Preference for Randomization*

Ambiguity Aversion and Inequality Aversion

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June 16, 2010

Abstract

In Anscombe and Aumann's (1963) domain, there are two types of mixtures. One is an *ex–ante mixture*, or a lottery on acts. The other is an *ex–post mixture*, or a state–wise mixture of acts. These two mixtures have been assumed to be indifferent under the *Reversal of Order axiom*. However, we argue that the difference between these two mixtures is crucial in some important contexts. Under *ambiguity aversion*, an ex–ante mixture could provide only *ex–ante hedging* but not *ex–post hedging*. Under *inequality aversion*, an ex–ante mixture could provide only *ex–ante equality* but not *ex–post equality*. For each context, we develop a model that treats a preference for ex–ante mixtures separately from a preference for *ex–post mixtures*. One representation is an extension

*The part of this paper on inequality aversion was originally circulated as Saito, K. (January 25, 2008) "Social Preference under Uncertainty," the University of Tokyo, COE Discussion Papers, No. F-217.

†I am indebted to my adviser Eddie Dekel for continuous guidance. I am also indebted to my former adviser Michihiro Kandori for his guidance when I was at the University of Tokyo. I am grateful to Marciano Siniscalchi, Kyuongwon Seo, Faruk Gul, Peter Klibanoff, Jeff Ely for many discussions which have led to the improvement of the paper. I would also like to thank Itzhak Gilboa, Ernst Fehr, Colin Camerer, Kenneth Binmore, Massimo Marinacci, Elchanan Ben–Porath, Pierpaolo Battigalli, John E. Roemer, Costis Skiadas, Alvaro Sandroni, Nabil Al–Najjar, Christoph Kuzumics, Kazuya Kamiya, Akihiko Matsui, and Hitoshi Matsushima for their insightful comments and discussions. I gratefully acknowledge financial support from the Center for Economic Theory of the Economics Department of Northwestern University.

of Gilboa and Schmeidler's (1989) *Maxmin preferences*. The other representation is an extension of Fehr and Schmidt's (1999) *Piecewise–linear preferences*. In both representations, a single parameter characterizes a preference for ex–ante mixtures. For the both representations, instead of the Reversal of Order axiom, we propose a weaker axiom, the *Indifference axiom*, which is a criterion, suggested in Raiffa's (1961) critique, for evaluating lotteries on acts. These models are consistent with much recent experimental evidence in each context.

Keywords: Ambiguity; randomization; Ellsberg paradox; other–regarding preferences; inequality; maxmin utility.

JEL Classification Numbers: D81, D03

1 Introduction

This paper investigates a *preference for randomization*. People exhibit such a preference as a form of *hedging* because of *ambiguity aversion*, as Raiffa (1961) suggests in his famous critique. In addition, in a social context, people exhibit such a preference because of *inequality aversion*, as in the case of "Machina's (1989) mom" who prefers flipping a coin to decide how to allocate an indivisible good among her children.[1]

Despite the importance of such preferences, little work has been done on this preference for randomization. Recently, however, experimental researchers have begun to study such a preference in each context of aversion. One important observation drawn from such experimental studies is that *timing* of randomization matters, as will be discussed in detail later. The purpose of this paper is to provide an axiomatic model, which describes such a preference in each context in a unified way.

In one sense, the seminal paper by Anscombe and Aumann (1963) addresses the issue of timing of randomization. They consider two types of randomization depending on timings; One is an *ex–ante mixture*, or a lottery on payoff profiles. For example, P in Figure 1 is the fifty–fifty ex–ante mixture of $(\$100, \$0)$ and $(\$0, \$100)$. This type of mixture is henceforth

[1]Diamond (1967) proposes a similar argument for this preference for randomization.

indicated by $\oplus$. The other is an *ex–post mixture*, or a state–wise mixture of payoff profiles.

$$P : .5(\$100, \$0) \oplus .5(\$0, \$100) \equiv \begin{array}{l} \overset{.5}{\nearrow} (\$100,\ \ \$0) \\ \underset{.5}{\searrow} (\$0\ ,\ \$100) \end{array}$$

$$l : .5(\$100, \$0) + .5(\$0, \$100) \equiv \left(\begin{array}{l} \overset{.5}{\nearrow} \$100 \\ \underset{.5}{\searrow} \$0 \end{array} , \begin{array}{l} \overset{.5}{\nearrow} \$100 \\ \underset{.5}{\searrow} \$0 \end{array} \right)$$

Figure 1: Ex–ante Mixture P and Ex–post Mixture l

For example, l in Figure 1 is the fifty–fifty ex–post mixture of $(\$100, \$0)$ and $(\$0, \$100)$. This type of mixture is henceforth indicated by $+$, as in conventional literature.

However, one difficulty inherent in using Anscombe and Aumann's (1963) approach for studying a preference for randomization is that under the *Reversal of Order axiom*, an ex–ante mixture is identified with its ex–post mixture, i.e., $\alpha f \oplus (1-\alpha)g \sim \alpha f + (1-\alpha)g$ for any payoff profiles f and g, and, $\alpha \in [0, 1]$. Hence, this axiom precludes the study of a preference for ex–ante mixtures separately from a preference for ex–post mixtures. For example, the Reversal of Order axiom implies that P and l are indifferent.

For the above reason, we do not assume the Reversal of Order axiom. Instead, we propose a weaker axiom, the *Indifference axiom.* To see the difference between these axioms, notice that one way to justify the Reversal of Order axiom is *state–wise* evaluation; if you look at P, state–wise, it offers the same lottery as l, at each state. There is, however, another natural way of evaluation; if you look at P, at each payoff profile in the support, it offers nonconstant payoff profiles, namely $(\$100, \$0)$ and $(\$0, \$100)$, which would be less attractive than the constant payoff profile l under ambiguity aversion as well as under inequality aversion;[2] this way of evaluation is called *support–wise* evaluation. The Indifference axiom states that if two lotteries on payoff profiles are indifferent according to *both the state–wise and support–wise* criteria, then the lotteries should be indifferent. As will be explained in Section 1.1.1, the axiom is a weaker formalization of Raiffa's (1961) argument in his famous critique of

[2] In a social context, under which inequality aversion matters, states are reinterpreted as individuals. So, the nonconstant payoff profiles entail ex post inequality.

ambiguity aversion; The axiom also has a natural interpretation in the context of inequality aversion, as will be explained in Section 1.1.2.

Using this new Indifference axiom together with standard axioms, we characterize *Ex–ante/Ex–Post (EAP) preferences* that capture a preference for ex–ante mixtures and also, but separately, a preference for ex–post mixtures; for any *ex–ante* mixture P on payoff profiles f,

$$V(P) = \delta\, U\big((P_s)_{s\in S}\big) + (1-\delta)\int_{\mathscr{F}} U(f)dP(f), \tag{1}$$

where a real–valued function U on the set $\mathscr{F}$ of payoff profiles captures a preference for ex–post mixtures. Moreover, it will be shown that a real number δ captures a preference for ex–ante mixtures. The payoff profile $(P_s)_{s\in S}$ offers *the marginal distribution* P_s of P in each state s. That is, if $P = P(f^1)f^1 \oplus \cdots \oplus P(f^n)f^n$, then $P_s = P(f^1)f^1_s + \cdots + P(f^n)f^n_s$, where f^i_s is the payoff on the state s in the payoff profile f^i and $P(f^i)$ is the probability assigned to f^i.

In particular, we propose two tractable special cases of EAP preferences for each context respectively; For ambiguity aversion, we axiomatize *EAP Maxmin preference*, in which U is Gilboa and Schmeidler's (1989) *Maxmin preference*, shown as (3) in Section 1.1.1. For inequality aversion, on the other hand, we axiomatize *EAP Piecewise–linear preference*, in which U is Fehr and Schmidt's (1999) *Piecewise–linear preference*, shown as (5) in Section 1.1.2. This is because, as many experimental studies have found, inequality averse preferences are nonmonotonic with respect to prize, so that Maxmin preferences are inconsistent with such preferences.[3] Although these two preferences have completely different representations in U, the axioms which characterize these preferences are similar. In addition, as noted, the same Indifference axiom is essential for both characterizations.

Note that if $\delta = 1$, then EAP preferences represented by (1) implies the Reversal of Order axiom. If $\delta < 1$, then the preferences can distinguish between P and l in Figure 1. Given

[3]In the paper we focus on an agent's inequality aversion, not on a social planner's inequality aversion, in response to recent rich experimental finding on the former. So, the robust experimental evidence, drawn from ultimatum games, that recipients reject positive but unfair offers shows the nonmonotonicity. On the other hand, for a social planner, it would be reasonable to assume monotonic preferences, so that EAP Maxmin preferences would be consistent with a social planner's inequality averse preferences.

that our purpose is to develop a model which can distinguish between P and l, one might wonder why it does not suffice to consider the simpler special case in which $\delta = 0$. However, it is easy to see that the special case implies the *Independence axiom* on ex–ante mixtures so that there is *no* strict preference for ex–ante mixtures.

The remainder of Section 1 is organized as follows: Section 1.1 provides an overview of the main results on ambiguity aversion and inequality aversion. After that, Section 1.2 reviews some experimental evidence on a preference for ex–ante mixtures under the two types of aversion. Next, in Section 1.3, the related literature is discussed. Section 2 then introduces the setup. The axioms that characterize EAP Maxmin preferences are in Section 3, while Section 4 presents the axioms that characterize EAP Piecewise–linear preferences. In Section 5, EAP Maxmin and EAP Piecewise–linear preferences are applied to games. Finally in Section 6, further relationships among the axioms of Anscombe and Aumann (1963), Seo (2009), and our model are discussed. All proofs are in the appendix.

1.1 Main Results

1.1.1 Ambiguity Aversion

The Ellsberg (1961) paradox has raised questions about subjective expected utility models. He proposed the following thought experiment. Consider two urns, one of which we call *objective* and the other of which we call *ambiguous*. Each urn contains 100 balls, each of which is either red or black. The objective urn contains 50 black and 50 red balls. There is no further information about the contents of the ambiguous urn. You first decide which urn you will draw from; then you bet on the color of the ball that you will draw, and you then draw a ball. If your bet turns out to be correct, you will get \$100. Typically, subjects strictly prefer the objective urn than the ambiguous urn. This behavior is called *ambiguity aversion*.

Widely–used preferences that are consistent with ambiguity aversion are *Maxmin preferences* proposed by Gilboa and Schmeidler (1989):

$$U(f) = \min_{\mu \in C} \int_S u(f_s) d\mu(s), \tag{2}$$

where S is the set of states, C is a subset of the set $\Delta(S)$ of all finitely additive probabilities on S, and u is a von Neumann–Morgenstern utility function.

Raiffa (1961) criticizes ambiguity averse preferences based on the state–wise evaluation: By flipping a coin to choose a color in the ambiguous urn, you can obtain an ex–ante mixture P on bets that is shown in Figure 2. If you look at P *state–wise*, it offers the same lottery

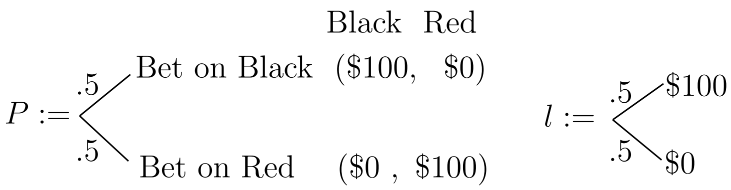


Figure 2: Raiffa's Critique

that the objective urn offers, namely, that shown as l in Figure 2. So, P and l should be indifferent. Hence, there is no reason why you strictly prefer the objective urn.

As Raiffa's (1961) argument suggests, some people might prefer flipping a coin and then deciding. Note that such people have a preference for ex–ante mixtures and violate the Independence axiom on such mixtures.[4] One conceivable justification for such a preference is that ex–ante mixtures provide *hedging* in the ex–ante expected payoffs. When a coin is flipped, the ex–ante expected payoff for each color becomes a constant \$50, although the decision maker finally ends up with the ambiguous bets ex post. We call this preference for the ex–ante mixtures *ex–ante ambiguity aversion*. In contrast, conventional ambiguity aversion constitutes a preference for ex–post mixtures. Henceforth, we call this conventional ambiguity aversion *ex–post ambiguity aversion*.

Recent experiments, reported in Spears (2008) and Dominiak and Schnedler (2009), have found that subjects often have different attitudes toward ex–ante and ex–post ambiguity.[5] This evidence contradicts the Reversal of Order axiom, which implies that attitudes toward ex–ante and ex–post ambiguity should be the same.

Using the Indifference axiom together with standard axioms used in Gilboa and Schmeidler (1989), we characterize *Ex–ante/Ex–post (EAP) Maxmin preferences* that capture attitudes

[4]The Independence axiom shows that if $(\$100, \$0) \sim (\$0, \$100)$, then $P \sim (\$100, \$0) \sim (\$0, \$100)$.

[5]We discuss the experimental data further in Section 1.2.1.

toward ex–ante ambiguity and also, but separately, attitudes toward ex–post ambiguity as follows:

$$V(P) = \delta \min_{\mu \in C} \int_S \Big(\int_{\mathscr{F}} u(f_s) dP(f) \Big) d\mu(s) + (1-\delta) \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f_s) d\mu(s) \Big) dP(f). \tag{3}$$

As will be shown in Section 3, the parameter δ is an *index of ex–ante ambiguity aversion.* In particular, the preferences exhibit strict ex–ante ambiguity aversion if and only if $\delta > 0$. The first term of (3) represents a concern for hedging in the ex–ante expected utilities. The second term of (3) represents a concern for hedging in the ex–post utilities.

Finally, we conclude this section by discussing further connection between the Indifference axiom and Raiffa (1961)'s critique. In addition to the state–wise evaluation, he also proposes another way for evaluating lotteries in Figure 2, which corresponds to *support–wise* criterion: if you look at the acts in the support of P, all acts are ambiguous bets, namely, $(\$100, \$0)$ and $(\$100, \$0)$, which are less attractive than l. So, l should be preferred over P, contrary to the conclusion that the lotteries should be indifferent, according to the state–wise criterion. Hence, he criticizes ambiguity averse preferences for this inconsistency in the preference on the lotteries. Note that the Indifference axiom does *not* lead to this inconsistency, because the axiom requires that these lotteries are indifferent according to the *both* criteria.

1.1.2 Inequality Aversion

Another kind of situation in which people would typically prefer ex–ante mixtures is a social environment in which inequality matters. Next is a brief review of *inequality averse* preferences and then our main results.

There is overwhelming evidence that a person's welfare is affected by *ex–post equality.* For example, in dictator games in which subjects have to allocate a prize between themselves and passive recipients, many studies have found that subjects offer, on average, 20 percent of the prize to the recipients; In ultimatum games, recipients can reject the offers, as opposed to dictator games, and in case of rejection, both receive nothing. Indeed, almost half of recipients, on average, reject offers of less than 20 percent of the prize. These behavior is called *inequality* *aversion*. (See Fehr and Schmidt (2005) for a survey.) The fact that recipients reject positive offers shows that inequality averse preferences are nonmonotonic with respect to prize. Hence, Maxmin preferences are not consistent with such preferences.

Widely used preferences that are consistent with inequality aversion are *Piecewise–linear preferences* proposed by Fehr and Schmidt (1999). Agents with these preferences rank allocations $f = (f_1, \ldots, f_S)$ among the set S of *individuals* according to the criterion

$$U(f) = f_1 - \sum_{s \neq 1} (\alpha_s \max\{f_s - f_1, 0\} + \beta_s \max\{f_1 - f_s, 0\}), \tag{4}$$

where $1 \in S$ denotes the decision maker and $\alpha_s, \beta_s \geq 0$ for all $s \neq 1$. The parameters α_s and β_s can be interpreted as indices of disutility from *envy* and *guilt* toward the individual s when the decision maker gets less and more, respectively, than the individual s.

Recently, several experimental papers have studied how risk affects inequality aversion; in

$$P := \begin{cases} \overset{.5}{\nearrow} & \text{Dictator Wins} \quad (\$100, \ \$0) \\ \underset{.5}{\searrow} & \text{Recipient Wins} \ (\$0\ , \ \$100) \end{cases} \qquad \begin{array}{c} \text{Dictator} \quad \text{Reciever} \end{array}$$

Figure 3: Flipping a Coin with a Dictator Game

probabilistic dictator games, in which dictators have to allocate chances to win a prize, these studies have found that a substantial fraction of subjects shared chances to win. For example, some dictators chose flipping a coin to decide the winner over being the winner for sure; in that case, he obtains an ex–ante mixture P on allocations, as in Figure 3. See, for the experiments, Karni, Salmon, and Sopher (2008), Bohnet, Greig, Herrmann, and Zeckhauser (2008), Bolton and Ockenfels (forthcoming), Krawczyk and Le Lec (2008), and Kircher, Luding, and Sandroni (2009).[6]

These observations suggest that decision makers have a preference for ex–ante mixtures to maintain *ex–ante equality*, or equality in the expected payoff.[7] We call this preference for

[6]We discuss other experimental results in Section 1.2.2.

