Collateralized Security Markets*

John Geanakoplos
Yale University

William R. Zame
UCLA

Abstract Much of the lending in modern economies is secured by some form of collateral: residential and commercial mortgages, corporate bonds, mortgage-backed securities, and collateralized debt obligations are familiar examples. This paper builds an extension of general equilibrium theory that incorporates durable goods, collateralized securities and the possibility of default to argue that the reliance on collateral to secure loans, the particular collateral requirements (chosen by the social planner or by the market), and the scarcity of collateral have a profound impact on prices, on allocations, on the structure of markets, and especially on the efficiency of market outcomes. Some of these findings provide useful insights into housing and mortgage markets, and into the sub-prime mortgage market in particular.

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1 Introduction

Recent events in financial markets provide a sharp reminder that much of the lending in modern economies is secured by some form of collateral: residential and commercial mortgages are secured by the mortgaged property itself, corporate bonds are secured by the physical assets of the firm, collateralized mortgage obligations and debt obligations and other similar instruments are secured by pools of loans that are in turn secured by physical property. The total of such collateralized lending is enormous: in 2007, the value of U.S. residential mortgages alone was roughly $10 trillion and the (notional) value of collateralized debt obligations was estimated to exceed $50 trillion. The reliance on collateral to secure loans is so familiar that it might be easy to forget that it is a relatively recent innovation: extra-economic penalties such as debtor's prisons, indentured servitude, and even execution were in widespread use in Western societies into the middle of the 19th Century.

Reliance on collateral to secure loans — rather than on extra-economic penalties — avoids the moral and ethical issues of imposing penalties in the event of bad luck, the cost of imposing penalties, and the difficulty of finding the defaulter in order to impose penalties at all. Such penalties represent a pure deadweight loss – to the borrower who defaults, to the lender who suffers the default, and to society as a whole (and are often triggered by miscalculation rather than by deliberate effort). Collateral, which simply transfers resources from one owner to another, is intended to avoid this deadweight loss. (In practice, seizure of collateral may involve deadweight losses of its own.) This paper argues that the reliance on collateral to secure loans and the particular levels of collateral chosen (by the government or by the market) have a profound impact on prices, on allocations, on the structure of financial institutions, and especially on the efficiency of market outcomes.

Several effects of collateral are perhaps the most important. The first and most obvious effect is that *collateral requirements limit borrowing*. The second, and more subtle effect is that *collateral requirements distort both choices and prices*. This distortion is reflected in the existence of some good used as collateral and some buyer of that good who pays a price strictly above his/her marginal utility for consuming that good – so prices *do not* equate marginal utilities of consumption. Thus, the equilibrium price of each collateral good reflects both a *consumption value* and a *collateral value*, reflecting what Fostel and Geanakoplos (2008) term a *liquidity wedge.* (When this distortion and liquidity wedge are absent, collateral equilibrium coincides with general equilibrium with incomplete markets.) The third effect is that *collateral requirements make it easier to borrow to buy goods but also increase competition between borrowers for the very same goods*; the net welfare effects are ambiguous.

Because different collateral requirements may lead to different equilibria, it is natural to ask about optimal collateral requirements. It might seem that society – or at least lenders – would prefer to set collateral requirements sufficiently high that there will be no default. However, although high collateral requirements make loans safer, they also inhibit borrowing – which may be bad for lenders as well as for borrowers. As we show, collateral requirements that lead to equilibrium default – even to crashes – with positive probability may be Pareto optimal, and so might be chosen by a benevolent social planner. Put differently: sub-prime mortgages may be socially optimal.

When all lending must be collateralized, the supply of collateral becomes an important financial constraint. If collateral is in short supply the necessity of using collateral to back promises creates incentives to create collateral and to stretch existing collateral. The state can (effectively) create collateral by issuing bonds that can be used as collateral and by promulgating law and regulation that make it easier to seize goods used as collateral.[1,2] The market's

[1]The home mortgage market in Israel provides a good example. Historically, government regulation made it easy to seize owner-occupied homes on which the mortgage was in default, but difficult to seize renter-occupied homes. This asymmetry provided an incentive for owners near default to rent their homes to close relatives at below-market prices. As a consequence, down payment requirements frequently exceeded 50% of the sale price and mortgages were difficult to obtain. In the 1980's, changes in government regulations made it easier to seize renter-occupied homes. As a consequence, down payment requirements fell to levels comparable to the U.S. mortgage market and mortgages became much easier to obtain.

[2]Similarly, state regulations concerning seizure can have an enormous influence on bankruptcies; see Lin and White (2001) and Fay, Hurst, and White (2002) for instance.

attempts to stretch collateral have driven much of the financial engineering that has rapidly accelerated over the last three-and-a-half decades (beginning with the introduction of mortgage-backed securities in the early 1970's) and that has been designed specifically to stretch collateral by making it possible for the same collateral to be used several times: allowing agents to collateralize their promises with other agents' promises (pyramiding) and allowing the same collateral to back many different promises (tranching). These two innovations are at the bottom of the securitization and derivatives boom on Wall Street, and have greatly expanded the scope of financial markets.

To make these points and others, we formulate an extension of intertemporal general equilibrium theory that incorporates durable goods, collateral and the possibility of default. To focus the discussion, we restrict attention to a pure exchange framework with two dates but many possible states of nature (representing the uncertainty at time 0 about exogenous shocks at time 1). As is usual in general equilibrium theory, we view individuals as anonymous price-takers.[3] For simplicity, we use a framework with a finite number of agents and divisible loans.[4]

Central to the model is that the definition of a security must now include not just its promised deliveries but also the collateral required to back that promise. The same promise backed by a different collateral constitutes a different security and might trade for a different price. We assume that collateral is held and used by the borrower and that forfeiture of collateral is the *only* consequence of default; in particular, there are no penalties for default other than forfeiture of the collateral, and there is no destruction of property in the seizure of collateral. As a result, borrowers will always deliver the minimum of what is promised and the value of the collateral. Lenders,

[3]Anonymity and price-taking might appear strange in an environment in which individuals might default. In our context, however, individuals will default when the value of promises exceeds the value of collateral and not otherwise; thus lenders do not care about the identity of borrowers, but only about the collateral they bring.

[4]The assumptions of anonymity and price-taking might be made more convincing by building a model that incorporates a continuum of individuals, and the realism of the model might be enhanced by allowing for indivisible loans, but doing so would complicate the model without qualitatively changing the conclusions.

knowing this, need not worry about the identity of the borrowers but only about the value of the collateral. Our basic model requires that each security be collateralized by a distinct bundle of physical goods; residential mortgages provide the canonical example of such securities.

Although default is suggestive of disequilibrium, our model passes the basic test of consistency: under the hypotheses on agent behavior and foresight that are standard in the general equilibrium literature, equilibrium always exists (Theorem 1). As we show, the existence of equilibrium rests on the fact that collateral requirements place an endogenous bound on short sales. (The reader will recall that it is the possibility of unbounded short sales that leads to non-existence of equilibrium in the standard model of general equilibrium with incomplete markets.)

The familiar models of Walrasian equilibrium (WE) and of general equilibrium with incomplete markets (GEI) tacitly assume that all agents keep all their promises, but ignore the question of *why* agents should keep their promises; implicitly the familiar models assume that there are infinite penalties for breaking promises – so that agents never intentionally fail to keep their promises – and that agents never make mistakes – so that agents never accidentally fail to keep their promises. We compare Collateral Equilibrium (CE) to WE and GEI as a way of investigating how equilibrium changes when we make the opposite assumption: that borrowers have no incentive to repay and that the only recourse for the lender is to confiscate collateral. As we show, modulo some technical assumptions, there are two sharp dichotomies. First: either CE is equivalent to GEI *or* there is distortion and a non-zero liquidity wedge (Theorem 2). Second: either CE is equivalent to WE and hence efficient (Pareto optimal) *or* it is inefficient (Theorem 3). To illustrate these ideas, we describe a simple mortgage market (Example 1) in an environment with no uncertainty, and compute equilibrium as a function of the wealth distribution and down payment requirements. We identify parameter regions where CE is or is not Pareto optimal. We also compute individual and social welfare, and show (as noted above) that the welfare impact of collateral requirements is ambiguous: lower collateral requirements make it possible for buyers to hold more houses but create more competition for the same houses, thereby driving up the prices. The last point suggests an important parallel with U.S. lending institutions and housing prices over the last hundred years. Before and shortly after World War I, mortgage down payment requirements were typically on the order of 50%. However, the rise of Savings and Loan institutions, later the VHA and FHA, and most recently the sub-prime mortgage market, have all made it easier for (some) consumers to obtain mortgages with much lower down payment requirements. Lower down payment requirements increase competition and drive up housing prices, so some (perhaps very substantial) portion of the boom in housing prices may have over this period should presumably be ascribed to these institutional changes in mortgage markets, rather than to a change in fundamentals. (Contrast Mankiw and Weil (1989).)

An extension of our simple mortgage market to an environment with uncertainty (Examples 2, 3) allows us to make a number of additional points. Perhaps the most striking of these is that collateral requirements that lead to default (with positive probability) in equilibrium may be *ex ante* Pareto optimal although *ex post* suboptimal (with positive probability). Moreover, if securities offering the same promise but backed by different collateral requirements are offered, the market may choose a collateral requirement that leads to default (with positive probability). This suggests an important implication for the subprime mortgage market which seems to have been ignored: even if it is true that defaults on subprime mortgages led to a crash *ex post*, such mortgages might well have been Pareto improving *ex ante*. Whether the market *always* chooses efficient collateral requirements or whether it can sometimes be welfare improving for government to restrict collateral requirements is a question to which we do not have an answer. We do show, however, that government action can improve social welfare only if it alters terminal prices (Theorem 4). Hence any valid welfare-based argument for regulation of down-payment requirements would seem to require that regulators can correctly forecast the price changes that would accompany such regulation.

To address the way the market stretches collateral we expand our model to include securities that are collateralized by bundles of commodities *and* bundles of other securities (pooling and pyramiding) and offer multiple payment streams (tranching). As in our basic model, the requirement that borrowing be collateralized implies an endogenous bound on short sales, so that equilibrium always exists (Theorem 5). Although the existence of more complicated securities expands the set of possible market outcomes, it may still fail to yield Walrasian allocations. In particular, no collateral equilibrium can ever achieve an allocation in which some agent's consumption in some terminal state has less value than his/her initial (unpledgeable) endowment in that state (Theorem 6). As a consequence, even with pooling, pyramiding and tranching, collateral equilibrium is robustly inefficient: given any array of consumer preferences and any social endowment, there is always an open set of distributions of that endowment with the property that collateral equilibrium from those endowments fails to be Pareto optimal (Theorem 7) – no matter what securities are available for trade. On the other hand, any Walrasian equilibrium in which every agent's consumption in each terminal state has greater value than his/her initial (unpledgeable) endowment in that state can be obtained as a collateral equilibrium whenever a complete set of tranched Arrow securities is available (Theorem 8). Absent tranching, this conclusion *does not* hold (Example 4); thus, tranching serves an important role in furthering social welfare. (As we will discuss, in our framework of perfect information, perfectly divisible goods and loans, and frictionless markets, pooling and pyramiding serve no function when a complete set of tranched Arrow securities is available, but will generally serve a useful function when fewer securities are available.)

Following a brief discussion of the literature below, Section 2 presents the basic model and Section 3 demonstrates that equilibrium exists in that model. Section 4 describes a simple mortgage market that illustrates the workings of the basic model and many of the points we want to make including some of the sources of inefficiency. Section 5 discusses distortion, efficiency and the liquidity wedge. Section 6 expands the first example to an uncertain environment to show that both the social planner and the market may choose collateral requirements that lead to default and that, at least in some circumstances, the market always chooses efficiently. Section 7 expands the basic model to allow for pooling, pyramiding and tranching and demonstrates that equilibrium exists in the expanded model as well. Section

8 shows what pooling pyramiding and tranching can accomplish and what they cannot. All proofs are collected in the Appendix.

Literature

Hellwig (1981) provides the first theoretical treatment of collateral and default in a market setting; the focus of that work is on the extent to which the Modigliani–Miller irrelevance theorem survives the possibility of default. Dubey, Geanakoplos, and Zame (1995) and Geanakoplos and Zame (1997, 2002), which are forerunners of the present work, provide the first general treatments of a market in which deliveries on financial securities are guaranteed by collateral requirements. Araujo, Pascoa, and Torres-Martinez (2002) use a version of the same basic model to show that collateral requirements rule out the possibility of Ponzi schemes in infinite-horizon models, and hence eliminate the need for the transversality requirements that are frequently imposed (Magill and Quinzii, 1994; Hernandez and Santos, 1996; Levine and Zame, 1996). Araujo, Fajardo, and Pascoa (2005) expand the model to allow borrowers to set their own collateral levels, and Steinert and Torres-Martinez (2007) expand the model to accommodate security pools and tranching.

Dubey, Geanakoplos, and Shubik (2005) is a seminal work in a somewhat different literature, which treats extra-economic penalties for default. (In that particular paper, extra-economic penalties are modeled as direct utility penalties; when penalties are sufficiently severe, that model reduces to the standard model in which enforcement is perfect — and costless, because penalties are never imposed in equilibrium). One of the central points of that paper, and of Zame (1993), which uses a very similar model, is that the possibility of default may promote efficiency (a point that is made here, in a different way, in Example 2). Kehoe and Levine (1993) builds a model in which the consequences of default are exclusion from trade in subsequent financial markets, but these penalties constrain borrowing in such a way that there is no equilibrium default. Sabarwal (2003) builds a model which combines many of these features: securities are collateralized, but the consequences of default may involve seizure of other goods, exclusion from subsequent financial markets and extra-economic penalties, as well as forfeiture of collateral. Kau, Keenan, and Kim (1994) provide a dynamic model of mortgages as options, but ignore the general equilibrium interrelationship between mortgages and housing prices.

