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NOTES AND COMMENTS

ON "REPUTATION" REFINEMENTS WITH HETEROGENEOUS BELIEFS

BY PIERPAOLO BATTIGALLI AND JOEL WATSON[1]

CONSIDER A REPEATED GAME with incomplete information in which a patient long run player, whose type is unknown, faces a sequence of short run opponents (as in Fudenberg and Levine (1989)). The standard "reputation" result is that the patient long run player can obtain an average long payoff almost equal to the Stackelberg payoff of the stage game by consistently playing as a Stackelberg leader. In the analysis, one generally takes as fundamental an assumption that the players have common prior beliefs on the states of the world and that behavior is consistent with the concept of Bayesian Nash equilibrium. However, these related suppositions have been called into question as unrealistic and too stringent (cf. Gul (1991)). In many games of incomplete information a player's prior probabilities on types (or states of the world) are best regarded as purely subjective psychological parameters, unknown to the modeler and to this player's opponents. Therefore, it is important to understand whether the standard reputation results (among others) are implied by weaker assumptions on the knowledge and behavior of the players. In fact, as Watson (1993) demonstrates, the reputation result does not require equilibrium. It is implied by a weak notion of rationalizability with some restrictions on the beliefs of the players.

Here we qualify Watson's (1993) study and extend the line of inquiry of Watson (1993) and Battigalli (1994) concerning settings in which reputations are effective. As Watson shows, two main conditions on the beliefs of the players, along with weak rationalizability, imply the reputation result. First, there must be a strictly positive and uniform lower bound on the subjective probability that players assign to the "Stackelberg type." Second, the conditional beliefs of the short run players must not be too dispersed. Watson (1993) does not explicitly indicate on what the updated beliefs of the short run players are conditioned. We make this explicit and show that it is necessary to assume that the conditional beliefs of the short run players satisfy a stochastic independence property (cf. Battigalli (1996)). We also comment on the dispensability of equilibrium regarding the reputation result in games with two long run players.

1. THE PERTURBED REPEATED GAME MODEL

A finite two-player stage game $G = \{A_1, A_2; u_1, u_2\}$ is infinitely repeated. Let $a^t = (a_1^t, a_2^t) \in A_1 \times A_2$ denote the pair of actions chosen in period t. A fixed individual with objective function $\sum_{t=1}^{\infty} \delta^{t-1} u_1(a^t)$ plays in the role of player 1. It is assumed that player 1 is patient (i.e. her discount factor δ is arbitrarily close to one). Player 1 faces a sequence of short run opponents 2_i, $i = 1, 2, \ldots$. Player 2_i's payoff function is $u_2(a^i)$. ("2" refers to the set of short run opponents.)

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Suppose for simplicity that player 2 has a single-valued best response function $BR: A_1 \to A_2$. Player 1's *Stackelberg (stage game) payoff* is

$$u_1^* = \max_{a_1 \in A_1} u_1(a_1, BR(a_1)).$$

Player 1's *Stackelberg action* is a fixed action a_1^* which attains the maximum above; i.e.

$$u_1^* = u_1(a_1^*, BR(a_1^*)).$$

Let H_t denote the set of histories through period t. That is, $H_t = (A_1 \times A_2)^t$. Let $H = \bigcup_{t=0}^{\infty} H_t$, where $H_0 = \{\phi\}$ is the singleton containing the empty initial history. Assuming perfect monitoring, a strategy for player 1 is a mapping $s_1: H \to A_1$. A strategy for player 2_i is a mapping $s_2^i: H_i \to A_2$. Player 1's strategy space is denoted S_1, while player 2_i's strategy space is denoted S_2^i. The set of strategy profiles is thus $S \equiv S_1 \times \prod_{t=1}^{\infty} S_2^t$. Let $H_\infty \equiv (A_1 \times A_2)^\infty$ be the set of infinite histories. Each profile $s \in S$ induces a unique history $h_\infty(s) \in H_\infty$. For any finite history $h_t \in H_t$, let $S_1(h_t)$ be the set of strategies for player 1 that are consistent with history h_t. That is, $s_1 \in S_1(h_t)$ if and only if there is a profile s_{-1} of strategies for the players 2 such that h_t is the t-period truncation of $h_\infty(s_1, s_{-1})$. We focus on the set $H_\infty^* \equiv (\{a_1^*\} \times A_2)^\infty$ of histories in which player 1 always plays the Stackelberg action. Let $H_t^* \equiv (\{a_1^*\} \times A_2)^t$ and let $H^* \equiv \bigcup_{t=0}^{\infty} H_t^*$.

At each information set in the game (after all histories), the players entertain beliefs about the strategies of the others. For example, conditional on a given history, each player 2_i forms a belief about the strategies of player 1 and the other players 2. We focus on each player 2_i's conjecture concerning the strategy of player 1. Formally, conditional on each history $h \in H$, player 2_i's conjecture is $\mu_i^h \in \Delta S_1$, a probability distribution over player 1's strategies.[2] Note that μ_i^ϕ is this player's conjecture at the beginning of the game. Player 2_i also forms conditional beliefs about the strategies of the other players 2, but we do not model these directly. Below we assume that the beliefs of the short run players obey Bayes' rule and have a stochastic independence property. (Our main contribution is in making the latter condition explicit.)