[7]Such subjects violate the Independence axiom on ex–ante mixtures. This is because the axiom implies that if they prefer winning to losing, they should allocate the probability 1 to winning.

ex–ante mixtures *ex–ante inequality aversion.* In contrast, conventional inequality aversion constitutes a preference for ex–post mixtures. Henceforth, we call conventional inequality aversion *ex–post inequality aversion.*

Using the Indifference axiom, again, together with standard axioms, we characterize *Ex–ante/Ex–post (EAP) Piecewise–linear preferences* that capture ex–ante inequality aversion and also, but separately, ex–post inequality aversion as follows:

$$\begin{aligned} V(P) = \delta\Bigg(E_P u(f_1) - \sum_{s\neq 1} \Big(\alpha_s \max\{E_P u(f_s) - E_P u(f_1), 0\} + \beta_s \max\{E_P u(f_1) - E_P u(f_s), 0\}\Big)\Bigg) \\ + (1-\delta)\int_{\mathscr{F}} \Bigg(u(f_1) - \sum_{s\neq 1} \Big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\}\Big)\Bigg) dP(f), \end{aligned} \tag{5}$$

where $E_P u(f_s) = \int_{\mathscr{F}} u(f_s) dP(f)$.[8] As will be shown in Section 4, the parameter δ is an *index of ex–ante inequality aversion.* In particular, the preferences exhibit strict ex–ante inequality aversion if and only if $\delta > 0$. The first term represents a concern about *ex–ante equality*, because the term depends on the differences in the expected utilities. The second term captures a concern about *ex–post equality*, because the term depends on the differences in the ex–post utilities. Under the assumption of the risk neutrality, EAP Piecewise–linear preferences reduce to Fehr and Schmidt's (1999) Piecewise–linear preferences, for degenerate lotteries on payoff profiles.

The Indifference axiom has a natural interpretation in a social context as well, under which *states are reinterpreted as individuals.* The criterion conditional *on states* (i.e., *on individuals*) corresponds to *ex–ante equality*, because *state–wise* (i.e., *individual–wise*) evaluation yields the ex–ante expected payoff for each individual. On the other hand, the support–wise criterion corresponds to *ex–post equality.* For example, according to the state–wise criterion, P in Figure 3 would be indifferent to a constant payoff profile, in which both dictator and recipient independently obtain the fifty–fifty lottery of \$100 and \$0. This is because both P and the payoff profile are equally desirable from the view point of ex–ante equality. According to the

[8] As noted, EAP Maxmin preferences can be consistent with a social planner's inequality averse preferences. It is easy to see that if $\delta \in (0, 1)$, the model can describe the choice of "Machina's (1989) mom" as follows: $.5(1, 0) \oplus .5(0, 1) \succ (1, 0) \sim (0, 1)$.

support–wise criterion, on the other hand, the constant profile would be preferred over P, because only P is not ex–post equal.

1.2 Experimental Evidence

1.2.1 Ambiguity Aversion

EAP Maxmin preferences represented by (3) are consistent with some recent experimental evidence. Dominiak and Schnedler (2009) have studied the relationship between attitudes toward ex–ante and ex–post ambiguity. The number in Table 1 shows the number of subjects who exhibited a corresponding attitude toward ex–ante and ex–post ambiguity.[9]

		Ex–post ambiguity		
		averse	neutral	
Ex–ante ambiguity	averse	6	0	$\delta < 0$
	neutral	17	12	$\delta = 0$
	loving	12	2	$\delta > 0$
		35	14	

Table 1: Attitudes toward Ex–ante and Ex–post Ambiguity

Dominiak and Schnedler's (2009) experimental result might be summarized by the following two points. First, subjects who are averse to ex–post ambiguity differ in their attitudes toward ex–ante ambiguity. Indeed, there are more ex–ante ambiguity loving subjects than ex–ante ambiguity averse subjects. This result is inconsistent not only with the Reversal of Order axiom but also with Raiffa's (1961) critique because his claim implies that all of the ex–post ambiguity averse decision makers should be ex–ante ambiguity averse as well. Second, however, most of the ex–post ambiguity neutral subjects are ex–ante ambiguity neutral as well. Although using a small sample, Spears (2008) has found similar tendencies to these.

The first observation is explained by the heterogeneity of the parameter δ as follows. Suppose EAP Maxmin preferences exhibit ex–post ambiguity aversion. Then, as will be

[9] The table excludes four subjects who exhibited ex–post ambiguity loving.

shown in Section 3.5, the preferences exhibit ex–ante ambiguity aversion, neutrality, and loving, if and only if $\delta > 0$, $\delta = 0$, and $\delta < 0$, respectively, which is consistent with Table 1. Hence, the heterogeneity observed in the experiment can be characterized simply by whether or not δ is positive.

The second observation is also consistent with EAP Maxmin preferences. As will be shown in Section 3.5, among EAP Maxmin preferences, ex–post ambiguity neutrality implies ex–ante ambiguity neutrality for any δ, which is also consistent with Table 1.

1.2.2 Inequality Aversion

EAP Piecewise–linear preferences represented by (5) are also consistent with some recent experimental evidence. Firstly, if $\delta > 0$, then the preferences are consistent with the experimental evidence drawn from probabilistic dictator games, that a substantial fraction of subjects shared chances to win because of the *ex–ante equality*.

Secondly, Kariv and Zame's (2009) experiment is also consistent with suitable values of the parameters α, β and δ. In their experiments, subjects are asked to divide a budget z into x and y such that $x + qy \leq z$, where q is a given price. After the decision, the payoff of the decision maker and a recipient are determined as x or y with the probability .5, so that the outcome is an ex–ante mixture $.5(x, y) \oplus .5(y, x)$. Hence, the subjects are required to make decisions under a *veil of ignorance*.

One of their main findings is that most of the subjects did not allocate all funds to the cheaper element. This fact is also consistent with EAP Piecewise-linear preferences. To see this, assume the risk neutrality, for simplicity. Then,

$$V\Big(.5(x, y) \oplus .5(y, x)\Big) \;=\; \frac{1}{2}\Big[(x + y) - (1 - \delta)(\alpha + \beta)|x - y|\Big]. \tag{6}$$

So, when $(1 - \delta)(\alpha + \beta)$ exceeds a certain level, the decision maker tries to equalize x and y even if the prices are not the same.[10]

[10] Suppose $q < 1$. If $1 - \delta > \frac{q-1}{(\alpha+\beta)(q+1)}$, then EAP Piecewise-linear preferences with such parameters are consistent with the experimental evidence that many subjects did not spend all the budget to the cheaper element, i.e., y.

Finally, EAP Piecewise-linear preferences are also consistent with seemingly contradictory experimental results on *efficiency versus inequality*. In particular, a number of papers have recently claimed that *efficiency*, or the sum of allocation across agents, has a stronger influence than inequality. See, for example, Charness and Rabin (2002) and Engelmann and Strobel (2004, 2006).

In particular, Charness and Rabin (2002) report that in a dictator game, almost 50 percent of their subjects chose an efficient but unequal allocation (in which the dictator obtained 375 points and the receiver obtained 750 points) to an inefficient but equal allocation (in which each player obtained 400 points). This behavior seems contradictory to any theory of ex–post inequality aversion including Fehr and Schmidt's (1999) Piecewise–linear preferences.

The key fact that can explain the contradiction is that in the experiments that are in favor of efficiency, each subject makes decisions as if he were a dictator, but the actual roles (i.e., dictator or receiver) are determined at random. So, the subjects face *risk over roles*. Indeed, in experiments of three–person dictator games, Bolton and Ockenfels (2006) found that under risk over roles, subjects tended to choose efficient but unequal allocations over inefficient but equal allocations.[11]

Under the risk over roles, each subject is facing a game with the other subjects. In Section 5.2, we study a game that describes the aforementioned dictator game under the risk over roles and show that in an equilibrium, subjects with EAP Piecewise-linear preferences choose the efficient but unequal allocation rather than the inefficient but equal allocation because of the ex–ante equality.

To understand this result intuitively, note that *risk over roles plays a role similar to that of veil of ignorance*. To see this, assume that there are two subjects. Suppose both of them decide to allocate x to themselves and y to the other. Then, under the risk over roles, what they obtain is the ex–ante mixture $.5(x, y) \oplus .5(y, x)$. Thus, the utility of each subject is

[11] In a three–person dictator game, a dictator decides an allocation of a prize among two passive receivers and the dictator himself.

determined as in (6); in other words,

$$V\Big(.5(x,y) \oplus .5(y,x)\Big) \;\; = \;\; \frac{1}{2}\Big[(\text{“efficiency”}) - (1-\delta)(\alpha+\beta)(\text{“inequality”})\Big].$$

Hence, even if a subject cares about ex–post equality (i.e., α and β are positive), if he weighs ex–ante equality heavily enough (i.e., δ is larger than a certain level), then his utility from choosing the efficient but unequal allocation becomes larger than his utility from choosing the inefficient but equal allocation, given that the other player chooses the same efficient allocation.[12] Therefore, it looks *as if* the subjects with EAP Piecewise–linear preferences care more about efficiency than about inequality.

1.3 Related Literature

To our knowledge, no other papers have studied a preference for ex–ante mixtures and also, but separately, a preference for ex–post mixtures.

In terms of axiomatic structures, however, the paper that is most closely related to the present paper is Seo (2009), in the sense that only his model and our model do not assume the Reversal of Order axiom.[13] One key feature of his model is that under his *Dominance axiom*, the Reversal of Order axiom and the *Reduction of Compound Lotteries axiom* become equivalent. This equivalence implies a negative result that, in his model, distinction between ex–ante mixtures and ex–post mixtures is impossible as long as we assume rational attitude toward reduction of compound lotteries. In contrast, under the Indifference axiom, this incompatibility does not arise, because, as will be shown in Section 6, the Indifference axiom is weker than the Reversal of Order axiom so as to allow the distinction between the two types mixtures, but is still, stronger than the Reduction of Compound Lotteries axiom. In addition, it will be shown in Section 6 that, under the Reduction of Compound Lotteries axiom, his Dominance axiom implies the Indifference axiom but not vice versa.

In terms of applications, the present paper is related with literature on game theory

[12]The observation suggests $\delta > 0$, while the observation reported by Kariv and Zame (2009) suggests $\delta < 1$. So, $\delta \in (0,1)$ can be consistent with both experiments.

[13]He assumes the Independence axiom on ex–ante mixtures, so that no preference for ex–ante mixture.

that studies ambiguity averse players, in which mixed strategies corresponds to lotteries on acts. The special cases of EAP Maxmin preferences and EAP Choquet preferences (Choquet counterpart of EAP Maxmin), where $\delta = 0$ or 1, have been used in the literature;[14] Klibanoff (1996) and Lo (1996) have applied EAP Maxmin preferences with $\delta = 1$; Eichberger and Kelsey (2000) have applied EAP Choquet preferences with $\delta = 0$; Mukerji and Shin (2002) have applied EAP Choquet preferences with $\delta = 0$ as well as with $\delta = 1$. In Section 5.1, it will be shown that, in some games, $\delta \in (0, 1)$ would predict more realistic behavior of ambiguity averse players than $\delta = 0$ and 1.

Finally, in terms of motivation associated with inequality aversion, the present paper also shed light on certain issues in the social choice literature regarding the trade–off between equality of opportunity and equality of outcome; these issues are addressed especially in Ben–Porath, Gilboa, and Schmeidler (1997) and Gajdos and Maurin (2004).[15] However, the models and motivations in both papers are different from ours. These papers have considered a social planner's preferences on matrices of real numbers that are utilities over a product space that consists of states and individuals. Hence, in their model, there is no conceptual counterpart of ex–ante mixtures.[16] In addition, as noted, our emphasis in the paper is an agent's inequality aversion, not a social planner's inequality aversion, in response to recent rich experimental evidence on the former; the present paper is the first paper apart from Saito (2008) to provide an axiomatization of Fehr and Schmidt's (1999) Piecewise-linear preferences.

2 Setup

For any topological space X, let $\Delta(X)$ be the set of distributions over X with finite supports. An element in $\Delta(X)$ is called a *lottery* on X. Let $\delta_x \in \Delta(X)$ denote a point mass on x.

Let S be a set of states and let Σ be an algebra of subsets of S. Let Z denote a set of

[14]Since Choquet expected utilities with convex capacity have Maxmin representation, it is trivial–given the axiomatization of EAP Maxmin preferences–to axiomatize EAP Choquet preferences with convex capacities.

[15]Ben–Porath et al. (1997) do not study axiomatization. Gajdos and Maurin (2004) axiomatize a weaker representation than the one used in Ben–Porath et al. (1997). The representations and axioms proposed by Gajdos and Maurin (2004) are different from ours.

[16]Mixing two matrices in their model conceptually corresponds to an ex–post mixture in our model.

outcomes. Both set S and set Z are assumed to be nonempty. A payoff profile f is called an *act* and defined to be a Σ-measurable bounded function from S into $\Delta(Z)$. For each act f, we write $f_s \in \Delta(Z)$, instead of $f(s)$. Let $\mathscr{F}$ be the set of all acts.

A preference relation $\succsim$ is defined on $\Delta(\mathscr{F})$. As usual, $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succsim$. A *constant act* is an act f such that $f_s = f'_s$ for all $s, s' \in S$. Elements in $\Delta(\mathscr{F})$ are denoted by P, Q, and R. For all $P \in \Delta(\mathscr{F})$, supp P is the support of P. Elements in $\mathscr{F}$ are denoted by f, g , and h. Elements in $\Delta(Z)$ are denoted by l, q, r and are identified as constant acts. For $f \in \mathscr{F}$, an element $l_f \in \Delta(Z)$ is a *certainty equivalent* for f if $f \sim l_f$.

Finally, ex–ante mixtures and ex–ante mixtures are formally defined as follows;

Definition: For all $\alpha \in [0,1]$ and $P, Q \in \Delta(\mathscr{F})$, $\alpha P \oplus (1-\alpha)Q \in \Delta(\mathscr{F})$ is a lottery on acts such that $(\alpha P \oplus (1-\alpha)Q)(f) = \alpha P(f) + (1-\alpha)Q(f) \in [0,1]$ for each $f \in \mathscr{F}$. This operation is called an *ex–ante mixture*. For degenerate lotteries on acts, we write $\alpha f \oplus (1-\alpha)g \in \Delta(\mathscr{F})$, instead of $\alpha\delta_f \oplus (1-\alpha)\delta_g$, for any $\alpha \in [0,1]$, and $f, g \in \mathscr{F}$.

Definition: For all $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$, $\alpha f + (1-\alpha)g \in \mathscr{F}$ is an act such that $(\alpha f + (1-\alpha)g)(s) = \alpha f_s + (1-\alpha)g_s \in \Delta(Z)$ for each $s \in S$. This operation is called an *ex–post mixture*.

3 Ex–ante/Ex–post Maxmin Preferences

We now discuss ambiguity averse preferences. Instead of Reversal of Order, we assume Indifference as well as the axioms used in Gilboa and Schmeidler (1989).

3.1 Axioms

The first six axioms are due to Gilboa and Schmeidler (1989). However, since Reversal of Order is not assumed, both Continuity and Certainty Independence are assumed for ex–ante mixtures and also, but separately, for ex–post mixtures.

Axiom 1 (Weak Order): $\succsim$ is complete and transitive.

Axiom 2 (Continuity):

(i) For all $P, Q, R \in \Delta(\mathscr{F})$, if $P \succ Q$ and $Q \succ R$, then there are α and β in $(0, 1)$ such that $\alpha P \oplus (1-\alpha)R \succ Q$ and $Q \succ \beta P \oplus (1-\beta)R$.

(ii) For all $f, g, h \in \mathscr{F}$, if $f \succ g$ and $g \succ h$, then there are α and β in $(0, 1)$ such that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$.

Axiom 3 (Monotonicity): For all $f, g \in \mathscr{F}$,

$$f_s \succsim g_s \text{ for all } s \in S \Rightarrow f \succsim g.$$

If a preference relation $\succsim$ satisfies Axioms 1–3, then each act $f \in \mathscr{F}$ admits a certainty equivalent $l_f \in \Delta(Z)$.

Axiom 4 (Nondegeneracy): There exist $z_+, z_- \in Z$ such that $z_+ \succ z_-$.

Axiom 5 (Ex–post Ambiguity Aversion): For all $\alpha \in [0, 1]$ and $f, g \in \mathscr{F}$,

$$f \sim g \Rightarrow \alpha f + (1-\alpha)g \succsim f.$$

Mixing constant acts, ex–ante as well as ex–post, does not provide any hedging. Hence,

Axiom 6 (Ex–ante/Ex–post Certainty Independence):

(i) For all $\alpha \in (0, 1]$, $P, Q \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$,

$$P \succsim Q \Leftrightarrow \alpha P \oplus (1-\alpha)l \succsim \alpha Q \oplus (1-\alpha)l.$$

(ii) For all $\alpha \in (0, 1]$, $f, g \in \mathscr{F}$, and $l \in \Delta(Z)$,

$$f \succsim g \Leftrightarrow \alpha f + (1-\alpha)l \succsim \alpha g + (1-\alpha)l.$$

The final axiom is a weaker formalization of Raiffa's (1961) critique. As we saw in Introduction, he proposes two criteria. One is *state–wise* and the other is *support–wise*. First, to formalize the *state–wise* criterion, a preliminary concept is defined here:

DEFINITION: For all $P \in \Delta(\mathscr{F})$ and $s \in S$,

$$P_s = P(f^1)f^1_s + \cdots + P(f^n)f^n_s,$$

where $P = P(f^1)f^1 \oplus \cdots \oplus P(f^n)f^n$.

