously differentiable). Then the game vector field

$$F : I \times I \to \mathbf{R}^2, \quad F(x) = -x + \nu(x)$$

has negative divergence. For by definition of the Nash map (Equation (3)),

$$F^i(x^1, x^2) = -x^i + \beta^i(x^{-i}), \quad i = 1, 2, \tag{33}$$

whence the divergence of F at x is

$$\frac{\partial F^1}{\partial x^1} + \frac{\partial F^2}{\partial x^2} = -2.$$

Therefore Liouville's formula (Hartman, 1964) shows

$$\frac{d}{dt}\mathrm{Det}(D\Phi_t(x)) = -2,$$

showing that Φ_t decreases areas for $t > 0$. The case where F is merely Lipschitz follows by approximating each component β^i by a C^1 map $I \to I$, and applying standard continuity theorems in differential equations. **QED**

Corollary 4.4 *No compact set invariant under Φ can separate the plane.*

Proof Suppose a compact invariant set $L \subset I \times I$ separates the plane. Its complement has at least one bounded connected component A. Then A is an open invariant set contained in $I \times I$, hence A has the same area as $\Phi_t A = A$, contradicting Lemma 4.3. **QED**

Proof of Theorem 2.2. We first prove that the limit set Λ of a state sequence $\{x_k\}$ is almost surely contained in $\mathcal{E}$.

By Theorem 3.3, Λ is almost surely a compact, connected attractor free invariant set. Consider the dimension $d \in \{0, 1, 2\}$ of Λ (see Hurewicz and Wallman (1948) for dimension theory). If $d = 1$ we apply a result of (Hirsch and Pugh, 1988), implying that if a 1-dimensional attractor-free set for a planar flow contains a nonstationary point, the set must separate the plane. Since we have seen that Λ cannot separate the plane, Λ consists entirely of stationary points. If Λ is 2-dimensional then it has nonempty interior, then the boundary of its interior contains an invariant 1-dimensional continuum which separates the plane, leading to a similar contradiction. If Λ is 0-dimensional then, being connected, it is a singleton (because a 0-dimensional set is totally disconnected); hence Λ is a singleton, necessarily an equilibrium by invariance. Therefore in every case Λ is a connected subset of the stationary set, proving part (i).

If the compact connected set Λ is finite or countably infinite, then it reduces to a singleton $p \in \mathcal{E}$.

Suppose Λ is not a singleton. By Equation (19), the set of equilibria is the intersection of two graphs:

$$\mathcal{E} = \{(x^2, x^1) : x^2 = \beta^2(x^1)\} \bigcap \{(x^1, x^2) : x^1 = \beta^1(x^2)\}.$$

Therefore the projection $I \times I \to I$ on the first (or second) factor maps Λ homeomorphically onto a compact connected subset of I. Therefore Λ is homeomorphic either to a point or a compact interval. This concludes the proof of assertions (a) and (b) of Theorem 2.2. Assertion (c) follows Theorem 4.1 and Remark 4.2 (b). **QED**

Equilibrium Selection, Local Stability, and Path Dependence

Even if we know that the final outcome of the game has to be an equilibrium, it is not obvious which equilibrium will be selected. This section addresses this important question of *equilibrium selection* and the related problem of *path dependence*.

We recall some standard terms from dynamical systems. A map is C^r, $r \geq 1$ if it is differentiable and its partial derivatives up to order r are continuous; and C^0 means continuous. The game vector field $F : S \to TS$ is C^r provided the perturbation matrices $\mathbf{E^i}$ in Equation (1) have C^{r-1} densities. Assume F is at least C^1. Let $x_* \in S$ be an equilibrium of F If the Jacobian matrix $DF(x_*)$ is invertible, x_* is called *simple*. If all eigenvalues of the Jacobian matrix $DF(x_*)$ have nonzero real parts, x_* is called *hyperbolic*. If all eigenvalues have negative real parts, x_* is *linearly stable*, while if some eigenvalue has positive real part x_* is *linearly unstable*.

Equilibrium x_* of F is called *asymptotically stable* if there exists a neighborhood $U \subset S$ of $x*$ such that $\lim_{t\to\infty} \Phi_t(y) = x_*$ uniformly in $y \in U$, where Φ denotes the flow of F (see Equations (29), (30)). In particular, linear stable equilibria are asymptotically stable.

Let $x \in S$ be a game state for an adaptive FK-game with any number μ of players. We say the game is *diffuse at x* if whenever the game state is x, every action profile a has a positive probability of being selected at the next play: For all $a \in A$,

$$\hat{\nu}(x)_a > 0.$$

If this holds for all $x \in S$, we say the game is *diffuse*.

The following theorem shows that unstable equilibria of the game vector field are eliminated as outcomes of diffuse adaptive FK-games with C^2 game vector fields:

Theorem 4.5 *Let $x_* \in S$ be a linearly unstable equilibrium of F. If F is C^2 in a neighborhood of x_*, and the game is diffuse at x_*, then:*

$$\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} = 0.$$

Proof We derive Theorem 4.5 from the following useful result due to Pemantle (1990):

Theorem 4.6 (Pemantle) *Consider the stochastic approximation process (31) in $\mathbf{R}^n$:*

$$x_{k+1} - x_k = \frac{1}{k+1}[F(x_k) + Z_{k+1}]$$

where the sequence $\{Z_{k+1}\}$ of $\mathbf{R}^n$-valued random variables is a priori *bounded with zero conditional expectations. Let x_* be a linearly unstable equilibrium of F. Assume F is C^2 in a neighborhood of x_*, and that there exists $c > 0$ and a neighborhood N of x_* such that for every unit vector $\Theta \in \mathbf{R}^n$, the following condition holds:*

$$\mathsf{E}(\max(0, \langle Z_{k+1}, \Theta\rangle) \mid x_k \in N) > c. \tag{34}$$

Then

$$\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} = 0.$$

Using the joint best response map, the game vector field can now be expressed as

$$F(x) = \sum_{a \in A} \hat{\nu}(x)_a \mathbf{a} - x,$$

by Equations (13) and (15). Since $F(x_*) = 0$ we have

$$x_* = \sum_{a \in A} \hat{\nu}(x_*)_a \, \mathbf{a}. \tag{35}$$

This exhibits x_* as a convex combination of all the extreme points a of S with strictly positive coefficients. Considering S as a convex body in $\mathbf{R}^n$, we have proved that *the diffuse equilibrium x_* is in the interior of S.*

