Combinatorial Voting*

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Abstract

We study elections that simultaneously decide multiple issues, where voters have independent private values over bundles of issues. The innovation is considering nonseparable preferences, where issues may be complements or substitutes. Voters face a political exposure problem: the optimal vote for a particular issue will depend on the resolution of the other issues. Moreover, the probabilities that the other issues will pass should be conditioned on being pivotal. We first prove equilibrium exists when distributions over values have full support or when issues are complements. We then study limits of symmetric equilibria for large elections. Suppose that, conditioning on being pivotal for an issue, the outcomes of the residual issues are asymptotically certain. Then limit equilibria are determined by ordinal comparisons of bundles. We characterize when this asymptotic conditional certainty occurs. Using these characterizations, we construct a nonempty open set of distributions where the outcome of either issue remains uncertain in all limit equilibria. Thus, predictability of large elections is not a generic feature of independent private values. While the Condorcet winner is not necessarily the outcome of the election, we provides conditions that guarantee the implementation of the Condorcet winner. Finally, we prove results that suggest transitivity and ordinal separability of the majority preference relation are conducive for ordinal efficiency and for predictability.

1 Introduction

Propositions 1A and 1B of the 2006 California General Election both aimed to increase funding for transportation improvements.[1] Suppose a voter prefers some increased funding and supports either proposition by itself, but given the state's fiscal situation, also prefers that both measures

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[1] Proposition 1A dedicated gasoline taxes for transportation improvements, at the exclusion of other uses, while Proposition 1B issued $20 billion in bonds to fund improvements. Both measures passed by large margins.

fail together than to have both pass together. She views the propositions as substitutes. However, the ballot does elicit her preferences over bundles of transportation measures, but only a separate up-down vote on each proposition. If she votes up on Proposition 1A while Proposition 1B passes, she contributes to the undesired passage of both measures. On the other hand, if Proposition 1B were to fail, she would like to see Proposition 1A pass to fund some transportation improvements.

How should she vote? Some subtle considerations complicate the answer to this question. What is the likelihood she is pivotal on either proposition or both? If she is pivotal on some proposition, what is the conditional likelihood that the other will pass or fail? The natural model for these questions is a game of incomplete information. The model begs other questions. Does equilibrium exist? What does it look like? Does it exhibit special properties in large elections? Are equilibrium outcomes predictable? Are these outcomes ordinally efficient? For elections with nonseparable issues, these basic questions are still undecided. To our knowledge, this paper is the first to follow the strategic implications of electoral complementarity or substitution to their equilibrium conclusions, and makes initial progress in addressing these concerns.

1.1 An example

The following example illustrates the strategic delicacy of elections with multiple issues. There are two issues, say Propositions 1 and 2. Each voter's private values for the four possible bundles $\emptyset, \{1\}, \{2\}, \{1,2\}$ can be represented as a four-dimensional type $\theta = (\theta_\emptyset, \theta_1, \theta_2, \theta_{12})$, where θ_A denotes the value for bundle A. Voters' types are independent and identically distributed with the following discrete distribution:

$$\theta = \begin{cases} (\delta, 0, 0, 1) & \text{with probability } 1 - 2\varepsilon \\ (1, 0, 0, 0) & \text{with probability } \varepsilon \\ (0, 1, 0, 0) & \text{with probability } \varepsilon \end{cases}$$

where $\delta, \varepsilon > 0$ are arbitrarily small. With high probability $1 - 2\varepsilon$, a voter wants both issues to pass, but slightly prefers both issues to fail than either issue pass to alone. With small probability ε, a voter is either type $(1, 0, 0, 0)$ and wants both issues to fail or type $(0, 1, 0, 0)$ and wants issue 1 to pass alone. In either case, she is indifferent between her less preferred alternatives. It is a dominant strategy for type $(1, 0, 0, 0)$ to vote down on both issues and for type $(0, 1, 0, 0)$ to support issue 1 and vote against issue 2. The question is how type $(\delta, 0, 0, 1)$ should vote.

A natural conjecture is that type $(\delta, 0, 0, 1)$ should vote up on both issues in any large election. Then the conjectured equilibrium strategy s^* as a function of types is

$$\begin{aligned} s^*(\delta, 0, 0, 1) &= \{1, 2\} \\ s^*(1, 0, 0, 0) &= \emptyset \\ s^*(0, 1, 0, 0) &= \{1\}, \end{aligned}$$

where $s^*(\theta)$ refers to the issues that type θ supports. When voters play this strategy, both issues will have majority support in large elections, which is efficient. The suggested strategy might appear incentive compatible, since $(\delta, 0, 0, 1)$ should vote up for either issue when she is confident that the other issue will pass.

However, the proposed strategy is *not* an equilibrium in large elections. This is because the conditional probability that the residual issue passes is starkly different from the unconditional probability. Consider a voter deciding whether to support issue 1. She correctly reasons that her support only matters when she is pivotal for issue 1. When the other votes on issue 1 are split, she is in the unlikely state of the world where half of the other voters are of type $(1, 0, 0, 0)$, since this is the only type who votes against issue 1. Moreover, in large elections, there will be some voters of type $(0, 1, 0, 0)$. Then voters of type $(\delta, 0, 0, 1)$ comprise a strict minority. Since these are the only types who support issue 2, this voter should conclude that issue 2 will surely fail whenever she is pivotal for issue 1 in a large election. Therefore, if the pivotal voter is of type $(\delta, 0, 0, 1)$, she should vote down on issue 1 because she prefers the bundle $\emptyset$ yielding utility δ to the bundle $\{1\}$ yielding utility 0. In fact, the only equilibrium in weakly undominated strategies is for type $(\delta, 0, 0, 1)$ to vote down on both issues, inducing the ex ante inefficient social outcome of the empty bundle in large elections.[2]

Finally observe that, had δ been equal to 0, then type $(\delta, 0, 0, 1) = (0, 0, 0, 1)$ would have had a dominant strategy to vote up on both issues. In this case, the suggested strategy where $s^*(\delta, 0, 0, 1) = \{1, 2\}$ would be an equilibrium and the efficient bundle would be implemented in large elections. So, a small amount of nonseparability, i.e. a slightly positive $\delta > 0$, is enough to remove efficiency and change the outcome of the election.

1.2 A political exposure problem

The basic complication for elections with nonseparable issues is the wedge between the unconditional probability that an issue will pass and the conditional probability when a voter is pivotal on another issue. This resonates with existing analyses of strategic voting on a single issue with interdependent values; for example, see Austen-Smith and Banks (1996) or Feddersen and Pesendorfer (1997). In these models, being pivotal provides additional information regarding other voters' signals about an unknown state of the world. The intuition there is analogous to the importance of strategic conditioning in common value auctions for a single item, where it leads to the winner's curse and strategic underbidding. In both single-object auctions and single-issue elections with common values, strategic conditioning complicates information aggregation and efficiency. This is because the expected value of the object or the proposal is different when the player conditions on being the winner of the auction or the pivotal voter of the election.

The intuition here also has a relationship with auction theory, but with a different branch.

[2] A related example on voting over binary agendas is due to Ordeshook and Palfrey (1988). There, being pivotal in the first round of a tournament changes the expected winner in later rounds. This reasoning can lead to inefficient sequential equilibria in their model.

Here, the wedge is related to the exposure problem in combinatorial auctions for multiple items, which exists even with private values. Suppose two items are sold in separate auctions. Consider a bidder with complementary valuations who desires only the bundle of both items. She must bid in both auctions to have any chance of obtaining this package. But, she should recognize that doing so exposes her to the risk of losing the second auction while winning the first, forcing her to pay for an undesired single item bundle. Moreover, the unconditional probability of winning the second auction is not appropriate in computing her exposure, but rather the conditional probability of winning the second auction assuming that she wins the first auction. Likewise, a voter in an election who desires a bundle of two issues to pass, but does not want either issue to pass alone, faces an exposure problem. In deciding her vote for issue 1, she should consider whether issue 2 will pass, but also condition this probability on the assumption that she is pivotal on issue 1.

This exposure problem disappears when values are separable across issues, in which case each issue can be treated like a separate election. However, with nonseparable preferences, the following intuitions from single-issue elections break down. First, with one issue, voting sincerely for the preferred outcome (pass or fail) is a weakly dominant strategy for every voter. In contrast, with nonseparable preferences, voting sincerely is never an equilibrium. Instead, a voter's equilibrium strategy must correctly condition the other voters' ballots on the assumption that she is pivotal for some issue. Second, with a single issue, there is a generic class of distributions over values for which the outcome is predictable in large elections. We assume independent private values, so the composition of preferences is known for large electorates. Nevertheless, there exists a nontrivial set of type distributions which generate unpredictable election outcomes. This aggregate *endogenous* uncertainty is despite the fact that there is no aggregate *primitive* uncertainty in large elections. Third, the Condorcet winner is always implemented in single-issue elections. With multiple issues, the Condorcet winning bundle can fail to be the outcome of large elections. Instead, additional assumptions are required to guarantee implementation of the Condorcet winner.

1.3 Outline

The paper proceeds as follows. Section 2 introduces the Bayesian game of voting over multiple issues. Section 3 shows the existence of equilibrium using two arguments. One is topological and converts the infinite-dimensional fixed point problem over strategies to a finite-dimensional problem over probabilities regarding which issues a voter is pivotal for and which issues will pass irrespective of her vote. This conversion yields later dividends in characterizing equilibrium. The second argument assumes complementarity between issues and shows the existence of a monotone equilibrium, where types with a stronger preference for passing more issues also vote for more issues. This proof relies on recent general monotone existence results due to Reny (2009).

Section 4 characterizes limit equilibria for large elections. In particular, we examine when the probability that issue y passes, conditional on a voter being pivotal on issue x, goes to zero or one. When this conditional asymptotic certainty holds, computation of best response is simplified. For every issue x, there exists some subset D_x of the residual issues such that a voter supports x if and only if the bundle $\{x\} \cup D_x$ is preferred to the bundle D_x. We characterize conditional certainty with an intuitive inequality on the limit strategy: the conditional probability that another voter supports issue $y \in D_x$, when the vote on x is split, is greater than a half. For the case of two issues, we identify an inequality on the primitive of the model, namely the distribution over types, which characterizes conditional certainty.

Section 5 leverages these limit characterizations to construct a nonempty open set of densities which exhibit aggregate uncertainty regarding the outcome of the election. Even though there is no uncertainty regarding the primitives of the model, unpredictability of the outcomes is required to maintain incentives in equilibrium. This establishes that predictability of outcomes is not a generic feature of large elections with multiple issues.

Section 6 uses the limit results to study the relationship between combinatorial voting and the majority preference relation. While the Condorcet winner is not generally the outcome of the election, we provide sufficient conditions for implementation of the Condorcet winner. Finally, we provide results suggest that ordinal separability of the majority preference relation is conducive to implementation of the Condorcet winner and hence to predictability.

Section 7 concludes and reviews open questions. Proofs are collected in the appendix.

1.4 Related literature

Several papers in political science recognize the potential problems introduced by nonseparable preferences over multiple issues. Brams, Kilgour, and Zwicker (1998) point out that the final set of approved issues may not match any single submitted ballot, which they call the "paradox of multiple elections."[3] Lacy and Niou (2000) construct an example with three strategic voters and complete information where the final outcome is not the Condorcet winner. Our model enriches this literature in two directions. First, while this literature largely focuses on sincere voting, we analyze the implications of strategic voting for this setting.[4] Second, we introduce uncertainty regarding other's preferences. This uncertainty is crucial for a voter with nonseparable preferences, whose optimal vote for a particular issue depends on her conjecture regarding the resolution of other issues.

While more specific comparisons are made as results are presented in the paper, we now highlight some differences between large elections under combinatorial rule and plurality rule. We focus on plurality rule as an alternative aggregation scheme because it shares the same space of ballots or messages as combinatorial voting.[5] As originally observed by Palfrey (1989), limit equilibria of

[3]The paradox is extended by Özkal-Sanver and Sanver (2007) and reinterpreted by Saari and Sieberg (2001).

[4]The exceptions are the mentioned example by Lacy and Niou (2000) and a single-person model of sequential survey responses by Lacy (2005).

[5]In our setting, the set of candidates is the power set 2^X of bundles. Combinatorial rule is not a scoring rule. In particular, combinatorial voting invokes the structure of the power set in an essential way, while this structure is irrelevant to a scoring rule. Moreover, general scoring rules require larger message spaces than combinatorial rule. For example, in this environment approval voting requires that the space of ballots be the power set of the power set, or 2^{2^X}. General treatments of scoring rules can be found in Myerson and Weber (1993) and Myerson (2002).

plurality rule typically satisfy Duverger's Law and involve two active candidates.[6] These equilibria have qualitatively different features than the limit equilibria of this model. First, predictability of the outcome is a generic feature of any Duvergerian equilibrium under plurality rule. There are multiple Duvergerian equilibria involving different pairs of candidates. But for any fixed equilibrium, the outcome is determinate for a generic set of type distributions. In contrast, combinatorial voting can yield unpredictability for an open set of type distributions. Second, plurality rule always has at least one limit equilibrium which selects a Condorcet winner when the winner exists. As the example shows, combinatorial rule can fail to have any limit equilibria which implement the Condorcet winner. Finally, in our view the strategic considerations under plurality rule are relatively simpler than under combinatorial rule. Once an equilibrium is fixed, each voter should support whichever of the two active candidates she prefers. In our model, if the voter assumes that she is pivotal for some issue, she must then condition her conjecture regarding the residual issues on that pivot event.

One feature common to both plurality rule and combinatorial rule is that the distribution of types conditional on being pivotal diverges from the ex ante distribution of types. Other multi-candidate models with independent private values and incomplete information also share this feature. The most closely related in terms of the strategic intuitions are models that have a dynamic element to the aggregation. In such models, the wedge between pivotal and unconditional probabilities can also lead to inefficiencies. The earliest example of which we are aware is the treatment of strategic voting on dynamic agendas by Ordeshook and Palfrey (1988); there the winner between alternatives a and b in the first round faces alternative c in the second round. With incomplete information, being pivotal for a against b in the first round can reverse the expected resolution of a vote between a and c. In particular, as in our initial example, this wedge can prevent a Condorcet winner from being the final outcome of the tournament. More recently, Bouton (2009) points out that, being pivotal for a candidate in the first round of a runoff election provides information on whether a runoff will take place and on which candidates will be active in the runoff. Consequently, there are equilibria where a Condorcet loser is the outcome of the election. While a wedge between pivotal and unconditional probabilities appears in existing work, to our knowledge this paper is the first to observe the wedge between pivotal and unconditional probabilities in the context of the exposure problem and nonseparabilities in multi-issue elections.

A natural application of our model is to simultaneous two-candidate elections for multiple political offices. Split tickets, such as those supporting a Republican president but a Democratic legislator, are increasingly common in such elections, constituting about a quarter of all ballots in recent presidential elections. In our model, the different tickets can be modeled as different bundles of Republican offices. Fiorina (2003) argues some voters have an inherent preference for divided government. Here, such voters would treat issues as substitutes.

Alesina and Rosenthal (1996) present a spatial model where voters split ticket to moderate policy location. Chari, Jones, and Marimon (1997) present a fiscal model where voters split tickets to

[6] Fey (1997) shows that only the Duvergerian limit equilibria are stable in a variety of senses.

increase local spending and restrain national taxation. These motivations provide foundations for nonseparable preferences, but also restrict the implied preferences and thus the implied predictions; Alesina and Rosenthal (1996) predict that all split tickets in a particular election support the same candidates, while Chari, Jones, and Marimon (1997) predict that all split tickets support a conservative president and a liberal legislator. While we are agnostic about the source of nonseparability, we allow arbitrary preferences over the composition of government, for example some voters may have a desire for unified government, and predict the full spectrum of split tickets. Finally, the existing models move directly to large elections with a continuum of voters. While we examine limits as they tend large, the finite electorates in this paper are essential to maintain the political exposure problem in equilibrium.