A substantial empirical literature examines the effect of bankruptcy and default rules (especially with respect to mortgage markets) on consumption patterns and security prices. Lin and White (2001), Fay, Hurst, and White (2002), Lustig and Nieuwerburgh (2005) and Girardi, Shapiro, and Willen (2008) are closest to the present work.

2 Basic Model

As in the canonical model of securities trading, we consider a world with two dates; agents know the present but face an uncertain future. At date 0 (the present) agents trade a finite set of commodities and securities. Between date 0 and date 1 (the future) the state of nature is revealed. At date 1 securities pay off and commodities are traded again.

2.1 Time & Uncertainty

There are two dates, 0 and 1, and S possible states of nature at date 1. We frequently refer to $0, 1, \ldots, S$ as *spots*.

2.2 Commodities, Markets & Prices

There are $L \geq 1$ commodities available for consumption and trade in spot markets at each date and state of nature; the commodity space is $\mathbb{R}^{L(1+S)} = \mathbb{R}^L \times \mathbb{R}^{LS}$. We interpret $x \in \mathbb{R}^{L(1+S)}$ as a claim to consumption at each date and state of the world. For a bundle $x \in \mathbb{R}^{L(1+S)}$ and indices s, ℓ, we write x_s for the vector of spot s consumption specified by x, and $x_{s\ell}$ for the quantity of commodity ℓ specified in spot s. We abuse notation and view $\mathbb{R}^L$ as the subspace of $\mathbb{R}^{L(1+S)}$ consisting of those vectors which are 0 in the last LS coordinates; thus we identify a vector $x \in \mathbb{R}^L$ with $(x, 0, \ldots, 0) \in \mathbb{R}^{L(1+S)}$. Similarly we view $\mathbb{R}^{LS}$ as the subspace of $\mathbb{R}^{L(1+S)}$ consisting of those vectors which are 0 in the first L coordinates. We write $\delta_{s\ell} \in \mathbb{R}^{L(1+S)}$ for the commodity bundle consisting of one unit of commodity ℓ in spot s and nothing else. We write $x \geq y$ to mean that $x_{s\ell} \geq y_{s\ell}$ for each s, ℓ; $x > y$ to mean that $x \geq y$ and $x \neq y$; and $x \gg y$ to mean that $x_{s\ell} > y_{s\ell}$ for each s, ℓ.

We depart from the usual intertemporal models by allowing for the possibility that goods are durable. If $x_0 \in \mathbb{R}^L$ is consumed (used) at date 0 we write $F_s(x_0)$ for what remains in state s at date 1. We assume the map

$F : S \times \mathbb{R}^L \to \mathbb{R}^L$ is continuous and is linear and positive in consumption. The commodity 0ℓ is *perishable* if $F(\delta_{0\ell}) \equiv 0$ and *durable* otherwise. It may be helpful to think of F as like a production function — except that inputs to production can also be consumed.

For each s, there is a spot market for consumption at spot s. Prices at each spot lie in $\mathbb{R}^L_{++}$, so $\mathbb{R}^{L(1+S)}_{++}$ is the space of spot price vectors. For $p \in \mathbb{R}^{L(1+S)}$, p_s are the prices in spot s and $p_{s\ell}$ is the price of commodity ℓ in spot s.

2.3 Consumers

There are I consumers (or types of consumers). Consumer i is described by a consumption set, which we take to be $\mathbb{R}^{L(1+S)}_+$, an endowment $e^i \in \mathbb{R}^{L(1+S)}_+$, and a utility function $u^i : \mathbb{R}^{L(1+S)}_+ \to \mathbb{R}$.

2.4 Collateralized Securities

A *collateralized security* (*security* for short) is a pair $\mathbf{A} = (A, c)$, where $A : S \times \mathbb{R}^L_{++} \times \mathbb{R}^L_{++} \to \mathbb{R}_+$ is the *promise* or *face value*, and $c \in \mathbb{R}^L_+$ is the *collateral requirement*. We allow for the possibility that the amount promised in each state depends on spot prices in that state and at date 0; hence A is a function (assumed continuous) of the state and of prices at date 0 and in that state at date 1. The collateral requirement c is a bundle of date 0 commodities; an agent wishing to sell one share of (A, c) must hold the commodity bundle c. (Recall that selling a security is borrowing.)

In our framework, the collateral requirement is the only means of enforcing promises. Hence, if agents optimize, the *delivery* per share of security (A, c) in state s will not be the face value $A_s(p_0, p_s)$ but rather the minimum of the face value and the value of the collateral in state s:

$$\text{DEL}((A, c), s, p) = \min\{A_s(p_0, p_s),\ p_s \cdot F_s(c)\}$$

The delivery on a portfolio $\theta = (\theta^1, \ldots, \theta^J) \in \mathbb{R}^J$ is

$$\textsc{Del}(\theta, s, p) = \sum_j \theta^j \textsc{Del}((A^j, c^j); s, p)$$

We take as given a finite (but perhaps very large) set of securities $\mathcal{A} = \{(A^1, c^1), \ldots, (A^J, c^J)\}$. Because deliveries never exceed the value of collateral, we assume without loss of generality that $F_s(c^j) \neq 0$ for some s. (Securities that fail this requirement will deliver nothing; in equilibrium such securities will have 0 price and purchases or sales of such securities will be irrelevant.) We find it convenient to distinguish between security purchases and sales; we typically write $\varphi, \psi \in \mathbb{R}^J_+$ for portfolios of security purchases and sales, respectively. We assume that buying and selling prices for securities are identical; we write $q \in \mathbb{R}^J_+$ for the vector of security prices. An agent who sells the portfolio $\psi \in \mathbb{R}^J_+$ will have to hold (and will enjoy) the collateral bundle $\textsc{Coll}(\psi) = \sum \psi^j c^j$.

Our formulation allows for nominal securities, for real securities, for options and for complicated derivatives. For ease of exposition, our examples focus on real securities.

2.5 The Economy

An *economy (with collateralized securities)* is a tuple $\mathcal{E} = \langle (e^i, u^i), \mathcal{A} \rangle$, where (e^i, u^i) is a finite family of consumers and $\mathcal{A} = \{(A^j, c^j)\}$ is a family of collateralized securities. (The set of commodities and the durable goods technology are fixed, so are suppressed in the notation.) Write $\bar{e} = \sum e^i$ for the social endowment. The following assumptions are always in force:

- Assumption 1 $\bar{e} + F(\bar{e}) \gg 0$
- Assumption 2 For each consumer i: $e^i > 0$
- Assumption 3 For each consumer i:
  (a) u^i is continuous and quasi-concave

(b) if $x \geq y \geq 0$ then $u^i(x) \geq u^i(y)$

(c) if $x \geq y \geq 0$ and $x_{s\ell} > y_{s\ell}$ for some $s \neq 0$ and some ℓ, then $u^i(x) > u^i(y)$

(d) if $x \geq y \geq 0$, $x_{0\ell} > y_{0\ell}$, and commodity 0ℓ is perishable, then $u^i(x) > u^i(y)$

The first assumption says that all goods are represented in the aggregate (keeping in mind that some date 1 goods may only come into being when date 0 goods are used). The second assumption says that that individual endowments are non-zero. The third assumption says that utility functions are continuous, quasi-concave, weakly monotone, strictly monotone in date 1 consumption of all goods and in date 0 consumption of perishable goods.[5]

2.6 Budget Sets

Given a set of securities $\mathcal{A}$, commodity prices p and security prices q, a consumer with endowment e must make plans for consumption, for security purchases and sales, and for deliveries against promises. In view of our earlier comments, we assume that deliveries are precisely the minimum of promises and the value of collateral, so we suppress the choice of deliveries. We therefore define the budget set $B(p, q, e, \mathcal{A})$ to be the set of *plans* (x, φ, ψ) that satisfy the budget constraints at date 0 and in each state at date 1 and the collateral constraint at date 0.

- **At date 0**

$$p_0 \cdot x_0 + q \cdot \varphi \leq p_0 \cdot e_0 + q \cdot \psi$$
$$x_0 \geq \textsc{Coll}(\psi)$$

That is, expenditures for consumption and security purchases do not exceed income from endowment and from security sales, and date 0 consumption includes collateral for all security sales.

[5]We do not require strict monotonicity in durable date 0 goods because we want to allow for the possibility that claims to date 1 consumption are traded at date 0; of course, such claims would typically provide no utility at date 0.

- **In state** s

$$p_s \cdot x_s + \textsc{Del}(\psi, s, p) \leq p_s \cdot e_s + p_s \cdot F_s(x_0) + \textsc{Del}(\varphi, s, p)$$

That is, expenditures for consumption and for deliveries on promises do not exceed income from endowment, from the return on date 0 durable goods, and from collections on others' promises.

If these conditions are satisfied, we frequently say that the portfolio (φ, ψ) *finances* x at prices p, q. Of course agents *know* date 0 prices but must *forecast* date 1 prices. Our equilibrium notion implicitly incorporates the requirement that forecasts be correct, so we take the familiar shortcut of suppressing forecasts and treating all prices as known to agents at date 0.[6]

Note that if security promises are independent of date 0 prices and homogeneous of degree 1 in state s prices — in particular, if securities are real (promise delivery of the value of some commodity bundle) — then budget constraints depend only on *relative prices.* In general — for instance, if security promises are nominal — budget constraints may depend on *price levels* as well as on relative prices.

2.7 Collateral Equilibrium

Given an economy $\mathcal{E} = \langle (e^i, u^i), \mathcal{A} \rangle$, a *collateral equilibrium* consists of commodity prices $p \in \mathbb{R}^{L(1+S)}_{++}$, security prices $q \in \mathbb{R}^J_+$ and consumer plans (x^i, φ^i, ψ^i) satisfying the usual conditions:

- **Commodity Markets Clear**

$$\sum_i x^i = \sum_i e^i + \sum_i F(e^i_0)$$

- **Security Markets Clear**

$$\sum_i \varphi^i = \sum_i \psi^i$$

[6] Barrett (2000) offers a model in which forecasts might be incorrect.

- **Plans are Budget Feasible**

$$(x^i, \varphi^i, \psi^i) \in B(p, q; e^i, \mathcal{A})$$

- **Consumers Optimize**

$$(x, \varphi, \psi) \in B(p, q, e^i, \mathcal{A}) \Rightarrow u^i(x) \leq u^i(x^i)$$

(As in a production economy, the market clearing condition for commodities incorporates the fact that some date 1 commodities come into being from date 0 activities.)

2.8 Walrasian Equilibrium and GEI Equilibrium

We will find it useful to compare collateral equilibrium with the benchmarks of Walrasian equilibrium and of GEI (incomplete markets) equilibrium.

We first recall the definition of Walrasian equilibrium in the present context; see Dubey, Geanakoplos, and Shubik (2005) for further details. We maintain the same structure of commodities and preferences. In particular, date 0 commodities are durable, and $F_s(x_0)$ is what remains in state s if the bundle x_0 is consumed at date 0. Suppressing commodities and the nature of durability, the data of a *durable goods economy* is thus a set (e^i, u^i) of consumers, specified by endowments and utility functions. We use notation in which a purchase at date 0 conveys the rights to what remains at date 1; hence if commodity prices are $p \in \mathbb{R}^{(1+S)L}_{++}$, the Walrasian budget set for a consumer whose endowment is e is

$$B^W(e, p) = \{x : p \cdot x \leq p \cdot e + p \cdot F(x_0)\}$$

A *Walrasian equilibrium* consists of commodity prices p and consumption choices x^i such that

- **Commodity Markets Clear**

$$\sum_i x^i = \sum_i e^i + \sum_i F(e^i_0)$$

- **Plans are Budget Feasible**

$$x^i \in B^w(e^i, p)$$

- **Consumers Optimize**

$$y^i \in B(e^i, p) \Rightarrow u^i(y^i) \leq u^i(x^i)$$

In the familiar GEI model, as in our collateral model, goods are traded on spot markets but only securities are traded on intertemporal markets. In the GEI context a *security* is a claim to income at each future state s as a function of prices p_0, p_s at date 0 and in state s; that is, a function $A : S \times \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}$. The data of a GEI economy consists of I consumers, characterized by utility functions u^i and endowments e^i, and J securities A^j.

To maintain the parallel with our collateral framework, it is convenient to continue to separate security purchases and sales. Given commodity spot prices $p \in \mathbb{R}^{L(1+S)}_{++}$ and security prices $q \in \mathbb{R}^J$, the budget set $B^{GEI}(p, q, e, \{A^j\})$ for a consumer with endowment e consists of consumption plans $x \in \mathbb{R}^{L(1+S)}_{+}$ and portfolios of security purchases and sales $\varphi, \psi \in \mathbb{R}^J$ that satisfy the budget constraints at date 0 and in each state at date 1:

- **At date 0**

$$p_0 \cdot x_0 + q \cdot \theta \leq p_0 \cdot e_0$$

- **In state s**

$$p_s \cdot x_s + \sum_j \psi_j \breve{A}^j_s(p_0, p_s) \leq p_s \cdot e_s + p_s \cdot F_s(x_0) + \sum_j \varphi_j \breve{A}^j_s(p_0, p_s)$$

Note that the GEI budget set differs from the collateral budget set in two ways: there is no collateral requirement at date 0, and security deliveries coincide with promises.

A *GEI equilibrium* consists of commodity spot prices $p \in \mathbb{R}^{L(1+S)}_{++}$, security prices $q \in \mathbb{R}^J$, consumption plans $x^i \in \mathbb{R}^{L(1+S)}_{+}$ and portfolio choices $\varphi^i, \psi^i \in \mathbb{R}^J_+$ such that

- **Commodity Markets Clear**

$$\sum_i x^i = \sum_i e^i + \sum_i F(e_0^i)$$

- **Security Markets Clear**

$$\sum_i \varphi^i = \sum_i \psi^i$$

- **Plans are Budget Feasible**

$$(x^i, \varphi^i, \psi^i) \in B(e^i, p, q, \{A^j\})$$

- **Consumers Optimize**

$$(x, \phi, \psi) \in B(e^i, p, q, \{A^j\}) \Rightarrow u^i(x) \leq u^i(x^i)$$

2.9 Rental Markets

In our formulation of the Walrasian economy, the purchase of a durable good at date 0 conveys the rights to what the durable becomes at date 1. Because date 1 commodities are marketed at date 0, the rental of a durable – the purchase of date 0 rights *only* – can be accomplished by a purchase of the durable together with the simultaneous sale of the rights to what the durable becomes at date 1. Thus, the rental price of $x_0 \in R_+^L$ is $p \cdot x_0 - p \cdot F(x_0)$.