We use this to verify Pemantle's hypothesis (34). The function of $x \in S$ defined as

$$\mathsf{E}(\max(0, \langle Z_{k+1}, \Theta\rangle) \mid x_k = x)$$

is independent of k and continuous in x. It therefore suffices to show:

$$\mathsf{E}(\max(0, \langle Z_{k+1}, \Theta\rangle) \mid x_k = x_*) > 0. \tag{36}$$

From equation (14) we have

$$\begin{aligned} Z_{k+1} &= \mathbf{a}_{k+1} - \mathsf{E}(\mathbf{a}_{k+1} \mid x_k) \\ &= \mathbf{a}_{k+1} - \sum_{a \in A} \hat{\nu}(x_k)_a \mathbf{a}. \end{aligned} \tag{37}$$

Fix a unit vector $\Theta \in TS$. Let A_+ denote the set of extreme points $a \in A$ for which

$$\langle \mathbf{a} - x_*, \Theta\rangle > 0.$$

Then A_+ is nonempty: for from Equation (35) and the identity $\sum_{a \in A} \hat{\nu}_a \equiv 1$ we get:

$$\sum_{a \in A} \langle \mathbf{a} - x_*, \Theta\rangle = \sum_{a \in A} \langle \hat{\nu}(x_*)_a (\mathbf{a} - \mathbf{a}), \Theta\rangle = 0.$$

From the definition (7) of TS there exists $a \in A$ such that $\langle a - x_*, \Theta\rangle \neq 0$, so the last equation implies A_+ is nonempty. We therefore have from (37):

$$\begin{aligned} \mathsf{E}(\max(0, \langle Z_{k+1}, \Theta\rangle) \mid x_k = x_*) &= \mathsf{E}(\max(0, \langle \mathbf{a}_{k+1} - \sum_{a \in A} \Xi_a(x_*)\mathbf{a}, \Theta\rangle \mid x_k = x_*) \\ &= \mathsf{E}(\max(0, \langle \mathbf{a}_{k+1} - x_*, \Theta\rangle \mid x_k = x_*) \quad (\text{by} \quad (35)), \\ &= \sum_{a \in A} \hat{\nu}(x_*)_a \max(0, \langle \mathbf{a} - x_*, \Theta\rangle) \\ &= \sum_{a \in A_+} \hat{\nu}(x_*)_a \langle \mathbf{a} - x_*, \Theta\rangle > 0. \end{aligned}$$

This verifies (36), and Theorem 4.5 follows from Pemantle's theorem. **QED**

The assumption that the game vector field is twice continuously differentiable is not always satisfied; it is used in existing proofs of Pemantle's theorem, but may be unnecessary there. Theorem 4.5 does not need the full generality of Pemantle's result, as it deals with a Markov process. There are in fact earlier results similar to Pemantle's, for more restricted classes of stochastic processes, which do not assume C^2 vector fields; but we do not find them entirely satisfactory for present purposes.

Theorem 4.7 *Consider a diffuse adaptive FK-game with any number of players, having state space S. For every attractor $A \subset S$ of the game vector field,*

$$\mathbf{P}\{\lim_{k\to\infty} \mathrm{dist}\,(\mathrm{x_k}, \mathrm{A}) = 0\} > 0.$$

In particular, if $x_* \in S$ is an asymptotically stable equilibrium then

$$\mathbf{P}\{\lim_{k\to\infty} x_k = x_*\} > 0.$$

For asymptotically stable equilibria, the proof is a consequence of a more or less well known general result for urn processes that exploits the countable cardinality of the state space (Arthur *et al.* (1987), Benaïm and Hirsch (1995a)). For general attractors the proof easily follows from Theorem 6.3 of Benaïm (1997). This result shows that for a diffuse adaptive FK game admitting several asymptotically stable equilibria, even if a particular equilibrium is more efficient than the others, at every stage there is positive probability that the state sequence converges toward a less efficient equilibrium. The coordination games considered in Section 2 is a good illustration of this phenomenon.

The following result shows that the probable action sequences in diffuse games are somewhat restricted. It implies that even if such rules entail convergence to a clearly optimal equilbrium, the players will occasionally, but infinitely often, play the worst possible actions.

Proposition 4.8 *Consider an adaptive FK-game with any number of players. Let x_* denote a zero of the game vector field. Suppose that x_* is a diffuse state, and that*

$$\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} > 0.$$

Then the conditional probability that every action profile $a \in A$ is played infinitely often, given that $x_k \to x_*$, is 1.

Proof Let $\hat{\nu}(x_*)_a = c > 0$. Let $\mathcal{F}_k$ denote the sigma field generated by $a^1, a^2, \ldots a^k$. By the generalized Borel-Cantelli lemma (Doob (1953), p.324) the following two sets of events coincide except for a set of measure zero:

$$\{a_k = a \text{ infinitely often}\},$$

and

$$\{\sum_k \mathsf{P}\{a_{k+1} = a \,|\mathcal{F}_k\} = \infty\}.$$

Observe also that:

$$\{\sum_k \mathsf{P}\{a_{k+1} = a \,|\mathcal{F}_k\} = \infty\} \;=\; \{\sum_k \hat{\nu}(x_k)_a\} = \infty\} \supset \{x_k \to x_*\}$$

where the last inclusion stands because $\Xi(x_n, a)$ converges toward $c > 0$ on the set of events $\{x_k \to x_*\}$. Therefore, modulo sets of measure zero:

$$\{x_k \to x_*\} \;\subset\; \{a_k = a \text{ infinitely often}\}.$$

QED

5 Beyond 2 × 2 Adaptive FK-Games

In this section we consider some adaptive FK games with more than 2 players.

First we treat an adaptive FK version of Jordan's 3 player matching game. It turns out that convergence depends on the noise, in quantifiable ways. For sufficiently concentrated noise (e.g., low variance if the noise is Gaussian), almost surely sample paths do not converge to the unique Nash distribution equilibrium; and for $n = 3$, almost surely sample paths cluster at a periodic orbit of the game vector field. For sufficiently diffuse noise, on the other hand, sample paths almost surely converge.

In the subsequent subsection we describe a larger class of $n \times 3$ generalized coordination games, and identify a subfamily where convergence is guaranteed.

Jordan's Nonconvergent Matching Game

It is known since Shapley (1964) that fictitious play in a deterministic context fails to converge for a family of 3×2 games (3 players, each with 2 pure strategies). Jordan (1993) exhibited a simple three-player game with the same property. Cowan (1992) gave examples of deterministic fictitious play in 2-player, 4-strategy games having rigorously proved "chaotic" behavior.