2 Model

There is a finite and odd set of I voters. They vote over a finite set of binary issues X, whose power set is denoted $\mathcal{X}$. Each voter i submits a ballot $A_i \in \mathcal{X}$, with $x \in A_i$ meaning that i votes "up" on issue x, and $y \notin A_i$ that she votes "down" on y. Issues can be interpreted as policy referenda which will pass or fail, or as elected offices decided between two political parties where one is labelled "up." The social outcome $F(A_1, \ldots, A_I) \in \mathcal{X}$ is decided by what we call combinatorial rule:

$$F(A_1, \ldots, A_I) = \{x \in X : \#\{i \in I : x \in A_i\} > I/2\}.$$

We assume each voter knows her own private values over outcomes, but allow uncertainty about others' values. Each voter has a (normalized) type space $\Theta_i = [0,1]^{\#\mathcal{X}}$, with typical element θ_i. Then $\theta_i(A)$ denotes type θ_i's utility for all the issues in A passing and all those in its complement failing: so θ_i's utility for the profile of ballots $(A_1, \ldots, A_I)$ is $\theta_i(F(A_1, \ldots, A_I))$. Let $\Theta = \prod_i \Theta_i$ denote the space of all type profiles, and $\Theta_{-i} = \prod_{j \neq i} \Theta_j$. Voter i's type realization follows the distribution $\mu_i \in \Delta\Theta_i$. We assume that μ_i admits a density. We also assume types are independent across voters, letting $\mu = \mu_1 \otimes \cdots \otimes \mu_I \in \Delta\Theta$ refer to the product distribution across voters.

A (pure) strategy s_i for each voter i is a measurable function $s_i : \Theta_i \to \mathcal{X}$ assigning a ballot to each of her types. The space of strategies for each voter is S_i. The space of strategy profiles is $S = \prod_i S_i$, and let $S_{-i} = \prod_{j \neq i} S_j$. Voter i's ex ante expected utility for the joint strategy profile $s(\theta) = (s_1(\theta_1), \ldots, s_I(\theta_I))$ is $EU_i(s) = \int_\Theta \theta_i(F(s(\theta))\, d\mu$.

Definition 1. A strategy profile s^* is a **voting equilibrium** if it is a Bayesian-Nash equilibrium in weakly undominated strategies.

A voter's values might exhibit certain structural characteristics. For example, she might view the issues as complements, as substitutes, or as having no interaction. These are captured by the following definitions.

Definition 2. θ_i is **supermodular** if, for all $A, B \in \mathcal{X}$:

$$\theta_i(A \cup B) + \theta_i(A \cap B) \geq \theta_i(A) + \theta_i(B).$$

θ_i is **submodular** if, for all $A, B \in \mathcal{X}$:

$$\theta_i(A \cup B) + \theta_i(A \cap B) \leq \theta_i(A) + \theta_i(B).$$

θ_i is **additively separable** if it is both supermodular and submodular.

3 Existence of equilibrium

We begin by proving existence of voting equilibria. We will present topological and lattice-theoretic arguments for existence.

3.1 General existence of equilibrium

Proposition 1. *Suppose μ_i admits a density function with full support. There exists a voting equilibrium s^*.*

Remark. The full support assumption can be replaced with the following weaker condition: for every bundle A, there is a positive measure of types whose unique weakly undominated strategy is to submit the ballot A. Alternatively, assuming a set of naive voters who submit the ballot A in all circumstances would also guarantee an equilibrium among the sophisticated voters.

The proof lifts the infinite-dimensional problem of finding a fixed point in the space of strategy profiles to a finite-dimensional space of probabilities. Specifically, when other voters submit the ballots A_{-i}, the strategically relevant information for voter i is summarized as the set of issues C for which voter i is pivotal and the set of issues D which will pass irrespective of voter i's ballot. The outcome of submitting the ballot A_i is that those issues which she supports and on which she is pivotal will pass, along with those issues which will pass no matter how she votes: $[A_i \cap C] \cup D$. The relevant uncertainty can therefore be summarized as a probability over the ordered disjoint pairs of subsets of X, which we write as $\mathcal{D} = \{(C, D) \in \mathcal{X} \times \mathcal{X} : C \cap D = \emptyset\}$. Each strategy profile $s \in S$ induces a probability $\pi_i(s) \in \Delta\mathcal{D}$ for each voter i over $\mathcal{D}$, where $\Delta\mathcal{D}$ denotes the space of probabilities on $\mathcal{D}$. Viewed as a function, $\pi_i : S \to \Delta\mathcal{D}$ is continuous by construction.[7]

In turn, each belief $P_i \in \Delta\mathcal{D}$ over these ordered pairs induces an optimal ballot $[\sigma_i(P_i)](\theta_i) \in \mathcal{X}$ for a voter with values θ_i, which is the ballot A_i that maximizes the expected utility $\sum_{\mathcal{D}} \theta_i(A_i \cap C \cup D) \cdot P_i(C, D)$. Observe that this expression for interim expected utility is a linear function with coefficients $P_i(C, D)$ on θ_i. Then the set of types for whom A_i is an optimal ballot are those where $\sum_{\mathcal{D}} \theta_i(A_i \cap C \cup D) \cdot P_i(C, D) \geq \sum_{\mathcal{D}} \theta_i(A_i' \cap C \cup D) \cdot P_i(C, D)$, which defines a finite intersection of half-spaces. Small changes in P_i induce small geometric changes in these half-spaces. The

[7] The topology on S_i is defined by the distance $d(s_i, s_i') = \mu_i(\{\theta_i : s_i(\theta_i) \neq s_i'(\theta_i)\})$.

density assumption implies that these small geometric changes also have small measure, proving that $\sigma_i : \Delta\mathcal{D} \to S_i$ is continuous.

Define the functions $\pi : S \to [\Delta\mathcal{D}]^I$ by $\pi(s) = (\pi_1(s), \ldots, \pi_I(s))$ and $\sigma : [\Delta\mathcal{D}]^I \to S$ by $\sigma(P_1, \ldots, P_I) = (\sigma_1(P_1), \ldots, \sigma_I(P_I))$. Then the composition $\pi \circ \sigma : [\Delta\mathcal{D}]^I \to S \to [\Delta\mathcal{D}]^I$ defines a continuous function between finite-dimensional spaces. However, before applying a fixed point theorem, we still need to prove that we can restrict attention to undominated strategies.

Consider a strategy profile s in weakly undominated strategies. The induced probability $\pi_i(s)$ that voter i will be pivotal for the issues in C while the issues in D pass is at least as large as the probability that half the other voter submit $C \cup D$ while the other half submits D. By the full support assumption, there is a strictly positive probability any voter submits C or $C \cup D$ in any weakly undominated strategy. The independence assumption allows us to multiply these probabilities across voters, to conclude that $[\pi_i(s)](C, D)$ is strictly positive. Then the probabilities induced by weakly undominated strategy profiles S^U lives in a compact subsimplex Δ^U in the interior of the entire $\Delta\mathcal{D}$, so $\pi_i(S^U) \subseteq \Delta^U \subseteq \text{int}(\Delta\mathcal{D})$. Since all strategically relevant events (C, D) have strictly positive probability in Δ^U, the induced best replies must also be weakly undominated. Therefore, the restriction $\pi \circ \sigma : [\Delta^U]^I \to S^U \to [\Delta^U]^I$ defines a continuous function from a compact subset of a finite-dimensional space to itself. By Brower's Theorem, there exists a fixed point P^* with strictly positive probabilities on all pairs. Then $\sigma(P^*)$ is a Bayesian-Nash equilibrium in weakly undominated strategies.

The key step in the proof, moving from the infinite-dimensional space of strategies to a finite-dimensional space of probabilities, is adapted from Oliveros (2007). The broad approach of reducing the fixed point problem to a finite simplex is reminiscent of the distributional approach of Radner and Rosenthal (1982) and Milgrom and Weber (1985). However, these results are not immediately applicable, as we require equilibrium in weakly undominated strategies. Beside the technical benefit, there is a methodological insight in conceptualizing equilibrium as a fixed point of probabilities over pivot and passing events. In subsequent sections, this approach will enable sharper characterizations of voting behavior in large elections.

A similar proof can be used to demonstrate existence without a common prior, as long as i's belief for the others' values is constant across her own types θ_i. Another variant can be used to show that if μ_i is identically distributed across voters, there exists a symmetric equilibrium.

Proposition 2. *Suppose μ admits a density function with full support and $\mu_i = \mu_j$ for all $i, j \in I$. There exists a symmetric voting equilibrium where $s_i^* = s_j^*$ for all $i, j \in I$.*

We will later focus attention to symmetric settings to obtain limit characterizations.

As mentioned in the introduction, sincere voting is not an equilibrium when μ_i has full support. The result has a simple intuition. Optimal voting is determined cardinally by utility differences across bundles, while sincere voting is determined ordinally by the best bundle.

Proposition 3. *If each μ_i admits a density with full support, then sincere voting, where*

$$s_i(\theta_i) = \arg\max_A \theta_i(A).$$

is not a voting equilibrium.

3.2 Existence of monotone equilibrium

With complementary issues, equilibrium can be sharpened to be monotone in the increasing differences order: those types who have a stronger preference for more issues passing will support more issues in equilibrium. Monotonicity of ballots with respect to types is useful for empirical identification. Monotonicity justifies the following inference: those who are observed to vote for more issues have a preference for larger bundles. For example, suppose X is a number of political offices and voting "up" corresponds to voting for the Republican candidate while voting "down" corresponds to voting for the Democratic candidate. If all voters prefer to have politicians of the same party in government, then we can infer than those who vote for more Republicans are more right-leaning than those who vote for fewer Republicans. However, if some voters are concerned with balancing party representation, i.e. if issues are substitutes, then this inference is no longer justified, as it confounds ideological centrism with a desire for party balance.

Consider the partial order $\geq$ on types defined as follows: $\theta_i' \geq \theta_i$ if the inequality $\theta_i'(A)-\theta_i'(B) \geq \theta_i(A)-\theta_i(B)$ holds for all $A \supseteq B$. This order captures the notion that a larger type θ' has a uniformly stronger preference for more issues to pass, as the difference in her utility between a larger bundle A and a smaller bundle B always dominates that difference for a smaller type θ. Going back to the ideology example, if "up" is coded as a Republican candidate for that office, the difference in utility between a more Republican (A) and a less Republican (B) legislature is greater for a right-leaning type θ' then it is for a left-leaning type θ. The following theorem demonstrates that assuming issues are complementary, i.e. that a more unified legislature is more desirable, suffices for the desired inference that more right-leaning types will vote for more Republican candidates.

Proposition 4. *Suppose μ admits a density whose support is the set of all supermodular type profiles. Define the increasing differences order $\geq$ on Θ_i by $\theta_i' \geq \theta_i$ if*

$$\theta_i'(A) - \theta_i'(B) \geq \theta_i(A) - \theta_i(B), \quad \forall A \supseteq B.$$

Then there exists a monotone voting equilibrium s^*, where $s_i^*(\theta_i') \supseteq s_i^*(\theta_i)$ whenever $\theta_i' \geq \theta_i$.

A topological proof similar to that of Proposition 1 would be enough to prove existence of a voting equilibrium. However, the key part of Propositon 4 is that the equilibrium is *monotone*, and lattice-theoretic arguments are essential.

Note that the election is not a supermodular game. Sufficiently large strategies by all voters guarantee that no voter is ever pivotal on any issue and eliminate the difference in interim utility between any of two strategies. Moreover, the restriction to weakly undominated strategies is important, since the trivial equilibrium where all voters play the same constant strategy is monotone. To handle these considerations, the proof relies on recent monotone existence results by Reny (2009) that improve earlier theorems by Athey (2001) and McAdams (2003) by allowing for general orders on types, such as the increasing differences order, and for restrictions on strategies, such as the exclusion of weakly dominated strategies.

4 Limit behavior

This section examines asymptotic voting behavior in large elections. The results in this section provide useful tools in later analyzing the predictability and ordinal efficiency of large elections. To this end, we assume that voters are identically distributed, $\mu_i = \mu_j$ for all voters i, j, and focus attention to symmetric equilibria where $s_i^* = s_j^*$. For notational ease, we henceforth drop the subscript as a reference to a particular player, and let s_I^* denote the equilibrium strategy for an anonymous player in the game with I voters.

4.1 Conditional certainty

Given a sequence of strategies, the following defines whether an issue becomes certain to pass or fail at the limit, i.e. whether the outcome of that issue is predictable in large elections.

Definition 3. Consider a sequence of strategies $s_I \to s$. Issue y is **unconditionally certain to pass (fail)** if

$$\mathbf{P}\left(\#\{i : y \in s_I^*(\theta_i)\} > (<) \frac{I}{2}\right) \to 1.$$

If issue y is neither unconditionally certain to pass or fail, we say issue y is **unconditionally uncertain**.

If every issue y is unconditionally certain to pass or to fail, we write that the set

$$A = \{y \in X : y \text{ is unconditionally certain to pass}\}$$

is a **limit outcome** of the election.

However, the unconditional probability of an issue passing is not the strategically relevant statistic. Rather, it is the probability that this issue passes when a voter is pivotal for some other issue. This motivates the following definition.

Definition 4. Fix a sequence of strategies $s_I \to s$ and an issue x. An issue $y \neq x$ is **conditionally certain to pass (fail) at** x if:

$$\mathbf{P}\left(\#\{j \neq i : y \in s_I(\theta_j)\} > (<) \frac{I-1}{2} \,\middle|\, \#\{j \neq i : x \in s_I(\theta_j)\} = \frac{I-1}{2}\right) \to 1 \tag{1}$$

If each issue $y \neq x$ is conditionally certain to either pass or fail at x, then we say the sequence exhibits **conditional certainty at $\boldsymbol{x}$**. In this case, we let D_x denote those issues which are conditionally certain to pass and write that **$\boldsymbol{D_x}$ is conditionally certain at $\boldsymbol{x}$**.

If the sequence exhibits conditional certainty at every issue x, then we simply write that it exhibits **conditional certainty**. If it does not exhibit conditional certainty, we write that it exhibits **conditional uncertainty.**

As the number of voters gets large, the probability of being pivotal for any single issue x becomes small. But, since each bundle is submitted with strictly positive probability in a weakly dominant strategy, the probability of being simultaneously pivotal on two issues vanishes at a much faster rate. So when a voter conditions on the unlikely event of being pivotal for issue x, she can ignore the doubly unlikely event of also being pivotal for another issue. Then the only relevant uncertainty is how the residual issues besides x are resolved, after appropriately conditioning on being pivotal for issue x.

Suppose that as the electorate grows, conditioning on a voter being pivotal for issue x, it becomes certain that the issues in $D_x \subseteq X \setminus \{x\}$ will pass while the other issues outside of this set will fail. A voter's decision to support issue x then reduces to whether the conditional outcome D_x of the residual issues is better for her with or without the addition of issue x, i.e. whether $D_x \cup \{x\}$ is better than D_x alone. So, the equilibrium strategy is determined separately on each issue. This substantially reduces the complexity of deciding a voter's equilibrium ballot, which is then determined by her ordinal ranking of bundles. If there is conditional certainty for each issue, then all types which share the same ordinal ranking of bundles will submit the same equilibrium ballot; if two voters are observed submitting different ballots, then their preferences must be ordinally distinct. The reduction also provides a useful tool in proving the later limit results. In characterizing limit equilibria, we will often invoke the following result and examine purely ordinal conditions.

Proposition 5. *Consider a sequence of equilibrium strategies $s_I^* \to s^*$. If D_x is conditionally certain at x, then*

$$x \in s^*(\theta) \Longleftrightarrow \theta(D_x \cup \{x\}) \geq \theta(D_x).$$

The proof of Proposition 5 exploits the probabilistic structure used to prove Theorem 1, the general existence result. There, the strategically relevant information was summarized by an induced probability $P(C, D)$ over ordered disjoint pairs (C, D) of subsets of issues, interpreting C as the issues for which voter is pivotal and D as the issues which will pass irrespective of her ballot. In deciding whether to support issue x, the relevant probability is the conditional probability $P(C, D \,|\, x \in C)$. The incentive condition for whether the ballot $A \cup \{x\}$ is better than the ballot A for a voter of type θ is a comparison of the following weighted sums:

$$\sum_{C,D \in \mathcal{D}} P(\{x\}, D \,|x \in C) \cdot \theta([A \cup \{x\} \cap C] \cup D) \geq \sum_{C,D \in \mathcal{D}} P(C, D \,|\, x \in C) \cdot \theta([A \cap C] \cup D).$$

Conditional certainty of D_x implies that the $P(\{x\}, D_x \,|\, x \in C) \to 1$. But this is the weight on the term $\theta(D_x \cup \{x\})$ on the left and the term $\theta(D_x)$ on the right. Because $|\theta(A) - \theta(B)|$ is uniformly bounded, these terms determine the inequality in large elections, except on a geometrically diminishing set of types. By the density assumption, this set is also approaching probability zero. So, except on a vanishing set of types, the decision as to whether to submit the ballot A or the ballot $A \cup \{x\}$ is mediated by the difference $\theta(D_x \cup \{x\}) - \theta(D_x)$.