In words, P_s is a reduced marginal distribution of P on s. Kreps (1988, p. 106) as well as Raiffa (1961) have proposed an act $(P_s)_{s \in S}$, which offers P_s at each state s, as a reasonable embedding of $P \in \Delta(\mathscr{F})$ to $\mathscr{F}$. Henceforth, we write $(P_s)_s$, instead of $(P_s)_{s \in S}$ for simplicity.

The next embedding corresponds to the *support–wise* criterion; remember that $l_f \in \Delta(Z)$ is a certainty equivalent for an act f.

DEFINITION: For all $P \in \Delta(\mathscr{F})$,

$$l_P = P(f^1)l_{f^1} + \cdots + P(f^n)l_{f^n},$$

where $P = P(f^1)f^1 \oplus \cdots \oplus P(f^n)f^n$.[17]

AXIOM 7 (Indifference): For all $P, Q \in \Delta(\mathscr{F})$,

$$\left\{ \begin{array}{l} \text{(i) } (P_s)_s \sim (Q_s)_s; \text{ and} \\ \text{(ii) } \quad l_P \sim l_Q \end{array} \right) \Rightarrow P \sim Q.$$

Indifference states that if two lotteries on acts are indifferent according to the two criteria *jointly*, then the lotteries should be indifferent. As will be shown in Section 6, a stronger axiom without the condition (ii), which will be called *State–wise Indifference*, is equivalent to Reversal of Order.

As noted, Raiffa (1961) applies the two criteria *independently* as opposed to Indifference. To see the differences formally, consider two lotteries on acts in Figure 4; Since $(P_s)_s = (Q_s)_s$, P and Q are indifferent according to State–wise Indifference. So, Raiffa would conclude that P and Q should be indifferent. According to the support–wise criterion, on the other hand,

[17]Note that, in general, it is not true that $P \sim l_P$. However, for a degenerate lottery on acts, $f \sim l_{\delta_f} \equiv l_f$. So, there is no contradiction in the notations.

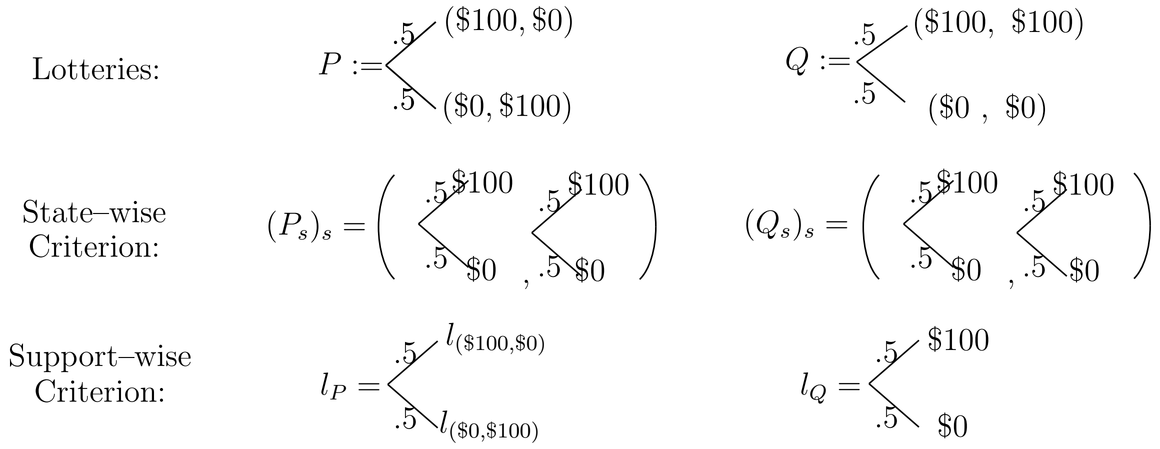


Figure 4: State–wise Criterion and Support–wise Criterion

Q is better than P, because, under ambiguity aversion, $l_Q = (\$100, .5;\ \$0, .5) \succ (\$100, \$0) \sim (\$0, \$100) \sim l_P$. Hence, Indifference does *not* require indifference between P and Q.[18]

3.2 Representation

Before stating the result, we mention that the topology to be used on the space of finitely additive set functions on Σ is the weak* topology.

Theorem 1: *For a preference relation $\succsim$ on $\Delta(\mathscr{F})$, the following statements are equivalent:*
(i) The preference relation satisfies Axioms 1–7.
(ii) There exist a real number δ, a nonempty convex closed set C of finitely additive probability measures on Σ, and a nonconstant mixture linear function $u : \Delta(Z) \to \mathbb{R}$, such that $\succsim$ is represented by the function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ of the form

$$V(P) = \delta \min_{\mu \in C} \int_S \Big(\int_{\mathscr{F}} u(f_s) dP(f) \Big) d\mu(s) + (1-\delta) \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f_s) d\mu(s) \Big) dP(f).$$

Definition: A preference relation $\succsim$ on $\Delta(\mathscr{F})$ is called an *Ex–ante/Ex–post (EAP) Maxmin* preference if it satisfies Axioms 1–7.

[18] Indeed, if Indifference is strengthened to apply (i) and (ii) independently as in Raiffa (1961) (that is, (i) *or* (ii) $\Rightarrow P \sim Q$), then Anscombe and Aumann's (1963) subjective expected utility is obtained in Theorem 1.

By Theorem 1, EAP Maxmin preferences can be represented by a triple (δ, C, u). Next, we give the uniqueness property of this representation.

Corollary 1: *The following two statements are equivalent:*

(i) Two triples (δ, C, u) and (δ', C', u') represent the same EAP Maxmin preference as in Theorem 1.

(ii) (a) $C = C'$, and there exist real numbers α and β such that $\alpha > 0$ and $u = \alpha u' + \beta$; and

(b) If C is nondegenerate, then $\delta = \delta'$.

3.3 Ex–ante and Interim Ambiguity Aversion and δ

The parameter δ has a direct behavioral characterization in terms of *Ex–ante Ambiguity Aversion* and *Interim Ambiguity Aversion*:

Axiom (Ex–ante Ambiguity Aversion): For all $\alpha \in (0,1)$ and $f, g \in \mathscr{F}$,

$$f \sim g \Rightarrow \alpha f \oplus (1-\alpha) g \succsim f.$$

Ex–ante Ambiguity Neutrality and *Ex–ante Ambiguity Loving* are defined in the same way by changing the right–hand side of the definition to $\alpha f \oplus (1-\alpha) g \sim f$ and to $f \succsim \alpha f \oplus (1-\alpha) g$, respectively.

Axiom (Interim Ambiguity Aversion): For all $\alpha \in (0,1)$ and $f, g \in \mathscr{F}$,

$$\alpha f + (1-\alpha) g \succsim \alpha f \oplus (1-\alpha) g.$$

Interim Ambiguity Aversion means that an ex–post mixture is preferred over its ex–ante mixture. This is because an ex–post mixture provides hedging in the ex–post utilities, whereas an ex–ante mixture provides hedging only in the ex–ante expected utilities. In addition, *Interim Ambiguity Neutrality* is defined in the same way by changing $\succsim$ to $\sim$.

Proposition 1: *Suppose $\succsim$ is EAP Maxmin preference with nondegenerate C.*

(i) $\succsim$ exhibits Ex–ante Ambiguity Aversion if and only if $\delta \geq 0$.

(ii) $\succsim$ exhibits Interim Ambiguity Aversion if and only if $\delta \leq 1$.

Note that given the representation, it is easy to see that EAP Maxmin preference with $\delta = 0$ and $\delta = 1$ satisfies Ex–ante Ambiguity Neutrality and Interim Ambiguity Neutrality, respectively.

3.4 Comparative Attitudes toward Ex–ante Ambiguity

We now study comparative attitudes toward ex–ante ambiguity.

DEFINITION: Given two preference relations $\succsim_1$ and $\succsim_2$, $\succsim_1$ is said to be *more Ex–ante Ambiguity Averse* than $\succsim_2$ if, for every $P \in \Delta(\mathscr{F})$ and every $f \in \mathscr{F}$,

$$P \succsim_2 f \Rightarrow P \succsim_1 f.$$

The next proposition shows that δ captures attitude toward ex–ante ambiguity.

PROPOSITION 2: *Suppose two EAP Maxmin preferences $\{\succsim_i\}_{i=1,2}$ are represented by $\{(\delta_i, C_i, u_i)\}_{i=1,2}$, where C_1 and C_2 are nondegenerate. Then, the following statements are equivalent:*
(i) $\succsim_1$ is more Ex–ante Ambiguity Averse than $\succsim_2$.
(ii) $\delta_1 \geq \delta_2$, $C_1 = C_2$, and there exist real numbers α and β such that $\alpha > 0$ and $u_1 = \alpha u_2 + \beta$.

Therefore, Proposition 2 says that a stronger Ex–ante Ambiguity Averse preference is characterized by *larger* values of δ. Therefore, δ can be interpreted as an *index of Ex–ante Ambiguity Aversion.*

3.5 Relationship between Attitudes toward Ex–ante and Ex–post Ambiguity

As Table 1 in Section 1.2.1 shows, Dominiak and Schnedler (2009) have found that among ex–post ambiguity averse subjects, the attitude toward ex–ante ambiguity is quite heterogeneous, but that most Ex–post Ambiguity Neutral subjects are Ex–ante Ambiguity Neutral as well. These results are formally described by EAP Maxmin preferences as follows:

PROPOSITION 3: *Suppose $\succsim$ is EAP Maxmin preference.*

(i) (a) Suppose $\delta > 0$. Then, $\succsim$ exhibits Ex–post Ambiguity Aversion if and only if $\succsim$ exhibits Ex–ante Ambiguity Aversion.

(b) Suppose $\delta < 0$. Then, $\succsim$ exhibits Ex–post Ambiguity Aversion if and only if $\succsim$ exhibits Ex–ante Ambiguity Loving.

(c) Suppose $\delta = 0$. Then, $\succsim$ exhibits Ex–ante Ambiguity Neutrality.

(ii) For any δ, if $\succsim$ exhibits Ex–post Ambiguity Neutrality, then $\succsim$ exhibits Ex–ante Ambiguity Neutrality.

Part (i) shows that the heterogeneity observed in the experiment can be described by whether or not δ is positive. Part (ii) shows that among EAP Maxmin preferences, Ex–post Ambiguity Neutrality implies Ex–ante Ambiguity Neutrality, as observed in the experiment.

4 Ex–ante/Ex–post Piecewise–linear Preferences

We now examine inequality averse preferences. Accordingly, the set S of states are assumed to be finite and *reinterpreted as individuals* including a decision maker, who is denoted by $1 \in S$.

4.1 Axioms

The axioms for EAP Maxmin preferences are now modified to capture inequality aversion. No modification is necessary for Indifference, and the first three modifications required are minor.

To capture inequality aversion, Monotonicity (Axiom 3) needs to be weakened as follows:

Axiom 3′ (Substitution): For all $f, g \in \mathscr{F}$,

$$f_s \sim g_s \text{ for all } s \in S \Rightarrow f \sim g.$$

Nondegeneracy (Axiom 4) is strengthened into Unboundedness, which requires that there are arbitrarily good and arbitrarily bad outcomes as follows:

Axiom 4′ (Unboundedness): There are $z_+, z_- \in Z$ such that for each $\alpha \in]0,1[$ there exist $z, z' \in Z$ such that $\alpha\delta_z + (1-\alpha)\delta_{z_-} \succ z_+ \succ z_- \succ \alpha\delta_{z'} + (1-\alpha)\delta_{z_+}$. [19]

Since Fehr and Schmidt's (1999) Piecewise–linear preferences imply the risk neutrality, the preferences satisfies Unboundedness.

The third minor change in the axioms is that the following axiom is assumed instead of Ex–post Ambiguity Aversion (Axiom 5). In order to define the axiom, preliminary notations are introduced; Let $l_0 = \frac{1}{2}\delta_{z_+} + \frac{1}{2}\delta_{z_-}$. For any lottery $l, r \in \Delta(Z)$ and $s \in S$, $(l, (r)_{-s})$ is an act which offers l for the individual s and offers r for the other individuals.

Axiom 5′ (Ex–post Inequality Aversion): For all $s \neq 1$,

(i) $(l_0, (l_0)_{-s}) \succsim (z_+, (l_0)_{-s})$; and

(ii) $(l_0, (l_0)_{-s}) \succsim (z_-, (l_0)_{-s})$.

Part (i) captures the disutility that results from *envy* toward the individual s when only the individual s is better off than the decision maker. Part (ii) captures the disutility that results from *guilt* toward the individual s when only the individual s is worse off than the decision maker.

The main axiom that requires a modification is Ex–ante/Ex–post Certainty Independence. Specifically, a new concept of *pointwise comonotonicity* needs to be defined, which is a weaker version of comonotonicity. Remember that $1 \in S$ denotes the decision maker.

Definition: Two acts $f, g \in \mathscr{F}$ are said to be *pointwise comonotonic* if for no $s \in S$, $f(s) \succ f(1)$ and $g(s) \prec g(1)$.

Suppose two acts f and g are pointwise comonotonic. Then, the rank of utilities of any individual with respect to the decision maker is not reversed between f and g.[20] Hence,

Axiom 6′ (Ex–ante/Ex–post Pointwise Comonotonic Independence):

(i) For all $\alpha \in (0,1]$ and $P, Q, R \in \Delta(\mathscr{F})$ such that $(P_s)_s, (R_s)_s$, and $(Q_s)_s, (R_s)_s$ are each

[19]This axiom is due to Cerreia–Vioglio, Maccheroni, Marinacci, and Montrucchio (2008).

[20]Schmeidler (1989, p. 586) has presented an interpretation of *comonotonicity* from the point of view of a *planner's social preference*: two income allocations f and g are comonotonic if the social rank of *any two agents* is not reversed between f and g. When we focus on an *agent's inequality averse preferences*, what is relevant to the agent is social rank *with respect to the agent himself*, not the social rank of *any two agents*.

pointwise comonotonic,

$$P \succsim Q \Leftrightarrow \alpha P \oplus (1-\alpha)R \succsim \alpha Q \oplus (1-\alpha)R.$$

(ii) For all $\alpha \in (0,1]$ and $f, g, h \in \mathscr{F}$ such that f, h, and g, h are each pointwise comonotonic,

$$f \succsim g \Leftrightarrow \alpha\, f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

As noted, no modification is necessary for Indifference. The interpretation of that axiom is straightforward here. The first criterion (i) corresponds to ex–ante equality, because each marginal distribution P_s yields the ex–ante expected payoff of the individual s. The second criterion (ii) corresponds to ex–post equality, because each certainty equivalent l_f reflects the ex–post equality of f. Hence, Indifference means that if P and Q are indifferent in the both ex–ante and ex–post equality, then P and Q should be indifferent.

4.2 Representation

THEOREM 2: *For a preference relation $\succsim$ on $\Delta(\mathscr{F})$, the following statements are equivalent:*
(i) The preference relation satisfies Axioms 1, 2, 3', 4', 5', 6', and 7.
(ii) There exist a real number δ, nonnegative numbers $\{\alpha_s, \beta_s\}_{s\neq 1}$, and a nonconstant mixture linear onto function $u : \Delta(Z) \to \mathbb{R}$ such that $\succsim$ is represented by the function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ of the form

$$\begin{aligned} V(P) = \delta\Bigg(& E_P u(f_1) - \sum_{s\neq 1}\Big(\alpha_s \max\{E_P u(f_s) - E_P u(f_1), 0\} + \beta_s \max\{E_P u(f_1) - E_P u(f_s), 0\}\Big)\Bigg) \\ & + (1-\delta)\int_{\mathscr{F}}\Bigg(u(f_1) - \sum_{s\neq 1}\Big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\}\Big)\Bigg) dP(f), \end{aligned}$$

where $E_P u(f_s) = \int_{\mathscr{F}} u(f_s) dP(f)$. Furthermore, the two quadruples $(\delta, \alpha, \beta, u)$ and $(\delta', \alpha', \beta', u')$ represent the same preference as in the above if and only if $(\alpha, \beta) = (\alpha', \beta')$, $\delta = \delta'$ if $(\alpha, \beta) \neq \mathbf{0}$, and there exist real numbers a and b such that $a > 0$ and $u = au' + b$.

DEFINITION: A preference relation $\succsim$ on $\Delta(\mathscr{F})$ is called an *Ex–ante/Ex–post (EAP) Piecewise–*

linear preference if it satisfies Axioms 1, 2, $3', 4', 5', 6'$, and 7.

4.3 Ex–ante and Interim Inequality Aversion and δ

The parameter δ has a direct behavioral characterization in terms of both *Ex–ante Inequality Aversion* and *Interim Inequality Aversion*, as follows:

Axiom(Ex–ante Inequality Aversion):For all $s \neq 1$ and $l_+, l_- \in \Delta(Z)$ such that $l_+ \succ l_0 \succ l_-$.

$$(l_+, (l_0)_{-s}) \sim (l_-, (l_0)_{-s}) \Rightarrow \frac{1}{2}(l_+, (l_0)_{-s}) \oplus \frac{1}{2}(l_-, (l_0)_{-s}) \succsim (l_+, (l_0)_{-s}).$$

With the allocation $(l_+, (l_0)_{-s})$, the decision maker feels envy toward the individual s because only the individual s is better off; while with the allocation $(l_-, (l_0)_{-s})$, the decision maker feels guilt toward the individual s because only the individual s is worse off. Ex–ante Inequality Aversion means that an ex–ante mixture of these allocations partly offsets these inequalities in the expected utilities. So, the ex–ante mixture becomes more desirable.[21] In addition, *Ex–ante Inequality Neutrality* is defined by changing the right–hand side of the definition above to $\frac{1}{2}(l_+, (l_0)_{-s}) \oplus \frac{1}{2}(l_-, (l_0)_{-s}) \sim (l_+, (l_0)_{-s})$.