Following Jordan, we describe here a family of n-player two-strategy games with a unique Nash equilibrium that includes his example. The unperturbed game is the n-players version of the matching pennies game considered by Jordan (1993) where $n \geq 3$ is an arbitrary number. There are n players choosing among two strategies (i.e. $A^i = \{1, 2\}$). Players are labeled modulo n (i.e., player $n+1$ =player 1). Player $i \in \{1, \ldots, n-1\}$ is rewarded for matching player $i+1$, that is, making the same action choice, but player n is rewarded for not matching player 1.

The payoffs are as follows. Let V denote the matrix:

$$V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For $i = 1, \ldots, n-1$, the payoff to player i is given by $V^i = V$: if players $i, i+1$ respectively play $k, l \in \{1, 2\}$ then the payoff to i is V^i_{kl}. Thus player i tries to match player $i+1$. The payoff matrix for player n is

$$V^n = -V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};$$

player n tries *not* to match player 1.

Consider now the infinitely repeated adaptive game defined by payoff noisy matrices

$$V^i_k = V^i + \mathbf{E}^i_k,\ i = 1, \ldots, n$$

where $\{\mathbf{E}^i_k\}_{k\in\mathbf{N}_+}$ is an IID sequence of random 2×2 matrices with zero means, having the form

$$\mathbf{E}^i_k = \begin{bmatrix} \eta^i_{1,k} & \eta^i_{1,k} \\ \eta^i_{2,k} & \eta^i_{2,k} \end{bmatrix}. \tag{38}$$

We assume the rows of $\mathbf{E}^i_k$ have probability distributions given by smooth (C^1) densities on $\mathbf{R}^2$. Then for each (i, k) the random variable $\eta^i_{2,k} - \eta^i_{1,k}$ also has mean zero, and has a smooth strictly positive density function $f^i : \mathbf{R} \to \mathbf{R}_+$. We denote its probability distribution function by

$$K^i : \mathbf{R} \to [-0, 1],$$

$$K^i(x) = \int_{-\infty}^{x} f^i(u)du. \tag{39}$$

Note that zero mean implies

$$K^i(0) = \frac{1}{2}, \tag{40}$$

while positivity of f^i implies

$$0 < K^i < 1.$$

Write $x^i_k = (p^i_k, 1 - p^i_k)$ where $p^i_k \in [0, 1]$ is the frequency with which player i has played strategy 1 in the first k games. After game k, player i observes his payoff matrix

$$V^i_{k+1} = V^i + \mathbf{E}^i_{k+1}$$

for game $k+1$ and the empirical frequency vector x^{i+1}_k for player $i+1$. Then in game $k+1$ he plays the pure strategy that maximizes his expected payoff, under the assumption that player $i+1$ plays the mixed strategy x^{i+1}_k. Thus the probability, conditioned on x^{i+1}_k, that player i plays strategy 1 in game $k+1$, is the same as the conditional probability

$$\mathbf{P}\{(V^i_{k+1}x^i_{k+1})_1 \geq (V^i_{k+1}x^i_{k+1})_2 \,|\, x^i_{k+1}\}.$$

From the assumptions on V^i_{k+1} this is easily calculated to be

$$\begin{array}{rcl} K^i(4p_k^{i+1} - 2) & \text{if} & i < n, \\ 1 - K^n(4p_k^1 - 2) & \text{if} & i = n. \end{array}$$

To compute the the game vector field using Equations (16) and (17), we define

$$h^i : [0,1] \to [0,1],$$

$$h^i(s) = K^i(4s - 2);$$

note that

$$0 < h^i < 1$$

and

$$h^{i\prime}(s) = 4K^{i\prime}(4s - 2) = 4f^i(4s - 2) > 0. \tag{41}$$

Then the game differential equation expressed in variables $p^1, \ldots, p^n$ takes the form

$$\begin{aligned} \frac{dp^i}{dt} &= F^i(p) \equiv -p^i + h^i(p^{i+1}), \;\; i = 1, \ldots, n-1; \\ \frac{dp^n}{dt} &= F^n(p) \equiv -p^n + 1 - h^n(p^1). \end{aligned} \tag{42}$$

Note that while the state space for the extended game is $[0,1]^n$, the game vector field F is defined in all of $\mathbf{R}^n$. It is easily seen that at F points into the interior Int $([0,1]^{\mathrm{n}})$ at boundary points of $[0,1]^n$. Therefore there is a compact attractor in Int $([0,1]^{\mathrm{n}})$ whose basin contains Int $([0,1]^{\mathrm{n}})$.

Lemma 5.1 *The extended game admits a unique Nash distribution equilibrium p_*, given by*

$$p^1_* = p^2_* = \ldots = p^n_* = \frac{1}{2}.$$

Proof Equation (40) implies that the right hand side of system (42) equals zero if $p = p_*$. If $(p^1, \ldots, p^n)$ is any equilibrium of (42), it must satisfy

$$p^1 = h^1 \circ \ldots \circ h^{n-1}(1 - h^n(p^1)).$$

Since the h^i is strictly increasing, $p^1 = 1/2$ is the unique solution to this fixed point equation. By induction it follows that $p^i = 1/2$ for all i. **QED**

Notice that the Nash distribution equilibrium p_* is also the unique Nash equilibrium of the unperturbed game.

The following two theorems illustrate how the noise densities f^i influence the game dynamics:

Theorem 5.2 *Assume*

$$\prod_{i=1}^{n} f^i(0) > \sup\{\frac{1}{4}|\cos(\frac{2k\pi}{n})|^{-n}, \quad k = 0, \ldots, n-1\}. \tag{43}$$

Then

(i) *The probability that the sequence of empirical frequencies $\{x_k\}$ converges to the unique Nash distribution equilibrium p_* is zero.*

(ii) *Assume n is odd and the K^i are analytic. Then there exists a closed curve ? $\in [0,1]^n$ which is an attracting periodic orbit of (42), such that the limit set $L\{x_k\}$ is ? with positive probability.*

(iii) *Assume $n = 3$ and the K^i are analytic. Then there exists only a finite number of periodic orbits of (42), and almost surely $L\{x_k\}$ is one of them.*

Remark 5.3 When the perturbations have the form $\mathbf{E^i}_{ml} = \varepsilon \eta^i_m$, then inequality (43) holds if the parameter ε is small enough.

