MERGING OF OPINIONS WITH INCREASING INFORMATION[1]

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1. History. One of us [1] has shown that if Zn, $n = 1, 2, \cdots$ is a stochastic process with D states, $0, 1, \cdots, D - 1$ such that $X = \sum_{n=1}^{\infty} Z_n/D^n$ has an absolutely continuous distribution with respect to Lebesgue measure, then the conditional distribution of $R_k = \sum_{n=1}^{\infty} Z_{k+n}/D^n$ given $Z_1, \cdots, Z_k$ converges with probability one as $k \to \infty$ to the uniform distribution on the unit interval, in the sense that for each λ, $0 < \lambda \leqq 1$, $P(R_k < \lambda \mid Z_1, \cdots, Z_k) \to \lambda$ with probability 1 as $k \to \infty$. It follows that the unconditional distribution of R_k converges to the uniform distribution as $k \to \infty$. If $\{Z_n\}$ is stationary, the distribution of R_k is independent of k, and hence uniform, a result obtained earlier by Harris [3]. Earlier work relevant to convergence of opinion can be found in [4, Chap. 3, Sect. 6].

Here we generalize these results and also show that the conditional distribution of R_k given $Z_1, \cdots, Z_k$ converges in a much stronger sense. All probabilities in this paper are countably additive.

2. Statement of the theorem. Let $\mathcal{B}_i$ be a σ-field of subsets of a set X_i, $i = 1, 2, \cdots$; and let $(X, \mathcal{B}) = (X_1 \times X_2 \times \cdots, \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots)$. Suppose $(X, \mathcal{B}, P)$ is a probability space and let P_n be the marginal distribution of $(X_1 \times \cdots \times X_n, \mathcal{B}_1 \times \cdots \times \mathcal{B}_n)$; that is, $P_n(A) = P(A \times X_{n+1} \times \cdots)$ for all $A \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$. The probability P is *predictive* if for every $n \geqq 1$, there exists a *conditional distribution* P^n for the *future* $X_{n+1} \times \cdots$ *given the past* $X_1, \cdots, X_n$; that is, if there exists a function $P^n(x_1, \cdots, x_n)(C)$ where $(x_1, \cdots, x_n)$ ranges over $X_1 \times \cdots \times X_n$ and C ranges over $\mathcal{B}_{n+1} \times \cdots$ with the usual three properties: $P^n(x_1, \cdots, x_n)(C)$ is $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$-measurable for fixed C; a probability distribution on $(X_{n+1} \times \cdots; \mathcal{B}_{n+1} \times \cdots)$ for fixed $(x_1, \cdots, x_n)$; and for bounded $\mathcal{B}$-measurable ϕ

$$\int \phi \, dP = \int [(\phi(x_1, \cdots, x_n, x_{n+1}, \cdots) \, dP^n (x_{n+1}, \cdots \mid x_1, \cdots, x_n)] \cdot dP_n (x_1, \cdots, x_n) \tag{1}$$

holds.

The assumption that P is predictive is mild and applies to all natural probabilities known to us. It is easy to verify that any probability which is absolutely continuous with respect to a predictive probability is also predictive.

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For any two probabilities μ_1 and μ_2 on the same σ-field $\mathfrak{F}$, the well known *distance* $\rho(\mu_1, \mu_2)$ between μ_1 and μ_2 is the least upper bound over $D \varepsilon \mathfrak{F}$ of $|\mu_1(D) - \mu_2(D)|$. Of course μ_i is absolutely continuous with respect to $(\mu_1 + \mu_2)/2 = m$ and has a density ϕ_i, so that $\rho(\mu_1, \mu_2) = \int_A (\phi_1 - \phi_2) dm = (1/2) \int |\phi_1 - \phi_2| \, dm$ where A is the set where $\phi_1 - \phi_2 > 0$.

MAIN THEOREM. *Suppose that P is a predictive probability on $(X, \mathfrak{B})$ and that Q is absolutely continuous with respect to P. Then for each conditional distribution P^n of the future given the past with respect to P, there exists a conditional distribution Q^n of the future given the past with respect to Q such that, with the exception of a set of histories $(x_1, \cdots, x_n, x_{n+1}, \cdots)$ of Q-probability 0, the distance between $P^n(x_1, \cdots, x_n)$ and $Q^n(x_1, \cdots x_n)$ converges to 0 as n converges to ∞.*

3. Martingale preliminaries. The proof of the theorem requires a slightly generalized martingale convergence theorem. Say that a sequence $\{y_n\}$ of random variables is *dominated in the sense of Lebesgue* if $\sup_n |y_n|$ has a finite expectation.