We consider a perturbed version of this repeated game in which player 1 may be a "crazy" type committed to play a given strategy. In particular, player 1 may be the *Stackelberg type* playing the strategy s_1^*, where $s_1^*(h) = a_1^*$ for all $h \in H$. The set of possible types is otherwise unrestricted. Since the opponents' payoffs are not directly affected by player 1's type, we do not need to model the type space explicitly. We implicitly assume that player 1's opponents have some subjective joint probability measure on player 1's types and strategies, which assigns positive probability to the Stackelberg type and we consider the marginal on the strategy space.[3] Furthermore, we assume that there is a *common lower bound* to the subjective prior probabilities of the Stackelberg strategy (type): there is some real number $\varepsilon \in (0,1)$, known to player 1, such that

(1) $\mu_i^\phi(\{s_1^*\}) \geq \varepsilon$ for each i.

[2] Endow S_1 with the σ-algebra induced by finite histories.

[3] For a discussion of how the crazy types are interpreted, see Watson (1994).

2. STOCHASTIC INDEPENDENCE

Note that, although the payoff of a short run player 2_i is not affected by the behavior of other short run players, in general player 2_i's expectations are given by probabilistic beliefs on the set of strategy profiles $S_1 \times S_2^1 \times \cdots \times S_2^{i-1} \times S_2^{i+1} \ldots$. These beliefs might exhibit correlation across players. Even if they are *ex ante* uncorrelated, conditional beliefs may exhibit correlation after a zero-probability history. This means that player 2_i's conditional probability that player 1 is playing a given strategy s_1 may be directly affected by previous actions of some players 2_j $(j<i)$. We exclude this possibility by assuming that expectations satisfy the following *stochastic independence* property: for each player 2_i, each history $h \in H$, and each $T_1 \subset S_1$,

(2) $$\mu_i^h(T_1)\mu_i^\phi(S_1(h)) = \mu_i^\phi(T_1 \cap S_1(h)).$$

This assumption states that, after histories consistent with his/her initial conjecture, player 2_i's conditional conjectures about player 1's strategy (a) do not depend upon expectations about the behavior of the other short run players and (b) are consistent with Bayes' rule. Obviously, this condition must be satisfied for every positive probability history if prior beliefs are uncorrelated. But stochastic independence requires that the condition hold even if h contains some unexpected actions by other short run players.[4]

3. REPUTATION

In this section we clarify the role of stochastic independence in extending the reputation result of Fudenberg and Levine (1989) to a nonequilibrium framework without common priors (Watson (1993)). Given player 2_i's conditional beliefs and any history $h \in H$, let π_i^h be the probability that this player assigns to player 1 selecting the Stackelberg action a_1^* after history h. (Formally, $\pi_i^h = \mu_i^h(S_1(h,(a_1^*, a_2)))$, where $a_2 \in A_2$ is arbitrary). For any set X, let $\#X$ denote the cardinality of X.

LEMMA 1 (cf. Fudenberg and Levine (1989)[5]): *Take any infinite history $h_\infty \in H_\infty^*$ and for each t let $h_t \in H_t^*$ be the t-period truncation of h_∞. If player 2_i's beliefs satisfy conditions* (1) *and* (2), *then for all $\zeta \in (0,1)$,*

(3) $$\#\{t \geq 0 \,|\, \pi_i^{h_t} < \zeta\} < \frac{\ln \varepsilon}{\ln \zeta}.$$

PROOF: Note that since $\mu_i^\phi(\{s_1^*\}) > 0$ and $s_1^* \in S_1(h_{t+1})$, it is the case that $\mu_i^\phi(S_1(h_{t+1})) > 0$. Using this, along with the stochastic independence property, we have that $\pi_i^{h_t} = \mu_i^{h_t}(S_1(h_{t+1})) = \mu_i^\phi(S_1(h_{t+1}))/\mu_i^\phi(S_1(h_t))$. Note that $S_1(h_0) = S_1$ and so $\mu_i^\phi(S_1(h_0)) = 1$. Thus, using the equation for $\pi_i^{h_t}$, we have $\prod_{t=0}^k \pi_i^{h_t} = \mu_i^\phi(S_1(h_{k+1}))$, for each positive integer k. Since $s_1^* \in S_1(h_{k+1})$ for all k, our lower bound assumption implies that $\mu_i^\phi(S_1(h_{k+1})) \geq \varepsilon$. Thus $\prod_{t=0}^\infty \pi_i^{h_t} \geq \varepsilon$. Suppose $\pi_i^{h_t} < \zeta$ for K integers. Since $\pi_i^{h_t} \in [0,1]$ for all t, it must be that $\zeta^K \geq \varepsilon$. Taking logarithms establishes the result. *Q.E.D.*

[4] This formulation is sufficient for the purposes of this note. A more complete formulation in terms of conditional probability systems is put forward in Battigalli (1994, 1996).