Proof Part (i). The characteristic polynomial $P(\lambda)$ of $DF(p_*)$ is easily computed to be

$$P(\lambda) = -(1+\lambda)^n - (4\rho)^n$$

where $\rho = \left[\prod_{i=1}^n f^i(0)\right]^{1/n}$. Therefore the eigenvalues of $DF(0)$ are

$$\lambda_k = -1 + 4\rho \exp(i\frac{2k\pi}{n}) : \ k = 0, \ldots, n-1; \ i = \sqrt{-1}.$$

Under the assumption of Theorem 5.2, all eigenvalues have nonzero real parts and some eigenvalues have positive real parts. Part (i) of the proposition follows from Theorem 4.5.

Part (ii): System (42) is a monotone cyclic feedback system, in the terminology of Mallet-Paret and Smith (1990), and it satisfies the hypothesis of Theorem 4.3 of that paper. According to this theorem, system (42) admits a periodic orbit ? which is an attractor. By an argument similar to the proof of Benaïm and Hirsch (1993, Theorem 2.5), one can show that the probability that $(L\{x_k\})$ is contained in any given attractor— in particular, ?— is positive. Because $L\{x_k\}$ is invariant under the flow of the game vector field (Theorem (3.3)), and ? is a single orbit, if $L\{x_k\} \subset$? then $L\{x_k\} =$?. Thus

$$\mathsf{P}\{L\{x_k\} = ?\} > 0.$$

Part (iii): Assume $n = 3$. Under the change of variables

$$y_1 = p1, y_2 = 1 - p_2, y_3 = p_3,$$

which amounts to relabeling the action sets for player 2, and doesn't change the coordinates of the equilibrium), System (42) becomes

$$\begin{aligned}\frac{dy^1}{dt} &= G^1(y) \equiv -y^1 + h^1(1-y^2) \\ \frac{dy^2}{dt} &= G^2(y) \equiv -y^2 - h^2(y^3) \\ \frac{dy^3}{dt} &= G^3(y) \equiv -y^3 + 1 - h^3(y^1).\end{aligned} \tag{44}$$

This is a *totally competitive system*, meaning that Jacobian matrices have no positive entries. Inequality (43) implies that $DF(p_*)$, and hence $(DG(p_*)$, has one negative eigenvalue and two complex eigenvalues with positive real parts. Moreover the negative eigenvalue has an eigenvector with all components positive, while the invariant linear subspace corresponding to the other eigenvalues is transverse to all positive vectors. (These are well-known implications of the Perron-Frobenius theorem applied to $-DF(p_*)$.)

A fundamental property of C^1 totally competitive systems is the existence of a globally attracting invariant surface S homeomorphic to an open subset of the plane; see Theorems 1.1 and 1.7 of Hirsch (1988); also Hirsch (1989). Moreover S is transverse to vectors in the positive octant $\mathbf{R}^3{}_+$, in the sense that $y - z \notin \mathbf{R}^3{}_+$ if $y, z \in S, y \neq z$. This implies that the 2-dimensional unstable manifold of x_* is a neighborhood of x_* in S. Therefore x_* is a repellor for the flow in S.

All chain recurrent points are contained in S, and Poincaré-Bendixson theory implies that the only connected chain recurrent sets are periodic orbits and the equilibrium. Real analyticity of the game vector field can be used to show that the nonstationary periodic orbits $?_i$ are finite in number. As $L\{x_k\}$ cannot be x_* by Pemantle's theorem 4.5, it must be one of the $?_i$ by the Limit Set Theorem (3.3).

The following result shows that if the noise is sufficiently diffuse, then there is convergence to the unique Nash distribution equilibrium. This is not surprising, but it is interesting to obtain a concrete estimate.

Theorem 5.4 *Assume that $n = 3$ and the densities f^i satisfy $f^i(s) < 1/2$ if $|s| < 2$. Then almost surely $\{x_k\}$ converges to p_*.*

Proof It is easy to see that now all eigenvalues of $DG(p_*)$ have negative real parts, so p_* is an attractor for the flow $\Phi = \{\Phi_t\}_{t \in \mathbf{R}}$ induced by G.

We use the coordinates y^i, described above, that make G totally competitive. As in part (iii) of Theorem 5.2, there is an invariant C^1 surface $S \subset \mathbf{R}^3$ that attracts all solutions. The restriction of the flow to S is denoted by $\Phi|S$.

Let $W \subset S$ denote the the basin of attraction of p_* for the flow in S. We show by contradiction that

$$W \supset [0,1]^3 \cap S. \tag{45}$$

Suppose this does not hold. Then the Poincaré-Bendixson theorem implies that the boundary in S of the open set W is a periodic orbit

$$\Lambda \subset \mathrm{Int}\,([0,1]^3) \cap (\mathrm{S} \setminus \mathrm{p}_*).$$

But this will imply that Λ is an attractor, which contradicts Λ being the boundary of W.

Fix $z \in \Lambda$ and a unit vector $u \in \mathbf{R}^3$. Recall that the curve

$$c : \mathbf{R} \to \mathbf{R}^3, \quad c(t) = D\Phi_t(z)u$$

satisfies the variational equation

$$\frac{dc}{dt} = DG(\Phi_t z)c(t).$$

Estimating dc/dt we obtain:

$$\begin{aligned} ||D\Phi_t(z)u|| &= ||D\Phi_0(z)u + t\frac{d}{dt}\,|_{t=0}\, D\Phi_t(z)u|| + o(t) \\ &= ||(I + tDG(z))u|| + o(t) \\ &\leq e^{t||DG(z)||} + o(t) \end{aligned}$$

where I is the identity matrix and $||A||$ denotes the operator norm of matrix A. From system (44) and the hypothesis of Theorem 5.4, we derive the estimate

$$||DG(z)|| < 1 + 4 \cdot \frac{1}{2} = 3,$$

whence

$$||D\Phi_t(z)u|| < e^{3t} + o(t).$$

Since $D\Phi_{-t}(z) = (D\Phi_t(z))^{-1}$, we also have

$$||D\Phi_t(z)u|| > e^{-3t} + o(t).$$

From the chain rule and invariance of $\mathrm{Int}\,([0,1]^3)$ under $D\Phi_t$ for $t \geq 0$, one then deduces

$$||D\Phi_t(z)u|| > e^{-3t}. \tag{46}$$

This implies that *every real eigenvalue of* $D\Phi_t(z)$ *is greater than* e^{-3t}.