If the right securities are available, then rental markets can be synthesized in our collateralized security market as well. Suppose that there is a vector of durable goods $x_0 = c$, and a security (A, c) that promises at least the value of the collateral in every state in period 1, $A_s \geq p_s \cdot F_s(c)\ \forall s \geq 1$. If q is the price of this security, then the rental price of the bundle $c \in R_+^L$ is $p_0 \cdot c - q$.

Of course date 0 purchases of date 1 goods can be synthesized as well, if the right securities exist, because purchasing only the date 1 rights (i.e, $F(x_0)$) to the durable x_0 amounts to the purchasing the security (A, c) above that promises delivery of at least the value of what the durable becomes in each state at date 1 ($A_s \geq p_s \cdot F_s(x_0)$) and is collateralized by the durable itself ($c = x_0$). However, date 0 *sales* of date 1 commodities usually *cannot* be synthesized through security markets, because selling only the date 1 rights to the durable x_0 amounts to selling the security (A, c) without holding the requisite collateral.

3 Equilibrium

Under the assumptions discussed in Section 2, collateral equilibrium always exists. We defer this and all other proofs to the Appendix.

Theorem 1 (Existence) *Under the maintained assumptions, every economy admits a collateral equilibrium.*

This may seem a surprising result, because we allow for real securities, options, derivatives and even more complicated non-linear securities; in the standard model of incomplete financial markets, the presence of any of these securities may be incompatible with existence of equilibrium.[7] In our framework, however, the requirement that security sales be collateralized places an *endogenous* bound on short sales. As in Radner (1972), a bound on short sales eliminates the discontinuity in budget sets that gives rise to non-existence and thus restores the existence of equilibrium.[8]

[7]See Hart (1975) for the seminal example of non-existence of equilibrium with real securities, Duffie and Shafer (1985) and Duffie and Shafer (1986) for generic existence with real securities, and Ku and Polemarchakis (1990) for robust examples of non-existence of equilibrium with options.

[8]Araujo, Pascoa, and Torres-Martinez (2002) exploit a similar idea to show that collateral requirements rule out Ponzi schemes in markets with an infinite horizon.

4 A Simple Mortgage Market

In this section we offer a simple example that illustrates the working of our model and some of the points described in the Introduction, and suggests some of the general results that follow. For the sake of simplicity, the various examples that follow are all variants of this simple example.

Example 1 [A Mortgage Market] Consider a world with no uncertainty ($S = 1$). There are two goods at each date: food F which is perishable and housing H which is perfectly durable. There are two (types of) consumers, with endowments

$$e^1 = (18 - w, 1; 9, 0)$$
$$e^2 = (w, 0; 9, 0)$$

We take $w \in (0, 18)$ as a parameter; we will be especially interested in the case $w = 7/2$. Consumer 1 finds food and housing to be perfect substitutes and has constant marginal utility of consumption; Consumer 2 finds date 0 housing and date 1 housing to be perfect substitutes, likes housing more than Consumer 1, but has decreasing marginal utility for date 0 food:

$$u^1 = x_{0F} + x_{0H} + x_{1F} + x_{1H}$$
$$u^2 = \log x_{0F} + 4x_{0H} + x_{1F} + 4x_{1H}$$

As a benchmark, we begin by recording the unique Walrasian equilibrium $\langle \tilde{p}, \tilde{x} \rangle$ (leaving the simple calculations to the reader). If we normalize so that $\tilde{p}_{0F} = 1$ then equilibrium prices are:

$$\tilde{p}_{0F} = 1 \ , \ \tilde{p}_{1F} = 1 \ , \ \tilde{p}_{0H} = 8 \ , \ \tilde{p}_{1H} = 4$$

equilibrium consumptions are:

$$\tilde{x}^1 = (17, 0; 18 - w, 0)$$
$$\tilde{x}^2 = (1, 1; w, 1)$$

and equilibrium utilities are

$$\tilde{u}^1 = 35 - w$$
$$\tilde{u}^2 = 8 + w$$

Consumer 2 likes housing much more than Consumer 1 and is rich in date 1, so, whatever her date 0 endowment, she buys all the date 0 housing — borrowing from her date 1 endowment if necessary, and of course repaying if she does so. Individual equilibrium utilities depend on w, but total utility is always 43 — which is the level it must be at any Pareto efficient allocation in which both agents consume food in date 1. (Because both agents have constant marginal utility of 1 for date 1 food, the economy has transferable utility in the range of allocations where both consume date 1 food.)

In the GEI world, in which securities always deliver precisely what they promise and security sales do not need to be collateralized, the Walrasian outcome will again obtain there are at least as many independent securities as states of nature — in this case, at least one security whose payoff is never 0. For comparison purposes, suppose exactly one security $\hat{A}_\alpha$ is available, delivering the value of $\alpha > 0$ units of food. Commodity and asset prices are

$$\tilde{p}_{0F} = 1 \ , \ \tilde{p}_{1F} = 1 \ , \ \tilde{p}_{0H} = 8 \ , \ \tilde{p}_{1H} = 4 \ , \ q_\alpha = \alpha$$

and equilibrium consumptions and utilities are:

$$\begin{array}{rclcrcl} \tilde{x}^1 & = & (17, 0; 18 - w, 0) & \qquad & \tilde{u}^1 & = & 35 - w \\ \tilde{x}^2 & = & (1, 1; w, 1) & & \tilde{u}^2 & = & 8 + w \end{array}$$

However, in the world of collateralized securities, no agent can make guarantees to pay without offering collateral, and Walrasian outcomes need not obtain. To the extent Consumer 2 can use housing as collateral, she will be able to buy more housing with borrowed money. However, competition also raises the price of housing. We can trace out the effects of these opposite forces across the range of security promises — equivalently, across the range of collateral requirements.

We assume that only one security $(A_\alpha, c) = (\alpha p_{1F}, \delta_{0H})$ is available for trade; (A_α, c) promises the value of α units of food in date 1 and is collateralized by 1 unit of date 0 housing. We take $w \in (0, 18)$ and $\alpha \in [0, 4]$

as parameters. (As we show below, delivery will never exceed 4, no matter what promises are, so that equilibrium when $\alpha > 4$ will reduce to equilibrium when $\alpha = 4$.)

The nature of collateral equilibrium depends in a complicated way on the parameters w, α. To begin the analysis, note first that we are free to normalize so that $p_{0F} = 1$. Moreover, because (A_α, c) is a real security we are also free to normalize so that $p_{1F} = 1$. It is easily seen that in every collateral equilibrium, Consumer 1 lends (buys the security) and Consumer 2 borrows (sells the security), that both consumers consume food in both dates, and that Consumer 2 acquires all the housing at date 1. Hence many of the equilibrium variables can be determined quickly from first order conditions. In particular:

$$\frac{MU^2_{1H}}{p_{1H}} = \frac{MU^2_{1F}}{p_{1F}} \tag{1}$$

$$\frac{MU^1_{0F}}{p_{0F}} = \frac{MU^1_{(A_\alpha,c)}}{q_\alpha} \tag{2}$$

It follows from (1) that $p_{1H} = 4$. Because $\alpha \in [0, 4]$, the date 1 value of collateral (weakly) exceeds the promise A_α, so $\text{DEL}(A_\alpha, p) = \alpha$; hence $MU^1_{(A_\alpha,c)} = 4$. Now (2) implies that $q_\alpha = \alpha$. Summarizing: for all $w \in (0, 18)$, all $\alpha \in [0, 4]$, and in every equilibrium we have

$$p_{0F} = 1\ ,\ p_{1F} = 1\ ,\ p_{1H} = 4\ ,\ q_\alpha = \alpha\ ,\ \psi^1 = 0\ ,\ \varphi^2 = 0 \tag{3}$$

As we shall see, the values of the remaining equilibrium variables — indeed the nature of equilibrium — depend sensitively on α, w. It is convenient to classify equilibrium according to the quantity of housing held and the fraction of borrowing capacity exercised by Consumer 2; in principle this leads to 9 possible types of equilibria, as in Table 1. Because the collateral requirement entails that $\psi^2 \le x^2_{0H}$, there are in fact no equilibria of type Ib or Ic; for the present functional forms, there are no equilibria of type IIb either. (But there would be equilibria of type IIb for some other functional forms.) For all the other types, we solve simultaneously for the equilibrium variables and the region in the parameter space in which an equilibrium of that type

Table 1: Types of Equilibrium

	$\psi^2/x^2_{0H} = 0$	$\psi^2/x^2_{0H} \in (0,1)$	$\psi^2/x^2_{0H} = 1$	
$x^2_{0H} = 0$	**Ia**	**Ib**	**Ic**	
$x^2_{0H} \in (0,1)$	**IIa**	**IIb**	**IIc**	
$x^2_{0H} = 1$	**IIIa**	**IIIb**	**IIIc**	

(unique in the present setting) obtains. We give details for types IIc and IIIc, leaving the calculations for other types to the reader.

We begin by analyzing equilibrium of type IIc. Consumer 1 holds food and housing at date 0, so he can trade housing for food or *vice versa*. Thus we have the first order condition:

$$\frac{MU^1_{0F}}{p_{0F}} = \frac{MU^1_{0H}}{p_{0H}} \tag{4}$$

Notice that $MU^1_{0H} = 5$: Consumer 1 enjoys 1 util from living in the house at date 0 and 4 more utils by selling the house at date 1 to buy 4 units of date 1 food. Hence $p_{0H} = 5$.

To solve for the remaining equilibrium variables we use Consumer 2's date 0 first order conditions — but the correct first order conditions may not be obvious. Because Consumer 2 holds food and housing at date 0, it might appear by analogy with the first order conditions for Consumer 1 that

$$\frac{MU^2_{0F}}{p_{0F}} = \frac{MU^2_{0H}}{p_{0H}} \tag{5}$$

$$\frac{MU^2_{0F}}{p_{0F}} = \frac{MU^2_{(A_\alpha,c)}}{q_\alpha} \tag{6}$$

Consumer 2 enjoys 4 utils from living in the house at each date, so $MU^2_{0H} = 8$. In view of our earlier calculations, it follows from (5) that $MU^2_{0F} = 8/5$ and from (6) that $MU^2_{0F} = 1$, which is nonsense.

The problem with the analysis above is that (5) and (6) are *not* the correct first order conditions for Consumer 2. Consumer 2 can borrow against date 1 income by selling the security, but selling the security requires holding collateral. By assumption, at equilibrium $x^2_{0H} = \psi^2$, so Consumer 2 is exercising all of her borrowing power; hence she cannot hold less housing without simultaneously divesting herself of some of the security and cannot sell more of the security without simultaneously acquiring more housing.

The correct first order conditions for Consumer 2 take borrowing and collateral into account. On the one hand, buying an additional infinitesimal amount ε of housing costs $p_{0H}\varepsilon$, but of this cost $\alpha\varepsilon$ can be borrowed by selling α units of the security, using the additional housing as collateral, so the net payment is only $(p_{0H} - \alpha)\varepsilon$. However, doing this will require repaying the loan in date 1, so the additional utility obtained will not be 8ε but rather $(8 - \alpha)\varepsilon$. On the other hand, selling an additional ε units of food generates income of εp_{0F} at a utility cost of $MU^2_{0F}\varepsilon$. Hence the correct first order condition for Consumer 2 is not (5), but rather

$$\frac{MU^2_{0F}}{p_{0F}} = \frac{8-\alpha}{p_{0H}-\alpha} \tag{7}$$

Consumer 2's date 0 budget constraint is

$$(5-\alpha)x_{0H} + x_{0F} = w \tag{8}$$

Solving yields

$$x^2_{0F} = \frac{5-\alpha}{8-\alpha}$$
$$x^2_{0H} = \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}$$

From this we can solve for all the equilibrium consumptions

$$x^1 = \left(18 - \frac{5-\alpha}{8-\alpha}, 1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}; 9 + \alpha\left[\frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right] + 4\left[1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right], 0\right)$$
$$x^2 = \left(\frac{5-\alpha}{8-\alpha}, \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}; 9 - \alpha\left[\frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right] - 4\left[1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right], 1\right)$$

and utilities

$$u^1 = 32 - w$$
$$u^2 = 8 + \log(5-\alpha) - \log(8-\alpha) + \left(\frac{8-\alpha}{5-\alpha}\right) w$$

(By definition, $\psi^2 = x^2_{0H}$ and $\varphi^1 = \psi^2$.)