4.2 Characterizing conditional certainty with strategies

This subsection characterizes conditional certainty with conditions on the limit equilibrium. The next proposition presents an inequality which essentially characterizes conditional certainty: it is sufficient in its strict form and necessary in its weak form.

Proposition 6. *Let $s^*_I \to s^*$. If*

$$\mu(y \in s(\theta) \,|\, x \in s(\theta)) > (<)\, \mu(y \notin s(\theta) \,|\, x \notin s(\theta)), \tag{$\star$}$$

then y is conditionally certain to pass (fail) at x.

Moreover, if y is conditionally certain to pass (fail) at x, then ($\star$) holds weakly.

A heuristic intuition for the inequality is straightforward: inequality ($\star$) is equivalent to

$$\frac{1}{2}\mu(y \in s(\theta) \,|\, x \in s(\theta)) + \frac{1}{2}\mu(y \in s(\theta) \,|\, x \notin s(\theta)) > \frac{1}{2}.$$

When a voter is pivotal on issue x, the conditional probability that someone else supports x or votes against x is a half, so the left hand side is the conditional probability that another supports issue y. The suggested inequality guarantees that the conditional probability another voter supports issue y is strictly larger than one half.

However, the result is not a simple application of the law of large numbers. Conditioning on being pivotal for x breaks the statistical independence across the other voters' ballots. Suppose voter i is pivotal on issue x. Then knowing that voter j supported issue x makes it more likely that another voter j' voted against it, since an equal number voted each for and against. From the pivotal voter's perspective, the votes on x are negatively dependent. This indirectly introduces statistical dependence for any issue which is strategically correlated with x, precluding a straightforward application of convergence results for independent sequences.

The proof handles this dependence by making an artificial conditioning assumption. This additional conditioning restores independence of votes across players, but has no effect on the conditional distribution of the sum. In particular, suppose that the highest-indexed voter I is pivotal on issue x. Assume that the lowest-indexed half of the other voters $1, \ldots, \frac{I-1}{2}$ supported issue x, while the higher-indexed voters $\frac{I+1}{2}, \ldots, I-1$ voted against it. When I is pivotal on x, the others' votes are no longer independent, but are still exchangeable. Moreover, we are interested only in the sum of supportive votes, and not in the identity of the supporters. This artificial assumption regarding the identities of the supporters has no effect on the distribution of the vote count on y.

However, this assumption breaks the correlation across ballots, because knowing player j's vote on issue x is no longer informative regarding player k's vote on x, since k's vote is now assumed to be known. We can then apply the law of large numbers separately to each artificial subsample, the sample of those assumed to support issue x or those assumed to vote against it.

4.3 Characterizing conditional certainty with primitives: two issues

The prior results characterized conditional certainty through inequalities on strategies. Turning to the simpler case with only two issues, the first part of this section expresses these inequalities on the primitive details of the model, namely the distribution of values over bundles. For either issue $x = 1, 2$, let $x' \neq x$ denote the complementary issue. Henceforth, for a fixed bundle A of issues, we let $A', A'' \neq X \setminus A$ denote its two neighbors; for the bundle $\{1\}$, its neighbors are $\{1, 2\}$ and $\emptyset$. For notational lightness, we will also write θ_A to mean $\theta(A)$.

Limiting attention to two issues allows the following generalization of Proposition 5, which will useful in proving characterization results. The equilibrium strategy s^* is summarized by two parameters $\alpha_1, \alpha_2 \in [0, 1]$ for large elections, where α_x corresponds to the asymptotic conditional probability that issue x passes when a voter is pivotal for issue x'. Proposition 5 is the special case where α_1 and α_2 are degenerately 0 or 1.

Lemma 1. *There exists $\alpha_x \in [0, 1]$ such that:*

$$x \in s^*(\theta) \Longleftrightarrow \alpha_x \theta_{12} + (1 - \alpha_x)\theta_x \geq \alpha_x \theta_x + (1 - \alpha_x)\theta_\emptyset.$$

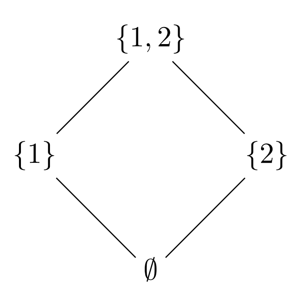
Restricting attention to two issues affords a notational simplification, since each issue can be conditionally certain only at its complement.

Definition 5. For the two issue case, the bundle $A \subseteq \{1, 2\}$ is **conditionally certain** if x is conditionally certain to pass at x' for every $x \in A$ and x is conditionally certain to fail at x' for every $x \notin A$.

So in the two-issue case, a sequence of equilibria exhibits conditional certainty if and only if there is a conditionally certain set A.

Recalling Proposition 5, if the bundle A is conditionally certain, then whether type θ votes up for issue x can be verified by testing whether $\theta(A \cup \{x\}) \geq \theta(A \setminus \{x\})$.

Proposition 7. *Consider the following graph:*



Let A', A'' denote the two nodes connected to A. If

$$\frac{\mu[\theta_A \geq \max\{\theta_{A'}, \theta_{A''}\}]}{\mu[\theta_A \leq \min\{\theta_{A'}, \theta_{A''}\}]} > \max\left\{\frac{\mu[\theta_{A'} \geq \theta_A \geq \theta_{A''}]}{\mu[\theta_{A''} \geq \theta_A \geq \theta_{A'}]}, \frac{\mu[\theta_{A''} \geq \theta_A \geq \theta_{A'}]}{\mu[\theta_{A'} \geq \theta_A \geq \theta_{A''}]}\right\}, \tag{†}$$

then there exists a sequence of equilibria $s_I^* \to s^*$ such that A is conditionally certain.

Moreover, if there exists a sequence of equilibria $s_I^* \to s^*$ such that A is conditionally certain, then (†) holds weakly.

When the sufficient inequality in Proposition 7 holds, computing an equilibrium for large elections is simple. By virtue of Proposition 5, the asymptotic equilibrium decision to support issue x is determined by

$$x \in s^*(\theta) \iff \theta(A \cup \{x\}) \geq \theta(A \setminus \{x\}).$$

The sufficiency of the inequality, which ignores the value for the bundle $\{1,2\} \setminus A$ that is unconnected to A, is suggestive of the scope for inefficiency or miscoordination in elections with nonseparable preferences.

Lemma 2. *The following conditions are equivalent:*

(i) *Inequality (†) in Proposition 7;*

(ii) $\mu(\theta_A \geq \theta_{A'} \,|\, \theta_{A''} \geq \theta_A) > \mu(\theta_{A'} \geq \theta_A \,|\, \theta_A \geq \theta_{A''})$ *and*
$\mu(\theta_A \geq \theta_{A''} \,|\, \theta_{A'} \geq \theta_A) > \mu(\theta_{A''} \geq \theta_A \,|\, \theta_A \geq \theta_{A'})$;

(iii) $\mu(\theta_A \geq \theta_{A'}) > \frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''}) + \mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}$ *and*
$\mu(\theta_A \geq \theta_{A''}) > \frac{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''}) + \mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}$.

Moreover, the weak versions of these conditions are equivalent.

To interpret the inequalities in condition (iii) of Lemma 2, consider the quantity on either side. On the left hand side is the statistical electoral advantage that A enjoys against its neighbor A' or A''. The right hand side is a ratio which measures, conditional on A being between its neighbors, the likelihood that the most preferred bundle among them is the opposite neighbor. This ratio is close to one if, for example, the likelihood that $\theta_{A''} \geq \theta_A \geq \theta_{A'}$ is much larger than $\theta_{A''} \geq \theta_A \geq \theta_{A'}$. This reflects a local asymmetry across the bundle A and its neighbors. So, two factors will make the sufficient inequalities more likely to carry:

(i) Local lectoral advantage, i.e. a large proportion of the population favors the bundle A to either of its neighbors (this increases the quantities on the left hand sides);

(ii) Local symmetry, i.e. the distribution of rankings treats the two neighbors as nearly identical (this decreases the quantities on the right hand sides).

5 Unpredictability

This section studies the predictability of outcomes in combinatorial elections. Since the distribution μ over types is fixed, there is no aggregate uncertainty in large elections about the proportion of types in the population. Nevertheless, we prove that the outcomes of large elections remain uncertain for a nontrivial set of type distributions. This unpredictability is not an artifact of primitive statistical uncertainty, but rather is necessary to maintain incentives in equilibrium.[8]

The unpredictability of outcomes under combinatorial rule is qualitatively distinct from the indeterminacy of equilibrium under plurality rule. Under plurality rule, there are multiple limit equilibria where different pairs of candidates are active. But for any *selection* of a specific equilibrium, the outcome is generically certain. In contrast, the unpredictability of outcomes under combinatorial rule is not due to multiplicity of equilibria. Rather, for any fixed limit equilibrium, the probability of an issue passing must be uniformly bounded away from 0 or 1 in large elections.

Unpredictability also contrasts nonseparable preferences from separable preferences in our model. Within the class of distribution with separable support, predictable outcomes are relatively generic. Excepting knife-edge distributions where voters are equally likely to prefer an issue's passage or its failure, the outcome of each issue is certain in large elections. Unpredictability in election outcomes is therefore difficult to reconcile with a model of costless voting with private separable values. With nonseparable preferences, the predictability of large elections depends on the type distribution μ.

We will focus on the following set of densities for the setting with two issues.

Example 1. Pick some small $\varepsilon > 0$.[9] Consider the class of densities $\mathcal{C}$ which satisfy the following restrictions:

$$\frac{1-\varepsilon}{4} < \mu(\theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) < \frac{1}{4}$$
$$\frac{1-\varepsilon}{4} < \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) < \frac{1}{4}$$
$$\frac{1-\varepsilon}{4} < \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) < \frac{1}{4}$$
$$\frac{1-\varepsilon}{4} < \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) < \frac{1}{4}$$

This class is open and nonempty.[10]

For all sequences of equilibria for all distributions in $\mathcal{C}$, the probability that either issue will pass is uniformly bounded away from 0 or 1. In other words, even given the exact distribution $\mu \in \mathcal{C}$ and an arbitrarily large number of voters that are independently drawn from that distribution, an observer would not be able to predict the outcome of either issue.

[8] In a two-period version of their model with aggregate uncertainty regarding the distribution of preference, Alesina and Rosenthal (1996) predict uncertain presidential winners for a nontrivial range of parameters. However, this assumes primitive uncertainty on the distribution of preferences. The unpredictability disappears in their basic model where the distribution of preferences is common knowledge.

[9] In fact, ε can be as large as $\frac{1}{16}$.

[10] It is open in both the sup and weak convergence topologies.

Proposition 8. *For every density in $\mathcal{C}$, all convergent sequences of equilibria exhibit unconditional uncertainty on both issues.*

There are two key features regarding the majority preference ranking in the example. The first feature is that a Condorcet cycle exists. The second feature is that the majority preference is not ordinally separable; a majority prefers issue 2 to pass if issue 1 were to pass ($\mu(\theta_{12} \geq \theta_1) > \frac{1}{2}$), but also prefer issue 2 to fail if issue 1 were to fail ($\mu(\theta_\emptyset \geq \theta_2) > \frac{1}{2}$). We will explore the extent to which transitivity and separability of the majority preference ensure predictability in Section 6.

The proof of Proposition 8 proceeds in two major steps. First, we establish that every density in $\mathcal{C}$ exhibits conditional uncertainty on both issues. Second, we demonstrate that conditional uncertainty on both issues implies unconditional uncertainty on both issues.

The proof of the first step begins by showing that there must be at least one conditionally uncertain issue. This is straightforward given our prior results. Proposition 7 provides a necessary inequality for a bundle A to be conditional certain. This inequality is violated for every bundle A by the construction of $\mathcal{C}$: the fraction $\frac{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)}{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}$ is large because the denominator $\mu(\theta_2 \geq \theta_{12} \geq \theta_1)$ is less than ε.

This forces conditional uncertainty on at least one issue. The proof of the first step proceeds by arguing that conditional uncertainty on one issue implies conditional uncertainty on the other. For example, suppose issue 1 is conditionally uncertain while issue 2 is conditionally certain to pass. We invoke Lemma 1 to parametrize the equilibrium decision to support issue 1 by some probability $\alpha \in (0, 1)$. In particular, in large elections a type θ will support issue 1 if and only if

$$\alpha\theta_{12} + (1 - \alpha)\theta_2 \geq \alpha\theta_1 + (1 - \alpha)\theta_\emptyset.$$

This parameterization identifies the subsets of types who support or oppose issue 2 in large elections. For example, any type θ where $\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset$ must vote support issue 2 regardless of the value of α, because $\theta_{12} \geq \theta_1$ and $\theta_2 \geq \theta_\emptyset$. We can also use Proposition 5 to identify the subsets of types who support or oppose issue 1. Namely, θ supports issue 1 if $\theta_{12} \geq \theta_2$, and opposes issue 1 otherwise. We can then bound these probabilities using the construction of the example, e.g. the probability that $\theta_{12} \geq \theta_2$ is at most $\frac{1}{4} + \varepsilon$. However, we can also use Proposition 6 to show that issue 2 is conditionally certain to pass only if certain inequalities on these probabilities hold. These required inequalities are precluded by the bounds we obtained after characterizing the limit equilibrium with α. Considering different cases yields similar contradictions. Therefore, we conclude that conditional uncertainty on one issue must be accompanied by conditional uncertainty on the other.

The first step proved that there must be *conditional* uncertainty on both issues when a voter is pivotal on the other. But the relevant uncertainty regards the *unconditional* uncertainty. The second step links the two uncertainties: in particular conditional uncertainty on both issues is equivalent to unconditional uncertainty on both issues. To provide some intuition for the equivalence, recall that assuming conditional uncertainty on both issues imposes restrictions on the limits of conditional probabilities, such as the probability issue 1 will pass when a voter is pivotal for issue 2. By reapplying the artificial conditioning argument used in the proof of Proposition 6 to work around the conditional dependence of the ballots, we can apply the central limit theorem to the conditional distribution of the vote count on issue 1. Therefore, the conditional vote count on issue 1 passing can be approximated by a normal cumulative distribution function. For there for to be conditional uncertainty on issue 1, the conditional probability that a random voter supports issue 1 when there is a split on issue 2 must converge to one half at rate faster than $\sqrt{I-1}$:

$$\sqrt{I-1}\left|\frac{1}{2}\mu(1 \in s_I^*(\theta) \,|\, 2 \in s_I^*(\theta)) + \frac{1}{2}\mu(1 \in s_I^*(\theta) \,|\, 2 \notin s_I^*(\theta)) - \frac{1}{2}\right| < \infty,$$

otherwise the distribution function will collapse too quickly and will be degenerately 0 or 1 at one half. A similar rate of convergence must hold for the conditional probability of supporting issue 2 given a split on issue 1.

To move from the conditional probabilities to the unconditional probabilities, observe that the unconditional probability of a voter supporting issue 1 can be written as a convex combination of the two conditional probabilities given her vote, either up or down, on issue 2. This can also be viewed as a linear equation. A similar linear equation can be written for the unconditional probability of a voter supporting issue 2 as a function of conditional probabilities. Jointly, the pair defines a system of two linear equations with two unknowns, namely the unconditional probabilities. The coefficients of the system are given by the conditional probabilities. The resulting solutions for the unconditional probabilities imply that root convergence to one half for both conditional probabilities is equivalent to root convergence to one half for both unconditional probabilities. In fact, conditional uncertainty on both issues is that on the only case where this conversion is possible, because this guarantees that the coefficients of the linear system are finite.

In general, characterizing unconditional uncertainty and limit outcomes faces two obstacles. First, a full characterization of the relevant limits would involve not only deciding whether there is convergence of equilibrium probabilities, but rather controlling the *rate* of this convergence. Moreover, while the characterizations of Section 4 are helpful in controlling the the *conditional* uncertainty, we are ultimately interested in the *unconditional* uncertainty. While there is a tight connection between these concepts when both issues are uncertain, this connection is lost in all other cases. For example, it is possible for both issues to be unconditionally certain but to have conditional uncertainty on one issue.

While a complete characterization is still outstanding, additional assumptions on the type distribution make it possible to provide simple and easily verifiable conditions which guarantee whether an issue is conditionally certain to pass or fail. One such assumption is supermodularity. This is because, in the two-issue case, supermodularity is equivalent to the inequality $\theta_{12}+\theta_\emptyset \geq \theta_1+\theta_2$. For example, suppose $\theta_1 \geq \theta_\emptyset$. Then, to maintain supermodularity, it must be that $\theta_{12} \geq \theta_2$. But such a voter must support issue 1, because its passage is beneficial regardless of whether issue 2 passes (since $\theta_{12} \geq \theta_2$) or fails (since $\theta_1 \geq \theta_\emptyset$). So if more than half the types satisfy $\theta_1 \geq \theta_\emptyset$, then more than a majority will support issue 1 in large elections. Similarly, if $\theta_2 \geq \theta_{12}$, then supermodularity forces $\theta_\emptyset \geq \theta_1$, so voting down on issue 1 is dominant.