Let Λ have period $T > 0$ and fix a point $q \in \Lambda$. Because the matrices $-DG(x)$ are nonnegative and irreducible, the matrix $D\Phi_{-T}(q)$ is strictly positive (Hirsch 1984, Kunze & Siegel 1994, Smith 1995). By the Perron-Frobenius theorem, $D\Phi_{-T}(q)$ has a simple eigenvalue $\rho > 0$ equal to the spectral radius of $D\Phi_{-T}(q)$, and corresponding to ρ there is unique positive unit eigenvector v for $D\Phi_{-T}(q)$. Since ρ^{-1} is an eigenvalue of $D\Phi_T(q)$, with eigenvector v, we have

$$\rho^{-1} > e^{-3T}. \tag{47}$$

Now $D\Phi_T(q)$ also has the eigenvector $F(q)$ for the eigenvalue 1. The two eigenvectors $F(q)$ and v are independent, because v is positive (all components are positive), while if $F(q)$ were positive or negative, then the forward orbit of q would converge (Selgrade 1979).

Therefore there must be a third eigenvector w of $D\Phi_T(q)$ such that $w, F(q)$ and v are linearly independent. Because q lies in the invariant surface S, it follows that w is tangent to S at q.

Let $\mu > 0$ be the third eigenvalue of $D\Phi_T(q)$. Because the determinant is the product of the eigenvalues,

$$\mathrm{Det}\, D\Phi_T(q) = \rho^{-1}\mu > e^{-3T}\mu. \tag{48}$$

On the other hand, Liouville's formula (Hirsch & Smale 1974)

$$\mathrm{Det}\, D\Phi_T(q) = \exp(\int_0^T \mathrm{Tr} DG(\Phi_t q)dt$$

where Tr denotes the trace of a matrix. It is easy to see from Equation (41) that

$$\mathrm{Tr} DG(\Phi_t q) = -3.$$

Therefore from Equation (48) we get

$$\mu < e^{3T}\,\mathrm{Det}\, D\Phi_T(q) = e^{3T}e^{-3T} = 1.$$

Thus the two eigenvalues for $D(\Phi_T|S)(q)$ are 1 and $\mu, 0 < \mu < 1$. This makes Λ an attractor for $\Phi|S$, leading to the desired contradiction. This proves 45.

It follows that $\{p_*\}$ is a global attractor for the flow in $[0,1]^3$. Therefore almost surely sample paths converge to p_*, by the Limit Set Theorem 3.3. **QED**

Convergence in 2×3 Generalized Coordination Games

In this section we consider a broad class of $n \times 2$ (n players, 2 strategies) adaptive FK games whose game vector fields have very convenient dynamical properties. For general n we show that for many coordination games there is at least a positive probability of convergence. For $n = 3$ we exhibit several classes of games, both coordination and anticoordination, in which convergence is guaranteed. In contrast, Jordan's matching game discussed above is an anticoordination game in which, for certain parameter values, there is no chance of convergence.

We keep the notation of section 1, assuming additionally that each player has exactly two pure strategies.

Consider first a classical 2×2 game with payoff matrix V^i for player $i \in \{1,2\}$. It is sometimes informally called a *coordination game* when the diagonal entries of V^i dominate columns, i.e.,

$$V^i_{11} > V^i_{12}, \quad V^i_{22} > V^i_{21},$$

because players do better if they both play the same action rather than different actions.

We call a 2×2 game a *generalized coordination game* if each payoff matrix has the weaker property that the sum of the diagonal entries is not less than the sum of the off-diagonal entries, that is,

$$V^i_{11} + V^i_{22} \geq V^i_{21} + V^i_{12}.$$

When the opposite inequality holds for both matrices, the game is called an *generalized anticoordination game*. We call use the term *strict* if the inequalities are strict.

An example of a 2×2 generalized coordination game is given by payoff matrices

$$V^1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix},\ V^2 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

There is a unique Nash equilibrium: It is clearly optimal for Player 1 to play action 2 and player 2 to play action 1. Thus the players do not necessarily "coordinate" their actions. Remark 5.6 gives a sense in which action choices tend to reinforce each other under fictitious play.

A $n\times 2$ game, $n \geq 3$, is a generalized coordination (respectively, anticoordination) game if each *partial* 2×2 game, meaning a game obtained by fixing the actions of all but two players, has the corresponding property. Such a game is called *irreducible* if for every pair (i, j) of distinct players there exists $m \geq 2$ and sequence of players $i_1, \ldots, i_m$ such that $i = i_1$, $j = i_m$, and the partial 2×2 games for players i_l and i_{l+1} are strict for $l = 1, \ldots, m$.

The next proposition gives conditions on the game vector field that are equivalent to generalized coordination and anticoordination.

Recall that a vector field G in Euclidean space is *cooperative* its Jacobian matrices have nonnegative off-diagonal terms, i.e., $\partial G_i/\partial x_j \geq 0$ for $i \neq j$. When the off-diagonal terms are nonpositive, G is called *competitive*. If the Jacobian matrices are irreducible, G is called irreducible.

Proposition 5.5 *Let F denote the game vector field of an $n \times 2$ adaptive FK game whose unperturbed game is denoted by ? . Then F is cooperative (respectively, competitive) if ? a generalized coordination (respectively, generalized anticoordination) game. In either case, F is irreducible provided ? is irreducible.*

Remark 5.6 This gives the following interpretation to an FK generalized coordination game: Suppose the actions of all players except i and j are kept fixed. Then at each round of play player i's probability of playing action $a \in \{1, 2\}$ is a nondecreasing function of player j's empirical frequency of past plays of a. For this is implied by $\partial F_i/\partial p_j \geq 0$ for $i \neq j$. Analogously for anticoordination.

Proof of Proposition 5.5 It is convenient to rephrase our definitions as follows. Let $I = (i_1, \ldots, i_m)$ be a sequence of m distinct players and $J = (b^1, \ldots, b^m) \in \{1, 2\}^m$ a sequence of m pure strategies. For any action profile $a \in A$, let $T^I_J(a)$ denote the action profile obtained from a by replacing a^{i_l} by b^l for $l = 1, \ldots, m$. It is clear that we have a coordination game if and only if for every pair of distinct players $i \neq j$,we have

$$V^i \circ T^{i,j}_{1,1} + V^i \circ T^{i,j}_{2,2} \geq V^i \circ T^{i,j}_{1,2} + V^i \circ T^{i,j}_{2,1}$$

and a generalized anticoordination game if and only if

$$V^i \circ T^{i,j}_{1,1} + V^i \circ T^{i,j}_{2,2} \leq V^i \circ T^{i,j}_{1,2} + V^i \circ T^{i,j}_{2,1}.$$

Consider now an $n \times 2$ adaptive FK game with action set $A^i = \{1, 2\}$ for each player, determined by the random perturbations $\mathbf{E^i}$ and equation 11 where we assume that

(i) $\mathbf{E}^i(a) = \eta^i(a^i)$ for each action profile $a = (a^1, \ldots, a^n) \in A = A^1 \times A^n$.