Finally, the region in which equilibria are of type IIc is defined by the requirement that $x^2_{0H} \in (0,1)$, so

$$\text{Region IIc } = \left\{(w,\alpha) : \frac{5-\alpha}{8-\alpha} < w < \frac{(5-\alpha)(9-\alpha)}{8-\alpha}\right\}$$

In equilibria of type IIIc, $x^2_{0H} = 1$ and $\psi^2/x^2_{0H} = 1$ so Consumer 1 no longer holds housing in date 0, and we cannot guess in advance what the price of housing will be in period 0, but must solve for it along with the other variables. Reasoning as above, we see that Consumer 2's date 0 first-order condition and budget constraint are

$$\frac{8-\alpha}{p_{0H} - \alpha} = \frac{1}{x_{0F}}$$
$$p_{0H} - \alpha + x_{0F} = w$$

Solving yields

$$x^2_{0F} = \frac{w}{9-\alpha}\ ,\ p_{0H} = \alpha + \left(\frac{8-\alpha}{9-\alpha}\right) w$$

and hence

$$p_{0H} = \alpha + \left(\frac{8-\alpha}{9-\alpha}\right) w$$

Equilibrium consumptions are

$$x^1 = (18 - \frac{w}{9-\alpha}, 0; 9+\alpha, 0)$$
$$x^2 = (\frac{w}{9-\alpha}, 1; 9-\alpha, 1)$$

and utilities are

$$
\begin{aligned}
u^1 &= 27 + \alpha - \frac{w}{9-\alpha} \\
u^2 &= \log\left(\frac{w}{9-\alpha}\right) + 17 - \alpha
\end{aligned}
$$

Finally, the region in which equilibria are of type IIIc is determined by the requirements that it be optimal for Consumer 2 to borrow the maximum amount possible, whence $x_{0F} \leq 1$, and that Consumer 1 not wish to buy housing, whence $p_{0H} \geq 5$. Putting these together yields:

$$
\text{Region IIIc } = \left\{(w,\alpha) : \left(\frac{5-\alpha}{8-\alpha}\right)(9-\alpha) \leq w \leq (9-\alpha)\right\}
$$

Summarizing these findings and similar calculations for the other regions, we find the regions and equilibria to be:

- **Type Ia**

$$
\begin{aligned}
p_{0H} &= 5 \\
x^1 &= (18-w, 1; 13, 0) \\
\varphi^1 &= 0 \\
x^2 &= (w, 0; 5, 1) \\
\psi^2 &= 0 \\
u^1 &= 32 - w \\
u^2 &= \log w + 9 \\
\text{Region Ia } &= \left\{(w,\alpha) : w \leq \frac{5-\alpha}{8-\alpha}\right\}
\end{aligned}
$$

- **Type Ib** none

- **Type Ic** none

- **Type IIa**

$$
\begin{aligned}
p_{0H} &= 5 \\
x^1 &= (18 - \frac{5}{8}, 1 - (\frac{w}{5} - \frac{1}{8}); 13 - 4(\frac{w}{5} - \frac{1}{8}), 0) \\
\varphi^1 &= 0 \\
x^2 &= (\frac{5}{8}, \frac{w}{5} - \frac{1}{8}; 5 + 4(\frac{w}{5} - \frac{1}{8}), 1) \\
\psi^2 &= 0 \\
u^1 &= 32 - w \\
u^2 &= \log(\frac{5}{8}) + 8 + \frac{8w}{5} \\
\text{Region IIa } &= \left\{ (w, \alpha) : \alpha = 0 \ , \ \frac{5}{8} < w < \frac{45}{8} \right\}
\end{aligned}
$$

- **Type IIb** none

- **Type IIc**

$$
\begin{aligned}
p_{0H} &= 5 \\
x^1 &= \left(18 - \frac{5-\alpha}{8-\alpha}, 1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}; 9 + \alpha \left[\frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha} \right] + 4 \left[1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha} \right], 0 \right) \\
\varphi^1 &- \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha} \\
x^2 &= \left(\frac{5-\alpha}{8-\alpha}, \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}; 9 - \alpha \left[\frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha} \right] - 4 \left[1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha} \right], 1 \right) \\
\psi^2 &= \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha} \\
u^1 &= 32 - w \\
u^2 &= 8 + \log(5-\alpha) - \log(8-\alpha) + \left(\frac{8-\alpha}{5-\alpha} \right) w \\
\text{Region IIc } &= \left\{ (w, \alpha) : \frac{5-\alpha}{8-\alpha} < w < \frac{(5-\alpha)(9-\alpha)}{8-\alpha} \right\}
\end{aligned}
$$

- **Type IIIa**

$$\begin{aligned}
p_{0H} &= \frac{8w}{9} \\
x^1 &= (18 - \frac{w}{9}, 0; 9, 0) \\
\varphi^1 &= 0 \\
x^2 &= (\frac{w}{9}, 1; 9, 1) \\
\psi^2 &= 0 \\
u^1 &= 27 - \frac{w}{9} \\
u^2 &= \log\left(\frac{w}{9}\right) + 17 \\
\text{Region IIIa } &= \{(w, \alpha) : w \geq 9\}
\end{aligned}$$

- **Type IIIb**

$$\begin{aligned}
p_{0H} &= 8 \\
x^1 &= (17, 0; 18 - w, 0) \\
\varphi^1 &= \frac{9 - w}{\alpha} \\
x^2 &= (1, 1; w, 1) \\
\psi^2 &= \frac{9 - w}{\alpha} \\
u^1 &= 35 - w \\
u^2 &= 8 + w \\
\text{Region IIIb } &= \{(w, \alpha) : 9 - \alpha < w < 9\}
\end{aligned}$$

- **Type IIIc**

$$
\begin{aligned}
p_{0H} &= \alpha + \left(\frac{8-\alpha}{9-\alpha}\right) w \\
x^1 &= (18 - \frac{w}{9-\alpha}, 0; 9+\alpha, 0) \\
\varphi^1 &= 1 \\
x^2 &= (\frac{w}{9-\alpha}, 1; 9-\alpha, 1) \\
\psi^2 &= 1 \\
u^1 &= 27 + \alpha - \frac{w}{9-\alpha} \\
u^2 &= \log\left(\frac{w}{9-\alpha}\right) + 17 - \alpha \\
\text{Region IIIc } &= \left\{(w,\alpha) : \left(\frac{5-\alpha}{8-\alpha}\right)(9-\alpha) \leq w \leq (9-\alpha)\right\}
\end{aligned}
$$

As we have already noted, whatever the parameters are, $p_{1H} = 4$ in every equilibrium. Thus, if $\alpha > 4$ an agent who sells (A_α, c) will default, and delivery will be 4 rather than α. Hence equilibrium when $\alpha > 4$ will coincide with equilibrium when $\alpha = 4$. $\diamondsuit$

We have chosen a formulation in which the security promise is specified exogenously and its price is determined endogenously. In the context of home mortgages, a more familiar formulation would specify the down payment requirement (as a fraction of the sale price) exogenously and the interest rate would be determined endogenously. Of course, the two formulations are equivalent: the down payment requirement d, interest rate r, house price p_{0H}, security price q_α and promise α are related by the obvious equations:

$$
\begin{aligned}
d &= \frac{p_{0H} - q_\alpha}{p_{0H}} \\
r &= \frac{\alpha - q_\alpha}{q_\alpha}
\end{aligned}
$$

This example illustrates a number of important points about collateral equilibrium.

- Collateral equilibrium may be inefficient, even though financial markets are "complete". In this example, inefficiency is easy to identify because, as we have already noted, at least over the set of allocations at which both consume date 1 food, the economy displays transferable utility: an allocation is Pareto efficient if and only if the sum of individual utilities is 43 (which is the maximum possible sum); these allocations are precisely those for which Consumer 2 holds all the housing in both dates and exactly one unit of date 0 food; i.e., $x^2_{0H} = x^2_{1H} = 1$ and $x^2_{0F} = 1$. Hence, collateral equilibrium is Pareto efficient only in the portion of region IIIa where $w = 9$, in all of region IIIb, and in the portion of region IIIc where $w = 9 - \alpha$, and when collateral equilibrium is efficient it coincides with Walrasian equilibrium. Moreover, there is an open set of endowment distributions from which *no* collateral equilibrium is efficient. We return to these points in Theorems 3 and 7 below.

- The inefficiency of collateral equilibrium has two sources. Most obviously, collateral requirements *limits* each consumer's borrowing power. This can be seen most extremely in the portion of region IIIa where $w > 9$: Consumer 2's equilibrium marginal utility of food is greater in date 1 than in date 0, so she would like to save, but she can only do so if Consumer 1 borrows — and Consumer 1 can only borrow by holding housing, which he does not wish to do.

  A little less obviously, collateral requirements *distort* consumption decisions, forcing agents who borrow to hold more of the collateral good than they would otherwise wish to do. For instance, fix $w = 7/2$. For $\alpha \in (0, 2)$ parameter values are in region IIc; for $\alpha \in [2, 4]$ parameter values are in region IIIc, but in either case, the collateral requirement leads Consumer 2 to hold excess housing. The simplest way to see this is to compare marginal utilities per dollar for date 0 food and date 0 housing: In region IIc Consumer 2's marginal utility per dollar for date 0 food is $(8-\alpha)/(5-\alpha)$ which is everywhere greater than her marginal utility per dollar from date 0 housing, which is $8/5$. In region IIIc Consumer 2's marginal utility per dollar for date 0 food is $(2/7)(9-\alpha)$

which is everywhere greater than her marginal utility per dollar from date 0 housing, which is $16/[7(\alpha + \frac{8-\alpha}{9-\alpha})]$.

- The same kind of distortion can be seen in prices. Again fix $w = 7/2$. To say that Consumer 2's marginal utility per dollar for date 0 food exceeds her marginal utility per dollar for date 0 housing is the say that the price of date 0 housing is too high. However, Consumer 2 is willing to pay the higher price of date 0 housing because holding housing enables her to borrow; that is, she derives a *collateral value* from housing as well as a consumption value. Consumer 2 finds the marginal utility per dollar for date 0 food to be higher than the marginal utility of making the payments on the security. Just as the price of the collateral is too high, so the price of the security is too high. She therefore sells the security, that is she borrows, up to her collateral limit. As we shall see in Theorem 2 below, , this phenomenon *always* occurs when the collateral requirement binds.

- As we have already noted, collateral requirements have welfare effects, but the directions of these effects may not be obvious. To make the point, fix $w = 7/2$ once again. For $\alpha \in [0, 2)$, increases in α (equivalently, decreases in the down payment requirement) make it possible for Consumer 2 to afford more housing; the net result is Pareto improving. However, for $\alpha \in [2, 4]$, further increases in α (equivalently, further decreases in the down payment requirement) make it possible for Consumer 2 to access more of his/her date 1 income to purchase houses at date 0; competition (of Consumer 2 with his/her self – or of consumers of the same type with each other) drives up the price of date 0 housing (from $p_{0H} = 5$ when $\alpha = 2$ to $p_{0H} = 34/5$ when $\alpha = 4$); this price increase makes Consumer 1 better off but makes Consumer 2 *worse off*.

5 Fundamental Values & the Liquidity Wedge

The purpose of this section is to identify the distortion induced by the necessity to hold collaterl: whenever the collateral constraints are binding, then there must be an agent who pays more for some collateral good, and borrows more by selling some security, than she thinks is merited by their respective "*fundamental values*".

To make this point, fix an economy $\mathcal{E} = \langle (e^i, u^i), \mathcal{A} \rangle$ and a collateral equilibrium $\langle p, q, (x^i, \varphi^i, \psi^i) \rangle$ for $\mathcal{E}$. Assume (for the remainder of this Section) that utility functions u^i are differentiable at the equilibrium consumptions and that each consumer's consumption is non-zero in each spot $s \geq 0$. Consider consumer i. For each state $s \geq 1$ and commodity k, consumer i's marginal utility for good sk is

$$MU^i_{sk} = \frac{\partial u^i(x^i)}{\partial x_{sk}}$$

By assumption, $x_s \neq 0$ so there is some ℓ for which $x^i_{s\ell} > 0$; for any such ℓ, consumer i's *marginal utility of income at state s* is

$$\mu^i_s = \frac{1}{p_{s\ell}} MU^i_{s\ell}$$

Durability means that i's utility for $0k$ has two parts: utility from consuming $0k$ at date 0 consumption and utility from the income derived by selling what $0k$ becomes at date 1; hence we can express marginal utility as:

$$MU^i_{0k} = \frac{\partial u^i(x^i)}{\partial x_{0k}} + \sum_{s=1}^{S} \mu^i_s \, [p_s \cdot F_s(\delta_{0k})]$$

Again, there is some ℓ for which $x^i_{0\ell} > 0$; for any such ℓ, marginal utility of income is at date 0 is

$$\mu^i_0 = \frac{1}{p_{0\ell}} MU^i_{0\ell}$$

The marginal utility of any security (A, c) to consumer i is the utility generated by actual deliveries at date 1

$$MU^i_{(A,c)} = \sum_{s=1}^{S} \mu^i_s \, \textsc{Del}_s(A, c)$$

We define the *fundamental value* of commodity $0k$ and security (A, c) to consumer i as

$$FV^i_{0k} = \frac{MU^i_{0k}}{\mu^i_0}$$
$$FV^i_{(A,c)} = \frac{MU^i_{(A,c)}}{\mu^i_0}$$

To explain the terminology, consider the incomplete markets economy $\check{\mathcal{E}} = \langle (e^i, u^i), \check{\mathcal{A}} \rangle$ with the same consumers but with securities whose promises are

$$\check{A}^j = \text{DEL}(A^j, c^j)$$

If $\langle p, q, (x^i, \varphi^i, \psi^i) \rangle$ were a GEI equilibrium for $\mathcal{E}$ then the first order conditions would imply immediately that for each consumer i, commodity prices weakly exceed fundamental values, with equality for those commodities for which consumption is strictly positive, and coincide with fundamental values:

$$p_{0k} \geq FV^i_{0k} \tag{9}$$
$$q_j = FV^i_{\check{A}^j} \tag{10}$$

Note that (9) holds with equality if i consumes commodity $0k$ but that (10) holds whether or not i is buys or sells security j.

However, in a collateral equilibrium that does not reduce to GEI equilibrium — that is, a collateral equilibrium in which some collateral constraints bind — some prices will *strictly exceed* fundamental values for some consumer. Indeed, there will be at least one consumer who pays more for some commodity than its fundamental value (because that consumer derives value from using that commodity as collateral) *and* who sells some security for more than its fundamental value (because that consumer finds money at date 0 to be more valuable than the deliveries at date 1).