[5] Fudenberg and Levine (1989) obtain equation (3) for Stackelberg histories on the Bayes-Nash equilibrium path, while we consider arbitrary Stackelberg histories which may have zero probability according to the subjective prior μ^i.

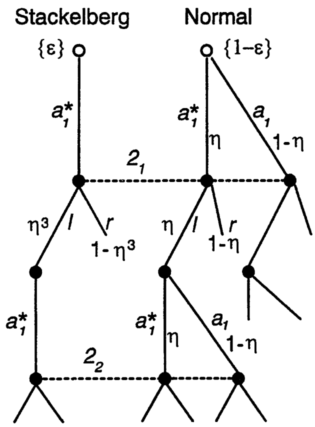

FIGURE 1.—Part of the extensive form of the multistage game. Beliefs are obtained in the limit as $\eta \to 0$.

The proof makes clear that under the stated assumptions, along any Stackelberg path, the conditional probability of the Stackelberg strategy (or type) is a nondecreasing function of time and the conditional probability of the Stackelberg action is never smaller than ε. The example depicted in Figure 1 shows that these properties need not hold if the stochastic independence assumption is violated. Consider the conditional probabilities obtained as the "correlated η-trembles" depicted in the figure become negligible.[6] Under these conditional probabilities, player 2_2 sees player 2_1's choice of action l as a signal that player 1 is the normal type. Player 1 may not be able to establish a reputation when the players 2 hold such (unreasonable) beliefs. Therefore, Watson's (1993) reputation result does not hold if the stochastic independence assumption is not satisfied. The version of Watson's result stated below makes this assumption explicit.

Let ν denote player 1's prior expectation regarding the strategies of the players 2. Then player 1's supremum expected payoff is given by

$$w_1(\nu) = \sup_{s_1 \in S_1} \left\{ \int_{S_{-1}} \left[(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a^t(s_1, s_{-1})) \right] \nu(ds_{-1}) \right\}$$

where $a^t(s)$ is the pair of actions induced by profile s in period t. The following theorem shows that if the short run opponents' conditional expectations are not too diverse and satisfy conditions (1) and (2), then a patient player 1 is able to get (almost) the Stackelberg payoff in the long run.

THEOREM 1 (cf. Watson (1993)): *Fix $\varepsilon > 0$. Let Λ be a set of systems of conditional probabilities $\mu = (\mu^h)_{h \in H}$ satisfying conditions* (1) *and* (2). *Furthermore, assume that Λ is compact with respect to the following quasimetric:*[7]

$$d(\mu, \underline{\mu}) = \sup_{h \in H^*} \{|\pi^h - \underline{\pi}^h|\}.$$

[6] Such conditional probabilities can be represented as part of an extensive form assessment (behavior strategies and beliefs) satisfying Bayes' rule wherever possible.

[7] The compactness assumption can be relaxed. See the discussion of a more general assumption in Watson (1993).

Assume that player 1 believes that each short run opponent maximizes his conditional expected payoff given some $\mu \in \Lambda$. Then there is a positive integer κ, which depends on Λ and ε, such that

$$w_1(\nu) \geq \delta^\kappa u_1^* + (1-\delta^\kappa) \min_{a_2 \in A_2} u_1(a_1^*, a_2). \tag{4}$$

PROOF: Watson's (1993) proof invokes the lemma of Fudenberg and Levine (1989) that we have generalized here. It is enough to notice that stochastic independence is implicit in Watson's analysis. One can then substitute our lemma for that of Fudenberg and Levine and then follow Watson's proof. *Q.E.D.*

4. REPUTATION WITH A LONG RUN OPPONENT

We wish to point out that the reputation result also holds in a weak rationalizability setting for games in which both players 1 and 2 are long run players. It is not difficult to extend the result of Schmidt (1993) on games of "conflicting interests" to the setting of weak rationalizability. In such games, player 1's Stackelberg action holds player 2's stage game payoff to the minmax level. Suppose player 2 believes with some probability $\mu_2^\phi(s_1^*) \geq \varepsilon$ that player 1 adopts the Stackelberg strategy. Further assume that player 1 knows this and believes that player 2 maximizes his/her expected payoff given his/her belief. Then, letting δ be the discount factor of player 1, we obtain inequality (4) again, where κ depends on ε and the discount factor of player 2.[8]

We should note that this result can also be generalized along the lines of Cripps, Schmidt, and Thomas (1993), who find bounds on equilibrium payoffs in general two-player repeated games. (Schmidt (1993, p. 344) describes a version of this extension.) The bounds are of the same form, in that a player can only establish a reputation for playing a particular action each period, but the bounds are weaker than the "Stackelberg" variety. Watson (1996) uses a different methodology to show that players can establish reputations for using more complicated strategies in a nonequilibrium context, when players use "forgiving" strategies.

Dept. of Economics, Princeton University, Princeton, NJ 08544-1021, U.S.A.
and
Dept. of Economics, University of California, San Diego, 9500 Gilman Dr., La Jolla, CA 92093-0508, U.S.A.

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[8] An earlier version of this paper contains the details.

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