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NOTE

A 2×2 Game without the Fictitious Play Property*

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The definition of fictitious play may depend on first move rules, initial beliefs, weights assigned to initial beliefs, and tie-breaking rules determining the particular best replies chosen at each stage. Using the original definition of Brown (1951) in which the first moves are chosen arbitrarily and no tie-breaking rules are assumed, we give an example of a fictitious play process in a 2×2 game that does not converge to equilibrium. *Journal of Economic Literature* Classification Numbers: C72 C73

1. INTRODUCTION

Consider two players engaged in a repeated play of a finite game in strategic (normal) form. Every player observes the actions taken in previous stages, forms beliefs about his opponent's next move, and chooses a myopic pure best reply against these beliefs. In a "fictitious play," proposed by Brown (1951), every player assumes that the other player is using a stationary (i.e., stage-independent) mixed strategy. Every player takes the empirical distribution of the other player's actions to be his belief about this player's mixed strategy. The definition of the fictitious play process may depend on first move rules, weights assigned to initial beliefs, and tie-breaking rules determining the particular best replies chosen at each stage. Other variations of the process can include deterministic perturbations of payoffs (see, e.g., Robinson, 1951) or stochastic perturbations (see, e.g., Fudenberg and Kreps, 1993). In this paper we stick to the original definition of fictitious play in which the first moves are chosen arbitrarily and no tie-breaking rules are assumed.

We say that the process converges to equilibrium if the sequence of beliefs

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(regarded as vectors of mixed strategies) is as close as we wish to the equilibria set after sufficient number of stages. We say that a game has the fictitious play property (FPP) if every fictitious play process converges to equilibrium. As was shown by Shapley (1964), not every game has the FPP. It is interesting therefore to identify classes of games with the FPP. Three classes of such games have been already found: zero-sum games, i.e., bimatrix games of the form $(A, -A)$ (Robinson, 1951); games with identical payoff functions, i.e., bimatrix games of the form (A, A) (Monderer and Shapley, 1996); and games that are strongly dominance solvable (Milgrom and Roberts, 1991). Obviously, every game that is best-reply equivalent in mixed strategies to a game with the FPP has the FPP.

It has been commonly thought that Miyasawa (1961) proved that every 2×2 game has the fictitious play property. However, Miyasawa assumed a particular tie-breaking rule. Monderer and Shapley (1996) showed that every 2×2 game that satisfies the diagonal property[1] has the FPP. In a 2×2 game that does not satisfy the diagonal property, at least one of the players has a strictly dominated strategy or identical strategies. In this note we show by an example that games with identical strategies for one of the players do not necessarily have the fictitious play property. The game we discuss has strategic complementarities and diminishing returns. Krishna (1991) showed that in such games every fictitious play process in which the players use a particular stationary tie-breaking rule converges to equilibrium.[2] This example shows therefore that Krishna's result depends on the tie-breaking rule. Note that the set of games that do not satisfy the diagonal property has a zero measure. So, generically every 2×2 game has the FPP. We do not know whether such a generic result holds for Krishna's games as well. Although the game we discuss is degenerate and therefore our result can be considered "technical," it may indicate that the fictitious play process is too sensitive to small changes of parameters and thus may not be the right choice for describing learning phenomena in social sciences. This conclusion is supported by Deschamps' example (Deschamps, 1973), where it is shown that small perturbations in a zero-sum game yield a game without the FPP, and it can support the conceptual objections to the usage of the process as a learning device discussed in Fudenberg and Kreps (1993). Another goal of this note is to clarify a commonly made mistake in quoting Miyasawa's theorem.[3]

[1] Let $G = (a(i, j), b(i, j))_{i,j=1}^{2}$ be a bimatrix game. G has the diagonal property if $\alpha \neq 0$ and $\beta \neq 0$, where

$$\begin{aligned} \alpha &= a(1,1) + a(2,2) - a(1,2) - a(2,1), \\ \beta &= b(1,1) + b(2,2) - b(1,2) - b(2,1). \end{aligned}$$

[2] Each player chooses the largest best reply in a given linear order of his strategy set.

[3] Such a mistake was made by the authors several times.

2. FICTITIOUS PLAY

Let $N = \{1, 2\}$ be the set of players. The set of strategies of Player i is denoted by Y^i and the payoff function of Player i is denoted by u^i. For $i \in N$ let $-i$ be the other player (i.e., Player $(3 - i)$). Let Δ^i be the set of mixed strategies of Player i and let U^i be the payoff function of Player i in the mixed extension game. For $i \in N$ and for $y^i \in Y^i$ we denote by $e_{y^i} \in \Delta^i$ the probability distribution concentrated on y^i. We call e_{y^i} a pure strategy and we will identify this pure strategy with the strategy y^i whenever it is convenient to do so.

For every sequence $(e(t))_{t=1}^{\infty}$ of pure strategy profiles in $Y = Y^1 \times Y^2$ we associate a sequence of beliefs $f = f(e) = (f(t))_{t=1}^{\infty}$ in $\Delta = \Delta^1 \times \Delta^2$, where $f(t) = (1/t \sum_{s=1}^{t} e(s)$ for every $t \geq 1$. $f^i(t) \in \Delta^i$ is interpreted as the belief of Player $-i$ on the $(t + 1)$th move of Player i. The sequence $e = (e(t))_{t=1}^{\infty}$ is a *fictitious play process* if for every player i, $e^i(t)$ is a best reply versus $f^{-i}(t - 1)$ for every $t > 1$. We say that the fictitious play process $(e(t))_{t=1}^{\infty}$ *converges to equilibrium* if every limit point of the associated belief sequence is an equilibrium profile. Equivalently, the process converges to equilibrium if for every $\varepsilon > 0$ there exists an integer T such that $f(t)$ is an ε-equilibrium for all $t \geq T$. We say that G has the *fictitious play property* if every fictitious play process in G converges to equilibrium.