Theorem 2 (Fundamental Values) *Let $\mathcal{E} = \langle e^i, u^i), \mathcal{A} \rangle$ be an economy with collateralized securities and let $\langle p, q, x^i, \varphi^i, \psi^i \rangle$ be an equilibrium for $\mathcal{E}$. Assume that each consumer's equilibrium consumption is non-zero in each spot and that utilities are differentiable at equilibrium consumptions. Then either* *(i) each consumer finds that all date 0 commodities he holds and all securities are priced at their fundamental values and* $\langle p, q, x^i, \varphi^i, \psi^i \rangle$ *is a GEI for the incomplete markets economy* $\langle (e^i, u^i, \check{\mathcal{A}} \rangle$

or

(ii) some consumer finds that some date 0 commodity he holds and some security are priced above *their fundamental value.*

Thus, when collateral constraints are binding, there is a consumer i, a commodity $0k$ and a security j for which the differences $p_{0k} - FV^i_{0k}$ and $q_j - FV^i_{\check{A}^j}$ are strictly positive. We identify this as a *liquidity wedge.*

One consequence of Theorem 2 is that, when collateral constraints are binding, arbitrage efficiency does not hold; that is, prices do not equate ratios of marginal utilities across all consumers. A second consequence is that efficient collateral equilibria are Walrasian.

Theorem 3 (Efficient Collateral Equilibria are Walrasian) *Let* $\mathcal{E} = \langle (e^i, u^i), \mathcal{A} \rangle$ *be an economy with collateralized securities and let* $\langle p, q, x^i, \varphi^i, \psi^i \rangle$ *be an equilibrium for* $\mathcal{E}$*. Assume that each consumer's equilibrium consumption is non-zero in each spot, that utilities are differentiable at equilibrium consumptions, and that at least one consumer's consumption of every good is strictly positive. Then either*

(i) (x^i) *is a Pareto optimal allocation and* $\langle p, q, x^i \rangle$ *is a WE for the economy* $\langle e^i, u^i \rangle$

or

(ii) (x^i) *is not a Pareto optimal allocation and some consumer finds that some date 0 commodity he holds and some security are priced* above *their fundamental value to her.*

6 Default, Efficiency and Crashes

This Section makes a number of related points. The first point is that default — although suggestive of inefficiency — may be welfare enhancing.[9] More precisely, as Example 2 shows, levels of collateral that are socially optimal may lead to default with positive probability. The second point is that there is a link between collateral requirements and future prices. Lower collateral requirements lead buyers to take on more debt; the difficulties of servicing this debt can lead to reduced demand and lower prices — even to crashes — in the future. Importantly, such a crash occurs precisely *because* lower collateral requirements encourage borrowers to take on more debt than they can service. As Example 2 also shows, despite such crashes, lower collateral requirements may be welfare enhancing. The third point is that although the set of securities available for trade is given exogenously as part of the data of the model, the set of securities that are actually traded is determined endogenously at equilibrium. Thus, we may view the financial structure of the economy as chosen by the competitive market. As Example 3 shows, the market may choose levels of collateral that lead to default with positive probability, and this choice may be efficient. Theorem 4 indentifies a context in which the market choice of securities (in particular collateral levels) is *necessarily* efficient.

Example 2 (A Mortgage Market with Uncertainty) We construct a variant on Example 1. Rather than present a full-blown analysis in the style of Example 1, we fix endowments and take only the security promise as a parameter, which makes it easier to focus on the points of interest.

There are two states of nature and two goods: Food, which is perishable, and Housing, which is durable. There are two (types of) consumers, with endowments:

$$e^1 = (29/2, 1; 9, 0; 9, 0)$$
$$e^2 = (7/2, 0; 9, 0; 5/2, 0)$$

[9] A similar point has been made, in different contexts, by Zame (1993), Sabarwal (2003) and Dubey, Geanakoplos, and Shubik (2005).

Consumer 1 has constant marginal utility of consumption for food and housing at each date/state; Consumer 2 has constant marginal utility for housing at each date/state but decreasing marginal utility for date 0 food; both consumers view the states as equally likely:

$$u^1 = x_{0F} + x_{0H} + \frac{1}{2}[x_{1F} + x_{1H}] + \frac{1}{2}[x_{2F} + 3x_{2H}]$$
$$u^2 = \log x_{0F} + 4x_{0H} + \frac{1}{2}[x_F + 4x_{1H}] + \frac{1}{2}[x_{2F} + 4x_{2H}]$$

(Note that endowments and preferences are similar to those in Example 1, except that Consumer 1's marginal utility for housing in state 2 is greater than in state 1 and that Consumer 2 is poor in state 2.)

Suppose that a single security $\mathbf{A}_\alpha = (\alpha p_{1F}, \alpha p_{2F}; \delta_{0H})$, promising the value of α units of food and collateralized by 1 unit of housing, is available for trade; we take $\alpha \in [0, 4]$ as a parameter.[10] (Equivalently, we could consider securities that promise to deliver the value of one unit of food and are collateralized by $1/\alpha$ units of housing.) We distinguish four regions; in each there is a unique equilibium. In Region I, α is sufficiently small that Consumer 2 cannot borrow enough to buy all the housing at date 0, but buys the remaining housing in date 1. In Region II, α is large enough that Consumer 2 can buy all the housing at date 0 but small enough that she will be able to honor her promises in both states at date 1 and retain all the housing at date 1. In Region III, Consumer 2 will honor her promises but will not be able to retain all the housing. In Region IV, Consumer 2 will default. Finally, at the boundary of Regions II and III, equilibrium consumptions are determinate but prices are indeterminate. The calculations in Regions I, II are almost identical to those in Example 1 and are omitted; the calculations for Regions III, IV follow the same method with the appropriate changes to incorporate default, and are sketched.

- **Region I:** $\alpha \in [\mathbf{0}, \mathbf{2})$
  Consumers 1 and 2 both hold date 0 housing; Consumer 2 honors her

[10] As before, the case $\alpha > 4$ reduces to the case $\alpha = 4$.

promises in both states at date 1.

$$\begin{aligned}
p &= (1,5;1,4;1,4) \\
q_\alpha &= \alpha \\
x^1 &= (18 - \frac{5-\alpha}{8-\alpha}, 1 - \frac{7}{2(5-\alpha)} + \frac{1}{8-\alpha}; 9+\alpha, 0; 9+\alpha, 0) \\
x^2 &= (\frac{5-\alpha}{8-\alpha}, \frac{7}{2(5-\alpha)} - \frac{1}{8-\alpha}; 9-\alpha, 1; \frac{5}{2}-\alpha, 1) \\
\varphi^1 &= \frac{7}{2(5-\alpha)} - \frac{1}{8-\alpha} \\
\psi^2 &= \frac{7}{2(5-\alpha)} - \frac{1}{8-\alpha}
\end{aligned}$$

- **Region II:** $\alpha \in [\mathbf{2}, \frac{\mathbf{5}}{\mathbf{2}})$
  Consumer 2 holds all the housing at both dates and honors her promises at both states in date 1.

$$\begin{aligned}
p &= \left(1, \alpha + \left(\frac{7}{2}\right)\left(\frac{8-\alpha}{9-\alpha}\right); 1,4;1,4\right) \\
q_\alpha &= \alpha \\
x^1 &= \left(18 - \left(\frac{7}{2}\right)\left[1 - \left(\frac{8-\alpha}{9-\alpha}\right)\right], 0; 9+\alpha, 0; 9+\alpha, 0\right) \\
x^2 &= \left(\left(\frac{7}{2}\right)\left[1 - \left(\frac{8-\alpha}{9-\alpha}\right)\right], 1; 9-\alpha, 1; \frac{5}{2}-\alpha, 1\right) \\
\varphi^1 &= 1 \\
\psi^2 &= 1
\end{aligned}$$

- **Boundary between Regions II, III:** $\alpha = \frac{\mathbf{5}}{\mathbf{2}}$
  Consumer 2 holds all the date 0 housing and honors her promise in date 1; in state 2 this leaves consumer 2 with all the housing and no

food, whence p_{2H} is indeterminate.

$$
\begin{aligned}
p &= \left(1, \frac{5}{2} + \left(\frac{7}{2}\right)\left(\frac{11}{13}\right); 1, 4; 1, p_{2H}\right) \\
p_{2H} &\in [3, 4] \\
q_\alpha &= \alpha \\
x^1 &= \left(9 + \left(\frac{7}{2}\right)\left(\frac{11}{13}\right), 0; \frac{23}{2}, 0; \frac{23}{2}, 0\right) \\
x^2 &= \left(\frac{5}{2} - \left(\frac{7}{2}\right)\left(\frac{11}{13}\right), 1; \frac{13}{2}, 1; 0, 1\right)
\end{aligned}
$$

- **Region III:** $\alpha \in (\mathbf{\frac{5}{2}}, \mathbf{3}]$
  Consumer 2 holds all the date 0 housing and honors her promise in the good state. In the bad state, the price of housing falls to $p_{2H} = 3$; Consumer 2 has assets of $\frac{5}{2} + 3$ (endowment plus housing) and liabilities of α (the security promise), so sells the house, repays her debt, and then buys all the housing she can afford at the price $p_{2H} = 3$; Consumer 1 buys the remaining housing.

$$
\begin{aligned}
p &= \left(1, \alpha + \left(\frac{7}{2}\right)\left(\frac{8-\alpha}{9-\alpha}\right); 1, 4, 1, 3\right) \\
q_\alpha &= \alpha \\
x^1 &= \left(\frac{29}{2} + \left(\frac{7}{2}\right)\left(\frac{8-\alpha}{9-\alpha}\right), 0; 9+\alpha, 0; \frac{23}{2}, 1 - \frac{\frac{11}{2} - \alpha}{3}\right) \\
x^2 &= \left(\frac{5}{2} - \left(\frac{7}{2}\right)\left(\frac{8-\alpha}{9-\alpha}\right), 1; 9-\alpha, 1; 0, \frac{\frac{11}{2} - \alpha}{3}\right)
\end{aligned}
$$

- **Region IV** $\alpha \in [\mathbf{3}, \mathbf{4}]$
  Consumer 2 holds all the date 0 housing and honors her promise in the good state, but defaults in the bad state (delivering the house instead of the promise α). After the default, Consumer 2 buys back all the housing she can afford (less than the available quantity of housing) at

the price $p_{2H} = 3$.

$$p = \left(1, \frac{\alpha+3}{2} + \left(\frac{5}{2}\right)\left(\frac{8-\frac{\alpha+3}{2}}{9-\frac{\alpha+3}{2}}\right); 1, 4; 1, 3\right)$$

$$q_\alpha = \frac{\alpha+3}{2}$$

$$x^1 = (\frac{29}{2} + \frac{5}{2}\left(\frac{8-\frac{\alpha+3}{2}}{9-\frac{\alpha+3}{2}}\right), 0; 9+\alpha, 0; \frac{23}{2}, \frac{1}{6}) \qquad x^2 = (\frac{5}{2} - \frac{5}{2}\left(\frac{8-\frac{\alpha+3}{2}}{9-\frac{\alpha+3}{2}}, 1; 9-\alpha, 1; 0, \frac{5}{6}\right)$$

Consumers 1 and 2 have constant and equal marginal utilities for food in state 1, so this is a transferable utility economy and we may identify social welfare with the sum of individual utilities. Direct computation shows that

- $0 \le \alpha < 2$: welfare of both Consumer 1 and Consumer 2 is increasing, and social welfare is increasing
- for $2 < \alpha \le 4$: welfare of Consumer 1 is increasing and welfare of Consumer 2 is decreasing, and social welfare is increasing

In particular, social welfare attains its maximum when $\alpha = 4$, so *collateral levels that lead to default with positive probability are socially efficient.* ♢

In our framework, the set of securities available for trade is given exogenously, but the set of securities actually traded is determined endogenously at equilibrium. Because the former set might be very large — conceptually, all conceivable securities — we can view the action of the market as determining the observed security structure. In the present context, the market chooses the most efficient collateral requirements even though those requirements lead to default in the bad state.

Example 3 (Which Securities are Traded?) We maintain the entire structure of Example 2, except that some set $\{\mathbf{A}_\alpha\}$ of securities is available for trade where as above $\mathbf{A}_\alpha = (\alpha p_{1F}, \alpha p_{2F}; \delta_{0H})$ promises the value of α units of food and is collateralized by one unit of housing (Equivalently: a finite number of collateral requirements are possible.) To be consistent with our framework, we assume the set of available securities (the range of collateral requirements offered) is finite, but, at least conceptually, we might imagine that *all possible* collateral requirements are offered. We assert that at any equilibrium, only the security whose promise is greatest — equivalently, the security with the lowest collateral requirement — is traded.

To see this, consider an environment in which A_β, A_γ are available, where $\beta < \gamma$, and suppose A_β is traded. Only Consumer 2 sells securities (borrows), so Consumer 2's equilibrium plan involves selling some positive amount of A_β. Consider the alternative plan for Consumer 2 which involves selling ε fewer shares of A_β and ε more shares of A_γ. Given the specified endowments, Consumer 2's equilibrium consumption must be non-zero in date 0 and in both states at date 1, so if ε is small enough, this alternative plan is feasible. Moreover, because securities are priced at their expected payoffs, this alternative plan is preferred if Consumer 2's marginal utility for income in date 0 exceeds his expected marginal utility for income in date 1.