Axiom (Interim Inequality Aversion): For all $s \neq 1$,

$$\frac{1}{2}(z_+, (l_0)_{-s}) + \frac{1}{2}(z_-, (l_0)_{-s}) \succsim \frac{1}{2}(z_+, (l_0)_{-s}) \oplus \frac{1}{2}(z_-, (l_0)_{-s}).$$

To interpret Interim Inequality Aversion, recall that $l_0 = \frac{1}{2}\delta_{z_+} + \frac{1}{2}\delta_{z_-}$. Hence, the ex–ante mixture in the right hand side could provide *ex–ante equality* but not *ex–post equality*, as opposed to the ex–post mixture in the left hand side. So, the ex–post mixture is preferred over the ex–ante mixture. In addition, *Interim Inequality Neutrality* is defined by changing $\succsim$ to $\sim$.

Corollary 2: *Suppose $\succsim$ is EAP Piecewise–linear preference with $(\alpha, \beta) \neq \boldsymbol{0}$.*

[21] Ex–ante Inequality Aversion is consistent with the experimental evidence, drawn from the probabilistic dictator games, that subjects who are indifferent between winning and losing tend to prefer flipping a coin to decide the winner.

(i) $\succsim$ *exhibits Ex–ante Inequality Aversion if and only if* $\delta \geq 0$.

(ii) $\succsim$ *exhibits Interim Inequality Aversion if and only if* $\delta \leq 1$.

Note that given the representation, it is easy to see that EAP Piecewise–linear preferences with $\delta = 0$ and $\delta = 1$ satisfy Ex–ante Inequality Neutrality and Interim Inequality Neutrality, respectively.

4.4 Comparative Attitudes toward Ex–ante Inequality

As mentioned in Introduction, a preference for ex–ante mixtures is due to Ex–ante Inequality Aversion in a social context, in contrast to ambiguous situations, in which such a preference is due to Ex–ante Ambiguity Aversion. So, the same definition of being *more Ex–ante Ambiguity Averse* in Section 3.4 is interpreted as the definition of being *more Ex–ante Inequality Averse* in a social context.

Hence, results analogous to those derived from Proposition 2 in Section 3.4 also hold for inequality aversion.

Corollary 3: *Suppose two EAP Piecewise–linear preferences* $\{\succsim_i\}_{i=1,2}$ *are represented by* $\{(\delta_i, \alpha^i, \beta^i, u_i)\}_{i=1,2}$, *where* $(\alpha^1, \beta^1) \neq \mathbf{0} \neq (\alpha^2, \beta^2)$. *Then the following statements are equivalent:*

(i) $\succsim_1$ *is more Ex–ante Inequality Averse than* $\succsim_2$.

(ii) $\delta_1 \geq \delta_2$, $(\alpha^1, \beta^1) = (\alpha^2, \beta^2)$, *and there exist real numbers* a *and* b *such that* $a > 0$ *and* $u_1 = au_2 + b$.

Therefore, Corollary 3 says that a stronger Ex–ante Inequality Averse preference is characterized by *larger* values of δ. Therefore, δ can be interpreted as an *index of Ex–ante Inequality Aversion* in a social context.

5 Games

In preceding sections, we saw how EAP Maxmin and EAP Piecewise–linear preferences are consistent with many experimental results, mainly on single-person decision making. In this section, EAP Maxmin and EAP Piecewise–linear preferences are applied to games in order to see the implications of the models in strategic situations.

5.1 EAP Maxmin Preferences in Games

As noted in Introduction, the special cases of EAP Maxmin and EAP Choquet preferences, where $\delta = 0$ or 1 have been used in the game theory literature on ambiguity averse players.[22] The following two symmetric games, Game I and Game II, suggest that $\delta \in (0, 1)$ would predict more realistic behavior of ambiguity averse players than $\delta = 0$ and 1, respectively. The numbers in the games are von Neumann–Morgenstern utilities and x is a positive number.

$1\backslash 2$	d	e
a	$2x$	0
b	0	$2x$

Game I

$1\backslash 2$	d	e
a	$2x$	0
b	0	$2x$
c	$x-\varepsilon$	$x-\varepsilon$

Game II

For both games, when they are played for the first time, the symmetry makes it difficult for each player to have a unique prior probability over the opponent's strategies. So, in Game I, the ambiguity averse players would prefer mixed strategies to pure strategies in order to hedge. In addition, in Game II, if a positive number ε is less than a certain threshold, player 1 would prefer strategy c, whose payoff is constant, to any mixed strategies over a and b.

EAP Maxmin preferences with $\delta \in (0, 1)$ can describe these reasonable behaviors in a *strict* equilibrium in each game.[23] However, EAP Maxmin preferences with $\delta = 0$ show that $a \sim b \Rightarrow .5a \oplus .5b \sim a$ and $d \sim e \Rightarrow .5d \oplus .5e \sim d$, so that for both players, there are no strict incentives to use the mixed strategies in Game I. In addition, EAP Maxmin preferences

[22]To see the relationship between the literature and our model, fix a game and a player. Then, the player's pure strategy corresponds to an act; the set of strategies of the other players corresponds to the set of states; hence, the player's mixed strategy corresponds to ex–ante mixtures on acts.

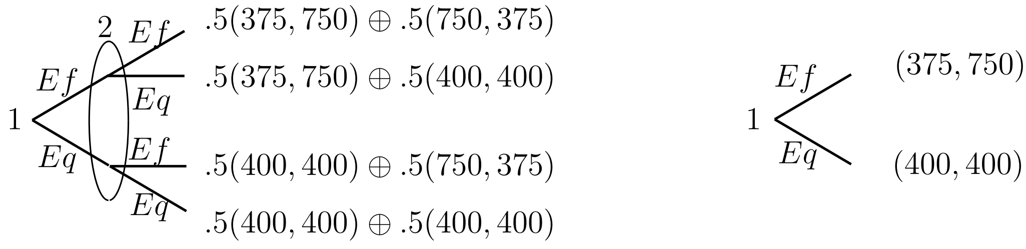
[23]See Klibanoff (1996) for a definition of an equilibrium with ambiguity averse players. He assumes $\delta = 1$ but the definition is easily applied to EAP Maxmin preferences with $\delta \neq 1$.

with $\delta = 1$ show that $.5a \oplus .5b \succ c$ for any small positive number ε, so strategy c will not be employed by player 1 in Game II.

5.2 EAP Piecewise–linear Preferences in Games

In this section, we show that EAP Piecewise–linear preferences can describe seemingly contradictory experimental results on efficiency versus inequality. As mentioned in Introduction, the key to resolving the putative contradiction is that it is only in the experiments that are strongly in favor of efficiency that subjects are under *risk over roles.* That is, in the experiments, each subject makes a decision as if he were a dictator, but actual roles (i.e., dictator or receiver) are determined at random.

We study a Bayesian game that describes the dictator game from Charness and Rabin (2002), mentioned in Section 1.2.2. In the game, they report that about 50 percent of the subjects chose the efficient but unequal allocation rather than the inefficient but equal allocation. Assume, for simplicity, there exist two players $\{1, 2\}$ and two types of players: *fair* (i.e., $\alpha^F, \beta^F > 0$ and $\delta^F > 0$) and *selfish* (i.e., $\alpha^S = 0 = \beta^S$). Player 1's set of actions is $\{(375, 750), (400, 400)\}$ and player 2's set of actions is $\{(750, 375), (400, 400)\}$, where the first and second coordinates show the material prizes for players 1 and 2, respectively. Denote the efficient but unequal payoff by Ef, and the inefficient but equal payoff by Eq. The game is



Game I Game II

Figure 5: Dictator Games *with and without Risk–over–roles*

described as Game I in Figure 5.

In Game I, the player's choice determines outcomes only if he turns out to be a dictator. Given that each role is determined with the probability .5, outcomes are fifty–fifty ex–ante mixtures on allocations. For example, if player 1 chooses action Ef and player 2 chooses Eq, the outcome is an ex–ante mixture that gives $(375, 750)$ and $(400, 400)$ with the probability .5. Game I is different from Game II, since there is no risk over roles in the latter. That is, in Game II, player 1 knows that he is a dictator for sure. Now the result can be stated as follows:

PROPOSITION 4: *Suppose*

(a) Players' preferences are EAP Piecewise–linear with $u(z) = \log z$ *for all* $z \in \mathbb{R}_+$.

(b) There exist two types, fair $(i.e., \alpha^F, \beta^F > 0, and\ \delta^F > 0)$ *and selfish* $(i.e., \alpha^S = 0 = \beta^S)$.

Let $\alpha^F = .2$, $\beta^F = .9$, *and* $\delta^F = .85$. *Then the following results hold:*

(i) In Game I, there exists a Bayesian Nash equilibrium in which the fair type choose the efficient payoff (Ef), the selfish type choose the equal payoff (Eq), and the common prior probability on the fair type is .5.

(ii) In Game II, for both types, choosing the equal payoff (Eq) strictly dominates choosing the efficient payoff (Ef).

Note that in the result (i), the common prior probability on the fair type is consistent with the experimental evidence found by Charness and Rabin (2002), who report that about 50 percent of the subjects chose E_f.

Having subjects make decisions under risk over roles is currently prevalent in experimental research because it makes the number of samples much larger. However, Proposition 4 shows that risk over roles makes subjects with EAP Piecewise–linear preferences tend to choose the efficient allocations, even if they do not have a preference for efficiency itself. This is consistent with the experimental evidence, found by Bolton and Ockenfels (2006), that under risk over roles, subjects tend to choose efficient but unequal allocations over inefficient but equal allocations.

6 Discussion

In this section, the relationships among the key axioms used in Anscombe and Aumann (1963), Seo (2009), and our model are discussed. In particular, we will show that Reversal of

Order implies Indifference but not vice versa, and Indifference, in turn, implies Reduction of Compound Lotteries but not vice versa.

First, Reversal of Order by Anscombe and Aumann (1963) is formally defined;

Axiom (Reversal of Order): For all $\alpha \in (0,1]$, and $f, g \in \mathscr{F}$,

$$\alpha f + (1-\alpha) g \sim \alpha f \oplus (1-\alpha) g.$$

As noted, Reversal of Order turns out to be equivalent to the following axiom:

Axiom (State–wise Indifference): For all $P, Q \in \Delta(\mathscr{F})$,

$$(P_s)_s \sim (Q_s)_s \Rightarrow P \sim Q.$$

Lemma 1: *Reversal of Order and State–wise Indifference are equivalent.*

Note that State–wise Indifference is a strengthening of Indifference by dropping the requirement of the support–wise criterion. Hence,

Corollary 4: *Reversal of Order implies Indifference.*

It is easy to see that the opposite of Corollary 4 is not true. However, Indifference implies Reversal of Order *among constant acts*. Formally,

Axiom (Reduction of Compound Lotteries): For all $\alpha \in [0,1]$ and $l, r \in \Delta(Z)$,

$$\alpha l + (1-\alpha) r \sim \alpha l \oplus (1-\alpha) r.$$

Lemma 2: *Indifference implies Reduction of Compound Lotteries.*

As noted, Seo (2009) also does not assume Reversal of Order and, instead, proposes an axiom of his own, *Dominance*. To present the axiom, we must first introduce preliminary notations. For each $f \in \mathscr{F}$ and $\mu \in \Delta(S)$, $\Psi(f, \mu) = \mu(s_1) f_{s_1} + \cdots + \mu(s_{|S|}) f_{s_{|S|}} \in \Delta(Z)$.[24] In addition, for each $P \in \Delta(\mathscr{F})$ and $\mu \in \Delta(S)$, $\Psi(P, \mu) = P(f^1)\Psi(f^1, \mu) \oplus \cdots \oplus P(f^n)\Psi(f^n, \mu)$, where $P = P(f^1) f^1 \oplus \cdots \oplus P(f^n) f^n$. Now, his axiom can be sated as follows:

[24] Seo (2009) assumes that the set of states is finite.

Axiom (Dominance, Seo (2009)): For all $P, Q \in \Delta(\mathscr{F})$,

$$\Psi(P,\mu) \succsim \Psi(Q,\mu) \text{ for all } \mu \in \Delta(S) \Rightarrow P \succsim Q.$$

Seo (2009, p. 1587, Lemma 5.1) shows that under Dominance, Reduction of Compound Lotteries and Reversal of Order are equivalent. This observation together with Corollary 4 imply the following result:

Corollary 5: *Under Reduction of Compound Lotteries, Dominance implies Indifference.*

Therefore, under Reduction of Compound Lotteries, Dominance together with the axioms used in Theorem 1 and 2 (except Indifference) respectively imply EAP Maxmin and EAP Piecewise–linear preferences, with $\delta = 0$.

Appendix: Proofs

Section A provides a sketch of the proofs of sufficiency for Theorems 1 and 2. Section B provides proofs for Lemmas. The proofs of Theorem 1 and related results are in Section C, while Section D presents the proofs of Theorem 2 and related results.

A Sketch of Proofs

By the standard argument, there exists a function V representing $\succsim$ on $\Delta(\mathscr{F})$. Ex–ante/Ex–post Certainty Independence and Indifference will show that V can be taken so that the restriction U of V on $\mathscr{F}$ has a Maxmin representation. That is, there exists a set C of priors and a mixture linear function u on $\Delta(Z)$ such that $U(f) = \min_{\mu\in C} \int_S u(f_s)d\mu(s)$.

Then, for all $P \in \Delta(\mathscr{F})$,

$$U((P_s)_s) = \min_{\mu\in C} \int_S \Big(\int_{\mathscr{F}} u(f_s)dP(f) \Big) d\mu(s); \quad U(l_P) = \int_{\mathscr{F}} \Big(\min_{\mu\in C} \int_S u(f_s)d\mu(s) \Big) dP(f). \quad (7)$$

Hence, it follows from Jensen's inequality that $U((P_s)_s) \geq U(l_P)$ for all $P \in \Delta(\mathscr{F})$. Define

$$\mathscr{C} = \Big\{(u(l), u(l)) \in \mathbb{R}^2 \;\Big|\; l \in \Delta(Z)\Big\}; \quad \mathscr{D} = \Big\{\big(U((P_s)_s), U(l_P)\big) \in \mathbb{R}^2 \;\Big|\; P \in \Delta(\mathscr{F})\Big\}. \tag{8}$$

We now can show that $\mathscr{C}$ consists of the upper boundary of $\mathscr{D}$ as in Figure 6. In addition, if $(x, y) \in \mathscr{D}$, $(c, c) \in \mathscr{C}$, and $\alpha \in [0, 1]$, then $\alpha(x, y) + (1 - \alpha)(c, c) \in \mathscr{D}$.

Define a binary relation $\hat{\succsim}$ on $\mathscr{D}$: for all $(x, y), (x', y') \in \mathscr{D}$,

$$(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow V(P) \geq V(Q),$$

where $P, Q \in \Delta(\mathscr{F})$, $\big(U((P_s)_s), U(l_P)\big) = (x, y)$, and $\big(U((Q_s)_s), U(l_Q)\big) = (x', y')$. Indifference will show that $\hat{\succsim}$ is a well–defined binary relation. The purpose of the proof is to show that there exists a real number δ such that for any (x, y) and $(x', y') \in \mathscr{D}$, $(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow \delta x + (1 - \delta)y \geq \delta x' + (1 - \delta)y'$. Together with the definition of $\hat{\succsim}$, this implies that

$$V(P) \geq V(Q) \Leftrightarrow \delta U((P_s)_s) + (1 - \delta)U(l_P) \geq \delta U((Q_s)_s) + (1 - \delta)U(l_Q).$$

Since V is unique up to positive affine transformation and both V and U coincide with u on $\Delta(Z)$, then $V(P) = \delta U((P_s)_s) + (1 - \delta)U(l_P)$ for all P, as desired.

In the following, we sketch how to show the existence of the desired real number δ.[25] It will be shown that $\hat{\succsim}$ satisfies completeness, transitivity, monotonicity on $\mathscr{C}$, and *certainty independence*:

$$(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow \alpha(x, y) + (1 - \alpha)(c, c) \mathrel{\hat{\succsim}} \alpha(x', y') + (1 - \alpha)(c, c). \tag{9}$$

If the set C of priors is degenerate, the existence of the δ is trivial.[26] So, suppose that C is nondegenerate. Then, there exist $f^*, g^* \in \mathscr{F}$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \succ f^*$. Let $(x^*, y^*) =$

[25]Note that, the continuity of $\hat{\succsim}$ does not follow directly from the continuity of $\succsim$. In addition, in $\mathbb{R}^2$, it is well–known that in general, additive linear representation requires more than Independence. (See Debrue (1960).) So, the standard argument might not show the existence of the desired δ directly.

[26]See Step 3 in the proof of Theorem 1 for details.