(ii) The probability density f^i of $\eta^i(2) - \eta^i(1)$ is smooth and strictly positive.

Let $0 \leq p^i \leq 1$ represent the mixed strategy $(p^i, 1 - p^i)$ for player i in which he plays action 1 with probability p^i and action 2 with probability $(1 - p^i)$.

Any function $\Psi : A \to \mathbf{R}$ of joint pure actions is extended to joint mixed strategy profiles as follows. For any strategy profile $p = (p^1, \ldots, p^n)$ we define $\Psi(p)$ to be the expected value of $\Psi(a)$ when a is a random variable with probability distribution p:

$$\Psi(p) = \sum_{a \in A} \Psi(a) \Pi_{k=1}^n \mathsf{P}(a^k)$$

where $\mathsf{P}(a^k) = p^k$ if $a^k = 1$ and $\mathsf{P}(a^k) = 1 - p^i$ if $a^k = 2$. With this notation the Nash map Equation (3) is given by

$$\nu^i(p) = K^i \left[(V^i \circ T^i_1 - V^i \circ T^i_2)(p) \right], \ i = 1, \ldots, n \tag{49}$$

where K^i is the probability distribution of $\eta^i(2) - \eta^i(1)$, (Equation (39)).

Recall that the associated vector field is $F(x) = -x + \nu(x)$. Thus a key property of F is that

$$\frac{\partial F_i}{\partial p^i} = -1$$

and

$$\frac{\partial F_i}{\partial p^j} = f^i [V^i \circ T^i_1 - V^i \circ T^i_2][V^i \circ T^{i,j}_{1,1} + V^i \circ T^{i,j}_{2,2} - V^i \circ T^{i,j}_{1,2} - V^i \circ T^{i,j}_{2,1}]$$

for $i \neq j$. The latter formula implies the proposition. **QED**

Remark that our definition of coordination obviously depends on the labeling of the actions. The following proposition shows that it is sometime possible to transform a given game into a generalized coordination game by a convenient relabeling.

An $n \times 2$ game is called *sign symmetric* if every 2×2 partial game corresponding to every pair of distinct players i, j has payoff matrices V^i, V^j such that the signs ($1, -1$ or 0) of the numbers α^i, α^j are the same, where

$$\alpha^i = (V^i_{11} + V^i_{22}) - (V^i_{21} + V^i_{12})$$

and

$$\alpha^j = (V^j_{11} + V^j_{22}) - (V^j_{21} + V^j_{12}).$$

To a sign symmetric $n \times 2$ game we associate a nonoriented signed graph $\mathcal{G}$ with vertices $\{1, \dots, n\}$, as follows. For distinct players i, j there is an edge with positive sign (respectively negative sign) if $\alpha^i + \alpha^j > 0$ (respectively, $\alpha^i + \alpha^j < 0$.

Now consider a loop in $\mathcal{G}$. The loop is called *frustrated* if it has an even number (possibly zero) of negative edges and *nonfrustrated* otherwise. The game is called frustrated if it has at least one frustrated loop, and nonfrustrated otherwise.

Proposition 5.7 *For any nonfrustrated sign symmetric game, there exists a relabeling of the strategies which transform the game into a game of coordination.*

The proof is left to the reader.

A similar proposition holds for generalized anticoordination games. For instance, Jordan's matching game of the preceding section can be transformed into a generalized anticoordination game provided the number of players is odd.

We have seen above in Theorem 5.2 that in an adaptive FK form of Jordan's 3×2 matching game, there is zero probability of convergence to the unique Nash distribution equilibrium. The following results shows that under quite broad conditions, this is not the case for generalized coordination games:

Theorem 5.8 *Consider an $n \times 2$ FK game, $n \geq 2$, that is an irreducible generalized coordination game. If either*

(a) *the set of Nash distribution equilibria is finite; or*

(b) *the maps K^i are analytic,*

then there exists a Nash distribution equilibrium p^* such that the sequence of empirical frequency vectors converge to p^* with positive probability.

Proof Proposition 5.5 says that the game vector field is cooperative and irreducible. Under hypothesis (a) or (b) F has dissipative dynamics (i.e., there is a global attractor), and therefore there exists at least one asympotically stable equilibrium for F by Hirsch (1985a) or Jiang (1991). Thus the result follows from Theorem 4.7 **QED**

We now assume there are $n = 3$ players. Let $\rho(p)$ denote the spectral radius of the Jacobian matrix $D\nu(p)$. In principle this can be calculated from (49).

Theorem 5.9 *Assume $n = 3$ and one the following conditions holds:*

(a) *The game is an irreducible generalized coordination game; or*

(b) *The game is an irreducible generalized anticoordination game, and $\rho(p) < 2$ for all $p \in [0, 1]^3$.*

Then:

(i) *The limit set of the sequence of empirical frequencies is almost surely a connected set of Nash distribution equilibria.*

(ii) *If the K^i are analytic, then the the set of Nash distribution equilibria is finite, and almost surely sequence of empirical frequencies converges to one of them.*

Proof From Equation (49) we obtain the game vector field $F = I - \nu$ and the game differential equation:

$$\frac{dp^i}{dt} = F^i(p) \equiv p^i + K^i\left[V^i(T_1^i(p)) - V^i(T_2^i(p))\right], \; i = 1, 2, 3. \tag{50}$$

Since p^i does not enter into $T_1^i(p)$ or $T_2^i(p)$, it follows that F has negative divergence; more precisely,

$$\frac{\partial F^i}{\partial p^i} = -1, \quad \mathrm{Trace\,DF(p)} = -3. \tag{51}$$

By the Limit Set Theorem 3.3 we may assume the limit set of a sample path is connected, compact and attractor-free.

We first deal with conclusion (i).