To see that this is indeed the case, first estimate marginal utility of income in each state. In state 1, prices are $p_{1F} = 1, p_{1H} = 4$ so Consumer 2's marginal utility of income is 1. In state 2, prices are $p_{2F} = 1, p_{2H} \geq 3$ so Consumer 2's marginal utility of income is at most 4/3. Hence expected marginal utility of income in date 1 is at most 7/6. Marginal utility of income in date 0 is the maximum of marginal utility per dollar for food and marginal utility per dollar for housing. The former exceeds 7/6 unless $x^2_{0F} \geq 6/7$. If Consumer 1 holds any housing at all, then $p_{0H} = 5$, so if $x^2_{0F} \geq 6/7$ then the marginal utility of a dollar of housing to Consumer 2 is greater than the marginal utility of a dollar of food, which is a contradiction. Hence Consumer 1 holds no housing, so that $x^2_{0H} = 1$ and

$$p_{0H} = p_{0H} x^2_{0H} \leq \frac{7}{2} + 4 - x^2_{0F} \leq \frac{15}{2} - \frac{6}{7} = \frac{93}{14}$$

Hence marginal utility of housing per dollar is at least $8/(93/14) = 112/93 > 7/6$. Hence marginal utility of income at date 0 is greater than expected marginal utility of income at date 1. Thus, the alternative plan is preferred, which contradicts optimality of equilibrium plans. We conclude that A_β is not traded, as asserted. $\diamondsuit$

In this environment at least, the market chooses efficient collateral levels — even though those collateral levels may lead to default. Characterizing economies when the market does or does not choose efficient collateral levels seems an important and difficult question, to which we do not know the answer. (Indeed, because multiple equilibria are possible, it is not entirely clear precisely how to formulate the question.) However, the answer is affirmative in at least one important case: if date 1 prices do not depend on collateral levels.

Theorem 4 (Constrained Optimality) *Every set of collateral equilibrium plans is Pareto optimal among all sets of plans that:*

(a) are socially feasible;

(b) given whatever date 0 decisions are assigned, respect each consumer's budget set at every state s at date 1 at the given *equilibrium prices;*

(c) call for deliveries on securities that are the minimum of the promise and the value of collateral.

In particular, sequestering securities cannot lead to a Pareto improvement unless date 1 prices change; if date 1 prices do not change, the market chooses the security structure efficiently. In particular, if only one good is available for consumption at date 1, then collateral equilibrium is constrained efficient and the market chooses the security structure efficiently; compare Kilenthong (2006).

7 Securitization

Securitization usually refers to the process of converting non-tradable assets into tradable securities through the repacking of their cash flows (Elul, 2005). More generally, we may think of securitization as the process of creating securities – we shall refer to them as *security pools* – that are collateralized by other securities. In general, the securities used as collateral might in turn be collateralized by other securites, and so forth through many layers, but for our purposes it shall be enough to allow for only a single layer; we leave the straightforward generalization to the interested reader. This section presents the formal model; discussion and applications to welfare are discussed in Section 8

Fix commodities and a family $\mathcal{A} = \{(A^1, c^1), \dots, (A^J, c^J)\}$ of collateralized securities. A *security pool* is a tuple $\mathbf{B} = (B^1, \dots, B^T; \chi)$ where each *tranche* B^t is a promise of delivery as a function of prices, and $\chi = (\chi_), \chi_1) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+$ (a bundle of commodities and a portfolio of securities) is the collateral requirement. It is convenient to write:

$$\textsc{Del}(\chi; s, p) = p \cdot \chi_0 + \textsc{Del}(\chi_1; s, p)$$

for the delivery of the collateral requirement $\chi = (\chi_0, \chi_1)$. We interpret the promise B^t as senior to B^{t+1}, so actual deliveries may be defined by induction:

$$\textsc{Del}(B^1; s, p) = \min\Big\{B^1(s, p_s),\ \textsc{Del}(\chi; s, p)\Big\}$$
$$\textsc{Del}(B^{t+1}; s, p) = \min\left\{B^{t+1}(s, p_s),\ \textsc{Del}(\chi; s, p) - \sum_{t'=1}^{t} \textsc{Del}(B^{t'}; s, p)\right\}$$

Note that the delivery on each of the promises lies (weakly) between 0 and the delivery on the collateral. There is no loss in assuming that all pools have the same number of tranches (because we can always add tranches that promise 0 delivery).

If $\mathcal{B} = \{\mathbf{B}^1, \dots, \mathbf{B}^{\mathbf{K}}\}$ is the set of available security pools, a portfolio of security tranches is a vector $\Theta \in \mathbb{R}^{KT}_+$; the delivery on Θ is

$$\textsc{Del}(\Theta; s, p) = \sum_k \sum_t \Theta^{kt} \textsc{Del}(\mathbf{B}^{kt}; s, p)$$

An *economy with collateralized securities and security pools* is a tuple $\mathcal{E} = \langle (e^i, u^i), \mathcal{A}, \mathcal{B} \rangle$.

For each k, t, we write Q^{kt} for the price of tranche B^{kt} of pool $\mathbf{B}^k$, and Φ^{kti}, Ψ^{kti} for consumer i's purchases and sales of this tranche. Given spot prices p, security prices q and tranche prices $Q = (Q^{kt})$, the *budget set* of a consumer whose endowment is e is the set of plans $(x, \varphi, \psi, \Phi, \Psi)$ (for consumption, security purchases, security sales, tranche purchases and tranche sales) that satisfy the budget constraints at date 0 and in each state at date 1 and the collateral constraints at date 0:

- **At date 0**

$$p_0 \cdot x_0 + q \cdot \varphi + Q \cdot \Phi \leq p_0 \cdot e_0 + q \cdot \psi + Q \cdot \Psi$$

$$x_0 \geq \sum_j \psi_j c^j$$
$$\varphi \geq \sum_k \max_t \Psi^{kt} \chi^k$$

  That is, expenditures for consumption, security purchases and pool purchases do not exceed income from endowment, security sales and pool sales, date 0 consumption includes collateral for all security sales and date 0 security purchases include collateral for all pool sales. Note that, as intended, holding the collateral χ^k is sufficient to collateralize sales of one unit of *each* tranche of pool $\mathbf{B}^k$.

- **In state s**

$$p_s \cdot x_s + \textsc{Del}(\varphi; s, p) + \textsc{Del}(\Phi; s, p) \leq p_s \cdot e_s + p_s \cdot F_s(x_0) + \textsc{Del}(\psi, s, p) + \textsc{Del}(\Psi; s, p)$$

That is, expenditures for consumption and deliveries on securities and pools do not exceed income from endowment, from the return on durable goods, and from deliveries on security promises and pool promises.

A *pool equilibrium* for such an economy consists of spot prices $p \in \mathbb{R}_+^{L(1+S)}$, security prices $q \in \mathbb{R}_+^J$, pool prices $Q \in \mathbb{R}_+^{KT}$ and consumer plans $(x^i, \varphi^i, \psi^i, \Phi^i, \Psi^i)$ satisfying the conditions:

- **Commodity Markets Clear**

$$\sum_i x^i = \sum_i e^i + \sum_i F(e_0^i)$$

- **Security Markets Clear**

$$\sum_i \varphi^i = \sum_i \psi^i$$

- **Pool Markets Clear**

$$\sum_i \Phi^i = \sum_i \Psi^i$$

- **Plans are Budget Feasible**

$$(x^i, \varphi^i, \psi^i) \in B(p, q, Q; e^i, \mathcal{A}, \mathcal{B})$$

- **Consumers Optimize**

$$(x, \varphi, \psi, \Phi^i, \Psi^i) \in B(p, q, Q; e^i, \mathcal{A}, \mathcal{B}) \Rightarrow u^i(x) \leq u^i(x^i)$$

It is natural to think of security pools as assembled by intermediaries who purchase all the collateral and then sells some of the tranches, holding the rest themselves.

Our model of security pools satisfies the basic consistency requirement that equilibrium exists.

Theorem 5 ***(Existence of Pool Equilibrium)*** *Every economy with collateralized securities and security pools, satisfying Assumptions 1-3 (in Section 2) admits an equilibrium.*

Our model incorporates three distinct processes: *pyramiding* (the use collateralized securities to collateralize further securities), *pooling* (the combining of bundling of collateral goods and securities to collateralize different loans) and *tranching* (the using collateral goods and securities to collateralize several securities). Section 8 shows how these processes operate when used together (in our environment) but a brief informal discussion may guide the reader.

- To see how pyramiding could be useful, imagine an economy with one consumption good and three states at date 1. Suppose there is a durable good (houses today) yields consumption in quantities (2,1,1) in the three states. Agent 0 has utility for date 0 housing, agent 1 only wants to consume in state 1, and agent 2 (who is very risk averse) wants to smooth consumption perfectly in date 1. Suppose further that in the initial condition of society, only riskless promises (i.e., promises of the form (a, a, a) can be written. If agent 0 owns the house and sells off a promise of (1,1,1) to agent 2, then agent 0 gets stuck consuming 1 in state 1 tomorrow. On the other hand, if agent 1 owns the house and sells of the promise (1,1,1) to agent 2, then the right agent gets consumption of 1 in state 1 tomorrow, but the house is in the wrong hands. With pyramiding, agent 0 could own the house and sell off promise (2,2,2) to agent 1. Agent 1 could use that promise – which delivers (2,1,1) – as collateral for a futher promise of (1,1,1) to agent 2. This achieves the efficient allocation of getting 0 to live in the house, agent 1 to consume 1 in state 1, and agent 2 to consume (1,1,1) in the three states tomorrow. (We might think of agent h=0 as a homeowner, agent 1 as a speculator, and agent 2 as the risk averse lender.) We see that pyramiding, combined with default, allows for a socially superior allocation.
- To see how pooling is useful, imagine a variant of the previous example

in which there are two houses and two potential homeowners 0 and 0'. Suppose the first house pays (1,1,0) and the second house pays (1,0,1) in the three states. The optimal allocation is achieved when 0 buys the first house and using it as collateral sells the promise (1,1,1), thereby delivering (1,1,0). Agent 0' buys the second house and using it as collateral sells the promise (1,1,1), delivering (1,0,1). Agent 1 buys both promises, pooling them together as collateral to back the promise (1,1,1), which delivers fully and is sold to agent 2, leaving agent 1 with the residual payoff of (1,0,0). Pooling the promises allowed for the diversification that made the pool able to fully cover the (1,1,1) promise. Note that the houses could not directly be pooled together, because they need to be owned by separate homeowners. This example illustrates the power of say subprime mortgage pools to enable homeowners to borrow the money to buy houses to live in, while dividing the mortgage cash flows between speculators and risk averse agents. In states 2 and 3 one homeowner defaults, but at the pool level the promise is kept.[11]

- Tranching allows the same collateral is used to back more than one loan or tranche. With more than one loan depending on the same collateral, a seniority is required to define the payoffs. Consider the first example in which the homeowner buys the house and using it as collateral issues a senior promise (first mortgage) promising (1,1,1) and a junior tranche (second mortgage) also promising (1,1,1). The senior tranche will fully deliver (1,1,1) and be bought by agent 2, and the junior tranche will deliver (1,0,0) and be bought by agent 1.

[11]An example in keeping with how events unfolded over the past two years, as opposed to how they were meant to unfold in theory, would involve a fourth state in which many both house payoffs are 0, forcing two defaults at the homeowner level as well as default at the pool level.

8 Securitization and Efficiency

We argue here that, in a world in which all securities must be collateralized, securitization promotes efficiency but that there are robust situations in which efficiency cannot be obtained. To make these points we begin with a simple observation.

Theorem 6 (Net Savers) *If $\langle p, q, Q, (x^i, \varphi^i, \psi^i, \Phi^i, \Psi^i \rangle$ is an equilibrium for the economy $\langle (e^i, u^i), \mathcal{A}, \mathcal{B} \rangle$ then each consumer's future expenditures must exceed his/her unpledgeable income in every future state; that is,*

$$p_s \cdot x_s^i \geq\geq p_s e_s^i$$

for each consumer i and state s.

This simple theorem has a striking negative consequence for efficiency: provided we rule out avoid corner solutions, inefficiency is a robust phenomenon – independently of consumer preferences and the availability of securities and security pools.

Theorem 7 (Robust Inefficiency) *Fix a positive social endowment $e \geq 0$ and smooth utility functions (u^i) that are strictly monotone and satisfy the boundary condition.*[12] *There is an non-empty open subset Ω of the set of endowment profiles $\{(e^i) : \sum e^i = e\}$ with the property that* no *collateral equilibrium from any endowment profile in Ω can be Pareto optimal, no matter what securities and security pools are available for trade.*

On the other hand, any allocation that can be supported as a Walrasian equilibrium and that Theorem 6 does not rule out as occurring in a collateral equilibrium can in fact be obtained whenever "enough' securities and security pools are available.[13]

[12] That is, indifference curves through any point in the strictly positive orthant lie entirely in the strictly positive orthant; Debreu (1972).

[13] As the proof shows, we need only a very simple set of security pools whose tranches are Arrow securities.

Theorem 8 (Supporting Walrasian Equilibrium) *If $\langle \tilde{p}, (x^i) \rangle$ is a Walrasian equilibrium for the economy $\langle (e^i, u^i) \rangle$, and each consumer is a net saver in the sense that*

$$\tilde{p}_s \cdot (x^i_s - e^i_s) \geq 0$$

for each consumer i and state s, then there is a family $\mathcal{A}^*$ of collateralized securities and a family $\mathcal{B}^*$ of security pools such that if $\mathcal{A}$ is any family of collateralized securities and $\mathcal{B}$ is any family of security pools for which $\mathcal{A} \supset \mathcal{A}^*$ and $\mathcal{B} \supset \mathcal{B}^*$ then there is an equilibrium $\langle p, q, Q, (x^i, \varphi^i, \psi^i, \Phi^i, \Psi^i) \rangle$ for the economy $\langle (e^i, u^i), \mathcal{A}, , \mathcal{B} \rangle$ with the same consumptions (and the same commodity prices) as the given Walrasian equilibrium.

A simple example illustrates Theorem 8 and the reason why tranching is required for efficency.