Proposition 9. *Suppose the support of μ is the set of supermodular types. Then the following hold for $x = 1, 2$:*

(i) *If $\mu(\theta_x \geq \theta_\emptyset) > \frac{1}{2}$, then issue x is unconditionally certain to pass.*

(ii) *If $\mu(\theta_{x'} \geq \theta_{12}) > \frac{1}{2}$, then issue x is unconditionally certain to fail.*

Note that the asymmetry in the tests in parts (i) and (ii) for being conditional to pass and to fail is due to the definition of supermodularity. An analogous version of Proposition 9 can be proven for submodular preferences.

The other assumptions which are useful in guaranteeing predictability are restrictions on the majority rule preference, which is the subject of the next section.

6 Condorcet orders and combinatorial rule

In this section, we examine the Condorcet consistency of majority rule with two issues. We make two points at the outset. First, since we assume independent private values, any Vickrey-Clarke-Groves mechanism will implement the utilitarian outcome in dominant strategies. However, the communication demands and the implied transfers of such mechanisms seem impracticable in many situations. Given a restriction to mechanisms where the message space is equivalent to the outcome space (such as in combinatorial or plurality rule), ordinal efficiency is the most that can be achieved.

Second, a distinguishing feature of combinatorial rule is its dependence on the structure of the power set of bundles. This manifests itself in the earlier characterizations, which all apeal exclusively to the relationship between a bundle A and its neighbors A' and A''. Consequently, the more useful concept is not whether a bundle A is preferred by a majority to all other bundles, but whether A is preferred by a majority to its neighboring bundles.

We now define the Condorcet order, or majority rule preference. The maximal and minimal bundles of this order are the Condorcet winner and loser. In addition, in combinatorial voting, local comparisons with neighboring bundles are particularly important. So, we also define a local Condorcet winner as a bundle which is preferred by a majority to its neighbors; an analogous notion defines a local Condorcet loser.

Definition 6. The **Condorcet order** $\succ_{\mathcal{C}}$ on $\mathcal{X}$ is defined by: $A \succ_{\mathcal{C}} B$ if $\mu(\theta_A \geq \theta_B) > \frac{1}{2}$.[11]

The bundle $A \in \mathcal{X}$ is a **Condorcet winner** $A \succ_{\mathcal{C}} B$ for all $B \neq A$. It is a **local Condorcet winner** if $A \succ_{\mathcal{C}} B$ for $B = A', A''$.

The bundle $A \in \mathcal{X}$ is a **Condorcet loser** if $B \succ_{\mathcal{C}} A$, for all $B \neq A$. It is a **local Condorcet winner** if $B \succ_{\mathcal{C}} A$ for $B = A', A''$.

[11]The inequality $\theta_A \geq \theta_B$ is weak to maintain notational consistency with the rest of the paper. Given the density assumption on μ, it is equivalent to the strict version, i.e. $A \succ_{\mathcal{C}} B$ if and only if $\mu(\theta_A > \theta_B) > \frac{1}{2}$.

The Condorcet criterion for ordinal efficiency is that an electoral mechanism implements the Condorcet winner whenever such a winner exists. Under plurality rule, there exists a limit equilibrium of plurality rule that selects the Condorcet winner. As the example in the introduction illustrates, there is no such guarantee under combinatorial rule. We now examine conditions which determine whether the Condorcet winner is implemented under combinatorial voting.

Proposition 10. *Suppose A is a local Condorcet winner. Suppose there exists $x = 1, 2$ such that either of the following statements holds:*

(i) *$x \in A$ and x is conditionally certain to pass, or*

(ii) *$x \notin A$ and x is conditionally certain to fail.*

Then A is unconditionally certain.

Moreover, if A is unconditionally certain, then there exists $x = 1, 2$ such that either (i) or (ii) holds.

So, to implement the Condorcet winner, it suffices for either issue to conditionally agree with the Condorcet winning bundle. In addition, a local Condorcet winner fails to be implemented if, and only if, either both issues are conditionally uncertain or one issue conditionally disagrees with A. In other words, if a local Condorcet winner is not the limit outcome, then either there is conditional uncertainty on at least one issue, or its complement be conditionally certain. For example, if $\{1, 2\}$ is the Condorcet winner but is not the limit outcome, it must be the case that neither issue 1 nor issue 2 is conditionally certain to pass. In the example in the introduction, this is indeed the case, since $\{1, 2\}$ is the Condorcet winner while the bundle $\emptyset$ is conditionally certain. A special case of Proposition 10 is that if a Condorcet winner is conditionally certain, then it is also unconditionally certain. Then the primitive conditions for the conditional certainty of A provided in Proposition 7 and Lemma 2 also suffice as primitive conditions for its *unconditional* uncertainty when A is a local Condorcet winner. In particular, by part (iii) of Lemma 2, if a local Condorcet winner A enjoys a sufficiently large electoral advantage over its neighbors and the type distribution μ is sufficiently symmetric across neighbors, then A is a limit outcome.

An appealing feature of the limit equilibria when the Condorcet winner is conditionally certain is that they require less strategic sophistication by the voters. This is because the same bundle, namely the Condorcet winner, is both conditionally and unconditionally certain. So even if voters fail to condition their strategies on being pivotal, they will still submit their equilibrium ballots as long as their conjectures regarding the unconditional limit outcome are correct. More precisely, consider a voter who somewhat naively votes up on issue 1 because she expects issue 2 to pass, without correcting for being pivotal on issue 1 or understanding the correlation in votes across issues 1 and 2. Such a voter will nonetheless submit her suggested equilibrium ballot.

We mentioned that plurality rule always has one limit equilibrium which enacts the Condorcet winner. However, plurality rule also yields other equilibria which fail to pass the Condorcet winner, e.g. when the Condorcet winner is not one of the two active candidates. This logic for inefficiency also exists under combinatorial rule. The assumption in Proposition 10 is that the bundle A is a *local* Condorcet winner. There can be two local Condorcet winners. For example, if $\succ_{\mathcal{C}}$ is transitive and $\{1,2\} \succ_{\mathcal{C}} \emptyset \succ_{\mathcal{C}} \{1\} \succ_{\mathcal{C}} \{2\}$, then both $\{1,2\}$ and $\emptyset$ are local Condorcet winners. Consequently, there can be two limit equilibria under combinatorial rule, one which selects $\{1,2\}$ and another which selects $\emptyset$, even though a majority prefers $\{1,2\}$ to $\emptyset$.

Because plurality rule always yields an efficient limit equilibrium, Condorcet inconsistency under plurality rule is a consequence of miscoordination or bad equilibrium selection. In contrast, combinatorial voting can fail to yield *any* limit equilibria that selects the Condorcet winner. So combinatorial rule generates other factors beyond miscoordination which exclude the existence of any efficient limit equilibria and can lead to ordinal inefficiency.

Another criterion for ordinal efficiency is that, when a Condorcet loser exists, it is not the outcome of the election. When a Condorcet loser exists, it cannot be a limit outcome of plurality rule for any sequence of equilibria. For combinatorial voting, the Condorcet loser is generically not a limit outcome.

Proposition 11. *If A is a local Condorcet loser and $\succ_{\mathcal{C}}$ is complete, then A is not a limit outcome.*

The Condorcet order $\succ_{\mathcal{C}}$ is generically complete. Therefore, Proposition 11 proves that, for a generic set of distributions, a Condorcet loser cannot be the determinate outcome of the election. However, this result leaves open whether a Condorcet loser can have strictly positive probability of being enacted; Proposition 11 only proves that this probability is generically less than one.

Recall Example 1, which is an open set of distributions with unpredictable election outcomes, had two features. The first is the lack of a Condorcet winner. The second is that the Condorcet order was not separable; a majority preferred issue 2 to pass if issue 1 were to pass, but also preferred issue 2 if issue 1 were to fail. We now examine whether excluding these two pathologies, cyclicity and nonseparability of the Condorcet order, generates predictability.[12] We first introduce an ordinal definition of separability.

Definition 7. A binary relation $\succ$ on $\mathcal{X}$ is **quasi-separable** if

$$A \succ A \cap B \Longleftrightarrow A \cup B \succ B.$$

Under either the assumption of conditional certainty or the assumption of supermodularity, quasi-separability of $\succ_C$ guarantees that the Condorcet winner the implemented when it exists. Consequently, these results also provide sufficient conditions for predictability.

When $\succ_{\mathcal{C}}$ is a quasi-separable weak order, conditional certainty of any bundle, even a bundle that is not the Condorcet winner, is sufficient for the Condorcet winner to be the outcome of the election.

[12] An open question is whether the existence of a Condorcet winner, by itself, implies predictability. Note that this is different than the question of Condorcet consistency; in the introductory example the Condorcet winner $\{1,2\}$ fails but the outcome of $\emptyset$ is still predictable.

Proposition 12. *Suppose $\succ_C$ is quasi-separable and there is conditional certainty on both issues. If a Condorcet winner A exists, then A is conditionally certain and A is a limit outcome.*[13]

There are two observations in proving Proposition 12. For concreteness, take the case where $\{1,2\}$ is the Condorcet winner. This first observation is that, if the Condorcet ranking is quasi-separable and $\{1,2\}$ is the Condorcet winner, then its complement $\emptyset$ must be the Condorcet loser. The second observation is that a Condorcet loser cannot be conditionally certain. Then since both issues cannot be conditionally certain to fail, one issue must be conditionally certain to pass. By Proposition 10, this is enough to insure that $\{1,2\}$ is conditionally certain.

Another condition which pairs with quasi-separability to ensure Condorcet consistency is common knowledge of supermodularity. So, if all voters agree that the issues are complements, then a quasi-separable $\succ_C$ suffices to make the Condorcet winner a limit outcome of the election. In this case, we can also conclude that the Condorcet winner is the unique outcome of the election, across all limit equilibria.

Proposition 13. *Suppose $\succ_C$ is quasi-separable and the support of μ is the set of supermodular (or submodular) types. If a Condorcet winner A exists, then it is the unique limit outcome.*

To obtain some intuition for Proposition 13, consider the case where $A = \{1,2\}$ is the Condorcet winner. Then quasi-separability of $\succ_C$ implies that its complement $\emptyset$ is the Condorcet loser. So, $\{1\} \succ_C \emptyset$, i.e. $\mu(\theta_1 \geq \theta_\emptyset) > \frac{1}{2}$. Since we assumed supermodularity, Proposition 9 implies that issue 1 is unconditionally certain to pass. Arguing similarly, issue 2 is also unconditionally certain to pass. Thus $\{1,2\}$ must be the limit outcome of the election.

A corollary of Propositions 12 and 13 is that if the Condorcet order is a quasiseparable weak order, then either conditional certainty or supermodularity would guarantee predictability of large elections. We conjecture that either assumption is dispensable, in other words that quasiseparability and transitivity of the majority preference are sufficient for predictability. However, the difficulties in proving this are similar to those in completely characterizing limit outcomes: the need to control rates of convergence and the need to pass these rates from conditional to unconditional probabilities.

7 Conclusion

This paper introduced and analyzed a model of elections with nonseparable preferences over multiple issues. We provided topological and lattice-theoretic proofs for existence of equilibrium. We characterized limit equilibria for large elections. With these characterizations, we showed that the predictability of elections outcomes is not generic and provided sufficient conditions for predictability. While the Condorcet winner is not generally the outcome of the election, we provided sufficient conditions for its implementation. We conclude by posing some open questions that have not yet been mentioned.

[13] We cannot conclude that the Condorcet winner is the unique limit outcome because there could be other limit equilibria without conditional certainty.

Multi-issue elections induce a political exposure problem which is analogous to the exposure problem in multi-unit auctions. This exposure problem was at the core of our strategic analysis. We assumed voters understand this exposure, and that they conditioned the exposure on being pivotal for some issue. Alternative assumptions regarding the sophistication of voters could generate different predictions.

Our results regarding predictability and ordinal efficiency were restricted to the setting with two issues. While two issues were enough to construct the negative counterexamples, it remains open whether our positive results have analogs with three or more issues. We suspect there is an extension of our methods, that compared bundles that are connected in the set-containment lattice, to more general settings.

In conjunction with existing results on large plurality elections, our findings provide an initial comparison of combinatorial rule and plurality rule. Plurality rule always has an equilibrium which selected a Condorcet winner, but the multiplicity of equilibria presents voters with a coordination problem. Like plurality rule, combinatorial rule can also present voters with a coordination problem among equilibria. More distinctively however, combinatorial rule can fail to have any equilibria which implement the Condorcet winner. On the other hand, in some cases the Condorcet winner is the unique limit outcome across equilibria. We feel like our general understanding of the comparison, especially across the general space of distributions, is still incomplete. That said, the worst-case distributions for combinatorial rule can be very inefficient in terms of aggregate utility.

This perhaps begs why combinatorial voting is so pervasive despite its potential inefficiency. We took the set of issues on the ballot as exogenous. In reality, the set of referendums or initiatives is a consequence of strategic decisions by political agents. For example, a new substitute measure can be introduced to siphon votes away from an existing measure. Or, two complementary policies can be bundled as a single referendum. The persistence of combinatorial rule might be due to the considered introduction or bundling of issues. If agents anticipate the electoral consequences of their decisions, our model provides a first step in the analysis of strategic ballot design.

A Appendix

A.1 Proof of Proposition 1

We first verify that every undominated strategy assigns an open set of types to each ballot. Additive separability is too strong for this purpose because the set of additively separable types is Lebesgue null. This motivates the following weaker notion of separability:

Definition 8. θ_i is **quasi-separable** if

$$\theta_i(A) \geq [>]\theta_i(A \cap B) \Longleftrightarrow \theta_i(A \cup B) \geq [>]\theta_i(B).$$

When θ_i is quasi-separable, the voter's preference for whether any issue x is voted up or down is invariant to which of the other issues in $A \setminus \{x\}$ pass or fail. The following observation is also made by Lacy and Niou (2000, Result 4).

Lemma 3. *Suppose θ_i is quasi-separable and $A_i^*(\theta_i) = \arg\max_{A\in\mathcal{X}} \theta_i(A)$ is unique. Then $s_i(\theta_i) = A_i^*(\theta_i)$ whenever s_i is weakly undominated.*

Proof. Suppose θ_i is quasi-separable and $A_i^* = A_i^*(\theta_i)$ is unique. We first prove that if $x \in A_i^*$, then s_i is weakly dominated whenever $x \notin s_i(\theta_i)$. Since $\theta_i(A_i^*) > \theta_i(A_i^* \setminus \{x\})$, we have $\theta_i(\{x\}) > \theta_i(\emptyset)$. Consider any strategy s_i where $x \notin s_i(\theta_i)$. Compare this to the strategy $s_i'(\theta_i) = s_i(\theta_i) \cup \{x\}$ for θ_i and equal for all other types. Now, for any fixed ballot profile A_{-i} for the other voters, either i is pivotal for issue x or is not. If not, then the same set of issues passes under both strategies, so there is no loss of utility to θ_i. If she is pivotal on x, then the set of issues $F(s_i(\theta_i), A_i) \cup \{x\}$ passes, which leaves her strictly better off by quasi-separability. So s_i is weakly dominated.