Assume hypothesis (a), so that F is a cooperative irreducible vector field by Proposition 5.5. We show L consists of equilibria. By (Hirsch 1988), L is unordered and lies in an invariant surface S homemorphic to an open subset of the plane. An argument similar to (Hirsch 1989) shows that L does not separate S when, as in this case, F has negative divergence. Therefore L consists entirely of equilibria by (Hirsch & Pugh 1988).

Now assume hypothesis (b). For every $q \in S$ let $\mathbf{e}(q) = \{e_1(q), e_2(q)\}$ be an orthonormal basis for the tangent plane T_qS to S at q. Let $A_q(t)$ denote the 2×2 matrix expressing the linear transformation

$$D\Phi_t(q)|T_q(S) : T_qS \to T_{\Phi_t q}S$$

in the bases $\mathbf{e}(q)$, $\mathbf{e}(D\Phi_t(q))$.

We now prove that *the flow $\{\Phi_t\}$ generated by F decreases area in S for $t > 0$*, by showing that $\mathrm{Det\,A_q(t)} < 1$ for $t > 0$. Fix $T > 0$. An argument similar to that in the proof of Theorem 5.4 shows that the Jacobian matrix $D\Phi_T(p)$ has a real eigenvector v transverse to S at q, with eigenvalue

$$\lambda > e^{-3T}. \tag{52}$$

By Equation (51),

$$\mathrm{Det\,D\Phi_T(q) = e^{-3T}}. \tag{53}$$

Set $\Phi_T q = p$. The 3×3 matrix M expressing the linear transformation $D\Phi_T(q) : \mathbf{R}^3 \to \mathbf{R}^3$ in the bases

$$\{v, e_1(q), e_2(q)\} \quad \text{and} \quad \{D\Phi_T(q)v, e_1(p), e_2(p)\}$$

has the form

$$M = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix $A(q)_T$. It follows from (52) and (53) that $\mathrm{Det\,A(q)_T} < 1$.

Because the flow in S decreases area, it follows that L consists of equilibria, by the same argument as in the proof of Theorem 2.2 in Section 4. This completes the proof of (i).

Now assume the K^i, and hence F, are analytic. Then the equilibrium set is a compact, real analytic variety $Z \subset \mathbf{R}^3$. We will prove Z is zero dimensional, whence it consists of isolated points and (ii) will follow from (i). We may assume dim $(Z) \leq 2$).

Let $X \subset Z$ be a connected component. One can show F is dissipative; then by (Jiang 1991), X is unordered. By (Hirsch 1988) this implies X lies in an invariant planar surface S; therefore X has dimension at most 2. It is known that S is a smooth surface (Tereščák 1994).

We use the fact that analytic varieties can be triangulated. Suppose X is one dimensional. It is known that every vertex must belong to at least two 1-simplices. Therefore Y is not a tree, that is, it contains loops, which implies that Y separates S. But under assumption (b), this contradicts the earlier conclusion that the flow in S decreases area. Under hypothesis (a) we use the fact that the flow is strongly monotone and volume decreasing; in this case (Hirsch 1988) contradicts the existence of an unordered loop of equilibria.

Suppose X is two dimensional. Then the boundary of a 2-simplex is a loop of equilibria, and we reach the same contradictions as above. **QED**

6 Proof of Theorem 2.8

As usual, the index i takes values $1, 2$.

By hypothesis $M^i \neq 0$ and $N^i \neq 0$. There are several different cases to consider, depending on the signs of M^i and N^i. We prove assertions (i), (ii) and (iv) for the case

$$M^i > 0 > N^i$$

(see Figure 2(iii)); the other cases are left to the reader.

For parts (i), (ii) and (iv) we work in interval coordinates. In the case under consideration, ? (0) has pure Nash equilibria are $(1, 1)$ and $(0, 0)$, and a mixed Nash equilibrium (p^1_c, p^2_c), where

$$p^1_c = \frac{-N^2}{M^2 - N^2}, \;\; p^2_c = \frac{-N^1}{M^1 - N^1}.$$

Note that then $0 < p^i_c < 1$.

The main task is to show: *For all sufficiently small $\varepsilon > 0$ there are three Nash distribution equilibria: two are linearly stable and are respectively close to the pure equilibria $(1,1)$ and $(0,0)$ of $?(0)$, while the third is linearly unstable and is close to the mixed equilibrium of $?(0)$.*

We rewrite the components of the Nash map as

$$b^1_\varepsilon(x^2) = H^1\left(\frac{D^1(x^2 - x^2_c)}{\varepsilon}\right), \qquad (54)$$

$$b^2_\varepsilon(x^1) = H^2\left(\frac{D^2(x^1 - x^1_c)}{\varepsilon}\right). \qquad (55)$$

where $D^i = M^i - N^i > 0$.

Fix a small number η in the range

$$0 < \eta < 1 - \max(p^1_c, p^2_c).$$

Since $\lim_{t\to\infty} H^i(t) = 1$ and $\lim_{t\to-\infty} H^i(t) = 0$, it is easy to see that there exists $r = r(\eta) > 0$ such that for all $\varepsilon > 0$ we have:

$$p^i > p^i_c + r\varepsilon \text{ implies } b^{i'}_\varepsilon(p^i) > 1 - \eta. \qquad (56)$$

It follows that if (p^1, p^2) is a solution to (24) such that $p_1 - p^1_c > r\varepsilon$ or $p_2 - p^2_c > r\varepsilon$, then both p^1 and p^2 must be in the interval $[1-\eta, 1]$.

We claim that for ε small enough (depending on η), the composite mapping $b^1_\varepsilon \circ b^2_\varepsilon$ restricts to a contraction from $[1-\eta, 1]$ into itself. To see this, compute the derivative of the composite mapping $b^1_\varepsilon \circ b^2_\varepsilon : [0,1] \to [0,1]$ by the chain rule:

$$(b^1_\varepsilon \circ b^2_\varepsilon)'(s) = b^{1\prime}_\varepsilon(b^2_\varepsilon(s))b^{2\prime}_\varepsilon(s),$$

and use Equations (54) and (55) to get:

$$b^{i\prime}(s) = \frac{D^i}{\varepsilon} f^i\left(\frac{D^i}{\varepsilon}(s - p^{i'}_c)\right). \qquad (57)$$

Take ε so small that $p^{i'}_c + r\varepsilon < 1 - \eta$. Set $t = \frac{D^i}{\varepsilon}(s - p^{i'}_c))$ and rewrite (57) as

$$b^{i\prime}(s) = \frac{1}{s - p^{i'}_c} t f^i(t).$$

Now $|s - p^{i'}_c|$ is bounded away from 0 when $s \in [1-\eta, 1]$, and $\lim_{|t|\to\infty} t f^i(t) = 0$ (Hypothesis 2.7). We conclude that by taking ε sufficiently small, we can make the derivative of $b^1_\varepsilon \circ b^2_\varepsilon$ arbitrarily small on $[1-\eta, 1]$, which makes it a contraction on that interval.