Example 4 (Security Pools and Walrasian Equilibrium) We consider another variant of Example 1. There are two states of nature, two goods (Food and Housing), and four consumers. Each consumer assigns equal probability to the two states in date 1. Consumer 1 owns the housing and is risk neutral; Consumer 2 likes housing much more than other consumers; Consumers 3, 4 care only about food and have an insurance motive. We take the supply of housing $h \in [0, 5]$ as a parameter. Endowments and utilities are

$$\begin{aligned}
e^1 &= (12, h; 16, 0; 16, 0) \\
u^1 &= x_{0F} + x_{0H} + \frac{1}{2}[x_{1F} + x_{1H}] + \frac{1}{2}[x_{2F} + x_{2H}] \\
e^2 &= (9, 0; 32, 0; 32, 0) \\
u^2 &= \log(x_{0F}) + 4x_{0H} + \frac{1}{2}[x_{1F} + 4x_{1H}] + \frac{1}{2}[x_{2F} + 4x_{2H}] \\
e^3 &= (12, 0; 8, 0; 0, 0) \\
u^3 &= \log(x_{0F}) + \frac{1}{2}\log(x_{1F}) + \frac{1}{2}\log(x_{2F}) \\
e^4 &= (12, 0; 0, 0; 8, 0) \\
u^4 &= \log(x_{0F}) + \frac{1}{2}\log(x_{1F}) + \frac{1}{2}\log(x_{2F})
\end{aligned}$$

Walrasian prices and utilities are unique but equilibrium allocations are indeterminate:

$$\begin{aligned}\tilde{p} &= (1, 8; \frac{1}{2}(1, 4); \frac{1}{2}(1, 4)) \\ x^1 &= (28, 0; 8h + \zeta, 0; 8h - \zeta, 0) \\ x^2 &= (1, h; 40 - 8h - \zeta, h; 40 - 8h + \zeta, h) \\ x^3 &= (8, 0; 8, 0; 8, 0) \\ x^4 &= (8, 0; 8, 0; 8, 0)\end{aligned}$$

for $\zeta \in [0, \min(8h, 40 - 8h)]$.

For which values of h, ζ can this equilibrium be supported as a collateral equilibrium for an appropriate choice of securities and security pools? In view of Theorems 6 and 8, it is necessary and sufficient that each consumer be a net saver. This requirement is automatically satisfied for Consumers 3 and 4; for Consumers 1, 2 the requirement imposes inequalities. In particular, we conclude that

- for $h \in [0, 2)$ and for $h \in (2, 5]$ there is *no* Walrasian equilibrium that can be supported as a pool equilibrium
- for $h = 2$ there is at least one Walrasian equilibrium that can be supported as a pool equilibrium (namely the one with $\zeta = 0$)
- for $h = 2$ there are also Walrasian equilibria that *cannot* be supported as a pool equilibrium (those with $\zeta \neq 0$)

(For $h \in (2, 5]$, Consumer 2 is too poor at date 0, and cannot borrow enough to buy all the housing. For $h \in [0, 2)$, Consumer 2 is too rich at date 0, and cannot save because saving requires that some other consumer borrow — but borrowing would require some other consumer to hold housing.) The pool equilibria that support Walrasian equilibria are easy to describe: Consumer 2 buys all the housing, using it to collateralize the loan; Consumer 1 uses the housing loans (i.e., the mortgages) to collateralize a security pool with two tranches, each promising to deliver the value of 8 units of food in each state; Consumers 3 and 4 each buy one of these tranches.

It is instructive to see why collateralized securities alone are not sufficient to support *any* of the Walrasian equilibria, including the symmetric equilibrium when $h = 2$. To support a Walrasian equilibrium, Consumers 3 and 4 must each buy insurance that pays 8 units of account in a diffferent state in date 1. Buying insurance amounts to lending, and loans must be collateralized; if the loans must be collateralized by durable goods then *each* of these loans must be collateralized by at least two houses — so three houses cannot collateralize both loans. Security pools "solve" this problem by making it possible for the same houses to collateralize both loans. $\diamondsuit$

9 Conclusion

Collateral requirements are almost omnipresent in modern economies, but the effects of these collateral requirements have received little attention except in circumstances where there is actual default. This paper has argued that collateral requirements ahve important effects on every aspect of the economy — even when there is no default. Collateral requirements inhibit lending, limit borrowing, and distort consumption decisions. The shortage of collateral leads to financial innovations that stretch the available collateral. But even after all possible financial innovations, in the presence of collateral requirements, robust inefficiency is an inescapable possibility.

The model offered here abstracts away from many transaction costs, informational asymmetries, and many other frictions that play an important role in real markets. It also restricts attention to a two-date world, and so does not address issues such as default at intermediate dates. All these are important questions for later work.

Appendix

Proof of Theorem 1 In constructing an equilibrium for $\mathcal{E} = ((e^i, u^i), \mathcal{A})$, we must confront the possibility that security promises, and hence deliveries, may be 0 at some commodity spot prices.[14] (An option to buy gold at \$400/ounce will yield 0 in every state if the the spot price of gold never exceeds \$400/ounce.) Because of this, the argument is a bit delicate. We construct, for each $\rho > 0$, an auxiliary economy $\mathcal{E}^\rho$ in which security promises are bounded below by ρ; in these auxiliary economies, equilibrium security prices will be different from 0. We then construct an equilibrium for $\mathcal{E}$ by taking limits as $\rho \to 0$.

For each $s = 0, 1, \ldots, S$, choose and fix an arbitary *price level* $\beta_s > 0$. (Because promises are functions of prices, choosing price levels is not the same thing as choosing price normalizations, and we *do not* assert that equilibrium is independent of the price levels — only that for every set of price levels there exists an equilibrium.) Write

$$\Delta_s = \{(p_{s\ell}) \in \mathbb{R}^L_{++} : \sum_\ell p_{s\ell} = \beta_s\}$$

$$\Delta = \Delta_0 \times \ldots \times \Delta_S$$

Write $\mathbf{1}_0 = (1, \ldots, 1) \in R^L_+$ and define

$$\mathbb{Q} = \{q \in \mathbb{R}^J_+ : 0 \leq q^j \leq 2\beta_0 \mathbf{1}_0 \cdot c^j\}$$

We construct equilibria (for the auxiliary economies and then for our original economy) with commodity prices in Δ and security prices in $\mathbb{Q}$.

For each $\rho > 0$, define an security $(A^{\rho j}, c^j)$ whose promise is:

$$A^{\rho j} = A^j + \rho$$

Let $\mathcal{A}^\rho = \{(A^{\rho 1}, c^1) \ldots, (A^{\rho J}, c^J)\}$. Define the auxiliary economy $\mathcal{E}^\rho = \langle (e^i, u^i), \mathcal{A}^\rho \rangle$, so $\mathcal{C}^\rho$ differs from $\mathcal{E}$ only in that security promises have been increased by ρ in every state and for all spot prices.

[14] Non-trivial collateral requirements imply that if promises are not 0 then deliveries cannot be 0 either.

We first construct truncated budget sets and demand and excess demand correspondences in this auxiliary economy. By assumption, collateral requirements for each security are non-zero. Choose a constant μ so large that, for each j,

$$\mu c^j \not\leq \bar{e}_0$$

(Thus, to sell μ units of the security $\mathbf{A}^{\rho j}$ would require more collateral than is actually available to the entire economy.) For each $(p,q) \in \Delta \times \mathbb{Q}$ and each consumer i, define the truncated budget set

$$B_0^i(p,q) = \{\pi \in B^i(p,q,e^i\tilde{\mathcal{A}}^\rho) : 0 \leq \varphi^{ij} \leq \mu I \ , \ 0 \leq \psi^{ij} \leq \mu I \text{ for each } j\}$$

and the individual truncated demand correspondence

$$d^i(p,q) = \{\pi = (x,\varphi,\psi) \in B_0^i(p,q) : \pi \text{ is utility optimal in } B_0^i(p,q)\}$$

(Note that truncated demand exists at every price (p,q), because we bound security purchases and sales. Absent such a bound, demands would certainly be undefined at some prices. For instance, if $q^j = 2\beta_0\mathbf{1}_0 \cdot c^j$, agents could sell $A^{\rho j}$ for enough to finance the purchase of its collateral requirement c^j, so there would be an unlimited arbitrage. Bounding security sales bounds the arbitrage.) Write

$$D(p,q) = \sum_i d^i(p,q)$$

for the aggregate demand correspondence.

For each plan π, we define security excess demand and commodity excess demands $z_s(\pi)$ in each spot :

$$z_a(\pi) = \varphi - \psi$$
$$z_s(\pi) = x_s - \bar{e}_s$$

Write

$$z(\pi) = (z_0(\pi), \ldots, z_S(\pi); z_a(\pi)) \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J$$

and define the aggregate excess demand correspondence

$$Z : \Delta \times \mathbb{Q} \to \mathbb{R}^{L(1+S)} \times \mathbb{R}^J$$
$$Z(p,q) = z(D(p,q))$$

It is easily checked that $Z(p,q)$ is non-empty, compact, and convex for each p,q and that the correspondence Z is upper hemi-continuous. Because consumptions security sales are bounded, Z is also bounded below. Because utility functions are monotone, a familiar argument (Debreu (1959)) shows that Z satisfies the usual boundary condition:

$$||Z(p,q)|| \to \infty \text{ as } (p,q) \to \text{bdy}\Delta \times \mathbb{Q}$$

(It doesn't matter which norm we use.)

Now fix $\varepsilon > 0$, and set

$$\Delta^\varepsilon = \{p \in \Delta : p_{s\ell} \geq \varepsilon \text{ for each } s, \ell\}$$

Because Z is an upper hemi-continuous correspondence, it is bounded on $\Delta^\varepsilon \times \mathbb{Q}$; set

$$\mathbb{Z}^\varepsilon = \{z \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J : ||z|| \leq \sup_{(p,q)\in\Delta^\varepsilon\times Q} ||Z(p,q)||\}$$

Define the correspondence

$$F^\varepsilon : \Delta^\varepsilon \times \mathbb{Q} \times \mathbb{Z}^\varepsilon \to \Delta^\varepsilon \times Q \times \mathbb{Z}^\varepsilon$$
$$F^\varepsilon(p,q,z) = \text{argmax}\,\{(p^*,q^*)\cdot z : (p^*,q^*) \in \Delta^\varepsilon \times Q\} \times Z(p,q)$$

For prices $(p,q) \in \Delta \times \mathbb{Q}$ and a vector of excess demands $z \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J$, $(p,q)\cdot z$ is the value of excess demands. We caution the reader that, in this setting, Walras' law need not hold for arbitrary prices: the value of excess demand need not be 0. We shall see, however, that the value of excess demand *is* 0 at the prices we identify as candidate equilibrium prices

Our construction guarantees that F^ε is an upper-hemicontinuous correspondence, with non-empty, compact convex values. Kakutani's theorem guarantees that F^ε has a fixed point. We assert that for some $\varepsilon_0 > 0$ sufficiently small, the correspondences $F^\varepsilon, 0 < \varepsilon < \varepsilon_0$ have a *common fixed point*. To see this, write $G^\varepsilon \subset \Delta^\varepsilon \times Q \times \mathbb{Z}^\varepsilon$ for the set of all fixed points of F^ε; G^ε is a non-empty compact set. We show that for some $\varepsilon_0 > 0$ sufficiently small, the sets G^ε are nested and decrease as ε decreases; that is, $G^{\varepsilon_1} \subset G^{\varepsilon_2}$ whenever $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$.

To see this, note first that security deliveries are bounded, because deliveries never exceed the value of collateral. Hence individual expenditures at budget feasible plans (and in particular at plans in the truncated demand set) are bounded, independent of prices (because income from endowments is bounded, security prices and sales are bounded, and security purchases and deliveries are bounded). Choose an upper bound $M > 0$ on individual expenditures at budget feasible plans. Because commodity demands are non-negative, individual excess demands are bounded below; choose a lower bound $-R < 0$ on individual exess demands.

Because excess demand is the sum of individual demands less the sum of endowments, it follows that if $z \in Z(p,q)$ then

$$\begin{array}{rcll} (p,q)\cdot z & \leq & MI & \\ z_{s\ell} & \geq & -RI & \quad \text{for each commodity } s\ell \end{array}$$

A familiar argument (based on strict monotonicity or preferences) shows that if commodity prices tend to the boundary of Δ then aggregate commodity excess demand blows up. If the price of some security tends to 0 but the value of its collateral does not, then deliveries on that security do not tend to 0, whence demand for that security and consequent aggregate commodity excess demand again blow up. Hence we can find $\varepsilon_0 > 0$ such that if $(p,q) \in \Delta \times \mathbb{Q}$, $z \in Z(p,q)$, and $p_{s_0\ell_0} < \varepsilon_0$ for some spot s_0 and commodity ℓ_0 then there is some spot s_1 and commodity ℓ_1 such that

$$z_{s_1\ell_1} > \frac{1}{\beta_{s_1} - (L-1)\varepsilon}\left[MI + \varepsilon R(L-1) + (\max_s \beta_s)RI\right] \tag{11}$$

We assert that if $0 < \varepsilon < \varepsilon_0$ then $G^\varepsilon \subset \Delta^{\varepsilon_0} \times \mathbb{Q} \times \mathbb{Z}^{\varepsilon_0}$. To see this, suppose that $(p,q,z) \in G^e p$ and $p \notin \Delta^{\varepsilon_0}$. Define $\tilde{p} \in \Delta$ by

$$\tilde{p}_{s\ell} = \begin{cases} \varepsilon & \text{if } s = s_0, \ell \neq \ell_0 \\ \beta_s - (L-1)\varepsilon & \text{if } s = s_0, \ell = \ell_0 \\ \beta_s / L & \text{otherwise} \end{cases}$$

Direct calculation using equation (11) shows that $(\tilde{p},0)\cdot z > MI$, which is a contradiction. We conclude that $p \in \Delta^{\varepsilon_0}$ and hence that $(p,q,z) \in G_0^\varepsilon$ as desired.

The definition of F^ε implies that if $0 < \varepsilon_1 < \varepsilon_2$ and $G^{\varepsilon_1} \subset \Delta^{\varepsilon_2} \times \mathbb{Q} \times \mathbb{Z}^{\varepsilon_2}$ then $G^{\varepsilon_1} \subset G^{\varepsilon_2}$. Hence, for $0 < \varepsilon < \varepsilon_0$ the sets G^ε are nested and decrease as ε decreases. A nested family of non-empty compact sets has a non-empty intersection so we may define the non-empty set G:

$$G = \bigcap_{\varepsilon < \varepsilon_0} G^\varepsilon$$

Let $(p, q, z) \in G$; we assert that $z = 0$ and that p, q constitute equilibrium prices for the economy $\mathcal{E}^\rho$.