Similarly, if $x \notin A_i^*$, then s_i is weakly dominated whenever $x \in s_i(\theta_i)$. Therefore $A_i^* = s_i(\theta_i)$ for every weakly undominated strategy s_i. □

We now begin the proof of existence. We endow each voter's strategy space S_i with the topology induced by the following distance: $d(s_i, s_i') = \mu_i(\{\theta_i : s_i(\theta_i) \neq s_i'(\theta_i)\})$ and endow the space of strategy profiles S with the product topology.[14] For a fixed strategy profile s, let the function $G^{s_{-i}} = (G_0^{s_{-i}}, G_+^{s_{-i}}) : \Theta_{-i} \to \mathcal{X} \times \mathcal{X}$ be defined by

$$G_0^{s_{-i}}(\theta_{-i}) = \left\{x \in X : \#\{j \neq i : x \in s_j(\theta_j)\} = \frac{I-1}{2}\right\},$$

i.e. the set of issues where voter i is pivotal, and

$$G_+^{s_{-i}}(\theta_{-i}) = \left\{x \in X : \#\{j \neq i : x \in s_j(\theta_j)\} > \frac{I-1}{2}\right\},$$

i.e. the set of issues which pass irrespective of voter i's ballot. Then, for type θ_i, her utility for a fixed ballot profile $(A_1, \ldots, A_I)$ is $\theta_i([A_i \cap G_0^{A_{-i}}] \cup G_+^{A_{-i}})$, i.e. the union of two sets: first, the set of issues where she is pivotal and she votes up; second, the set of issues which are passed irrespective of her ballot. Let $\mathcal{D} \subseteq \mathcal{X} \times \mathcal{X}$ denote the space of ordered disjoint pairs of sets of issues, $\mathcal{D} = \{(C, D) \in \mathcal{X} \times \mathcal{X} : C \cap D = \emptyset\}$. For a type θ_i, her expected utility for a strategy profile s is

$$\sum_{(C,D)\in\mathcal{D}} \theta_i([s_i \cap C] \cup D) \times \mu_{-i}\left([G^{s_{-i}}]^{-1}(C, D)\right).$$

An important observation is that the type's expected utility for a ballot depends only on her belief about for which issues she will be pivotal and which issues will pass irrespective of her ballot. Let $\Delta\mathcal{D}$ denote the probability distributions over $\mathcal{D}$. For $P, P' \in \Delta\mathcal{D}$, define the sup metric $\|P - P'\| = \max_{(C,D)\in\mathcal{D}} |P(C, D) - P'(C, D)|$.

Define the probability $\pi_i(s) \in \Delta\mathcal{D}$ by

$$[\pi_i(s)](C, D) = \mu_{-i}\left([G^{s_{-i}}]^{-1}(C, D)\right). \tag{2}$$

In words, $[\pi_i(s)](C, D)$ is the probability, from voter i's perspective, that she will be pivotal on the issues in C and that the issues in D will pass no matter how she votes, given that the strategy s is being played by the other voters. Fix $(C, D) \in \mathcal{D}$. If s_j is weakly undominated, by Lemma 3 there exists some quasi-separable type θ_j for whom $s_j(\theta_j) = D$. Moreover, these conditions are satisfied in an open neighborhood U^D of θ_i. By the full support assumption, there is some strictly positive probability $\mu_j(U^D) > 0$ of a type for j

[14] To be precise, this is defined over equivalence classes of strategies whose differences are μ_i-null.

with $s_j(U^D) = D$, and similarly $\mu_j^{C \cup D}$ of a set of types $U^{C \cup D}$ for which $s_j(U^{C \cup D}) = C \cup D$. Enumerate $I \setminus \{i\} = \{j_1, \ldots, j_{I-1}\}$. By independence of μ across voters, for any weakly undominated strategy profile the joint probability that D is submitted for the first $\frac{I-1}{2}$ other voters and $C \cup D$ is submitted by the second $\frac{I-1}{2}$ other voters is at least

$$L_i(C, D) = \prod_{k=1}^{\frac{I-1}{2}} \mu_{j_k}^D \times \prod_{k=\frac{I+1}{2}}^{I-1} \mu_{j_k}^{C \cup D} > 0.$$

Thus $[\pi_i(s)](C, D) \geq L_i(C, D)$ for all $(C, D) \in \mathcal{D}$, whenever s is weakly undominated. Let

$$L = \min\{L_i(C, D) : i \in I, (C, D) \in \mathcal{D}\}$$

and define the following compact convex subset of probabilities:

$$\Delta^U = \left\{ P \in \Delta\mathcal{D} : \min_{(C,D) \in \mathcal{D}} P(C, D) \geq L \right\}.$$

So, letting S^U denote the space of weakly undominated strategy profiles, we can consider the function $\pi_i : S^U \to \Delta^U$. By independence of μ,

$$\begin{aligned} [\pi_i(s)](C, D) &= \mu_{-i}\left([G^{s_{-i}}]^{-1}(C, D)\right) \\ &= \sum_{\{A_{-i} \in \mathcal{X}^{I-1} : G_i^{A_{-i}} = (C,D)\}} \mu_{-i}(\{\theta_{-i} : s_{-i}(\theta_{-i}) = A_{-i}\}) \\ &= \sum_{\{A_{-i} \in \mathcal{X}^{I-1} : G_i^{A_{-i}} = (C,D)\}} \left[\prod_{j \neq i} \mu_j(\{\theta_j : s_j(\theta_j) = [A_{-i}]_j\}) \right]. \end{aligned}$$

The last expression is a sum of products of probabilities $\mu_j(\{\theta_j : s_j(\theta_j) = [A_{-i}]_j\})$ which, considered as functions dependent on S^U, are immediately continuous in the defined topology on S^U. Hence π_i is continuous. Then the function $\pi : S^U \to [\Delta^U]^I$ defined by $\pi(s) = (\pi_1(s), \ldots, \pi_I(s))$ is continuous.

Fix a belief $P_i \in \Delta^U$. Then the set of types for voter i for which it is optimal to submit the ballot A_i is defined by

$$A_i(P_i) = \bigcap_{A_i' \in \mathcal{X}} \left\{ \theta_i : \sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i(C, D) \geq \sum_{\mathcal{D}} \theta_i([A_i' \cap C] \cup D) \times P_i(C, D) \right\}.$$

Fix an enumeration $\mathcal{X} = \{A^1, \ldots, A^{|\mathcal{X}|}\}$ and define the function $\sigma_i : \Delta\mathcal{D} \to S$ as follows. Let A^0 denote the set of types which are not quasi-separable or do not have a unique $\arg\max_{A \in \mathcal{X}} \theta_i(A)$. For all $\theta_i \in A^k(P_i) \setminus [A^0 \cup \ldots A^{k-1}]$, let $[\sigma_i(P_i)](\theta_i) = A^k$.[15] Since P_i is in the interior of $\Delta\mathcal{D}$, $\sigma_i(P_i)$ is not weakly dominated: $\sigma_i(P_i) \in S_i^U$. Observe that the set of types θ_i which play A_i is a convex polytope (with open and closed faces).

We now prove that $\sigma_i : \Delta^U \to S_i^U$ is continuous. Since $P_i \in \Delta^U$ is strictly bounded away from zero, the set of types which have multiple optimal ballots given belief P_i is of strictly lower dimension than Θ_i, hence μ_i-null since μ_i admits a density. Then $\mu_i(A_i(P_i) \setminus [\sigma(P_i)]^{-1}(A_i)) = 0$, so it suffices to show that $\mu_i(A_i(P_i))$

[15]This construction is to avoid ambiguous assignments on the μ_i-null set of types with multiple optimal ballots given P_i. Alternatively, one can consider the space S modulo differences of μ-measure zero, in which case the ambiguous assignment is irrelevant.

is continuous in P_i. Fix $\varepsilon > 0$. The set $A_i(P_i)$ is nonempty, since there exists a nonempty neighborhood of quasi-separable types which submit A_i in any undominated strategy. By outer regularity of μ_i, the probability of the closed set $A_i(P_i)$ is arbitrarily well approximated by the probabilities of its neighborhoods (cf. Parthasarathy 1967, Theorem 1.2), i.e. there exists some δ-neighborhood of $A_i(P_i)$, denoted $U_\delta[A_i(P_i)]$, such that $\mu_i(U_\delta[A_i(P_i)]) < \mu_i(A_i(P_i)) + \varepsilon$. Moreover, there exists a sufficiently small $\gamma > 0$ such that if, for all $A'_i \in \mathcal{X}$,

$$\sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i(C, D) \geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P_i(C, D) - \gamma,$$

then $\theta_i \in U_\delta[A_i(P_i)]$; this because both sides of the inequality are continuous in θ_i. Suppose $\|P_i - P'_i\| < \gamma/2$. The difference in expected utility for any action across the two probabilities is bounded by $\gamma/2$, since values were normalized to live in the unit interval. Then, fixing $\theta_i \in A_i(P'_i)$, i.e. a type θ_i for whom A_i is optimal given conjecture P'_i, we have for all $A'_i \in \mathcal{X}$:

$$\begin{aligned}
\sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P_i(C, D) &\geq \sum_{\mathcal{D}} \theta_i([A_i \cap C] \cup D) \times P'_i(C, D) - \gamma/2 \\
&\geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P'_i(C, D) - \gamma/2 \\
&\geq \sum_{\mathcal{D}} \theta_i([A'_i \cap C] \cup D) \times P_i(C, D) - \gamma.
\end{aligned}$$

So, $A_i(P'_i)$ is contained in $U_\delta[A_i(P_i)]$. Then $\mu_i(A_i(P'_i) \setminus A_i(P_i)) \leq \mu_i(U_\delta[A_i(P_i)] \setminus A_i(P_i)) < \varepsilon$. Similarly, there exists a sufficiently small distance $\gamma' > 0$ such that if $|P_i - P'_i| < \gamma'$, then $\mu_i(A_i(P_i) \setminus A_i(P'_i)) < \varepsilon$. But

$$\begin{aligned}
\mu_i\left(\{\theta_i : [\sigma_i(P_i)](\theta_i) \neq [\sigma_i(P'_i)](\theta_i)\}\right) &\leq \sum_{A_i \in \mathcal{X}} \mu_i(A_i(P_i) \,\Delta\, A_i(P'_i)) \\
&= \sum_{A_i \in \mathcal{X}} \left(\mu_i(A_i(P'_i) \setminus A_i(P_i)) + \mu_i(A_i(P_i) \setminus A_i(P'_i))\right) \\
&< 2|\mathcal{X}|\varepsilon
\end{aligned}$$

whenever $\|P_i - P'_i\| < \min\{\gamma/2, \gamma'\}$. So the function $\sigma : [\Delta^U]^I \to S^U$ defined by $\sigma(P_1, \dots, P_I) = (\sigma_1(P_1), \dots, \sigma_I(P_I))$ is continuous.

Then the composition $\pi \circ \sigma : [\Delta^U]^I \to S^U \to [\Delta^U]^I$ is continuous, hence yields a fixed point $(P_1^*, \dots, P_I^*)$ by Brower's Theorem. Then $s^* = \sigma(P_1^*, \dots, P_I^*)$ is, by construction, a best response to itself, hence the desired equilibrium in weakly undominated strategies.

A.2 Proof of Proposition 3

We maintain the notation from the proof of Theorem 1. Let s^0 denote the sincere voting profile, where $s_i^0(\theta_i) \in \arg\max_A \theta_i(A)$. By full support assumption, for all A and i, there is a strictly positive measure of types θ_i where the sincere ballot is A. Hence the induced probability $P_i^0 = \pi_i(s^0)$ is in the interior of $\Delta\mathcal{D}$. Therefore $P_i^0(C, D) > 0$. Consider θ_i with $\theta_i(\emptyset) = 1$ and $\theta_i(\{1\}) = 0$, and $\theta_i(A) = 1 - \delta$ whenever $A \neq \emptyset, \{1\}$. When $2 \in C$ and $D = \{1\}$

$$\theta_i([\{2\} \cap C] \cup \{1\}) - \theta_i([\emptyset \cap C] \cup \{1\}) = \theta_i(\{1, 2\}) - \theta_i(\{1\}) = 1 - \delta.$$

For all other (C, D), the difference $\theta_i([\{2\} \cap C] \cup D) - \theta_i([\emptyset \cap C] \cup D)$ is either 0 or $-\delta$. Since P_i^0 has full support, we have

$$\sum_{\mathcal{D}} \theta_i([\{2\} \cap C] \cup D) \times P_i^0(C, D) > \sum_{\mathcal{D}} \theta_i([\emptyset \cap C] \cup D) \times P_i^0(C, D)$$

for sufficiently small $\delta > 0$. So submitting the ballot $\{2\}$ is a strictly better reply than the sincere ballot $\emptyset$ for this θ_i. This property is maintained in a neighborhood of θ_i, so by the full support assumption sincere voting is not a best reply for a strictly positive measure of types.

A.3 Proof of Proposition 4

We use recent results due to Reny (2009). Namely, we will verify the assumptions of Theorem 4.1 and 4.2, which we summarize in the following statement.

Theorem 1 (Reny 2008)**.** *Suppose that, for every player i:*

G.1 Θ_i is a complete separable metric space endowed with a measurable partial order

G.2 μ_i assigns probability zero to any Borel subset of T_i having no strictly ordered points

G.3 A_i is a compact locally-complete metric semilattice

G.4 $u_i(\cdot, \theta) : A \to \mathbb{R}$ is continuous for every $\theta \in \Theta$.

For each i, let C_i be a join-closed, piecewise-closed, and pointwise-limit-closed subset of pure strategies containing at least one monotone pure strategy, such that the intersection of C_i with i's set of monotone best replies is nonempty whenever every other player j employs a monotone pure strategy in C_j. Then there exists a monotone (pure strategy) equilibrium in which each player i's pure strategy is in C_i.

We first show that the election is weakly quasi-supermodular and obeys single-crossing in $\geq$, which will be useful later.

Lemma 4. *The voting game is weakly quasi-supermodular in actions, i.e.*

$$\int_{\Theta_{-i}} \theta_i(A_i, s_{-i}(\theta_{-i}))d\mu_{-i} \geq \int_{\Theta_{-i}} \theta_i(A_i \cap B_i, s_{-i}(\theta_{-i}))d\mu_{-i}$$
$$\implies \int_{\Theta_{-i}} \theta_i(A_i \cup B_i, s_{-i}(\theta_{-i}))d\mu_{-i} \geq \int_{\Theta_{-i}} \theta_i(A_i, s_{-i}(\theta_{-i}))d\mu_{-i}$$

Proof. We show that supermodularity *in outcomes* of the ex post utilities implies weak quasi-supermodularity *in actions* of the interim utilities. So, suppose the hypothesis inequality holds. Carrying the notation from the proof of Theorem 1, this can be rewritten as

$$\sum_{C,D \in \mathcal{D}} \theta_i([A_i \cap C] \cup D) \times [\pi_i(s)](C, D) \geq \sum_{C,D \in \mathcal{D}} \theta_i([A_i \cap B_i \cap C] \cup D) \times [\pi_i(s)](C, D).$$

Applying supermodularity of θ_i to the sets $[A_i \cap C] \cup D$ and $[B_i \cap C] \cup D$:

$$\sum_{C,D \in \mathcal{D}} [\theta_i([A_i \cap C] \cup D) - \theta_i([A_i \cap B_i \cap C] \cup D)] \times [\pi_i(s)](C, D) \geq 0$$

implies

$$\sum_{C,D\in\mathcal{D}}[\theta_i([(A_i\cup B_i)\cap C]\cup D)-\theta_i([A_i\cap C]\cup D)]\times[\pi_i(s)](C,D)\geq 0,$$

which can be rewritten as the desired conclusion. □

Lemma 5. *The voting game satisfies weak single-crossing in* $\geq$*, i.e. if* $\theta_i'\geq\theta_i$ *and* $A_i'\supseteq A_i$*, then*

$$\int_{\Theta_{-i}}\theta_i(F(A_i',s_{-i}(\theta_{-i})))d\mu_{-i}\geq\int_{\Theta_{-i}}\theta_i(F(A_i,s_{-i}(\theta_{-i})))d\mu_{-i}$$
$$\implies\int_{\Theta_{-i}}\theta_i'(F(A_i',s_{-i}(\theta_{-i})))d\mu_{-i}\geq\int_{\Theta_{-i}}\theta_i'(F(A_i,s_{-i}(\theta_{-i})))d\mu_{-i}$$

for any profile s_{-i} *of monotone strategies by the other voters.*

Proof. Suppose $\theta_i'\geq\theta_i$ and fix a monotone strategy profile s_{-i} for the other voters. Suppose $A_i'\supseteq A_i$. Then $F(A_i',s_{-i}(\theta_{-i}))\supseteq F(A_i,s_{-i}(\theta_{-i}))$ for any $\theta_{-i}\in\Theta_{-i}$. By construction of the partial order $\geq$,

$$\theta_i'(F(A_i',s_{-i}(\theta_{-i})))-\theta_i'(F(A_i,s_{-i}(\theta_{-i})))\geq\theta_i(F(A_i',s_{-i}(\theta_{-i})))-\theta_i(F(A_i,s_{-i}(\theta_{-i}))).$$

This inequality is preserved by integration:

$$\int_{\Theta_{-i}}\theta_i'(F(A_i',s_{-i}(\theta_{-i})))d\mu_{-i}-\int_{\Theta_{-i}}\theta_i'(F(A_i,s_{-i}(\theta_{-i})))d\mu_{-i}$$
$$\geq\int_{\Theta_{-i}}\theta_i(F(A_i',s_{-i}(\theta_{-i})))d\mu_{-i}-\int_{\Theta_{-i}}\theta_i(F(A_i,s_{-i}(\theta_{-i})))d\mu_{-i}.$$

Then if

$$\int_{\Theta_{-i}}\theta_i(F(A_i',s_{-i}(\theta_{-i})))d\mu_{-i}\geq\int_{\Theta_{-i}}\theta_i(F(A_i,s_{-i}(\theta_{-i})))d\mu_{-i},$$

the inequality implies

$$\int_{\Theta_{-i}}\theta_i'(F(A_i',s_{-i}(\theta_{-i})))d\mu_{-i}\geq\int_{\Theta_{-i}}\theta_i'(F(A_i,s_{-i}(\theta_{-i})))d\mu_{-i}.$$ □

We can now check the assumptions in Reny's theorem. The technical conditions G.1 to G.4 are straightforward to verify. We will restrict attention to a space of strategies which will induce weakly undominated best responses. Let R_i be the subset of strategies for player i such that (μ_i almost surely): if θ_i is quasi-separable and $A_i^*(\theta_i)=\arg\max_{A\in\mathcal{X}}\theta_i(A)$ is unique, then $s_i(\theta_i)=A_i^*(\theta_i)$. This space is join-closed, pointwise-limit-closed, and piecewise-closed because it is the intersection of two measurable order inequalities (c.f. Reny 2008, Remark 4). To see this, let

$$f_i(\theta)=\begin{cases}A_i^*(\theta_i) & \text{if } \theta_i \text{ is quasi-separable and } A_i^*(\theta_i)=\arg\max_{A\in\mathcal{X}}\theta_i(A) \text{ is unique}\\ \emptyset & \text{otherwise}\end{cases},$$

and similarly

$$g_i(\theta)=\begin{cases}A_i^*(\theta_i) & \text{if } \theta_i \text{ is quasi-separable and } A_i^*(\theta_i)=\arg\max_{A\in\mathcal{X}}\theta_i(A) \text{ is unique}\\ X & \text{otherwise}\end{cases}.$$

And $R_i = \{s_i \in S_i : f_i(\theta_i) \subseteq s_i(\theta_i) \subseteq g_i(\theta_i),\ \mu_i\text{-a.s.}\}$.