The contracting map theorem now shows that for ε sufficiently small, there is a unique fixed point $w^1(\varepsilon) \in [1-\eta, 1]$ for $b^1_\varepsilon \circ b^2_\varepsilon$.

Setting $w^2(\varepsilon) = b_\varepsilon^2(w^1(\varepsilon))$, we see that the following holds: Given $0 < \eta < 1 - \max(p_c^1, p_c^2)$, there exists $r > 0$ and $\varepsilon_0 > 0$ with the following property. For any $\varepsilon \in (0, \varepsilon_0]$, the Nash map ν_ε has a unique fixed point $w(\varepsilon) = (w^1(\varepsilon), w^2(\varepsilon))$ in the region

$$\{x \in [0,1] \times [0,1] : x^1 > p_c^1 + r\varepsilon \text{ or } x^2 > p_c^2 + r\varepsilon\},$$

and $w(\varepsilon) \in (1-\eta, 1] \times (1-\eta, 1]$. Furthermore $w(\varepsilon)$ is a linearly stable Nash distribution equilibrium for $?(\varepsilon)$ (see (20)). Lastly, $w(\varepsilon)$ lies in the η-neighborhood of the pure Nash equilibrium $(1,1)$ of $?(0)$.

A similar analysis shows that if $0 < \eta < \min(p_c^1, p_c^2)$, we can choose ε_0 and r to have the following additional property: For $0 < \varepsilon \le \varepsilon_0$, the Nash map ν_ε also has a unique fixed point $v(\varepsilon)$ in the set

$$\{(x \in [0,1] \times [0,1] : x^1 < p_c^1 - r\varepsilon \text{ or } x^2 < p_c^2 - r\varepsilon$$

and $v(\varepsilon) \in [0, \eta) \times [0, \eta)$. This fixed point is also a linearly stable Nash distribution equilibrium for $?(\varepsilon)$, and it lies in the η neighborhood of the pure Nash equilibrium $(0,0)$ of $?(0)$.

From what has been shown about $b^{i'}$ it follows that for small enough ε we have $0 < b^{i'}(w^i(\varepsilon)) < 1$. This implies that at $w(\varepsilon)$, the curve $y = b^1(x)$ has smaller slope than the curve $x = b^2(y)$. The same is true at the fixed point $v(\varepsilon)$. This implies the two fixed points are stable Nash distribution equilibria.

This also implies that the two curves must meet at at at least one other fixed point for the Nash map. Moreover we proved earlier that all fixed points u other than $w(\varepsilon)$ and $v(\varepsilon)$ must satisfy

$$|u^i(\varepsilon) - p_c^i| \le r\varepsilon. \tag{58}$$

To conclude the analysis of fixed points, we show that we can choose ε_0 to satisfy the following further condition: If $0 < \varepsilon \le \varepsilon_0$, then every fixed point for ν_ε with both coordinates in the interval $[p_c^i - r\varepsilon, p_c^i + r\varepsilon]$ is linearly unstable. This will show that there is exactly one fixed point in that interval, and that it is linearly unstable.

To this end we use Hypothesis 2.7(ii) to find a strict lower bound $\kappa > 0$ for the set

$$\{f^i(s) : |s| \le rD^i\}.$$

Then from Equation (57) for sufficiently small ε for all $s \in [p_c^i - r\varepsilon, p_c^i + r\varepsilon]$, we have

$$b^{i'}(s) \ge \frac{D^i}{\varepsilon}\kappa.$$

Now take $0 < \varepsilon_0 < D^i\kappa$. Then if $0 < \varepsilon \le \varepsilon_0$, every fixed point satisfying (58) is linearly unstable, as required.

Assertions (i) and (iv) of Theorem 2.8 now follow from Theorem 2.6.

We pass to the proof of assertion (iii). We work in simplicial coordinates. Denote the payoff matrix to player i for the kth play by

$$\mathbf{U}^i(k) = V^i + \varepsilon \mathbf{E}^i(k), \; k = 1, 2 \ldots$$

where

$$\mathbf{E}^i(k)_{lm} = \eta^i_l(k).$$

Recall that $a^i_k \in A^i = \{1, 2\}$ denotes player i's action in game k. We employ the notational convention given at the begining of Section 1: numbers a^i_j in square brackets are subscripts in vectors or matrices. The cumulative average payoff to player i in the first k games is:

$$P^i_k(\varepsilon) = \frac{1}{k}\sum_{j=1}^{k} V^i[a^1_j, a^2_j] + \varepsilon\frac{1}{k}\sum_{j=1}^{k}\eta^i(j)[a^i_j],$$

Thus

$$\begin{aligned}\limsup_{k\to\infty} |P^i_k(\varepsilon) - \frac{1}{k}\sum_{j=1}^{k} V^i[a^1_j, a^2_j]| &\leq \varepsilon \limsup_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k}(|\eta^1(j)_1| + |\eta^2(j)_2|) \\ &= \varepsilon \mathsf{E}(|\eta^1_1| + |\eta^2_2|) = O(\varepsilon), \qquad (59)\end{aligned}$$

using the Law of Large Numbers.

On the other hand, we have from the definition (10) of the empirical joint frequency matrix C_n for game n:

$$\frac{1}{n}\sum_{j=1}^{n} V^i[a^1_j, a^2_j] = \sum_{m,l=1}^{2} V^i[m, l]C_n[m, l].$$

According to Corollary 2.4, in interval coordinates we have

$$\lim_{k\to\infty} C_k[m, l] = x_*(\varepsilon)^1_m x_*(\varepsilon)^2_l.$$

Therefore from Equation (59) we have:

$$\limsup_{k\to\infty} |P^i_k(\varepsilon) - \sum_{m,l=1}^{2} V^i_{ml} x^1_*(\varepsilon)_m (x^2_*(\varepsilon)_l)| = O(\varepsilon).$$

Letting ε go to zero, and using Equations (25), (26) we obtain

$$\limsup_{n\to\infty} P^i_n(\varepsilon) = P^i(x_*(0)).$$

The proof for $\liminf$ is similar. **QED**

7 References

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