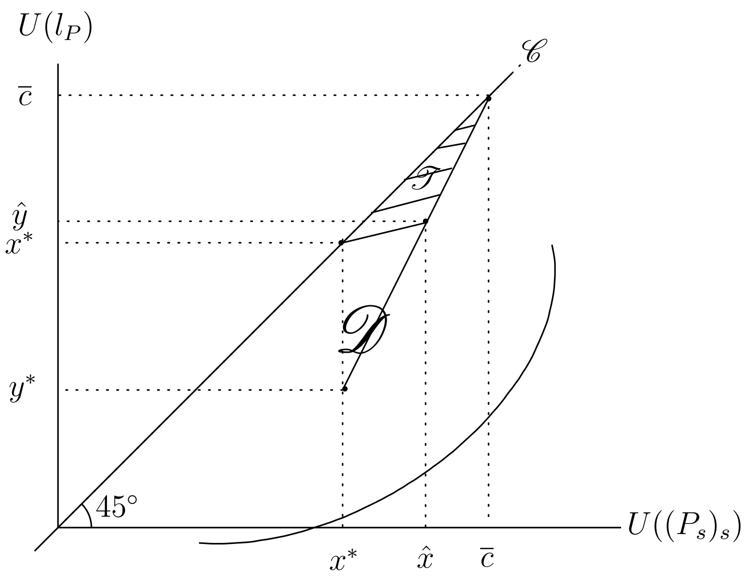


Figure 6: Indifference Curves of $\hat{\succsim}$.

$(\zeta,\eta)(\frac{1}{2}f^* \oplus \frac{1}{2}g^*) \in \mathscr{D}$. Then $x^* > y^*$. Now, consider the case where $\frac{1}{2}f^* + (1-\alpha)g^* \succsim \frac{1}{2}f^* \oplus (1-\alpha)g^*$.[27] This implies that $(x^*, x^*) \hat{\succsim} (x^*, y^*)$. Without loss of generality, assume there exists $\overline{c} > x^*$ such that $(\overline{c}, \overline{c}) \in \mathscr{D}$. Then, it follows from the monotonicity that $(\overline{c}, \overline{c}) \hat{\succ} (x^*, x^*)$. Hence, Continuity of $\succsim$ will show the existence of $\overline{\alpha}$ such that $(x^*, x^*) \hat{\sim} \overline{\alpha}(x^*, y^*) + (1-\overline{\alpha})(\overline{c}, \overline{c})$.

Define $(\hat{x}, \hat{y}) = (\overline{\alpha}x^* + (1-\overline{\alpha})\overline{c}, \overline{\alpha}y^*(1-\overline{\alpha})\overline{c})$. Then, let $\mathscr{T}$ be a triangle, including the interior, which consists of the vertices $(\overline{c}, \overline{c})$, (x^*, x^*), and $(\hat{x}, \hat{y})$. It follows that $\mathscr{T} \subset \mathscr{D}$ and $\mathscr{T}$ is not degenerate. Certainty Independence of $\hat{\succsim}$ and the Carathéodory's Theorem show that the indifference curves on $\mathscr{T}$ are parallel, as shown in Figure 6. Since $(x^*, x^*) \hat{\sim} (\hat{x}, \hat{y})$, the δ is determined to be $1 - (\hat{x} - x^*)/(\hat{x} - \hat{y})$. Finally, given that $\mathscr{C}$ consists of the upper boundary of both $\mathscr{D}$ and $\mathscr{T}$, certainty independence of $\hat{\succsim}$ again will show that the indifference curves are expanded over the whole domain $\mathscr{D}$. This completes the proof of Theorem 1.

In Theorem 2, the sufficiency of axioms is shown in an analogous way. First, we show that $\succsim$ restricted on $\mathscr{F}$ has Fehr and Schmidt's (1999) piecewise–linear utility representation. This step requires Unboundedness. Given the representation on $\mathscr{F}$, the rest of the proof is the same as the proof of Theorem 1.

[27] In the other case where $\frac{1}{2}f^* \oplus (1-\alpha)g^* \succsim \frac{1}{2}f^* + (1-\alpha)g^*$, analogous argument holds. See footnote 30 for details.

B Proof of Lemmas

Several notations are introduced as follows: $\hat{\sum}$ is a summation by ex–ante mixtures and $\sum$ is a summation by ex–post mixtures. That is, for any set $\{f^i\}_{i=1}^n$ of acts and any set of nonnegative numbers $\{\alpha_i\}_{i=1}^n$ such that $\sum_{i=1}^n \alpha_i = 1$, define $\hat{\sum}\alpha_i f^i \equiv \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$ and $\sum \alpha_i f^i \equiv \alpha_1 f^1 + \cdots + \alpha_n f^n$.

B.1 Proof of Lemma 1

To see that Reversal of Order implies State–wise Indifference, fix $P, Q \in \Delta(\mathscr{F})$ such that $(P_s)_s \sim (Q_s)_s$. Then, there exist sets $\{f^i\}$ and $\{g^j\}$ of acts and sets of nonnegative numbers $\{\alpha_i\}$ and $\{\beta_j\}$ such that $\sum_i \alpha_i = 1 = \sum_{j=1} \beta_j$, $P = \hat{\sum}_i \alpha_i f^i$, and $Q = \hat{\sum}_j \beta_j g^j$. Then, Reversal of Order shows $P \sim \sum_i \alpha_i f^i = (P_s)_s \sim (Q_s)_s = \sum_j \beta_j g^j \sim Q$.

To see that State–wise Indifference implies Reversal of Order, fix $f, g \in \mathscr{F}$ and $\alpha \in [0,1]$. Let $P = \alpha f + (1-\alpha) g$ and $Q = \alpha f \oplus (1-\alpha) g$. Then for all $s \in S$, $P_s = \alpha f_s + (1-\alpha) g_s = Q_s$, so that $(P_s)_s \sim (Q_s)_s$. Then, State–wise Indifference shows $P \sim Q$.

B.2 Proof of Lemma 2

To see that Indifference implies Reduction of Compound Lotteries, fix $l, r \in \Delta(Z)$ and $\alpha \in [0,1]$. Let $P = \alpha l + (1-\alpha) r$ and $Q = \alpha l \oplus (1-\alpha) r$. Then, $P_s = \alpha l + (1-\alpha) r = Q_s$ for all $s \in S$, so that $(P_s)_s = (Q_s)_s$, so that condition (i) is satisfied. In addition, $l_P = \alpha l + (1-\alpha) r = l_Q$, so that condition (ii) is also satisfied. Hence, Indifference implies $P \sim Q$.

C Proof of Theorem 1

The necessity of axioms is easy to check. To show Continuity, note that the set of finitely additive probabilities measures is compact under the product topology. So, the closed subset C of the set of finitely additive probabilities is compact. Hence, the Berge's Theorem can be applied.

In the following, we will prove the sufficiency. Suppose that a preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem 1. Then, by Lemma 2, $\succsim$ satisfies Reduction of Compound Lotteries as well.

The first step shows Reversal of Order between generic acts and constant acts, which will be used in the next step.

Step 1: For all $\alpha \in [0,1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$, $\alpha f \oplus (1-\alpha)l \sim \alpha f + (1-\alpha)l$.

Proof of Step 1: Fix $\alpha \in [0,1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$. Let $P = \alpha f \oplus (1-\alpha)l$ to show $P \sim \alpha f + (1-\alpha)l$. Then by Reduction of Compound Lotteries, for all $s \in S$, $P_s = \alpha f_s \oplus (1-\alpha)l \sim \alpha f_s + (1-\alpha)l$. So, by Monotonicity, $(P_s)_s \sim \alpha f + (1-\alpha)l$, so that the condition (i) in Indifference is satisfied. In addition, since $l_f \sim f$, Ex–post Certainty Independence shows $l_P = \alpha l_f + (1-\alpha)l \sim \alpha f + (1-\alpha)l$, so that the condition (ii) in Indifference is satisfied as well. Hence, Indifference shows $P \sim \alpha f + (1-\alpha)l$. ■

Step 2: There exists a function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ such that

(i) V represents $\succsim$ on $\Delta(\mathscr{F})$,

(ii) for all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $V(\alpha P \oplus (1-\alpha)l) = \alpha V(P) + (1-\alpha)V(l)$,

(iii) V is unique up to positive affine transformation.

(iv) Let U be the restriction of V on $\mathscr{F}$. There exists a nonempty convex closed set C of finitely additive probability measures on Σ, and a mixture linear function $u : \Delta(Z) \to \mathbb{R}$ such that $U(f) = \min_{\mu \in C} \int_S u(f_s)d\mu(s)$.

Proof of Step 2: From the implication of the von Neumann–Morgenstern's Theorem, there exists a mixture linear function $u : \Delta(Z) \to \mathbb{R}$ representing $\succsim$ restricted to $\Delta(Z)$. In addition, u is unique up to positive affine transformation. So, choose u such that $u(z_+) = 1$ and $u(z_-) = -1$.

For an arbitrary $P \in \Delta(\mathscr{F})$, define

$$M_P = \{\alpha P \oplus (1-\alpha)l | l \in \Delta(Z) \text{ and } \alpha \in [0,1]\}.$$

Thus, M_P is the set of ex–ante mixtures of P and the constant acts. Using the von Neumann–Morgenstern's Theorem again, there is a function $V_P : M_P \to \mathbb{R}$ representing $\succsim$ restricted to

M_P, which is linear with respect to the ex–ante mixtures. In addition, again, V_P is unique up to positive affine transformation. So, choose V_P such that $V_P(z_+) = 1$ and $V_P(z_-) = -1$.

For all $l, r \in \Delta(Z)$ $V_P(l) \geq V_P(r) \Leftrightarrow l \succsim r \Leftrightarrow u(l) \geq u(r)$. Hence, there exists an increasing function $v : u(\Delta(Z)) \to \mathbb{R}$ such that $V_P(l) = v(u(l))$ for all $l \in \Delta(Z)$. Moreover, since V_p and u are mixture linear, it follows from Reduction of Compound Lotteries that v is also mixture linear.[28] In addition, by the normalization, $v(1) = 1$ and $v(-1) = -1$. Hence, we can conclude that v is the identity function, so that $V_P(l) = u(l)$.

Now, we define a real valued function V on $\Delta(\mathscr{F})$ which represents $\succsim$ by $V(P) = V_P(P)$ for all $P \in \Delta(\mathscr{F})$. Note that V is well–defined, because if $R \in M_P \cap M_Q$, then $V_P(R) = V_Q(R)$. In addition, $V(\alpha P \oplus (1-\alpha)l) = \alpha V(P) + (1-\alpha)V(l)$ for all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$. Hence, parts (i), (ii), and (iii) hold.

Finally, to show (iv), let U be the restriction of V on $\mathscr{F}$. Fix $\alpha \in [0,1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$. Then by Step 1, $\alpha f + (1-\alpha)l \sim \alpha f \oplus (1-\alpha)l$. Hence, $U(\alpha f + (1-\alpha)l) = V(\alpha f \oplus (1-\alpha)l) = \alpha V(f) + (1-\alpha)V(l) = \alpha U(f) + (1-\alpha)U(l)$, where the second equality is by Step 2 (ii). Hence, by Ex–post Ambiguity Aversion and Continuity (ii), part (iv) follows from Gilboa and Schmeidler (1989). ■

Step 3: If C is degenerate then there exists a real number δ such that for all $P \in \Delta(\mathscr{F})$, $V(P) = \delta U((P_s)_s) + (1-\delta)\int_{\mathscr{F}} U(f)dP(f)$.

Proof of Step 3: Suppose $C = \{\mu^*\}$ for some $\mu^* \in \Delta(S)$. Then for all $P \in \Delta(\mathscr{F})$, $U((P_s)_s) = \int_S \int_{\mathscr{F}} u(f_s)dP(f)d\mu^*(s) = \int_{\mathscr{F}} \int_S u(f_s)d\mu^*(s)dP(f) = U(l_P)$, where the second equality holds by the Fubini's Theorem. Therefore, $(P_s)_s \sim l_P$. Hence, Indifference shows that for all $P \in \Delta(\mathscr{F})$, $P \sim (P_s)_s \sim l_P$. Therefore, by Step 2, $V(P) = U((P_s)_s) = \int_{\mathscr{F}} U(f)dP(f)$. So, the result holds. ■

Henceforth, consider the case where C is nondegenerate. First, to make notations simple, for all $P \in \Delta(\mathscr{F})$, define

$$\zeta(P) = U\big((P_s)_s\big); \quad \eta(P) = U(l_P).$$

[28]Choose $a, b \in u(\Delta(Z))$ and $\alpha \in [0,1]$ to show $v(\alpha a + (1-\alpha)b) = \alpha v(a) + (1-\alpha)v(b)$. There exist l, r such that $u(l) = a$ and $u(r) = b$. Then by Reduction of Compound Lotteries, $v(\alpha a + (1-\alpha)b) = v(u(\alpha l + (1-\alpha)r)) = V_P(\alpha l + (1-\alpha)r) = V_P(\alpha l \oplus (1-\alpha)r) = \alpha V_P(l) + (1-\alpha)V_P(r) = \alpha v(a) + (1-\alpha)v(b)$.

The next step shows the property of the functions ζ and η as follows:

Step 4:
(i) For all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $\zeta(\alpha P \oplus (1-\alpha)l) = \alpha\zeta(P) + (1-\alpha)\zeta(l)$ and $\eta(\alpha P \oplus (1-\alpha)l) = \alpha\eta(P) + (1-\alpha)\eta(l)$.
(ii) For all $l \in \Delta(Z)$, $\zeta(l) = u(l) = \eta(l)$.
(iii) For all $P \in \Delta(\mathscr{F})$, $\zeta(P) \geq \eta(P)$.
Proof of Step 4: Parts (i) and (ii) follow from Step 2 (iv). To show (iii), for all $x \in \mathbb{R}^S$ define $F : \mathbb{R}^S \to \mathbb{R}$ by $F(x) = \min_{\mu \in C} \int_S x_s \mu(s)$. Then F is concave. Therefore, for all $P \in \Delta(\mathscr{F})$, Jensen's Inequality (Hiriart–Urruty and Lemaréchal (1949, p. 76, Theorem 1.1.8)) shows $\zeta(P) = F\big(\big(\int_{\mathscr{F}} u(f_s) dP(f)\big)_{s \in S}\big) \geq \int_{\mathscr{F}} F\big(\big(u(f_s)\big)_{s \in S}\big) dP(f) = \eta(P)$. ■

Subsets $\mathscr{C}$ and $\mathscr{D}$ of $\mathbb{R}^2$ are defined by (8) in Section A. The next step shows that $\mathscr{C}$ and $\mathscr{D}$ are as in Figure 6 in Section A as follows:

Step 5:
(i) $\mathscr{C} \subset \partial\mathscr{D}$, where $\partial\mathscr{D}$ is the boundary of $\mathscr{D}$.
(ii) For all $(x, y) \in \mathscr{D}$, $(c, c) \in \mathscr{C}$, and $\alpha \in [0, 1]$, $\alpha(x, y) + (1-\alpha)(c, c) \in \mathscr{D}$.
Proof of Step 5: By Step 4 (iii), for all $(x, y) \in \mathscr{D}$, $x \geq y$. Hence, $\mathscr{C} \subset \partial\mathscr{D}$. Now we will show (ii). Choose any $(x, y) \in \mathscr{D}$, $(c, c) \in \mathscr{C}$, and $\alpha \in [0, 1]$. Then, there exist $P \in \Delta(\mathscr{F})$ and $l \in \Delta(Z)$ such that $(x, y) = (\zeta(P), \eta(P))$and $\zeta(l) = c = \eta(l)$. Hence, by Step 4 (i), $\zeta(\alpha P \oplus (1-\alpha)\delta_l) = \alpha\zeta(P) + (1-\alpha)\zeta(l) = \alpha x + (1-\alpha)c$ and $\eta(\alpha P \oplus (1-\alpha)\delta_l) = \alpha\eta(P) + (1-\alpha)\zeta(l) = \alpha y + (1-\alpha)c$. Therefore, $\alpha(x, y) + (1-\alpha)(c, c) \in \mathscr{D}$. ■

To define a binary relation $\hat{\succsim}$ on $\mathscr{D}$, first define $v : \mathscr{D} \to \mathbb{R}$ by for all $(x, y) \in \mathscr{D}$,

$$v(x, y) = V(P),$$

where $P \in \Delta(\mathscr{F})$ and $\zeta(P) = x$ and $\eta(P) = y$.

Step 6: v is well-defined, i.e., if $v(x, y) \neq v(x', y')$, then $(x, y) \neq (x', y')$.
Proof of Step 6: Choose any $(x, y), (x', y') \in \mathscr{D}$ such that $v(x, y) \neq v(x', y')$. Assume to the contrary that $(x, y) = (x', y')$. Then, by definition, there exist $P, Q \in \Delta(\mathscr{F})$ such

that $(\zeta(P), \eta(P)) = (x, y)$ and $(\zeta(Q), \eta(Q)) = (x', y')$. Hence, $(\zeta(P), \eta(P)) = (\zeta(Q), \eta(Q))$. Hence, $U((P_s)_s) = \zeta(P) = \zeta(Q) = U((Q_s)_s)$, so that the condition (i) in Indifference is satisfied. In addition, $U(l_P) = \int_{\mathscr{F}} U(f)dP(f) = \eta(P) = \eta(Q) = \int_{\mathscr{F}} U(f)dQ(f) = U(l_Q)$, so that the condition (ii) in Indifference is satisfied as well. Therefore, Indifference shows $v(x, y) = V(P) = V(Q) = v(x', y')$, which is a contradiction. Hence, $(x, y) \neq (x', y')$. ■

Now, define a binary relation $\hat{\succsim}$ on $\mathscr{D}$ by for all $(x, y), (x', y') \in \mathscr{D}$,

$$(x, y) \hat{\succsim} (x', y') \Leftrightarrow v(x, y) \geq v(x', y').$$

The next step shows the property of $\hat{\succsim}$ as follows:

Step 7: $\hat{\succsim}$ satisfies completeness, transitivity, monotonicity on $\mathscr{C}$, and certainty independence defined by (9) in Section A.