We first show that excess security demand $z_a = 0$. If the excess demand for security j were strictly positive, the requirement that (p, q) maximize the value of excess demand would imply that q_j is as big as possible: $q_j = 2\beta_0 \mathbf{1}_0 \cdot c^j$. But then agents could sell $A^{\rho j}$ for enough to finance the purchase of the collateral requirement, whence the excess demand for $A^{\rho j}$ would be negative, a contradiction. We conclude that security excess demand must be non-positive. If the excess demand for security j were strictly negative, the requirement that (p, q) maximize the value of excess demand would imply that q_j is as small as possible: $q_j = 0$. But if the price of $A^{\rho j}$ were 0 then every agent would wish to buy it because its delivery would be $\min\{\rho, p_s \cdot F_s(c^j)\} > 0$. Hence the excess demand for A^ρ_j must be positive, a contradiction.[15] We conclude that $z_a = 0$.

We show next that Walras' law holds for the prices p, q and the excess demand z: $(p, q) \cdot z = 0$. To see this, choose individual demands $\pi^i \in d^i(p, q)$ with the property that the corresponding aggregate excess demand is z:

$$Z(\sum_i \pi^i) = z$$

For each agent i, the plan π^i lies in the budget set at prices (p, q), so the date 0 expenditure required to carry out the plan π^i is no greater than the value of date 0 endowment. Because utility is strictly monotone in date 0 perishable commodities and in all commodities in state s, optimization implies that all

[15]Note that we could not obtain this conclusion in the original economy, because, at the prices (p, q) the security A^j might promise 0 in every state.

individuals spend all their income at date 0, so we conclude that the date 0 expenditure required to carry out the plan π^i is precisely equal to the value of date 0 endowment. Put differently, the value of date 0 excess demand is 0 for each individual. Summing over all individuals, we conclude that the value of date 0 aggregate excess demand is 0: $p_0 \cdot z_0 + q \cdot z_a = 0$. Now consider any state $s \geq 1$ at date 1. For individual i, we can argue exactly as above to conclude that the value of individual excess demand is equal to the net of deliveries on purchases and sales of securities. Thus, the value of aggregate excess demand in state s is the net of deliveries on aggregate purchases and sales of securities. However, $z_a = 0$ so aggregate purchases and sales of securities are equal. We conclude that the value of aggregate excess demand in state s is 0. Summing over all spots we conclude that $(p, q) \cdot z = 0$, as asserted.

We show next that $z = 0$. If not, Walras' law entails that excess demand for some commodity is positive; say $z_{s_0 \ell_0} > 0$. Define commodity prices $\tilde{p}$ by:

$$\tilde{p}_{s\ell} = \begin{cases} p_{s\ell} & \text{if} \quad s \neq s_0 \\ \varepsilon & \text{if} \quad s = s_0, \ell \neq \ell_0 \\ 1 - (L-1)\varepsilon & \text{if} \quad s = s_0, \ell = \ell_0 \end{cases}$$

Because $(p, q) \cdot z = 0$ and $z_{s_0 \ell_0} > 0$, $(\tilde{p}, q) \cdot z$ will be strictly positive if ε is small enough. However, this would contradict our assumption that $(p, q, z) \in G$ and hence is a fixed point of F^ε for *every* sufficiently small ε. We conclude that $z = 0$. It is clear that the prices p, q and plans (π^i) identified above constitute an equilibrium for the economy $\mathcal{E}^\rho$.

It remains to construct an equilibrium for the original economy $\mathcal{E}$. To this end, let $p(\rho), q(\rho), (\pi^i(\rho))$ be equilibrium prices and plans for $\mathcal{E}^\rho$ and let $\rho \to 0$. By construction, prices and plans lie in bounded sets, so we may choose a sequence $(\rho_n) \to 0$ for which the corresponding prices and plans converge; let the limits be $p, q, (\pi^i)$. Commodity prices p do not lie on the boundary of Δ (for otherwise the excess demands at prices $p(\rho_n), q(\rho_n)$ would be unbounded, rather than 0). It follows that $\pi^i(\rho)$ is utility optimal in consumer i's budget set at prices (p, q). Because the collection of plans (π^i) is the limit of collections of socially feasible plans, it follows that they are socially feasible and hence that the artificial bounds on security purchases and sales do not bind at the prices p, q. Hence $p, q, (\pi^i)$ constitute an equilibrium for $\mathcal{E}$. ■

Proof of Theorem 2 Fix a CE. As we have already observed, if this CE reduces to GEI then the fundamental asset pricing equations must all hold. Conversely, if the fundamental asset pricing equations hold then the first-order conditions for GEI hold. Because utility functions are quasi-concave and the budget and market-clearing conditions for GEI imply the budget and market-clearing conditions for CE, it follows that we are at a GEI. Put differently: if the CE does not reduce to GEI then at least one of the fundamental pricing equations must fail; we must show that the failure(s) are of the type(s) specified.

Not first that, because any agent can always buy more of any good or of any security, both commodity prices and security prices must weakly exceed fundamental value for every agent. Thus if CE does not reduce to GEI, there must be some agent i and some durable good he holds or some promise he is selling for which price strictly exceeds fundamental value.

This proof seems incomplete, or at least mysterious. What if there are NO perishable goods? What if ALL goods held by i are used as collateral?

On the other hand, any agent can always reduce or increase all his holdings of durable goods and all the promises sold by a common $\varepsilon\%$ without violating a collateral constraint, moving the resulting revenue into or out of perishable consumption. This marginal move must yield zero marginal utility if the agent was optimizing. Since every exchange of durable consumption for perishable consumption either yields zero marginal utility or positive marginal utility, and every reduction of promises sold either yields zero marginal utility or negative marginal utility, the across the board reduction must involve at least one durable good for which price was above fundamental value and at least one sold promise for which price was also above fundamental value, or agent i. ■

Proof of Theorem 3 If the allocation (x^i) is Pareto optimal, then ratios of marginal utilities are equal so prices must coincide with fundamental values, whence CE coincides with GEI. It follows exactly as in Elul (1999) that an efficient GEI in which some agent consumes a strictly positive amount of each good is Walrasian.[16] ■

Proof of Theorem 4 Let $\langle p, q, (x^i, \varphi^i, \psi^i)\rangle$ be an equilibrium, and suppose that $(\hat{x}^i, \hat{\varphi}^i, \hat{\psi}^i)$ is a family of plans meeting the given conditions that Pareto dominates the equilibrium set of plans. By assumption, all the alternative plans are feasible, meet the budget constraints at each state at date 1, and call for deliveries that are the minumum of promises and the value of collateral, Optimality of the equilibrium plans at prices p, q means, therefore, that *all* the alternative plans $(\hat{x}^i, \hat{\varphi}^i, \hat{\psi}^i)$ fail the budget constraints at date 0. Because the alternative set of plans is socially feasible, summing over consumers yields a contradiction. ■

Proof of Theorem 5 The proof follows exactly as the proof of Theorem 1 with the obvious addition of pools, pool prices, and pool purchases and sales. We leave the details to the reader. ■

Proof of Theorem 6 If $p_s \cdot x^i_s < p_s \cdot e^i_s$ for some consumer i and state s, then in state s, consumer i could default on all the promises of the securities s/he sold at date 0, surrender the collateral backing these promises, and still afford *more* than x^i_s. This would contradict the requirement that i's equilibrium plan be optimal in i's budget set. Hence $p_s \cdot x^i_s \geq p_s \cdot e^i_s$, as asserted. ■

Proof of of Theorem 7 Write $E = \{(e^1, \ldots, e^N) : \sum e^i = e\}$ for the set of endowment profiles that sum to the given social endowment e. Let $\Pi \subset \mathbb{R}^{(1+S)L}_{++}$ be the set of Walrasian prices from some endowment profile in E, normalized so that $p \cdot e = 1$, and let $\overline{\Pi}$ be its closure. A straightforward argument shows that $\overline{\Pi}$ is compact.

Fix a consumer, say consumer 1, a state, say state 1, and real numbers α, α' with $0 < \alpha < \alpha' < 1$. For each $w \in [\alpha, \alpha']$, write $X(w, p) \subset \mathbb{R}^{(1+S)L}_{+}$

[16] Elul (1999) treats the the standard model with no durable goods; the adaptation to the present context is entirely straightforward.

for consumer 1's demand at wealth w and prices p and let $X_1(w,p)$ be the component of $X(w,p)$ in state 1. The boundary condition implies that $p_1 \cdot X_1(w,p) < w$. A straightforward compactness argument implies there exists some $\beta < 1$ such that

$$p_1 \cdot X_1(w,p) < \beta w \text{ for all } w \in [\alpha, \alpha'], p \in \Pi$$

Set

$$\Omega = \{(e^1, \ldots, e^N) \in E : \alpha < p \cdot e^1 < \alpha', p_1 \cdot e^1_1 > \beta p \cdot e^1 \text{ for all } p \in \Pi\}$$

Because $e \gg 0$, Note that Ω is a non-empty (because $e \gg 0$) open set.

We claim that no collateral equilibrium from any endowment profile in Ω can be Pareto optimal. To see this, fix $(e^1, \ldots, e^N)\Omega$. In view of Theorem 3, every Pareto optimal collateral equilibrium is Walrasian, so it suffices to show that if $\langle p, (x^i)\rangle$ is a Walrasian equilibrium from the endowment profile $(e^1, \ldots, e^N)$ then the consumption allocation (x^i) cannot be supported in a collateral equilibrium. To see this, note that our construction guarantees that $p \cdot e^1 \in [\alpha, \alpha']$, $p_1 \cdot x^1_1 \leq \beta p \cdot e^1$ and $p_1 \cdot e^1_1 > \beta p \cdot e^1$, whence $p_1 \cdot x^1_1 < p_1 \cdot e^1_1$. Because utility functions are smooth and consumptions (x^i_1) are interrior, state 1 prices in any collateral equilibrium and in any Walrasian equilibrium must be collinear with marginal rates of substitution, so it follows from Theorem 6 that the consumption allocation $(x^i))\rangle$ cannot be supported in a collateral equilibrium. ■

Proof of Theorem 8 Suppose each consumer is a net saver. For $\ell = 1, \ldots, L$ let $\mathbf{B}^\ell = (B^{\ell 1}, \ldots, B^{\ell S}; \delta_{0\ell})$ be the security pool which is collateralized by one unit of the commodity 0ℓ and for which the s tranche $B^{\ell s}$ promises to deliver in state s the value of what one unit of commodity 0ℓ becomes in state s and promises to deliver nothing in states $\sigma \neq s$. That is:

$$B^{\ell s}_\sigma = \begin{cases} p_s \cdot F_s(\delta_{0\ell}) & \text{if} \quad \sigma = s \\ 0 & \text{if} \quad \sigma \neq s \end{cases}$$

Define prices for commoditiesand tranches as follows:

$$p_{0\ell} = \tilde{p}_{0\ell} + \sum_s \tilde{p}_s \cdot F_s(\delta_{0\ell})$$
$$p_{s\ell} = \tilde{p}_{s\ell}$$
$$Q^{\ell s} = p_s \cdot F_s(\delta_{0\ell}) \tag{12}$$

For each consumer i and each state s define

$$r^i_s = p_s \cdot [x^i_s - F_i(x^i_0)] - p_s \cdot [e^i_s - F_s(e^i_0)]$$

Note that this quantity could be positive, negative or zero. For each consumer i define the portfolios Φ^i, Ψ^i of purchases and sales of tranches as follows:

$$\varphi^i = x^i_0$$
$$\psi^i = x^i_0$$
$$\Phi^{i\ell s} = \frac{x^1_0}{p_s \cdot F_s(x^1_0} (r^1_s)^+$$
$$\Psi^{i\ell s} = \frac{x^1_0}{p_s \cdot F_s(x^1_0} (-r^1_s)^+ \tag{13}$$

We claim $\langle p, , Q, (x^i, \Phi^i, \Psi^i) \rangle$ is an equilibrium for the economy $\mathcal{E} = \langle (e^i, u^i), (\mathbf{F}^\ell) \rangle$.

To see this note first that deliveries on tranches coincide with promises: this follows immediately from the definitions. Moreover, for each consumer i the plan (x^i, Φ^i, Ψ^i) is in consumer i's collateral budget set $B(p, Q; e^i)$: this follows immediately by substituting the definitions of prices (12) and of portfolios (13) into the Walrasian budget constraints.[17] We assert that, for each i, all consumption plans that that can be financed by purchases and sales of security pools are in the Walrasian budget set $B^W(\tilde{p}; e^i)$. To see this, suppose $(\hat{x}^i, \hat{\varphi}^i, \hat{\Phi}^i, \hat{\Psi}^i)$ is in consumer i's budget set $B(p, Q; e^i)$. The date 0

[17]We do *not* assert that *every* consumption plan in the Walrasian budget sets can be financed by appropriate portfolios of security purchases and sales – and in general, this is not so – but only that these *particular* consumption plans can be so financed.

and state s budget constraints are

$$\begin{aligned} p_0 \cdot \hat{x}_0 + Q \cdot \hat{\Phi} &\leq p_0 \cdot e_0 + Q \cdot \hat{\Psi} \\ p_s \cdot \hat{x}_s + \textsc{Del}(\hat{\Phi}; s, p) &\leq p_s \cdot e_s + p_s \cdot F_s(\hat{x}_0) \\ &\quad + \ \textsc{Del}(\hat{\Psi}; s, p) \end{aligned}$$

Substituting the definitions of spot prices and security deliveries, summing and doing some algebra yields

$$\tilde{p}_0 \cdot \hat{x}_0 + \sum \tilde{p}_s \cdot \hat{x}_s \leq \tilde{p}_0 \cdot e_0 + \sum \tilde{p}_s \cdot e_s$$

which is the Walrasian budget constraint. ■

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