THEOREM 2. *Suppose that $\{y_n\}$, $n = 1, 2, \cdots$, a sequence of random variables dominated in the sense of Lebesgue, converges almost everywhere to a random variable y. Then for every monotone increasing or monotone decreasing sequence of σ-fields $\mathfrak{U}_j$, $j = 1, 2, \cdots$ converging to a σ-field $\mathfrak{U}$,*

$$\lim_{\substack{j \to \infty \\ n \to \infty}} E[y_n \mid \mathfrak{U}_j] = E[y \mid \mathfrak{U}], \tag{2}$$

almost everywhere and in L_1.

In this note we are primarily interested in the weaker conclusion that $\lim_{n \to \infty} E[y_n \mid \mathfrak{U}_n] = E[y \mid \mathfrak{U}]$. The two important special cases in which either y_n or $\mathfrak{U}_n$ is independent of n are in [2].

PROOF OF THEOREM 2. Let $g_k = \sup y_n$ for $n \geqq k$. Equalities and inequalities below are asserted to hold with probability 1. Fix k for a moment and let $n \geqq k$. Then $y_n \leqq g_k$ and

$$E[y_n \mid \mathfrak{U}_i] \leqq E[g_k \mid \mathfrak{U}_i]. \tag{3}$$

Letting

$$\begin{aligned} z &= \lim_j \sup_{\substack{i \geqq j \\ n \geqq j}} E[y_n \mid \mathfrak{U}_i], \\ x &= \lim_j \inf_{\substack{i \geqq j \\ n \geqq j}} E[y_n \mid \mathfrak{U}_i], \end{aligned} \tag{4}$$

you conclude from (3) and a usual form of martingale convergence theorem [For example, see 2, Theorem 4.3, Chap. VII] that

$$z \leqq \lim_j \sup_{i \geqq j} E[g_k \mid \mathfrak{U}_i] = \lim_i E[g_k \mid \mathfrak{U}_i] = E[g_k \mid \mathfrak{U}]. \tag{5}$$

Therefore $z \leqq \lim E[g_k \mid \mathfrak{U}] = E[y \mid \mathfrak{U}]$ by Lebesgue's theorem suitably generalized so as to apply to conditional expectations. [See, for example, 2, CE_5 Section 8, Chap. 1]. Similarly, $x \geqq E[y \mid \mathfrak{U}]$, and the proof of almost everywhere convergence is complete. The proof of L_1 convergence is routine and omitted.

COROLLARY 1. *Suppose that with probability* 1, *only a finite number of the events* $E_1, E_2, \cdots$ *occur. Then for any monotone sequence of* σ*-fields* $\mathfrak{U}_1, \mathfrak{U}_2, \cdots$

$$P[\bigcup_{k \geqq n} E_k \mid \mathfrak{U}_j] \quad \text{and} \quad P[E_n \mid \mathfrak{U}_j] \to 0. \tag{6}$$

almost surely as n *and* $j \to \infty$.

COROLLARY 2. *If* f_n *is any sequence of random variables that converges almost everywhere to* 0 *and* $\mathfrak{U}_j$ *is a monotone sequence of* σ*-fields, then with probability* 1, *for all* $\epsilon > 0$,

$$P[\sup_{k \geqq n} |f_k| > \epsilon \mid \mathfrak{U}_j], \quad \text{and} \quad P[|f_n| > \epsilon \mid \mathfrak{U}_j] \tag{7}$$

converge to 0 *as* n *and* j *converge to* ∞.

COROLLARY 3. *Let* $q \geqq 0$ *be a density function for which* $Q(B) = \int_B q\,dP$ *for all* $B \in \mathfrak{B}$; *let*

$$q_n(x_1, \cdots, x_n) = \int q(x_1, \cdots, x_n, x_{n+1}, \cdots)\, dP^n(x_{n+1}, \cdots \mid x_1, \cdots, x_n); \tag{8}$$

and let

$$d_n(x_1, \cdots, x_n, x_{n+1}, \cdots) = q(x_1, \cdots, x_n, x_{n+1}, \cdots)/ q_n(x_1, \cdots, x_n) \quad \text{or} \quad 1, \tag{9}$$

according as $q_n(x_1, \cdots, x_n) \neq 0$ *or not. Then, with* P*-probability* 1, *for all* $\epsilon > 0$,

$$P[d_n - 1 > \epsilon \mid x_1, \cdots, x_n] \to 0 \quad \text{as} \quad n \to \infty, \tag{10}$$

and with Q*-probability* 1, *for all* $\epsilon > 0$,

$$Q[|d_n - 1| > \epsilon \mid x_1, \cdots, x_n] \to 0 \quad \text{as} \quad n \to \infty. \tag{11}$$