THE COUNTEREXAMPLE. Let

$$G = \begin{pmatrix} (0, 1) & (0, 0) \\ (0, 0) & (0, 1) \end{pmatrix}.$$

The rows are labeled by a and b so are the columns. We proceed to prove that this game does not have the FPP.

Proof. Note that Player 1 is always indifferent between the rows. Player 2 chooses a at stage $(T + 1)$ if $f_a^1(T) > \frac{1}{2}$, he chooses b if $f_a^1(T) < \frac{1}{2}$, and he is indifferent between the two columns when $f_a^1(T) = \frac{1}{2}$.

Let $T_0 = S_0 = 1$. Both players play a at $t = 1$. We choose integers $1 < T_1 < S_1 < T_2 < S_2 < T_3 < \cdots$ in a way that is described below. Player 1 plays a for $T_{2k} < t \leq T_{2k+1}$, $k \geq 0$. He plays b otherwise. The integers are chosen so that $f_a^1(T_{2k}) = \frac{1}{4}$, $f_a^1(T_{2k+1}) = \frac{3}{4}$, and $f_a^1(S_k) = \frac{1}{2}$ for every $k \geq 1$. Consequently, in a fictitious play process, Player 2 plays a for every $S_{2k} < t \leq S_{2k+1}$ for $k \geq 0$, and he plays b otherwise. We show that

$$\liminf_{k\to\infty} f_a^2(T_{2k+2}) \geq \frac{1}{4}. \tag{2.1}$$

Hence the sequence of beliefs has a limit point (f^1, f^2) with $f_a^1 = \frac{1}{4}$ and $f_a^2 \geq \frac{1}{4}$. Note that the equilibrium set of the game consists of all pairs (f^1, f^2) such that either $f_a^1 < \frac{1}{2}$ and $f_a^2 = 0$, or $f_a^1 > \frac{1}{2}$ and $f_a^2 = 1$, or $f_a^1 = \frac{1}{2}$. Therefore the sequence of beliefs has a limit point not in equilibrium.

Let $J > 1$ be an integer.

Define recursively a sequence of positive integers $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3, \ldots$. For $k = 1$, $a_1 = J$, $b_1 = 2J$, c_1 is defined by the equation

$$\frac{J + c_1}{J + a_1 + b_1 + c_1} = \frac{1}{2},$$

and d_1 is defined by the equation

$$\frac{J + c_1 + d_1}{J + a_1 + b_1 + c_1 + d_1} = \frac{3}{4}.$$

At stage $k > 1$, a_k is defined by the equation

$$\frac{J + \sum_{j=1}^{k-1}(c_j + d_j)}{\sigma(a_k)} = \frac{1}{2},$$

where

$$\sigma(a_k) = J + \sum_{j \leq k-1} (a_j + b_j + c_j + d_j) + a_k.$$

b_k is defined by the equation

$$\frac{J + \sum_{j=1}^{k-1}(c_j + d_j)}{\sigma(b_k)} = \frac{1}{4},$$

where $\sigma(b_k) = \sigma(a_k) + b_k$. c_k is defined by the equation

$$\frac{J + c_k + \sum_{j=1}^{k-1}(c_j + d_j)}{\sigma(c_k)} = \frac{1}{2},$$

where $\sigma(c_k) = \sigma(b_k) + c_k$. d_k is defined by the equation

$$\frac{J + c_k + d_k + \sum_{j=1}^{k-1}(c_j + d_j)}{\sigma(d_k)} = \frac{3}{4},$$

where $\sigma(d_k) = \sigma(c_k) + d_k$.

Finally define $T_1 = J$, and for $k \geq 1$,

$$S_{2k-1} = T_{2k-1} + a_k, \quad T_{2k} = S_{2k-1} + b_k, \quad S_{2k} = T_{2k} + c_k, \quad T_{2k+1} = S_{2k} + d_k.$$

We proceed to establish (2.1). Note that

$$f_a^2(T_{2k+2}) = \frac{1}{T_{2k+2}} \left(S_1 + \sum_{j=1}^{k} (S_{2j+1} - S_{2j}) \right). \tag{2.2}$$

As $S_1 > T_1$ and $a_{k+1} \geq c_k$ for every $k \geq 1$, (2.2) implies

$$f_a^2(T_{2k+2}) \geq \frac{1}{T_{2k+2}} \left(T_1 + \sum_{j=1}^{k} (T_{2j+1} - T_{2j}) \right) = f_a^1(T_{2k+1}) \frac{T_{2k+1}}{T_{2k} + 2}.$$

As $\lim_{k\to\infty}(T_{2k+1}/T_{2k+2}) = \frac{1}{3}$, we obtain (2.1). ∎

3. REMARKS

2×2 games without the diagonal property can be easily classified according to the fictitious play property. As this is a degenerate class of games and because of the next remark, we do not think that such a classification is important.

Almost every "natural" tie-breaking rule that is incorporated into the definition of the fictitious play process will make Miyasawa's theorem valid; e.g., we can require that a player never switch from a best-reply strategy to another best-reply strategy, or we can use Miyasawa's tie breaking rule which, in contrast, assumes that a player switches to a new best-reply strategy as soon as possible.

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