Proof of Step 7: Since v is a well-defined real valued function, the completeness and transitivity are trivial. First, we will show the monotonicity on $\mathscr{C}$. Choose any $(c, c), (c', c') \in \mathscr{C}$. Then there exist $l, l' \in \Delta(Z)$ such that $u(l) = c$ and $u(l') = c'$. Hence, $(c, c) \hat{\succsim} (c', c') \Leftrightarrow v(u(l), u(l)) \geq v(u(l'), u(l')) \Leftrightarrow V(l) \geq V(l') \Leftrightarrow u(l) \geq u(l') \Leftrightarrow c \geq c'$.

Next, we will show the certainty independence. Choose any $(x, y), (x', y'), (c, c) \in \mathscr{D}$ and $\alpha \in [0, 1]$. By Step 5 (ii), $\alpha(x, y) + (1 - \alpha)(c, c), \alpha(x', y') + (1 - \alpha)(c, c) \in \mathscr{D}$. Then, there exist $P, Q \in \Delta(\mathscr{F})$ and $l \in \Delta(Z)$ such that $(x, y) = (\zeta(P), \eta(P))$, $(x', y') = (\zeta(Q), \eta(Q))$, and $(c, c) = (\zeta(l), \eta(l))$. By Step 4 (i), $\alpha(x, y) + (1-\alpha)(c, c) = (\zeta(\alpha P \oplus (1-\alpha)l), \eta(\alpha P \oplus (1-\alpha)l))$ and $\alpha(x', y') + (1 - \alpha)(c, c) = (\zeta(\alpha Q \oplus (1 - \alpha)l), \eta(\alpha Q \oplus (1 - \alpha)l))$. Therefore,

$$\begin{aligned}
&(x, y) \hat{\succsim} (x', y') \\
&\Leftrightarrow v\big(\zeta(P), \eta(P)\big) \geq v\big(\zeta(Q), \eta(Q)\big) \\
&\Leftrightarrow V(P) \geq V(Q) \\
&\Leftrightarrow V(\alpha P \oplus (1 - \alpha)l) \geq V(\alpha Q \oplus (1 - \alpha)l) && (\because \text{Step 2 (ii)}) \\
&\Leftrightarrow v\big(\zeta(\alpha P \oplus (1 - \alpha)l), \eta(\alpha P \oplus (1 - \alpha)l)\big) \geq v\big(\zeta(\alpha Q \oplus (1 - \alpha)l), \eta(\alpha Q \oplus (1 - \alpha)l)\big) \\
&\Leftrightarrow \alpha(x, y) + (1 - \alpha)(c, c) \hat{\succsim} \alpha(x', y') + (1 - \alpha)(c, c).
\end{aligned}$$

■

Because of the nondegeneracy of C, there exist $f^*, g^* \in \mathscr{F}$ such that $f^* \sim g^*$, $\frac{1}{2}f^* + \frac{1}{2}g^* \succ f^*$.[29] Define $(x^*, y^*) = \big(\zeta(\frac{1}{2}f^* \oplus \frac{1}{2}g^*), \eta(\frac{1}{2}f^* \oplus \frac{1}{2}g^*)\big) \in \mathscr{D}$. Hence, $x^* = \zeta(\frac{1}{2}f^* \oplus \frac{1}{2}g^*) = U(\frac{1}{2}f^* + \frac{1}{2}g^*) > U(f^*) = \eta(\frac{1}{2}f^* \oplus \frac{1}{2}g^*) = y^*$. By Nondegeneracy of $\succsim$, there exist $\overline{c}$ or $\underline{c}$ such that $\overline{c} > x^*$ or $x^* > \underline{c}$. By the mixture linearity of u, without loss of generality, assume $\overline{c} > x^* > \underline{c}$.

To define the set $\mathscr{T}$ as in Section A, the next step is proved.

Step 8:
(i) If $(x^*, x^*) \mathrel{\hat{\succsim}} (x^*, y^*)$, then there exist $\overline{\alpha} > 0$ such that $(x^*, x^*) \mathrel{\hat{\sim}} \overline{\alpha}(x^*, y^*) + (1-\overline{\alpha})(\overline{c}, \overline{c})$.
(ii) If $(x^*, y^*) \mathrel{\hat{\succsim}} (x^*, x^*)$, then there exist $\underline{\alpha} > 0$ such that $(x^*, x^*) \mathrel{\hat{\sim}} \underline{\alpha}(x^*, y^*) + (1-\underline{\alpha})(\underline{c}, \underline{c})$.

Proof of Step 8: We will show (i). By the monotonicity, $(\overline{c}, \overline{c}) \mathrel{\hat{\succ}} (x^*, x^*) \mathrel{\hat{\succsim}} (x^*, y^*)$. Then there exist $\overline{l} \in \Delta(Z)$ such that $u(\overline{l}) = \overline{c}$ and $\overline{l} \succsim \frac{1}{2}f^* + \frac{1}{2}g^* \succsim \frac{1}{2}f^* \oplus \frac{1}{2}g^*$. Then by Continuity of $\succsim$, there exists $\overline{\alpha} \in [0,1]$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \sim \overline{\alpha}(\frac{1}{2}f^* \oplus \frac{1}{2}g^*) \oplus (1-\overline{\alpha})\overline{l}$. Let $\hat{f} = \overline{\alpha}f^* + (1-\overline{\alpha})\overline{l}$ and $\hat{g} = \overline{\alpha}g^* + (1-\overline{\alpha})\overline{l}$. Then $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ \hat{f} \sim \hat{g}$ and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim \frac{1}{2}f^* + \frac{1}{2}g^*$. Hence, $\overline{\alpha}(x^*, y^*) + (1-\overline{\alpha})(\overline{c}, \overline{c}) = (\zeta, \eta)(\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g}) \hat{\sim} (\zeta, \eta)(\frac{1}{2}f^* + \frac{1}{2}g^*) = (x^*, x^*)$. Part (ii) is proved in the same way. ■

Henceforth, consider the case where $(x^*, x^*) \mathrel{\hat{\succsim}} (x^*, y^*)$. Denote $\big(\overline{\alpha c} + (1-\overline{\alpha})x^*, \overline{\alpha c} + (1-\overline{\alpha})y^*\big)$ by $(\hat{x}, \hat{y})$. [30] Then Step 8 shows $(\hat{x}, \hat{y}) \sim (x^*, x^*)$.

Define

$$\mathscr{T} = \Big\{(x,y) \in \mathbb{R}^2 \ \mid x \geq y,\ \big\langle (x^* - \hat{x}, \hat{y} - x^*), (x,y) - (x^*, x^*)\big\rangle \geq 0, \\ \text{and } \big\langle (\overline{c} - \hat{x}, \hat{y} - \overline{c}),\ \ (x,y) - (\overline{c}, \overline{c})\big\rangle \geq 0 \Big\}. \tag{10}$$

where $\langle \cdot, \cdot \rangle$ is a inner product. The set $\mathscr{T}$ is a triangle including the interior which consists of the vertices $(\overline{c}, \overline{c})$, (x^*, x^*), and $(\hat{x}, \hat{y})$ as shown in Figure 6 in Section A.

Step 9: $\mathscr{T}$ is nondegenerate and $\mathscr{T} \subset \mathscr{D}$.

[29] Otherwise, $f \sim g \Rightarrow \frac{1}{2}f + \frac{1}{2}g \sim f$ for all $f, g \in \mathscr{F}$. This implies the subjective expected utility, so C become degenerate.

[30] In the other case where $(x^*, y^*) \hat{\succsim} (x^*, x^*)$, denote $\big(\underline{\alpha c} + (1-\underline{\alpha})x^*, \underline{\alpha c} + (1-\underline{\alpha})y^*\big)$ by $(\tilde{x}, \tilde{y})$. Then, instead of the triangle $\mathscr{T}$ defined by (10), consider a triangle, including the interior, which consists of the vertices $(\tilde{c}, \tilde{c})$, $(\underline{c}, \underline{c})$, and (x^*, x^*). Then, the rest of the proof goes through exactly in the same way.

PROOF OF STEP 9: Since $x^* > y^*$ and, in addition, $\overline{\alpha} > 0$, then $\hat{x} > \hat{y}$. Therefore, $(x^*, x^*) \neq (\hat{x}, \hat{y}) \neq (\overline{c}, \overline{c})$. Hence, $\mathscr{T}$ is not degenerate. Choose any $(x, y) \in \mathscr{T}$ to show $(x, y) \in \mathscr{D}$. Since $\mathscr{T}$ is the triangle, the Carathéodory's Theorem (Hiriart–Urruty and Lemaréchal (1949, p. 29, Theorem 1.3.6)) shows that there exist $\alpha, \beta \in [0, 1]$ such that $(x, y) = \alpha(\overline{c}, \overline{c}) + \beta(x^*, x^*) + (1 - \alpha - \beta)(\hat{x}, \hat{y})$. Now, let $c = \frac{\alpha}{\alpha+\beta}\overline{c} + \frac{\alpha}{\alpha+\beta}x^*$. Then, $(x, y) = (\alpha + \beta)(c, c) + (1 - \alpha + \beta)(\hat{x}, \hat{y})$. Therefore, since $(\hat{x}, \hat{y}) \in \mathscr{D}$ and $(c, c) \in \mathscr{C}$, it follows from Step 5 (ii) that $(x, y) \in \mathscr{D}$. ■

The next step shows the existence of the desired real number δ on the restricted domain $\mathscr{T}$ as follows:

STEP 10: There exists a real number δ such that for any $(x, y), (x', y') \in \mathscr{T}$, $(x, y) \hat{\succsim} (x', y') \Leftrightarrow \delta x + (1 - \delta)y \geq \delta x' + (1 - \delta)y'$.
PROOF OF STEP 10:

SUBSTEP 10.1: For all $(x, y) \in \mathscr{T}$, there exists a unique number $\alpha \in [0, 1]$ such that $(x, y) \sim \alpha(\overline{c}, \overline{c}) + (1 - \alpha)(x^*, x^*)$.
PROOF OF SUBSTEP 10.1: Choose any $(x, y) \in \mathscr{T}$. Since $\mathscr{T}$ is the triangle, the Carathéodory's Theorem, again, shows that there exist $\alpha, \beta \in [0, 1]$ such that $(x, y) = \alpha(\overline{c}, \overline{c}) + \beta(x^*, x^*) + (1 - \alpha - \beta)(\hat{x}, \hat{y})$. Since $(\hat{x}, \hat{y}) \hat{\sim} (x^*, x^*)$, the transitivity and the certainty independence shows $(x, y) \hat{\sim} \alpha(\overline{c}, \overline{c}) + (1 - \alpha)(x^*, x^*)$. Since $\overline{c} > x^*$, the monotonicity of $\hat{\succsim}$ on $\mathscr{C}$ shows that α is unique.

For all $(x, y) \in \mathscr{T}$, define $c(x, y) = \alpha\overline{c} + (1 - \alpha)x^*$, where α is as in Substep 10.1.

SUBSTEP 10.2: For all $(x, y) \in \mathscr{T}$, $\dfrac{x - c(x, y)}{x - y} = \dfrac{\hat{x} - x^*}{\hat{x} - \hat{y}}$.
PROOF OF SUBSTEP 10.2: Choose any $(x, y) \in \mathscr{T}$. By the proof of Substep 10.1, there exist $\alpha, \beta \in [0, 1]$ such that $(x, y) = \alpha(\overline{c}, \overline{c}) + \beta(x^*, x^*) + (1 - \alpha - \beta)(\hat{x}, \hat{y})$. Then,

$$\begin{aligned}
\frac{x - c(x, y)}{x - y} &= \frac{x - \alpha\overline{c} - (1 - \alpha)x^*}{x - y} && (\because c(x, y) = \alpha\overline{c} + (1 - \alpha)x^*) \\
&= \frac{\hat{x} - x^*}{\hat{x} - \hat{y}}. && (\because (x, y) = \alpha(\overline{c}, \overline{c}) + \beta(x^*, x^*) + (1 - \alpha - \beta)(\hat{x}, \hat{y}))
\end{aligned}$$

Define $\delta = 1 - \dfrac{\hat{x} - x^*}{\hat{x} - \hat{y}}$.

SUBSTEP 10.3: For any $(x, y), (x', y') \in \mathscr{T}$, $(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'$.
PROOF OF SUBSTEP 10.3: Choose any $(x, y), (x', y') \in \mathscr{T}$. Then

$$\begin{aligned}
(x, y) \mathrel{\hat{\succsim}} (x', y') &\Leftrightarrow (c(x, y), c(x, y)) \mathrel{\hat{\succsim}} (c(x', y'), c(x', y')) && (\because \text{Substep 10.1}) \\
&\Leftrightarrow c(x, y) \geq c(x', y') && (\because \text{Step 7}) \\
&\Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'. && (\because \text{Substep 10.2})
\end{aligned}$$

■

The next step shows the existence of the desired δ on $\mathscr{D}$ as follows:

STEP 11: For all $(x, y), (x', y') \in \mathscr{D}$, $(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'$.
PROOF OF STEP 11: Choose any $(x, y), (x', y') \in \mathscr{D}$. Let $c^* = \frac{1}{2}\overline{c} + \frac{1}{2}x^*$. Since $\mathscr{T}$ is a nondegenerate triangle, there exists a positive number ε such that $\{(x, y) \in \mathscr{D} | \|(x, y) - (c^*, c^*)\| < \varepsilon\} \subset \mathscr{T}$. Hence, there exists $\alpha \in (0, 1]$ such that $\alpha(x, y) + (1-\alpha)(c^*, c^*)$ and $\alpha(x', y') + (1-\alpha)(c^*, c^*)$ belong to $\{(x, y) \in \mathscr{D}'' | \|(x, y) - (c^*, c^*)\| < \varepsilon\} \subset \mathscr{T}$. Therefore,

$$\begin{aligned}
(x, y) \mathrel{\hat{\succsim}} (x', y') &\Leftrightarrow \alpha(x, y) + (1-\alpha)(c^*, c^*) \mathrel{\hat{\succsim}} \alpha(x', y') + (1-\alpha)(c^*, c^*) && (\because \text{Step 7}) \\
&\Leftrightarrow \delta(\alpha x + (1-\alpha)c^*) + (1-\delta)(\alpha y + (1-\alpha)c^*) && (\because \text{Step 10}) \\
&\qquad \geq \delta(\alpha x' + (1-\alpha)c^*) + (1-\delta)(\alpha y' + (1-\alpha)c^*) \\
&\Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'.
\end{aligned}$$

■

STEP 12: For all $P, Q \in \Delta(\mathscr{F})$, $P \succsim Q \Leftrightarrow \delta\zeta(P) + (1-\delta)\eta(P) \geq \delta\zeta(Q) + (1-\delta)\eta(Q)$.
PROOF OF STEP 12: For all $P, Q \in \Delta(\mathscr{F})$, $P \succsim Q \Leftrightarrow V(P) \geq V(Q) \Leftrightarrow v(\zeta(P), \eta(P)) \geq v(\zeta(Q), \eta(Q)) \Leftrightarrow (\zeta(P), \eta(P)) \mathrel{\hat{\succsim}} (\zeta(Q), \eta(Q)) \Leftrightarrow \delta\zeta(P) + (1-\delta)\eta(P) \geq \delta\zeta(Q) + (1-\delta)\eta(Q)$, where the last equivalence is by Step 11. ■

Step 12 shows that $\delta\zeta + (1-\delta)\eta$ represents $\succsim$ on $\Delta(\mathscr{F})$. Also by Step 4 (ii), $V = u = \delta\zeta + (1-\delta)\eta$ on $\Delta(Z)$. Since V is unique up to positive affine transformation, Hence, $V = \delta\zeta + (1-\delta)\eta$. This completes the proof of Theorem 1.

C.1 Proof of Corollary 1

It is easy to see that (ii) implies (i). So, we will show that (i) implies (ii). Fix $\succsim$ on $\Delta(\mathscr{F})$. Let (δ, C, u) and (δ', C', u') represent $\succsim$ as in Theorem 1, then u and u' are affine representations of $\succsim$ restricted on $\Delta(Z)$. Hence, by the standard uniqueness results, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u = \alpha u' + \beta$. The uniqueness of C follows from Gilboa and Schmeidler (1989). So, $C = C'$.

To show $\delta = \delta'$, let V and V' be as in Theorem 1 defined by (δ, C, u) and (δ', C', u'), respectively. Let U and U' be the restriction of V and V' on $\mathscr{F}$, respectively. Since C is nondegenerate, there exist $f^*, g^* \in \mathscr{F}$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \succ f^* \sim g^*$.