PROOF OF COROLLARY 3. With respect to P measure,

$$E[q \mid x_1, \cdots, x_n] = q_n(x_1, \cdots, x_n), \tag{12}$$

so that according to Doob's martingale convergence theorem, $q_n(x_1, \cdots, x_n)$ converges to $q(x_1, \cdots, x_n, x_{n+1}, \cdots)$ almost surely with respect to P. Consequently, $\overline{\lim}\, d_n \leq 1$ a.s. P and $d_n \to 1$ a.s. Q since $q > 0$ a.s. Q. An application of Corollary 2 completes the proof.

4. Proof of main theorem.

Define

$$Q^n(x_1, \cdots, x_n)(C) = \int_C d_n(x_1, \cdots, x_n, x_{n+1}, \cdots)\, dP^n(x_{n+1}, \cdots \mid x_1, \cdots, x_n), \tag{13}$$

for all $C \in \mathfrak{B}_{n+1} \times \cdots$.

It is routine to verify that Q^n is a conditional distribution for the future given the past. Let $u = (x_1, \cdots, x_n)$ and $v = (x_{n+1}, \cdots)$, and compute thus:

$$
\begin{aligned}
\rho(P^n(x_1, \cdots, x_n), Q^n(x_1, \cdots, x_n)) & \\
&= \rho(P^n(u), Q^n(u)) \\
&= \textstyle\int (d_n(u, v) - 1) dP^n(v \mid u) \text{ over } v{:}d_n(u, v) - 1 > 0 \\
&\leqq \epsilon + \textstyle\int d_n(u, v) dP^n(v \mid u) \text{ over } v{:}d_n(u, v) - 1 > \epsilon \qquad (14) \\
&= \epsilon + Q^n(u)(v{:}d_n(u, v) - 1 > \epsilon) \\
&= \epsilon + Q[d_n - 1 > \epsilon \mid x_1, \cdots, x_n] \\
&= \epsilon + \epsilon
\end{aligned}
$$

for all but a finite number of n with Q-probability 1, according to (11). This completes the proof.

5. Interpretation. Usually, there is essentially only one conditional distribution Q^n of the future given the past. Therefore, our theorem may be interpreted to imply that if the opinions of two individuals, as summarized by P and Q, agree only in that $P(D) > 0 \leftrightarrow Q(D) > 0$, then they are certain that after a sufficiently large finite number of observations $x_1, \cdots, x_n$, their opinions will become and remain close to each other, where close means that for every event E the probability that one man assigns to E differs by at most ϵ from the probability that the other man assigns to it, where ϵ does not depend on E. Leonard J. Savage observed that our theorem applies to the particularly interesting case in which P and Q are symmetric (or exchangeable). That is, if the measures P and Q on the sequences x_i are those that arise when the x_i are, for a fixed parameter value, independent and identically distributed observations, with prior distributions p and q on the parameter, then the relations of absolute continuity between P and Q are precisely those between p and q.

6. Caution. Though the conditional distributions of the future P^n and Q^n merge as n becomes large, this need not happen to the unconditional distributions of the future. That is, let $P(n)(D) = P(X_1 \times \cdots \times X_n \times D)$ for all $D \varepsilon \mathcal{B}_{n+1} \times \cdots$, and let $Q(n)$ be similarly defined. The following is a simple example of two probabilities P and Q absolutely continuous with respect to each other for which $P(n)$ and $Q(n)$ do not merge with increasing n. Let R be the probability on infinite sequences $x_1, x_2, \cdots$ of 0's and 1's determined by independent tosses of a coin which has probability r of success, and let S be the probability determined if the coin has probability s for success, with $0 \leqq r \leqq 1$, $0 \leqq s \leqq 1$, and $r \neq s$. Now let $0 < p < q < 1$ and let P and Q be mixtures of R and S: $P = pR + (1 - p)S$, $Q = qR + (1 - q)S$. Since $P(n) = P$ and $Q(n) = Q$ for all n, there is no tendency for $P(n)$ and $Q(n)$ to merge.

7. An application. By viewing the unit interval as a product of two point spaces, the interested reader will see that the main theorem yields information about the local behavior of positive integrable functions $q(x)$ defined for $0 \leqq x \leqq 1$.

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