We next show that there exists a monotone strategy in R_i. Define the following strategy:

$$s_i(\theta_i) = \bigcup_{\theta_i' \leq \theta_i} f_i(\theta_i').$$

By construction, s_i is monotone. Now suppose $\theta_i' \geq \theta_i$ are quasi-separable with respective unique maximizers $A_i^*(\theta_i'), A_i^*(\theta_i')$. By repeated application of quasi-separability, we have

$$\theta_i(A_i^*(\theta_i) \cup A_i^*(\theta_i')) - \theta(A_i^*(\theta_i')) \geq 0.$$

Considering the definition of $\geq$,

$$\theta_i'(A_i^*(\theta_i') \cup A_i^*(\theta_i)) - \theta_i'(A_i^*(\theta_i')) \geq 0.$$

Since $A_i^*(\theta_i')$ is the unique maximizer of θ_i', this forces $A_i^*(\theta_i') \cup A_i^*(\theta_i) = A_i^*(\theta_i')$, i.e. $A_i^*(\theta_i') \supseteq A_i^*(\theta_i)$. So if θ_i' is separable and has a unique maximizer $A_i^*(\theta_i')$, then $f_i(\theta_i') \supseteq f_i(\theta_i)$ for all $\theta \leq \theta_i'$. Hence $s_i(\theta_i') = A_i^*(\theta_i)$. So $s_i \in R_i$.

Finally, we prove that any monotone strategy in R_{-i} will induce a monotone best reply in R_i. Since R_i is a superset of the weakly undominated strategies for i, clearly for any strategy profile s_{-i} of the other voters, there is some element of R_i which is a best response. Moreover, any best reply to a strategy profile from R must be weakly undominated. This is because the quasi-separable types with unique maximizer A constitute a relatively open subset of the supermodular types, so every ballot has positive probability for each voter by the full support assumption. Standard lattice arguments show that weak quasi-supermodularity and weak single-crossing imply that the pointwise join of each type's best replies in weakly undominated strategies constitutes a monotone best reply itself; for example, see the proof of Corollary 4.3 in Reny (2008). Since R_i is join-closed, this monotone best reply lives in R_i. So, there exists an equilibrium in strategies in R_i, and by construction this equilibrium must be in weakly undominated strategies.

A.4 Proof of Proposition 5

Fix $x \in X$ and take any set $B \subseteq X \setminus \{x\}$ and consider the strategies $A = B \cup \{x\}$ and B. Recall, from the proof of Theorem 1, that for a fixed electorate size I, the expression (2) for $[\pi_i(s)](C, D)$ reflects the probability induced by strategy profile s that voter i will be pivotal on the issues in C and the issues in D will pass irrespective of her vote. For ease of notation, let $P_I^* = \pi_i(s_I^*)$ in the election with I voters.

Then the incentive condition for A being a better reply than A' to s_I^* is

$$\sum_{C,D \in \mathcal{D}} P_I^*(C, D) \cdot \theta([A \cap C] \cup D) \geq \sum_{C,D \in \mathcal{D}} P_I^*(C, D) \cdot \theta([A' \cap C] \cup D) \tag{3}$$

However, since if $x \notin C$, we have $C \cap A \cup D = C \cap A' \cup D$, the only relevant components of the sums on both sides of this inequality are:

$$\sum_{C,D \in \mathcal{D}: x \in C} P_I^*(C, D) \cdot \theta_i([A \cap C] \cup D) \geq \sum_{C,D \in \mathcal{D}: x \in C} P_I^*(C, D) \cdot \theta_i([A' \cap C] \cup D)$$

Dividing both sides by $\sum_{(C,D): x \in C} P_I^*(C, D) > 0$, we can replace the unconditional probabilities P_I^* with the conditional probabilities $P_I^*(C, D \mid x \in C)$:

$$\sum_{C,D\in\mathcal{D}:x\in C} P_I^*(C, D \mid x \in C) \cdot \theta_i([C \cap A] \cup D) \geq \sum_{C,D\in\mathcal{D}:x\in C} P_I^*(C, D \mid x \in C) \cdot \theta_i([C \cap A'] \cup D). \tag{4}$$

We can rewrite the left hand side as:

$$\begin{aligned}&\sum_{C,D\in\mathcal{D}:x\in C} P_I^*(C, D \mid x \in C) \cdot \theta_i([C \cap A] \cup D) \\ &\quad = P_I^*(\{x\}, D_x \mid x \in C)\theta_i(\{x\} \cup D_x) + \sum_{C,D:x\in C,D\neq D_x} P_I^*(C, D \mid x \in C) \cdot \theta_i([C \cap A] \cup D)\end{aligned}$$

Similarly rewriting the right hand side, the incentive inequality (4) can be rewritten:

$$\begin{aligned}&P_I^*(\{x\}, D_x \mid x \in C) \cdot \theta_i(\{x\} \cup D_x) + \sum_{C,D:x\in C,D\neq D_x} P_I^*(C, D \mid x \in C) \cdot \theta_i([C \cap A] \cup D) \\ &\quad \geq P_I^*(\{x\}, D_x \mid x \in C) \cdot \theta_i(D_x) + \sum_{C,D:x\in C,D\neq D_x} P_I^*(C, D \mid x \in C) \cdot \theta_i([C \cap A'] \cup D)\end{aligned}$$

This is rearranged as

$$\theta(\{x\} \cup D_x) \geq \theta_i(D_x) + \Delta(x, I) \tag{5}$$

where

$$\Delta_I = \frac{\sum_{C,D:x\in C,D\neq D_x} P_I^*(C, D \mid x \in C) \left[\theta_i([C \cap A'] \cup D) - \theta_i([C \cap A] \cup D)\right]}{P_I^*(x, D_x \mid x \in C)}.$$

If $|\theta_i(\{x\} \cup D_x) - \theta_i(D_x)| > \Delta_I$ for all $x \in X$, her best response A to $s_I(\theta)$ is determined by

$$x \in A \iff \theta(\{x\} \cup D_x) \geq \theta(D_x).$$

To see this, suppose not. Then there exists some alternate ballot A' such that either $x \in A$ but $\theta(D_x) > \theta(\{x\} \cup D_x)$, or $x \notin A$ but $\theta(\{x\} \cup D_x) > \theta(D_x)$. Assume the former, the second case being entirely similar. Since $|\theta(\{x\} \cup D_x) - \theta(D_x)| > \Delta_I$, the inequality is stronger: $\theta(D_x) > \theta(\{x\} \cup D_x) + \Delta_I$. Now consider the alternative $A = \{x\} \cup A'$. Then, since inequality (5) was shown to be equivalent to inequality (3), the inequality $\theta(\{x\} \cup D_x) > \theta(D_x) + \Delta_I$ implies A is a strictly better response than A', a contradiction.

We finally show that this equivalence holds for an arbitrarily large measure of types at the limit. Whenever $D \neq D_x$, then

$$\begin{aligned}P_I^*(C, D|x \in C) &= \mathbf{P}\left(\begin{array}{ll} \#\{j \neq i : y \in s_I^*(\theta_i)\} = \frac{I-1}{2} & \forall y \in C \\ \#\{j \neq i : y \in s_I^*(\theta_i)\} > \frac{I-1}{2} & \forall y \in D \\ \#\{j \neq i : y \in s_I^*(\theta_i) < \frac{I-1}{2} & \forall y \notin C \cup D \end{array} \middle| \#\{j \neq i : x \in s_I^*(\theta_i)\} = \frac{I-1}{2}\right) \\ &\leq 1 - \mathbf{P}\left(\#\{j \neq i : y \in s_I^*(\theta_i)\} > \frac{I-1}{2} \forall y \in D_x \middle| \#\{j \neq i : x \in s_I^*(\theta_i)\} = \frac{I-1}{2}\right) \\ &\to 0\end{aligned}$$

Noticing that $|\theta(\cdot)| < 1$, we have $\Delta_I \to 0$ as $I \to \infty$.

Observe that the set of types for which $|\theta(\{x\} \cup D_x) - \theta(D_x)| > \Delta_I$ for all $x \in X$ is of full Lebesgue measure at the limit, since $\Delta_I \to 0$. Invoking the density assumption, this set also has full μ measure at the limit.

A.5 Proof of Proposition 6

Without loss of generality, suppose voter I is pivotal on issue 1 and consider whether issue 2 is conditionally certain to pass or fail. Let X_k^{Ii} ($I = 1, 3, \ldots$; $i = 1, \ldots, I-1$; $k = 1, 2$) denote the triangular array of indicator functions on the events $\{\theta_i : k \in s_I^*(\theta_i)\}$. While X_2^{Ii} are unconditionally rowwise independent, this independence is broken once we condition on voter I being pivotal on issue 1. This precludes a straightforward application of the law of large numbers to the array and the proof requires more delicacy.

The basic logic is to split the sample of $I-1$ other voters into two subsamples: those $\frac{I-1}{2}$ who voted for issue 1, and those $\frac{I-1}{2}$ who did not. Within each subsample, the votes on issue k are conditionally identical and independent because the votes on issue 1 are fixed. However, by exchangeability, the particular identity of voters in each subsample is irrelevant, so we can proceed without loss of generality by assuming the first half of other other voters constitute the first subsample while the remainder constitute the second.

Formally, consider the following arrays of rowwise independent binary random variables:

$$Y^{Ii} = \begin{cases} 1 & \text{with probability } \mu(2 \in s_I(\theta_i) | 1 \notin s_I(\theta_I)) \\ 0 & \text{with probability } \mu(2 \notin s_I(\theta_i) | 1 \notin s_I(\theta_I)) \end{cases}$$

and

$$Z^{Ii} = \begin{cases} 1 & \text{with probability } \mu(2 \in s_I(\theta_i) | 1 \in s_I(\theta_I) \\ 0 & \text{with probability } \mu(2 \notin s_I(\theta_i) | 1 \in s_I(\theta_I) \end{cases}$$

Lemma 6. *The distribution of $\sum_{i=1}^{I-1} X_2^{Ii}$ conditional on $\sum_{i=1}^{I-1} X_1^{Ii} = \frac{I-1}{2}$ is identical to the distribution of the sum*

$$\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii}.$$

Proof. Suppressing the I superscript for the size of the electorate and fixing any integer n:

$$\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i=1}^{I-1} X_1^i = \frac{I-1}{2}\right) \\
&= \sum_{A \subset I-1 : \#A = \frac{I-1}{2}} \left[\mathbf{P}\left(\sum_{i \in A} X_2^i = 0 \,\middle|\, \sum_{i=1}^{I-1} X_1^i = \frac{I-1}{2}\right) \mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i \in A} X_1^i = 0, \sum_{j \notin A} X_1^j = \frac{I-1}{2}\right)\right]
\end{aligned}$$

By exchangeability across voters, the particular identities of the voters in the set A that voted up on issue 1 is irrelevant. In other words, we can assume without loss that the first $\frac{I-1}{2}$ other voters included 1 in their ballots and the last $\frac{I-1}{2}$ other voters excluded 1 from their ballots. The prior expression is therefore equal to:

$$\begin{aligned}
&= \sum_{A \subset I-1 : \#A = \frac{I-1}{2}} \left[\mathbf{P}\left(\sum_{i=1}^{\frac{I-1}{2}} X_2^i = 0 \,\middle|\, \sum_{i=1}^{I-1} X_1^i = \frac{I-1}{2}\right) \mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i=1}^{\frac{I-1}{2}} X_1^i = 0, \sum_{j=\frac{I+1}{2}}^{I-1} X_1^j = \frac{I-1}{2}\right)\right] \\
&= \mathbf{P}\left(\sum_{i=1}^{I-1} X_2^i = n \,\middle|\, \sum_{i=1}^{\frac{I-1}{2}} X_1^i = 0, \sum_{j=\frac{I+1}{2}}^{I-1} X_1^j = \frac{I-1}{2}\right) \\
&= \sum_{m=0}^{\frac{I-1}{2}} \left[\mathbf{P}\left(\sum_{i=1}^{\frac{I-1}{2}} X_2^i = m \,\middle|\, \sum_{i=1}^{\frac{I-1}{2}} X_1^i = 0\right) \mathbf{P}\left(\sum_{j=\frac{I+1}{2}}^{I-1} X_2^j = n - m \,\middle|\, \sum_{j=\frac{I+1}{2}}^{I-1} X_1^j = \frac{I-1}{2}\right)\right]
\end{aligned}$$

$$= \sum_{m=0}^{\frac{I-1}{2}} \left[\mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} = m \right) \mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} = n - m \right) \right]$$

$$= \mathbf{P} \left(\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} = n \right) \qquad \square$$

Lemma 7. *The normalized sum* $\frac{\sum_{i=1}^{I-1} X_2^{Ii}}{I-1}$ *conditional on* $\sum_{i=1}^{I-1} X_1^{Ii} = \frac{I-1}{2}$ *converges in probability to* $\frac{1}{2}\mu(2 \in s_I^*(\theta) \,|\, 1 \in s_I^*(\theta)) + \frac{1}{2}\mu(2 \in s_I^*(\theta) \,|\, 1 \notin s_I^*(\theta))$.