Case 1: $\frac{1}{2}f^* \oplus \frac{1}{2}g^* \succsim \frac{1}{2}f^* + \frac{1}{2}g^*$. By Step 8 in the proof of Theorem 1, there exist $\hat{f}, \hat{g} \in \mathscr{F}$ such that $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ \hat{f} \sim \hat{g}$ and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim \frac{1}{2}f^* + \frac{1}{2}g^*$. Hence, $U(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) > U(\hat{f})$ and $U(\frac{1}{2}f^* + \frac{1}{2}g^*) = \delta U(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) + (1-\delta)U(\hat{f})$. So, $\delta = \frac{U(\frac{1}{2}f^* + \frac{1}{2}g^*) - U(\hat{f})}{U(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U(\hat{f})} = \frac{U'(\frac{1}{2}f^* + \frac{1}{2}g^*) - U'(\hat{f})}{U'(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U'(\hat{f})} = \delta'$, where the second equality holds because $U = \alpha U' + \beta$.

Case 2: $\frac{1}{2}f^* + \frac{1}{2}g^* \succsim \frac{1}{2}f^* \oplus \frac{1}{2}g^*$. The proof is the same as Case 1.

C.2 Proof of Proposition 1

Suppose $\succsim$ is EAP Maxmin preference represented by V as in Theorem 1 with nondegenerate C. Let U be the restriction of V on $\mathscr{F}$. Choose $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$. Then $V(\alpha f \oplus (1-\alpha)g) = \delta U(\alpha f + (1-\alpha)g) + (1-\delta)(\alpha U(f) + (1-\alpha)U(g))$ and $U(\alpha f + (1-\alpha)g) \geq \alpha U(f) + (1-\alpha)U(g)$. By the nondegeneracy of C, there exist $f^*, g^* \in \mathscr{F}$ such that $U(\frac{1}{2}f^* + \frac{1}{2}g^*) > U(f^*) = U(g^*)$.

To show (i), assume $f \sim g$. Then, $V(\alpha f \oplus (1-\alpha)g) \geq U(f) \Leftrightarrow \delta U(\alpha f + (1-\alpha)g) \geq \delta U(f) \Leftrightarrow \delta \geq 0$. Hence, $\succsim$ satisfies Ex–ante Ambiguity Aversion if and only if $\delta \geq 0$. Part (ii) is proved as follows: $U(\alpha f + (1-\alpha)g) \geq V(\alpha f \oplus (1-\alpha)g) \Leftrightarrow (1-\delta)U(\alpha f + (1-\alpha)g) \geq (1-\delta)(\alpha U(f) + (1-\alpha)U(g)) \Leftrightarrow \delta \leq 1$. Hence, $\succsim$ satisfies Interim Ambiguity Aversion if and only if $\delta \leq 1$.

C.3 Proof of Proposition 2

Fix two EAP Maxmin preferences $\{\succsim_i\}_{i=1,2}$. Let (δ_i, C_i, u_i) represent $\succsim_i$ as in Theorem 1. Suppose C_i is nondegenerate. Let V_i be as in Theorem 1 defined by (δ_i, C_i, u_i). Let U_i be the restriction of V_i on $\mathscr{F}$.

First, we will prove that (i) implies (ii). Suppose $\succsim_1$ is more Ex–ante Ambiguity Averse than $\succsim_2$. Then, there exist $\alpha > 0$ and β such that $u_1 = \alpha u_2 + \beta$. A straightforward argument shows $U_1 = U_2$. Hence, $C_1 = C_2$.[31] In the following, we will show $\delta_1 \geq \delta_2$.

Case 1: $\frac{1}{2}f^* \oplus \frac{1}{2}g^* \succsim_2 \frac{1}{2}f^* + \frac{1}{2}g^*$. By Step 8 in the proof of Theorem 1, there exist $\hat{f}, \hat{g} \in \mathscr{F}$ such that $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ_i \hat{f} \sim_i \hat{g}$ and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim_2 \frac{1}{2}f^* + \frac{1}{2}g^*$. Since $\succsim_1$ is more ex–ante ambiguity averse than $\succsim_2$, $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \succsim_1 \frac{1}{2}f^* + \frac{1}{2}g^*$. Hence, $U_i(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) > U_i(\hat{f}) = U_i(\hat{g})$, $V_i(\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g}) = \delta_i U_i(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) + (1-\delta_i)U_i(\hat{f})$, $V_2(\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g}) = U_2(\frac{1}{2}f^* + \frac{1}{2}g^*)$, and $V_1(\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g}) \geq U_1(\frac{1}{2}f^* + \frac{1}{2}g^*)$. Therefore, $\delta_1 \geq \frac{U_1(\frac{1}{2}f^*+\frac{1}{2}g^*)-U_1(\hat{f})}{U_1(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g})-U_1(\hat{f})} = \frac{U_2(\frac{1}{2}f^*+\frac{1}{2}g^*)-U_2(\hat{f})}{U_2(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g})-U_2(\hat{f})} = \delta_2$, where the second equality holds because $U_1 = \alpha U_2 + \beta$.

Case 2: $\frac{1}{2}f^* + (1-\frac{1}{2})g^* \succsim_2 \frac{1}{2}f^* \oplus (1-\frac{1}{2})g^*$. The proof is the same as Case 1.

Next, we will prove that (ii) implies (i). Suppose $\delta_1 \geq \delta_2$, $C_1 = C_2$, and there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that $u_1 = \alpha u_2 + \beta$. Then, $U_1 = \alpha U_2 + \beta$. Fix any $P \in \Delta(\mathscr{F})$ and $f \in \mathscr{F}$ such that $P \succsim_2 f$ to show $P \succsim_1 f$.

Case 1: $U_1((P_s)_s) = \int_{\mathscr{F}} U_1(g)dP(g)$. Then $V_1(P) = U_1((P_s)_s) \geq U_1(f)$. Since $U_1 = \alpha U_2 + \beta$, $V_2(P) = U_2((P_s)_s) \geq U_2(f)$, as desired.

Case 2: $U_1((P_s)_s) \neq \int_{\mathscr{F}} U_1(g)dP(g)$. Since $U_1 = \alpha U_2 + \beta$, $U_2((P_s)_s) \neq \int_{\mathscr{F}} U_2(g)dP(g)$. Since $P \succsim_2 f$, then $\delta_1 \geq \delta_2 \geq \frac{U_2(f)-\int_{\mathscr{F}} U_2(g)dP(g)}{U_2((P_s)_s)-\int_{\mathscr{F}} U_2(g)dP(g)} = \frac{U_1(f)-\int_{\mathscr{F}} U_1(g)dP(g)}{U_1((P_s)_s)-\int_{\mathscr{F}} U_1(g)dP(g)}$, where the last equality holds because $U_1 = \alpha U_2 + \beta$. So, $V_1(P) = \delta_1 U_1((P_s)_s) + (1-\delta_1)\int_{\mathscr{F}} U_1(g)dP(g) \geq U_1(f)$, as desired. ■

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[31] By a normalization, $u_1 = u_2$. Let, $l_0 = \frac{1}{2}\delta_{z_+} + \frac{1}{2}\delta_{z_-}$. Suppose to the contrary that $U_1 \neq U_2$. Without loss of generality assume that there exists $f \in \mathscr{F}$ such that $U_1(f) > U_2(f)$ and $1 \geq U_2(f) \geq -1$. Then $z_+ \succsim_2 f \succsim_2 z_-$. Since $\succsim_1$ is more Ex–ante Ambiguity Averse than $\succsim_2$, $z_+ \succsim_1 f \succsim_1 z_-$. Fix a positive number ε such that $\varepsilon < U_1(f) - U_2(f)$. Define $l = (U_2(f)+\varepsilon)\delta_{x_+} + (1-U_2(f)-\varepsilon)l_0$. Then $U_i(l) = U_2(f)+\varepsilon < U_1(f)$. Therefore, $l \succ_2 f$ but $f \succ_1 l$. This is a contradiction. Hence, $U_1 = U_2$, so that $C_1 = C_2$.

C.4 Proof of Proposition 3

Suppose $\succsim$ is EAP Maxmin preference represented by V as in Theorem 1. Let U be the restriction of V on $\mathscr{F}$. It is easy to see that (ii) holds. So, we will show (i). To show (a), suppose $\delta > 0$. It is easy to see that Ex–post Ambiguity Aversion implies Ex–ante Ambiguity Aversion. To see the opposite direction, suppose that $\succsim$ satisfies Ex–ante Ambiguity Aversion. Fix $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$ such that $f \sim g$ to show $\alpha f + (1-\alpha)g \succsim f$. Since $\alpha f \oplus (1-\alpha)g \succsim f$, then $U(f) \leq V(\alpha f \oplus (1-\alpha)g) = \delta U(\alpha f + (1-\alpha)g) + (1-\delta)U(f)$, so that $\delta U(f) \leq \delta U(\alpha f + (1-\alpha)g)$. Since $\delta > 0$, then $U(\alpha f + (1-\alpha)g) \geq U(f)$. Part (b) is proved in the same way. It is easy to see that (c) holds.

D Proof of Theorem 2

The necessity of axioms is easy to check. To show Continuity, note that EAP Piecewise–linear preference is a weighted sum of max functions and a mixture linear function u.

In the following, we will prove the sufficiency. Suppose that a preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem 2. Then, by Lemma 2, $\succsim$ satisfies Reduction of Compound Lotteries as well.

As noted in the sketch in Section A, after proving that $\succsim$ restricted on $\mathscr{F}$ has the Fehr and Schmidt's (1999) piecewise linear utility representation, the rest of the proof is the same as the proof of Theorem 1.

The first step shows Reversal of Order among pointwise comonotonic acts; remember $\hat{\sum}$ denotes a summation by ex–ante mixtures, while $\sum$ denotes a summation by ex–post mixtures.

Step 1: For any set $\{f^i\}_{i=1}^n$ of acts and any set of nonnegative numbers $\{\alpha_i\}_{i=1}^n$ such that $\sum_{i=1}^n \alpha_i = 1$, if any pair of acts among $\{f^i\}_{i=1}^n$ are pointwise comonotonic, then $\hat{\sum}_{i=1}^n \alpha_i f^i \sim \sum_{i=1}^n \alpha_i f^i$.

Proof of Step 1: By Induction on n. For $n = 1$, the statement is trivial.

First, we will prove the statement for $n = 2$. Fix $\alpha \in [0,1]$ and $f^1, f^2 \in \mathscr{F}$. Let

$P = \alpha f^1 \oplus (1-\alpha) f^2$ to show $P \sim \alpha f^1 + (1-\alpha) f^2$. Then by Reduction of Compound Lotteries, $P_s = \alpha f_s^1 \oplus (1-\alpha) f_s^2 \sim \alpha f_s^1 + (1-\alpha) f_s^2$ for all $s \in S$. So, by Substitution, $(P_s)_s \sim \alpha f^1 + (1-\alpha) f^2$, so that the condition (i) in Indifference is satisfied. In addition, since f^1 and f^2 are pointwise comonotonic, Ex–post Pointwise Comonotonic Independence shows $l_P = \alpha l_{f^1} + (1-\alpha) l_{f^2} \sim \alpha f^1 + (1-\alpha) f^2$, so that the condition (ii) in Indifference is satisfied as well. Hence, Indifference shows $P \sim \alpha f^1 + (1-\alpha) f^2$.

Let $P \equiv \hat{\sum}_{i=1}^{n-1} \frac{\alpha_i}{1-\alpha_n} f^i$ and $g \equiv \sum_{i=1}^{n-1} \frac{\alpha_i}{1-\alpha_n} f^i$. Suppose the statement is true for $n-1$. Then $P \sim g$. Now, we will show the statement for n. Since any pair of acts among $\{f^i\}_{i=1}^n$ are pointwise comonotonic, any pair among $(P_s)_s, g$, and f^n are pointwise comonotonic. Therefore, $\hat{\sum}_{i=1}^n \alpha_i f^i \equiv (1-\alpha_n) P \oplus \alpha_n f^n \sim (1-\alpha_n) g \oplus \alpha_n f^n \sim (1-\alpha_n) g + \alpha_n f^n \equiv \sum_{i=1}^n \alpha_i f^i$, where the second equivalence is by Ex–ante Pointwise Comonotonic Independence and the third equivalence is by the statement for $n = 2$. ■

STEP 2: There exists a function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ such that

(i) V represents $\succsim$ on $\Delta(\mathscr{F})$,

(ii) for all $\alpha \in [0,1]$ and $P, Q \in \Delta(\mathscr{F})$ such that $(P_s)_s$ and $(Q_s)_s$ are pointwise comonotonic, $V(\alpha P + (1-\alpha) Q) = \alpha V(P) + (1-\alpha) V(Q)$,

(iii) V is unique up to positive affine transformation.

(iv) Let u be the restriction of V on $\Delta(Z)$. Then $u : \Delta(Z) \to \mathbb{R}$ is mixture linear onto function such that $u(z_+) = 1$, $u(z_-) = -1$, and $u(l_0) = 0$.

PROOF OF STEP 2: In the same way as Step 2 in the proof of Theorem 1, there exist a real valued function V on $\Delta(\mathscr{F})$ satisfying (i) and (iii) and a mixture linear function u on $\Delta(Z)$. In addition, for all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $V(\alpha P \oplus (1-\alpha) l) = \alpha V(P) + (1-\alpha) V(l)$ and $V(l) = u(l)$.

Lemma 69 of Cerreia–Vioglio et al. (2008) shows that $\succsim$ satisfies Unboundedness if and only if u is onto function. Normalize u by $u(z_+) = 1$ and $u(z_-) = -1$. So, part (iv) holds.

Finally to show (ii), choose any $P, Q \in \Delta(\mathscr{F})$ such that $(P_s)_s$ and $(Q_s)_s$ are pointwise comonotonic. Since u is onto, there exists $l \in \Delta(Z)$ such that $l \sim P$. Hence, Ex–ante Pointwise Comonotonic Independence shows $\alpha l \oplus (1-\alpha) Q \sim \alpha P \oplus (1-\alpha) Q$. Hence, $V(\alpha P \oplus (1-\alpha) Q) = V(\alpha l \oplus (1-\alpha) Q) = \alpha V(l) + (1-\alpha) V(Q) = \alpha V(P) + (1-\alpha) V(Q)$. ■

STEP 3: For all $f \in \mathscr{F}$, there exists an act $\tilde{f} \in \mathscr{F}$ such that $\frac{1}{2}f_1 + \frac{1}{2}\tilde{f}_s \sim \frac{1}{2}f_s + \frac{1}{2}l_0$ for all $s \in S$, and $\frac{1}{2}f_1 + \frac{1}{2}\tilde{f} \sim \frac{1}{2}f + \frac{1}{2}l_0$.

PROOF OF STEP 3: Choose any $f \in \mathscr{F}$ and $s \in S$. Since u is onto, there exists an element $l_s \in \Delta(Z)$ such that $u(l_s) = u(f_s) - u(f_1)$. Define an act $\tilde{f} \in \mathscr{F}$ by $\tilde{f}(s) = l_s$ for all $s \in S$. Then by definition, $\frac{1}{2}f_1 + \frac{1}{2}\tilde{f}_s \sim \frac{1}{2}f_s + \frac{1}{2}l_0$ for all $s \in S$. Hence, Substitution implies $\frac{1}{2}f_1 + \frac{1}{2}\tilde{f} \sim \frac{1}{2}f + \frac{1}{2}l_0$. ■

Let U be the restriction of V on $\mathscr{F}$. For all $s \neq 1$, define

$$\alpha_s = -U((z_+, (l_0)_{-s})); \quad \beta_s = -U((z_-, (l_0)_{-s})).$$

STEP 4: $\{\alpha_s, \beta_s\}_{s \in S \setminus \{1\}}$ are nonnegative numbers such that for all $f \in \mathscr{F}$,

$$U(f) = u(f_1) - \sum_{s \neq 1} \Big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\} \Big).$$

PROOF OF STEP 4: By Ex–post Inequality Aversion, $\alpha_s, \beta_s \geq 0$ for all $s \neq 1$. Fix $f \in \mathscr{F}$ and let $\tilde{f}$ as in Step 3.

SUBSTEP 4.1: For all $(l_s)_{s \neq 1} \subset \Delta(Z)$, any pair in $\{(l_s, (l_0)_{-s})\}_{s \neq 1}$ are pointwise comonotonic.

PROOF OF SUBSTEP 4.1: A straightforward argument will show the result.

SUBSTEP 4.2: $\frac{1}{|S|}f \oplus \frac{|S|-1}{|S|}l_0 \sim \frac{1}{|S|}f_1 \oplus \hat{\sum}_{s \neq 1} \frac{1}{|S|}(\tilde{f}_s, (l_0)_{-s})$.

PROOF OF SUBSTEP 4.2: Since $l_0 \in \Delta(Z)$ is pointwise comonotonic with any acts, Step 1 shows $\frac{1}{|S|}f \oplus \frac{|S|-1}{|S|}l_0 \sim \frac{1}{|S|}f + \frac{|S|-1}{|S|}l_0$. By Substep 4.1, any pair among $\{(\tilde{f}_s, (l_0)_{-s})\}_{s \neq 1}$ are pointwise comonotonic. Since $f_1 \in \Delta(Z)$ is also pointwise comonotonic with any acts, Step 1 again shows $\frac{1}{|S|}f_1 \oplus \hat{\sum}_{s \neq 1} \frac{1}{|S|}(\tilde{f}_s, (l_0)_{-s}) \sim \frac{1}{|S|}f_1 + \sum_{s \neq 1} \frac{1}{|S|}(\tilde{f}_s, (l_0)_{-s})$.

Now, let $g \equiv \frac{1}{|S|}f + \frac{|S|-1}{|S|}l_0$ and $h \equiv \frac{1}{|S|}f_1 + \sum_{s \neq 1} \frac{1}{|S|}(\tilde{f}_s, (l_0)_{-s})$ to show $g_s \sim h_s$ for all $s \in S$. For all $s \in S$, $u(g_s) = \frac{2}{|S|}u\Big(\frac{1}{2}f_s + \frac{1}{2}l_0\Big) = \frac{2}{|S|}u\Big(\frac{1}{2}f_1 + \frac{1}{2}\tilde{f}_s\Big) = u\Big(\frac{1}{|S|}f_1 + \frac{1}{|S|}\tilde{f}_s + \frac{|S|-2}{|S|}l_0\Big) = u(h_s)$,where the second equality holds because $\frac{1}{2}f_s + \frac{1}{2}l_0 \sim \frac{1}{2}f_1 + \frac{1}{2}\tilde{f}_s$ by Step 3. Then Substitution shows $g \sim h$. Hence, $\frac{1}{|S|}f \oplus \frac{|S|-1}{|S|}l_0 \sim \frac{1}{|S|}f_1 \oplus \hat{\sum}_{s \neq 1} \frac{1}{|S|}(\tilde{f}_s, (l_0)_{-s})$.

SUBSTEP 4.3: Let $\overline{S} = \{s \in S | f_s \succ f_1\}$ and $\underline{S} = \{s \in S | f_1 \succ f_s\}$.

(i) for all $s \in \overline{S}$, $V((\tilde{f}_s, (l_0)_{-s})) = -\alpha_s \max\{u(f_s) - u(f_1), 0\}$,

(ii) for all $s \in \underline{S}$, $V((\tilde{f}_s, (l_0)_{-s})) = -\beta_s \max\{u(f_1) - u(f_s), 0\}$,

(iii) for all $s \in S \setminus (\overline{S} \cup \underline{S})$, $V((\tilde{f}_s, (l_0)_{-s})) = 0$.

Proof of Substep 4.3: We will show (i). Fix $s \in \overline{S}$. Then $f_s \succ f_1$. So, $\tilde{f}_s \succ l_0$. Hence, by Continuity (ii), there exists $n \in \mathbb{Z}_+$ such that $z_+ \succ \frac{1}{n}\tilde{f}_s + \frac{n-1}{n} l_0 \succ l_0$. By Step 2 (ii) and the normalization, $1 > V\big(\frac{1}{n}\tilde{f}_s + \frac{n-1}{n} l_0\big) = \frac{1}{n}u(\tilde{f}_s) > 0$. Hence, $\frac{1}{n}\tilde{f}_s + \frac{n-1}{n} l_0 \sim \big(\frac{1}{n}u(\tilde{f}_s)\big)\delta_{z_+} + \big(1 - \frac{1}{n}u(\tilde{f}_s)\big)l_0$. Hence, by Substitution, $\frac{1}{n}(\tilde{f}_s, (l_0)_{-s}) + \frac{n-1}{n} l_0 \sim \big(\frac{1}{n}u(\tilde{f}_s)\big)(z_+, (l_0)_{-s}) + \big(1 - \frac{1}{n}u(\tilde{f}_s)\big)l_0$. Therefore, by Step 2 (ii), again, $V((\tilde{f}_s, (l_0)_{-s})) = n(\frac{1}{n}u(\tilde{f}_s)V((z_+, (l_0)_{-s}))) = -\alpha_s u(\tilde{f}_s) = -\alpha_s \max\{u(f_s) - u(f_1), 0\}$. Part (ii) is proved in the same way. Finally, we will show (iii). For all $s \in S \setminus (\overline{S} \cup \underline{S})$, $\tilde{f}_s \sim l_0$, Hence, by Substitution, $(\tilde{f}_s, (l_0)_{-s}) \sim l_0$. Hence, $V((\tilde{f}_s, (l_0)_{-s})) = 0$.

By Substep 4.1–4.3, therefore,

$$\begin{aligned}
U(f) &= |S| V\big(\tfrac{1}{|S|} f \oplus \tfrac{|S|-1}{|S|} l_0\big) && (\because \text{Step 2 (ii)})\\
&= |S| V\big(\tfrac{1}{|S|} f_1 \oplus \hat{\textstyle\sum}_{s \neq 1} \tfrac{1}{|S|}(\tilde{f}_s, (l_0)_{-s})\big) && (\because \text{Substep 4.2})\\
&= V(f_1) + \textstyle\sum_{s \neq 1} V((\tilde{f}_s, (l_0)_{-s})) && (\because \text{Step 2 (ii) \& Substep 4.1})\\
&= u(f_1) - \textstyle\sum_{s \neq 1} \big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\}\big). && (\because \text{Substep 4.3})
\end{aligned}$$

■

If $(\alpha, \beta) = \mathbf{0}$, then the theorem holds trivially. So, henceforth, we consider the case where $(\alpha, \beta) \neq \mathbf{0}$. To make notations simple, for all $P \in \Delta(\mathscr{F})$, define

$$\zeta(P) = U\big((P_s)_s\big); \quad \eta(P) = U(l_P).$$

Then by Step 4, for all $P \in \Delta(\mathscr{F})$,

$$\begin{aligned}
\zeta(P) &= U(P_1) - \sum_{s \neq 1} \Big(\alpha_s \max\{u(P_s) - u(P_1), 0\} + \beta_s \max\{u(P_1) - u(P_s), 0\}\Big);\\
\eta(P) &= \int_{\mathscr{F}} \Big(u(f_1) - \sum_{s \neq 1} \Big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\}\Big)\Big) dP(f).
\end{aligned}$$

The next step shows the functions ζ and η have the same property as in Theorem 1.

Step 5:

(i) For all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $\zeta(\alpha P \oplus (1-\alpha)l) = \alpha\zeta(P) + (1-\alpha)\zeta(l)$ and $\eta(\alpha P \oplus (1-\alpha)l) = \alpha\eta(P) + (1-\alpha)\eta(l)$.

(ii) For all $l \in \Delta(Z)$, $\zeta(l) = u(l) = \eta(l)$.

(iii) For all $P \in \Delta(\mathscr{F})$, $\zeta(P) \geq \eta(P)$.

Proof of Step 5: Parts (i) and (ii) hold in the same way as Step 4 in the proof of Theorem 1. To show (iii), fix $P \in \Delta(\mathscr{F})$. Fix $s \neq 1$. For all $x \in \mathbb{R}^S$, define $G_s : \mathbb{R}^S \to \mathbb{R}$ by $G_s(x) = -\alpha_s \max\{x_s - x_1, 0\} - \beta_s \max\{x_1 - x_s, 0\}$, where x_t is the t–th element of x. Then G_s is concave.[32] For all $x \in \mathbb{R}^S$, define $F : \mathbb{R}^S \to \mathbb{R}$ by $F(x) = x_1 + \sum_{s\neq 1} G_s(x)$. Since F is a sum of concave functions, F is also concave. Therefore, for all $P \in \Delta(\mathscr{F})$, Jensen's Inequality shows $\zeta(P) = F\big(\big(\int_{\mathscr{F}} u(f_s)dP(f)\big)_{s\in S}\big) \geq \int_{\mathscr{F}} F\big(\big(u(f_s)\big)_{s\in S}\big)dP(f) = \eta(P)$. ■

Note that Ex–ante/Ex–post Pointwise Comonotonic Independence implies Ex–ante/Ex–post Certainty Independence. Given Step 5 above, the same argument as Step 5–12 in the proof of Theorem 1 shows the existence of the desired real number δ.

Finally, the next step shows the uniqueness property of the representation.

Step 6: The following two statements are equivalent:

(i) Two triples $(\delta, \alpha, \beta, u)$ and $(\delta', \alpha', \beta', u')$ represent the same preference $\succsim$ as in Theorem 2.

(ii) (a) $(\alpha, \beta) = (\alpha', \beta')$ and there exist $a > 0, b \in \mathbb{R}$ such that $u = au' + b$; and

(b) If $(\alpha, \beta) \neq \mathbf{0}$ then $\delta = \delta'$.

Proof of Step 6: It is easy to see that (ii) implies (i). So, we will show that (i) implies (ii). Choose any two triples $(\delta, \alpha, \beta, u)$ and $(\delta', \alpha', \beta', u')$ represent the same preference $\succsim$ as in Theorem 2. We will show (a). Given (a), part (b) is proved in the same way as Corollary 1. (α, β, u) and (α', β', u') represent the same preference $\succsim$ restricted to $\mathscr{F}$ as in Step 4. Then u and u' are affine representation of the restriction of $\succsim$ on $\Delta(Z)$. Hence, by the standard uniqueness results, there exist $a > 0, b \in \mathbb{R}$ such that $u = au' + b$. Suppose to the contrary that $(\alpha, \beta) \neq (\alpha', \beta')$. Then, there exists at least one element $s \neq 1$ such that $\alpha_s \neq \alpha'_s$ or $\beta_s \neq \beta'_s$. Without loss of generality, assume $\alpha_s > \alpha'_s$. Let V and V' have the representations on $\mathscr{F}$ as in Step 4 defined by (α, β, u) and (α', β', u'), respectively. Take n large enough to hold $\frac{\alpha_s}{n} < 1$.

[32] If $f : \mathbb{R}^S \to \mathbb{R}$ is convex, then $\max\{f(\cdot), 0\}$ is also convex. If $f, g : \mathbb{R}^S \to \mathbb{R}$ are convex and a, b are nonnegative numbers, then $af + bg$ is also convex.

Then, $V(\frac{1}{n}(z_+,(l_0)_{-s})+\frac{n-1}{n}l_0)=-\frac{1}{n}\alpha_s= V\big(\frac{\alpha_s}{n}z_-+(1-\frac{\alpha_s}{n})l_0\big)$ and $V'(\frac{1}{n}(z_+,(l_0)_{-s})+\frac{n-1}{n}l_0)$ $= b-\frac{1}{n}\alpha'_s a= V'\big(\frac{\alpha'_s}{n}z_-+(1-\frac{\alpha'_s}{n})l_0\big)$. Hence, $\frac{1}{n}(z_+,(l_0)_{-s})+\frac{n-1}{n}l_0\sim\frac{\alpha_s}{n}z_-+(1-\frac{\alpha_s}{n})l_0\succ \frac{\alpha'_s}{n}z_-+(1-\frac{\alpha'_s}{n})l_0\sim\frac{1}{n}(z_+,(l_0)_{-s})+\frac{n-1}{n}l_0$, where the second strict relation is by $\alpha_s>\alpha'_s$ and $l_0\succ z_-$. This is a contradiction. Hence, $(\alpha,\beta)=(\alpha',\beta')$. ■

This completes the proof of Theorem 2.

D.1 Proof of Corollary 2

Suppose $\succsim$ is EAP Piecewise–linear preference represented by V as in Theorem 2. Since $(\alpha,\beta)\neq\mathbf{0}$. Without loss of generality, assume $\alpha_s>0$ for some $s\neq 1$.

First, we will show (i). Choose any $l_+,l_-\in\Delta(Z)$ such that $l_+\succ l_0\succ l_-$ and $(l_+,(l_0)_{-s})\sim(l_-,(l_0)_{-s})$. Then, $U(l_+,(l_0)_{-s})=-\alpha_s u(l_+)$ and $U(l_-,(l_0)_{-s})=\beta_s u(l_-)$. Since $(l_+,(l_0)_{-s})\sim(l_-,(l_0)_{-s})$, then $-\alpha_s u(l_+)=\beta_s u(l_-)$. So, $\beta_s>0$.

Case 1: $\frac{1}{2}u(l_+)+\frac{1}{2}u(l_-)\geq 0$.[33] Then $V(\frac{1}{2}(l_+,(l_0)_{-s})\oplus\frac{1}{2}(l_-,(l_0)_{-s}))-U(l_+,(l_0)_{-s})=\delta\alpha_s\frac{1}{2}(u(l_+)-u(l_-))$. Hence, $\frac{1}{2}(l_+,(l_0)_{-s})\oplus\frac{1}{2}(l_-,(l_0)_{-s})\succsim(l_+,(l_0)_{-s})$ if and only if $\delta\geq 0$.

Case 2: $\frac{1}{2}u(l_+)+\frac{1}{2}u(l_-)\leq 0$. The proof is exactly the same as Case 1.

Next, we will show (ii). $V(\frac{1}{2}(z_+,(l_0)_{-s})\oplus\frac{1}{2}(z_-,(l_0)_{-s}))=-\frac{1}{2}(1-\delta)(\alpha_s+\beta_s)$. Hence, $(l_0,(l_0)_{-s})\succsim\frac{1}{2}(x_+,(l_0)_{-s})\oplus\frac{1}{2}(x_-,(l_0)_{-s})$ if and only if $\delta\leq 1$.

D.2 Proof of Corollary 3

It is easy to see that (ii) implies (i) in the same way as Proposition 2. So, we will show that (i) implies (ii). Fix two EAP Piecewise–linear preferences $\{\succsim_i\}_{i=1,2}$. Let V_i be as in Theorem 2 defined by $(\delta_i,\alpha^i,\beta^i,u_i)$. Let U_i be the restriction of V_i on $\mathscr{F}$. By the same argument in Proposition 2, there exist $\alpha>0$ and β such that $u_1=\alpha u_2+\beta$; Under the normalization of u_i by $u_1(z_+)=1=u_2(z_+)$ and $u_1(z_-)=-1=u_2(z_-)$, a straightforward argument will show $U_1=U_2$. Hence, $(\alpha^1,\beta^1)=(\alpha^2,\beta^2)$.

Suppose $(\alpha^i,\beta^i)\neq\mathbf{0}$ to show $\delta_1\geq\delta_2$. Without loss generality assume $\alpha^i_s\neq 0$ for some $s\neq 1$. Define $P=\frac{1}{2}(z_+,(l_0)_{-s})\oplus\frac{1}{2}(z_-,(l_0)_{-s})$. Then, $V_i(P)=-\frac{1}{2}(1-\delta_i)(\alpha^i_s+\beta^i_s)$. Since

[33] In this case, $V(\frac{1}{2}(l_+,(l_0)_{-s})\oplus\frac{1}{2}(l_-,(l_0)_{-s}))=-\delta\alpha_s\big[\frac{1}{2}u(l_+)+\frac{1}{2}u(l_-)\big]-(1-\delta)\frac{1}{2}\big[\alpha_s u(l_+)-\beta_s u(l_-)\big]$.

$u(Z) = \mathbb{R}$, there exists $z \in Z$ such that $V_2(P) = u_2(z)$. Then, $-\frac{1}{2}(1-\delta_2)(\alpha_s^2 + \beta_s^2) = u_2(z)$. If $\succsim_1$ is more ex–ante inequality averse than $\succsim_2$, then $V_1(P) \geq u_1(z)$. Hence, $-\frac{1}{2}(1-\delta_1)(\alpha_s^1 + \beta_s^1) = V_1(P) \geq u_1(z)$. Therefore, $\delta_1 \geq 1 + 2\left(\frac{u_1(z)}{\alpha_s^1+\beta_s^1}\right) = 1 + 2\left(\frac{u_2(z)}{\alpha_s^2+\beta_s^2}\right) = \delta_2$.

D.3 Proof of Proposition 4

Suppose (a) and (b) hold. Let $\alpha^F = .2$, $\beta^F = .9$, $\delta^F = .85$, and $u(z) = \log z$ for all $z \in \mathbb{R}_+$. Let V^t be as in Theorem 2 defined by $(\alpha^t, \beta^t, \delta^t, u)$ for all $t \in \{F, S\}$.

Part (ii) holds because $V^t(400, 400) = \log 400 > \log 375 \geq V^t(375, 750)$ for all t. In the following, we will show (i). The payoff function of player i with type t is denoted by Π_i^t. Let s_i^* be a strategy of player i such that the fair type (type F) play Ef and the selfish type (type S) play Eq. Given that the probability of fair type is .5, the payoff of player i given the opponent strategy s_j^* is defined as follows:

$$\Pi_i^F(Ef|s_j^*) = V^F\big(.5(375, 750) \oplus .25(750, 375) \oplus .25(400, 400)\big) \geq \delta^F\ 7.52 + (1-\delta^F)\ 5.89,$$
$$\Pi_i^F(Eq|s_j^*) = V^F\big(.5(400, 400) \oplus .25(750, 375) \oplus .25(400, 400)\big) \leq \delta^F\ 7.5 + (1-\delta^F)\ 6.$$

Hence, if $\delta^F \geq .85$ then $\Pi_i^F(Ef|s_j^*) \geq \Pi_i^F(Eq|s_j^*)$. Also, since

$$\begin{aligned}
\Pi_i^S(Ef|s_j^*) &= V^S\big(.5(375, 750) \oplus .25(750, 375) \oplus .25(400, 400)\big) &\leq 7.6,\\
\Pi_i^S(Eq|s_j^*) &= V^S\big(.5(400, 400) \oplus .25(750, 375) \oplus .25(400, 400)\big) &\geq 7.64,
\end{aligned}$$

then, $\Pi_i^S(Eq|s_j^*) > \Pi_i^S(Ef|s_j^*)$. Therefore, (s_1^*, s_2^*) and the prior probability consist a Bayesian Nash equilibrium.

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