Proof. We can apply the Strong Law of Large Numbers for triangular arrays to Y^{Ii} and Z^{Ii} to conclude that

$$\left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} \to \mu\left(2 \in s_I^*(\theta_i) | 1 \notin s_I^*(\theta_i)\right)$$

and

$$\left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} \to \mu(2 \in s_I^*(\theta_i) | 1 \in s_I^*(\theta_i))$$

almost surely, hence in distribution. By the Continuous Mapping Theorem, the sum

$$\frac{1}{2} \left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \frac{1}{2} \left(\frac{I-1}{2} \right)^{-1} \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii} \tag{6}$$

converges in distribution to the constant

$$\tfrac{1}{2}\mu(2 \in s_I^*(\theta) \,|\, 1 \in s_I^*(\theta)) + \tfrac{1}{2}\mu(2 \in s_I^*(\theta) \,|\, 1 \notin s_I^*(\theta)).$$

Since, by Lemma 6, the conditional distribution of $\frac{\sum_{i=1}^{I-1} X_2^{Ii}}{I-1}$ shares the distribution of (6), it also converges in distribution to the same constant. As convergence in distribution to a constant implies convergence in probability, this delivers the desired conclusion. $\square$

To prove the Proposition, suppose $\mu(2 \in s(\theta) \,|\, 1 \in s(\theta)) > \mu(2 \notin s(\theta) \,|\, 1 \notin s(\theta))$; the argument for the opposite strict inequality is symmetric. Then:

$$\begin{aligned} \mu(2 \in s(\theta) \,|\, 1 \in s(\theta)) &> \mu(2 \notin s(\theta)) \,|\, 1 \notin s(\theta)) \\ \mu(2 \in s(\theta) \,|\, 1 \in s(\theta) + 1 - \mu(2 \notin s(\theta) \,|\, 1 \notin s(\theta)) &> 1 \\ \mu(2 \in s(\theta)) \,|\, 1 \in s(\theta) + \mu(2 \in s(\theta) \,|\, 1 \notin s(\theta)) &> 1 \\ \tfrac{1}{2}\mu(2 \in s(\theta) \,|\, 1 \in s(\theta) + \tfrac{1}{2}\mu(2 \in s(\theta)) \,|\, 1 \notin s(\theta)) &> \tfrac{1}{2} \end{aligned}$$

Let $E = \frac{1}{2}\mu(2 \in s(\theta) \,|\, 1 \in s(\theta)) + \frac{1}{2}\mu(2 \in s(\theta) \,|\, 1 \notin s(\theta))$ and pick a strictly positive $\delta < E - \frac{1}{2}$. By Lemma 7, the probability that the normalized vote count on issue 2, conditional on voter I being pivotal on 1, is greater than $E - \delta > \frac{1}{2}$ approaches one. Thus, the conditional probability that 2 passes converges to one.

The necessity of the weak inequality follows from the contraposition of the sufficiency claim, i.e. if y is conditionally certain to pass then it is not conditionally certain to fail.

A.6 Proof of Proposition 7

It suffices to prove the case $A = \{1, 2\}$. Other cases then follow by appropriately permuting the direction of "pass" or "fail" on the ballot. For example, when $A = \{1\}$, consider the following permutation:

$$\{1, 2\} \mapsto \{1\}, \{1\} \mapsto \{1, 2\}, \{2\} \mapsto \emptyset, \emptyset \mapsto \{2\}.$$

Similar permutations apply for $A = \{2\}$ and $A = \emptyset$.

We first prove the sufficiency of the strict inequality. So let

$$\begin{aligned}
\Theta_{12} &= \{\theta : \theta_{12} \geq \max\{\theta_1, \theta_2\}\} \\
\Theta_1 &= \{\theta : \theta_1 \geq \theta_{12} \geq \theta_2\} \\
\Theta_2 &= \{\theta : \theta_2 \geq \theta_{12} \geq \theta_1\} \\
\Theta_\emptyset &= \{\theta : \theta_{12} \leq \min\{\theta_1, \theta_2\}\}
\end{aligned}$$

These four sets of types cover Θ. Since μ has full support and admits a density, they have strictly positive probability but null pairwise intersections. By assumption,

$$\frac{\mu(\Theta_{12})}{\mu(\Theta_\emptyset)} > \max\left\{\frac{\mu(\Theta_1)}{\mu(\Theta_2)}, \frac{\mu(\Theta_2)}{\mu(\Theta_1)}\right\}.$$

Let

$$\mathcal{P}_n = \left\{P \in \Delta^U : P(x, x'|x \in C) \geq 1 - \tfrac{1}{n}, \forall x = 1, 2\right\}$$

Recall that $P(C, D)$ is the probability that an anonymous voter is pivotal on the issues in C and that the issues in D will pass irrespective of her ballot. Let $A \subset \{x'\}$ and consider $A' = \{x\} \cup A$. The incentive condition for A' being a better reply than A given the belief $P \in \mathcal{P}_n$ over pivotal and passing events is:

$$\theta_{12} \geq \theta_x + \Delta_n(\theta)$$

where

$$\Delta_n(\theta) = \frac{P(x, \emptyset \,|\, x \in C)\,[\theta_\emptyset - \theta_x] + P(\{1, 2\}, \emptyset \,|\, x \in C)[\theta_A - \theta_{A'}]}{P(x, x' \,|\, x \in C)}.$$

Observe that $\Delta_n(\theta) \to 0$ as $n \to \infty$. Let $\Theta_n = \{\theta : |\theta_{12} - \theta_x| > \Delta_n\}$ and notice $\mu(\Theta_n) \to 1$. As $n \to \infty$, the proportion of types in Θ_{12} which include x in their optimal ballots covers the entire subset Θ_{12}, while the proportion of types in $\Theta \setminus \Theta_{12}$ which include x becomes null. Similarly arguing for $\Theta_1, \Theta_2, \Theta_0$, we have

$$\sigma_I(P_n) \to s$$

where $s_A(\theta_A) = A$ for all $\theta_A \in \Theta_A$, for any sequence of selections $P_n \in \mathcal{P}_n$. Then

$$\frac{\mu([\sigma_I(P)](\theta) = \{1, 2\})}{\mu([\sigma_I(P)](\theta) = \emptyset)} - \max\left\{\frac{\mu([\sigma(P)](\theta) = \{1\})}{\mu([\sigma(P)](\theta) = \{2\})}, \frac{\mu([\sigma_I(P)](\theta) = \{2\})}{\mu([\sigma_I(P)](\theta) = \{1\}}\right\}$$

which is arbitrarily close to

$$\frac{\mu(\Theta_{12})}{\mu(\Theta_\emptyset)} - \max\left\{\frac{\mu(\Theta_1)}{\mu(\Theta_2)}, \frac{\mu(\Theta_2)}{\mu(\Theta_1)}\right\} > 0$$

for any $P \in \mathcal{P}_n$ as $n \to \infty$. So, there exists some n_0 such that if $n > n_0$:

$$\frac{\mu([\sigma_I(P)](\theta) = \{1,2\})}{\mu([\sigma_I(P)](\theta) = \emptyset)} > \max\left\{\frac{\mu([\sigma_I(P)](\theta) = \{1\})}{\mu([\sigma_I(P)](\theta) = \{2\})}, \frac{\mu([\sigma_I(P)](\theta) = \{2\})}{\mu([\sigma_I(P)](\theta) = \{1\}}\right\}$$

for all $P \in \mathcal{P}_n$.

So, let $n > n_0$ and consider the sequence of strategies $s_I = \sigma_I(P)$ for any $P \in \mathcal{P}_n$. Fix I and let $\mu_A = \mu([\sigma_I(P)](\theta) = A)$ Then:

$$\frac{\mu_{12}}{\mu_\emptyset} > \frac{\mu_1}{\mu_2}$$
$$\mu_{12}\mu_2 > \mu_\emptyset\mu_1$$
$$\mu_{12}(\mu_2 + \mu_\emptyset) > \mu_\emptyset(\mu_1 + \mu_{12})$$
$$\frac{\mu_{12}}{\mu_1 + \mu_{12}} > \frac{\mu_\emptyset}{\mu_2 + \mu_\emptyset}$$
$$\mu\left(2 \in [\sigma_I(P)](\theta) \,|\, 1 \in [\sigma_I(P)](\theta)\right) > \mu(2 \notin [\sigma_I(P)](\theta) \,|\, 1 \notin [\sigma_I(P)](\theta))$$

By Proposition 5 (see also the remark immediately following the proof), for I sufficiently large, the probability $\pi_I(\sigma_I)$ satisfies

$$[\pi_I(\sigma_I(P))](x, x'|x \in C) \geq 1 - \tfrac{1}{n}$$

for $x = 1, 2$. This means for sufficiently large $I > I_0$, the image $[\pi \circ \sigma_I](\mathcal{P}_n) \subseteq \mathcal{P}_n$. It therefore admits a fixed point $P_I^* \in \mathcal{P}_n$ which defines an equilibrium s_I^*, for all sufficiently large $I > I_0(n)$.

Finally, define $n(I) = \max\{n : I > I_0(n)\} \vee I$. Observe that as $I \to \infty$, we have $n(I) \to \infty$. For each I, select a fixed point $P_I^* \in [\pi \circ \sigma](\mathcal{P}_{n(I)})$. The induced equilibrium strategy s_I^* satisfies

$$\mu\left(\sigma_I^*(\theta) \,|\, 1 \in [\sigma_I(P)](\theta)\right) > \mu(2 \notin [\sigma_I(P)](\theta) \,|\, 1 \notin [\sigma_I(P)](\theta))$$

By Propositon 5, the set $\{1, 2\}$ is conditionally certain. Also, recalling the construction, $s_I^* \to s^*$ where $s^*(\Theta_A) = A$.

We now prove the necessity of the weak inequality. Suppose each issue is conditionally certain to pass. In particular, 2 is conditionally certain to pass at 1. By Proposition 6:

$$\mu[2 \in s^*(\theta_i) \,|\, 1 \in s^*(\theta_i)] \geq \mu[2 \notin s^*(\theta_i) \,|\, 1 \notin s^*(\theta_i)].$$

For notational convenience, let $\mu_A^* = \mu(\{\theta_i : s^*(\theta_i) = A\})$. Since 2 is conditionally certain to pass at 1, we have:

$$\frac{\mu_{12}^*}{\mu_{12}^* + \mu_1^*} \geq \frac{\mu_\emptyset^*}{\mu_\emptyset^* + \mu_2^*}$$
$$\mu_{12}^*(\mu_\emptyset^* + \mu_2^*) \geq \mu_\emptyset^*(\mu_{12}^* + \mu_1^*)$$
$$\mu_{12}^*\mu_2^* \geq \mu_\emptyset^*\mu_1^*$$
$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_1^*}{\mu_2^*}$$

Symmetrically, since 1 is conditionally certain to pass at 2:

$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_2^*}{\mu_1^*}$$

The prior two inequalities imply

$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \max\left\{\frac{\mu_1^*}{\mu_2^*}, \frac{\mu_2^*}{\mu_1^*}\right\} \tag{7}$$

By Proposition 5, we have

$$x \in s^*(\theta_i) \Longleftrightarrow \theta_i(\{1,2\}) \geq \theta(\{x'\})$$

Thus:

$$s^*(\theta) = \begin{cases} \{1,2\} & \text{if } \theta_{12} \geq \max\{\theta_1, \theta_2\} \\ \{1\} & \text{if } \theta_1 \geq \theta_{12} \geq \theta_2 \\ \{2\} & \text{if } \theta_2 \geq \theta_{12} \geq \theta_1 \\ \emptyset & \text{if } \theta_{12} \leq \min\{\theta_1, \theta_2\} \end{cases}$$

Substituting these cases into condition (7) delivers the result.

A.7 Proof of Lemma 2

We first prove that (i) and (ii) are equivalent. Observe that the following equality holds:

$$\mu\left(\theta_A \geq \max\left\{\theta_{A'}, \theta_{A''}\right\}\right) = \mu\left(\theta_A \geq \theta_{A''} \geq \theta_{A'}\right) + \mu\left(\theta_A \geq \theta_{A'} \geq \theta_{A''}\right)$$

This can be rewritten as:

$$\begin{aligned} &\mu\left(\theta_A \geq \max\left\{\theta_{A'}, \theta_{A''}\right\}\right) + \mu\left(\theta_{A''} \geq \theta_A \geq \theta_{A'}\right) \\ &\qquad = \mu\left(\theta_{A''} \geq \theta_A \geq \theta_{A'}\right) + \mu\left(\theta_A \geq \theta_{A''} \geq \theta_{A'}\right) + \mu\left(\theta_A \geq \theta_{A'} \geq \theta_{A''}\right). \end{aligned}$$

This is equivalent to:

$$\mu\left(\theta_A \geq \max\left\{\theta_{A'}, \theta_{A''}\right\}\right) = \mu\left(\theta_A \geq \theta_{A'}\right) - \mu\left(\theta_{A''} \geq \theta_A \geq \theta_{A'}\right).$$

Reasoning analogously, we obtain the following four equations:

$$\begin{aligned} \mu\left(\theta_A \geq \max\left\{\theta_{A'}, \theta_{A''}\right\}\right) &= \mu\left(\theta_A \geq \theta_{A'}\right) - \mu\left(\theta_{A''} \geq \theta_A \geq \theta_{A'}\right) \\ \mu\left(\theta_A \geq \max\left\{\theta_{A'}, \theta_{A''}\right\}\right) &= \mu\left(\theta_A \geq \theta_{A''}\right) - \mu\left(\theta_{A'} \geq \theta_A \geq \theta_{A''}\right) \\ \mu\left(\theta_A \leq \min\left\{\theta_{A'}, \theta_{A''}\right\}\right) &= \mu\left(\theta_A \leq \theta_{A'}\right) - \mu\left(\theta_{A''} \leq \theta_A \leq \theta_{A'}\right) \\ \mu\left(\theta_A \leq \min\left\{\theta_{A'}, \theta_{A''}\right\}\right) &= \mu\left(\theta_A \leq \theta_{A''}\right) - \mu\left(\theta_{A'} \leq \theta_A \leq \theta_{A''}\right) \end{aligned}$$

Then condition (i) can be rewritten as the following two inequalities:

$$\frac{\mu(\theta_A \geq \theta_{A''}) - \mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_A \leq \theta_{A''}) - \mu(\theta_{A'} \leq \theta_A \leq \theta_{A''})} > \frac{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})} \tag{8}$$

$$\frac{\mu(\theta_A \geq \theta_{A'}) - \mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_A \leq \theta_{A'}) - \mu(\theta_{A''} \leq \theta_A \leq \theta_{A'})} > \frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}. \tag{9}$$

Inequality (8) can be expressed as any of the following equivalent inequalities:

$$\frac{\mu(\theta_A \geq \theta_{A''})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})} > \frac{\mu(\theta_A \leq \theta_{A''})}{\mu(\theta_{A'} \leq \theta_A \leq \theta_{A''})}$$

$$\frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})} > \frac{1 - \mu(\theta_A \geq \theta_{A''})}{\mu(\theta_A \geq \theta_{A''})}$$

$$\frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A''} \geq \theta_A)} > \frac{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_A \geq \theta_{A''})}$$

$$\mu(\theta_A \geq \theta_{A'} \mid \theta_{A''} \geq \theta_A) > \mu(\theta_{A'} \geq \theta_A \mid \theta_A \geq \theta_{A''}).$$

This is the first inequality in condition (ii). Similarly, inequality (9) can be rewritten as the second inequality in condition (ii).

We now prove that (ii) and (iii) are equivalent. First observe that the first inequality in (ii) is equivalent to the second inequality in (iii) through the following steps:

$$\frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A''} \geq \theta_A)} > \frac{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_A \geq \theta_{A''})}$$

$$\frac{\mu(\theta_A \geq \theta_{A''})}{\mu(\theta_{A''} \geq \theta_A)} > \frac{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}$$

$$\frac{\mu(\theta_{A''} \geq \theta_A)}{\mu(\theta_A \geq \theta_{A''})} < \frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}$$

$$\frac{1 - \mu(\theta_A \geq \theta_{A''})}{\mu(\theta_A \geq \theta_{A''})} < \frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}$$

$$\frac{1}{\mu(\theta_A \geq \theta_{A''})} - 1 < \frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}$$

$$\frac{1}{\mu(\theta_A \geq \theta_{A''})} < \frac{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'}) + \mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}$$

$$\mu(\theta_A \geq \theta_{A''}) > \frac{\mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}{\mu(\theta_{A''} \geq \theta_A \geq \theta_{A'}) + \mu(\theta_{A'} \geq \theta_A \geq \theta_{A''})}.$$

Similarly, the second inequality in condition (ii) is equivalent to the first inequality in condition (iii).

A.8 Proof of Lemma 1

Without loss of generality, consider the case $x = 2$. Let

$$\alpha_I = \mathbf{P}\left(\#\{j \neq i : 2 \in s_I^*(\theta_k)\} > \frac{I-1}{2} \,\middle|\, \#\{j \neq i : 1 \in s_I^*(\theta_k)\} = \frac{I-1}{2}\right).$$

and

$$\beta_I = \mathbf{P}\left(\#\{j \neq i : 2 \in s_I^*(\theta_k)\} < \frac{I-1}{2} \,\middle|\, \#\{j \neq i : 1 \in s_I^*(\theta_k)\} = \frac{I-1}{2}\right).$$

By the full support assumption, the conditional probability of being pivotal on issue 2 when pivotal on issue 1 vanishes, so $\alpha_I + \beta_I \to 1$.

Fix a voter i with type θ and consider a ballot $A \subseteq \{1\}$ which does not include 2. The incentive condition for $\{2\}$ being a better reply than $\{2\} \cup A$ to the strategy s_I^* is:

$$\alpha_I \theta_{12} + \beta_I \theta_2 + (1 - \alpha_I - \beta_I)\theta_{2\cup A} \geq \alpha_I \theta_1 + \beta_I \theta_\emptyset + (1 - \alpha_I - \beta_I)\theta_A.$$

Passing to a subsequence if necessary, there exists an α such that $\alpha_I \to \alpha$. The incentive inequality can be rewritten as

$$\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset + \Delta_I$$

where

$$\Delta_I = [\alpha_I - \alpha](\theta_1 - \theta_{12}) + [\beta_I - (1-\alpha)](\theta_\emptyset - \theta_2) + [1 - \alpha_I - \beta_I](\theta_A - \theta_{2\cup A}).$$

However, $\Delta_I \to 0$. From here, we can replicate the arguments which conclude the proof of Proposition 5 to conclude that, at the limit, the set of types which support issue 2 is characterized by the inequality

$$\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset.$$

A.9 Proof of Propostion 8

Suppose $\varepsilon < \frac{1}{16}$. Let μ be any density in the class described in Example 1. We first prove that there must exist be at least a single issue which exhibits conditional uncertainty.

Lemma 8. *For any density in $\mathcal{C}$, there is no sequence of equilibria that exhibits conditional certainty.*

Proof. To see that $\{1,2\}$ cannot be conditionally certain, observe that

$$\frac{\mu(\theta_{12} \geq \max\{\theta_1, \theta_2\})}{\mu(\theta_{12} \leq \min\{\theta_1, \theta_2\})} \leq \frac{\frac{1}{4} + \varepsilon}{\frac{1}{4} - \varepsilon}.$$

For small ε, this ratio approximates one. On the other hand,

$$\frac{\mu(\theta_2 \geq \theta_{12} \geq \theta_1)}{\mu(\theta_1 \geq \theta_{12} \geq \theta_2)} \geq \frac{\frac{1}{2} - \varepsilon}{\varepsilon}.$$

For small ε, this ratio becomes arbitrarily large. This precludes the required inequality the necessity direction of Proposition 7 for $A = \{1,2\}$. An entirely similar argument proves that the inequality also fails for $A = \{1\}, \{2\}, \emptyset$. By Proposition 7, there cannot be an equilibrium with conditional certainty. □

By Lemma 8, we can assume that there is some issue with conditional uncertainty. We now prove that this implies the other issue must also be conditionally uncertain.

Lemma 9. *For every density in $\mathcal{C}$, all convergent sequences of equilibria exhibit conditional uncertainty on both issues.*

Proof. The proof shows that assuming one issue is conditionally certain while the other is conditionally uncertain leads to a contradiction. So either both issues are conditionally certain or both issues are conditionally uncertain. By Lemma 8, it must be the latter case. We now demonstrate that if issue 1 is conditionally uncertain, then issue 2 cannot be conditionally certain to pass. The other cases can be argued symmetrically.

So, suppose issue 2 is conditionally certain to pass. Recall μ_A^* denotes the probability that an anonymous voter submits the ballot A when playing the limit strategy s^*. Since 2 is conditionally certain to pass, $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_1^*}{\mu_2^*}$. Since 1 is conditionally uncertain, it is not conditionally certain to fail. We therefore conclude $\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \frac{\mu_2^*}{\mu_1^*}$. So, we have the following inequality:

$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \geq \max\left\{\frac{\mu_1^*}{\mu_2^*}, \frac{\mu_2^*}{\mu_1^*}\right\}, \tag{10}$$

In view of Proposition 5 and Lemma 1, there exists some $\alpha \in (0,1)$ such that the following describes the limit strategy in terms of types:

$$s^*(\theta) = \begin{cases} \{1,2\} & \text{if } \theta_{12} \geq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \\ \{1\} & \text{if } \theta_{12} \geq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \\ \{2\} & \text{if } \theta_{12} \leq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \\ \emptyset & \text{if } \theta_{12} \leq \theta_2 \text{ and } \alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \end{cases}$$

Let

$$\begin{aligned} \varphi_{12}(\alpha) &= \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \,|\, \theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) \\ \varphi_\emptyset(\alpha) &= \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \,|\, \theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1). \end{aligned}$$

Observe that, since μ has full support and admits a density, φ_{12} and $\varphi_\emptyset$ are increasing and continuous functions with $\varphi_{12}(0) = \varphi_\emptyset(0) = 0$ and $\varphi_{12}(1) = \varphi_\emptyset(1) = 1$.

Now we can rewrite the limit probability of voting for both issues as:

$$\begin{aligned} \mu_{12}^* &= \mu(\theta_{12} \geq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \,|\, \theta_{12} \geq \theta_2) \\ &\leq \mu(\theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2)\mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \,|\, \theta_{12} \geq \theta_1 \geq \theta_\emptyset \geq \theta_2) + \varepsilon \\ &= \frac{1}{4}\varphi_{12}(\alpha) + \varepsilon. \end{aligned}$$

Likewise, the limit probability of voting down on both issues can be rewritten as:

$$\begin{aligned} \mu_\emptyset^* &= \mu(\theta_{12} \leq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \mid \theta_2 \geq \theta_{12}) \\ &\geq \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12})\mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \,|\, \theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) \\ &\quad + \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1)\mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \,|\, \theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\ &\quad + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset)\mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \,|\, \theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \\ &= \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) + \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1)(1 - \varphi_\emptyset(\alpha)) \\ &\geq \frac{1}{4} + \frac{1}{4}(1 - \varphi_\emptyset(\alpha)) - \varepsilon \end{aligned}$$

So

$$\frac{\mu_{12}^*}{\mu_\emptyset^*} \leq \frac{\frac{1}{4}\varphi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \varphi_\emptyset(\alpha)) - \varepsilon} \tag{11}$$

Inequality (10) provides that $\frac{\mu^*_{12}}{\mu^*_\emptyset} \geq \frac{\mu^*_2}{\mu^*_1}$. So, (11) implies:

$$\frac{\mu^*_2}{\mu^*_1} \leq \frac{\frac{1}{4}\varphi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \varphi_\emptyset(\alpha)) - \varepsilon} \tag{12}$$

Inequality (10) provides that $\frac{\mu^*_{12}}{\mu^*_\emptyset}$ is larger than a fraction and its reciprocal. So we have $\frac{\mu^*_{12}}{\mu^*_\emptyset} \geq 1$. Therefore (11) also implies:

$$\begin{aligned}
\frac{1}{4}\varphi_{12}(\alpha) + \varepsilon &\geq \frac{1}{4} + \frac{1}{4}(1 - \varphi_\emptyset(\alpha)) - \varepsilon \\
\varphi_{12}(\alpha) &\geq 2 - \varphi_0(\alpha) - 8\varepsilon \\
\varphi_{12}(\alpha) &> 1 - 8\varepsilon. \qquad (13)
\end{aligned}$$

Arguing symmetrically,

$$\begin{aligned}
\frac{1}{4}\varphi_{12}(\alpha) + \varepsilon &\geq \frac{1}{4} + \frac{1}{4}(1 - \varphi_\emptyset(\alpha)) - \varepsilon \\
\varphi_{12}(\alpha) &\geq 2 - \varphi_0(\alpha) - 8\varepsilon \\
\varphi_0 &\geq 2 - \varphi_1 2(\alpha) - 8\varepsilon \\
\varphi_0(\alpha) &> 1 - 8\varepsilon. \qquad (14)
\end{aligned}$$

We can rewrite the limit probability of voting only for issue 2 as:

$$\begin{aligned}
\mu^*_2 &= \mu(\theta_{12} \leq \theta_2) \cdot \mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \mid \theta_{12} \geq \theta_2) \\
&\geq \mu(\theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12})\mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \mid \theta_1 \geq \theta_\emptyset \geq \theta_2 \geq \theta_{12}) \\
&\quad + \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1)\mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \geq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \mid \theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1) \\
&\quad + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset)\mu(\alpha\theta_{12} + (1-\alpha)\theta_2 \leq \alpha\theta_1 + (1-\alpha)\theta_\emptyset \mid \theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \\
&= \mu(\theta_\emptyset \geq \theta_2 \geq \theta_{12} \geq \theta_1)\varphi_\emptyset(\alpha) + \mu(\theta_2 \geq \theta_{12} \geq \theta_1 \geq \theta_\emptyset) \\
&\geq \frac{1}{4}\varphi_0(\alpha) + \frac{1}{4} - \varepsilon.
\end{aligned}$$

Similarly, rewriting the probability of voting only for issue 1:

$$\mu^*_1 \leq \frac{1}{4}(1 - \varphi_{12}(\alpha)) + \varepsilon.$$

So,

$$\frac{\mu^*_2}{\mu^*_1} \geq \frac{\frac{1}{4}\varphi_0(\alpha) + \frac{1}{4} - \varepsilon}{\frac{1}{4}(1 - \varphi_{12}(\alpha)) + \varepsilon} \tag{15}$$

Combining (12) and (15):

$$\frac{\frac{1}{4}\varphi_{12}(\alpha) + \varepsilon}{\frac{1}{4} + \frac{1}{4}(1 - \varphi_\emptyset(\alpha)) - \varepsilon} \geq \frac{\frac{1}{4}\varphi_0(\alpha) + \frac{1}{4} - \varepsilon}{\frac{1}{4}(1 - \varphi_{12}(\alpha)) + \varepsilon}.$$

This can be rewritten as

$$(\varphi_{12}(\alpha) + 4\varepsilon)(1 - \varphi_{12}(\alpha) + 4\varepsilon) \geq (\varphi_0(\alpha) + 1 - 4\varepsilon)(2 - \varphi_0(\alpha) - 4\varepsilon). \tag{16}$$

At the same time, recalling earlier inequalities:

$$\begin{aligned}\varphi_{12}(\alpha) + 4\varepsilon &< 1 + 4\varepsilon \\ &< 2 - 12\varepsilon, \text{ since } \varepsilon < \tfrac{1}{16} \\ &< \varphi_0(\alpha) + 1 - 4\varepsilon, \text{ by (14)}.\end{aligned}$$

And:

$$\begin{aligned}1 - \varphi_{12}(\alpha) + 4\varepsilon &< 1 - (1 - 8\varepsilon) + 4\varepsilon, \text{ by (13)} \\ &= 12\varepsilon \\ &< 1 - 4\varepsilon, \text{ since } \varepsilon < \tfrac{1}{16} \\ &< 2 - \varphi_0(\alpha) - 4\varepsilon.\end{aligned}$$

But the prior two series of inequalities contradict (16), since they imply the left hand side of (16) is the product of strictly smaller positive quantities than those in the product on the right hand side (16). □

We will now prove that if there is conditional uncertainty on both issues, then there is unconditional uncertainty on both issues. Given Lemma 9, this will suffice to show that there is unconditional uncertainty on both issues for every density in $\mathcal{C}$.

For notational ease, we now define the following. Let

$$\mu^I(x|x') = \mu\left(x \in s_I(\theta_i) \mid x' \in s_I(\theta_i)\right)$$

and

$$\mu^I(x|\neg x') = \mu\left(x \in s_I(\theta_i) \mid x' \notin s_I(\theta_i)\right).$$

Let

$$\mu^I(x) = \mu\left(x \in s_I(\theta_i)\right).$$

Lemma 10. *Issue x is conditionally uncertain if and only if*

$$\lim_{I\to\infty} \left| \sqrt{(I-1)}\left(\mu^I(x|x') + \mu^I(x|\neg x') - 1\right) \right| < \infty.$$

Proof. Take $x = 2$; the case $x = 1$ is identical. Recall the two arrays defined in the proof of Proposition 6, rowwise independent binary random variables Y^{Ii} and Z^{Ii} whose success probabilities are $\mu(2 \in s_I(\theta_i)|1 \in s_I(\theta_I))$ and $\mu(2 \in s_I(\theta_i)|1 \notin s_I(\theta_I))$. In Lemma 6 of that proof, we demonstrated that the conditional distribution of the vote count on issue 2 is equal to the distribution of $\sum_{i=1}^{\frac{I-1}{2}} Y^{Ii} + \sum_{i=1}^{\frac{I-1}{2}} Z^{Ii}$. Let $W^{Ii} = Y^{Ii} + Z^{Ii}$. As Y^{Ii} and Z^{Ii} are mutually independent, the array W^{Ii} defines a rowwise independent array of random variables. We can write that

$$\mathbf{P}\left(\#\{j \neq i : 2 \in s_I^*(\theta_j)\} > \frac{I-1}{2} \middle| \#\{j \neq i : 1 \in s_I^*(\theta_j)\} = \frac{I-1}{2}\right) = \mathbf{P}\left(\sum_{i=1}^{\frac{I-1}{2}} W^{Ii} > \frac{I-1}{2}\right)$$

Recalling the definition of the binary random variables $Y^{Ii}(\theta)$ and $Z^{Ii}(\theta)$ we have that

$$\mathbf{E}\left(W^{Ii}\right) = \mu^I(2|1) + \mu^I(2|\neg 1)$$

and

$$\mathbf{Var}\left(W^{Ii}\right) = \mu^I(2|1)\left[1-\mu^I(2|1)\right] + \mu^I(2|\neg 1)\left[1-\mu^I(2|\neg 1)\right].$$

Applying the Central Limit Theorem for triangular arrays:

$$\mathbf{P}\left(\frac{\sum_{i=1}^{\frac{I-1}{2}} W^{Ii} - \left(\frac{I-1}{2}\right)\left[\mu^I(2|1)+\mu^I(2|\neg 1)\right]}{\sqrt{\left(\frac{I-1}{2}\right)\left(\mu^I(2|1)\left[1-\mu^I(2|1)\right]+\mu^I(2|\neg 1)\left[1-\mu^I(2|\neg 1)\right]\right)}} < y\right) \longrightarrow \Phi(y), \tag{17}$$

where Φ denotes the standard normal cumulative distribution function.

The conditional probability that issue i fails is $\mathbf{P}\left(\sum_{i=1}^{\frac{I-1}{2}} W^{Ii} < \frac{I-1}{2}\right)$. By manipulation of (17), this converges to:

$$\Phi\left(\frac{1}{2}\cdot\frac{\sqrt{(I-1)}\left(1-\left(\mu^I(2|1)+\mu^I(2|\neg 1)\right)\right)}{\sqrt{\mu^I(2|1)\left[1-\mu^I(2|1)\right]+\mu^I(2|\neg 1)\left[1-\mu^I(2|\neg 1)\right]}}\right).$$

Therefore $\lim_{I\to\infty}\left|\sqrt{(I-1)}\left(\mu^I(2|1)+\mu^I(2|\neg 1)-1\right)\right| < \infty$ is necessary and sufficient for issue 2 to be conditional uncertain. □

Lemma 11. *Issue x is unconditionally uncertain if and only if* $\lim_{I\to\infty}\left|\sqrt{I}\left(\mu_k^I - \frac{1}{2}\right)\right| < \infty$

Proof. Define the binary random variable

$$V^{Ii} = \begin{cases} 1 & \text{with probability } \mu^I(x) \\ 0 & \text{with probability } 1-\mu^I(x) \end{cases}$$

with mean $\mu^I(x)$ and variance $\mu^I(x)\left(1-\mu^I(x)\right)$. The probability that issue k will pass (fail) is

$$\mathbf{P}\left(\sum_{i=1}^{I} V_k^{Ii} > (<)\, \frac{I}{2}\right).$$

Arguing as in the proof of Lemma 10, we have that the asymptotic (unconditional) probability that issue x will pass is equal to

$$\Phi\left(\frac{\sqrt{I}\left(\frac{1}{2}-\mu^I(x)\right)}{\sqrt{\mu^I(x)\left(1-\mu^I(x)\right)}}\right).$$

Therefore

$$\lim_{I\to\infty}\left|\sqrt{I}\left(\frac{1}{2}-\mu^I(x)\right)\right| < \infty$$

is necessary and sufficient for unconditional uncertainty. □

Lemma 12. *There is unconditional uncertainty for both issues if and only if there is conditional uncertainty for both issues*

Proof. Let

$$
\begin{aligned}
x^I &= \mu^I(1) & a^I &= \mu^I(1|2)\\
y^I &= \mu^I(2) & b^I &= \mu^I(1|\neg 2)\\
& & c^I &= \mu^I(2|1)\\
& & d^I &= \mu^I(2|\neg 1)
\end{aligned}
$$

We have a system of two equations with two unknowns, x^I and y^I:

$$x^I = a^I y^I + b^I\left(1 - y^I\right) \tag{18}$$

$$y^I = c^I x^I + d^I\left(1 - x^I\right) \tag{19}$$

The corresponding solutions for x and y are:

$$x^I = \frac{(a^I - b^I)d^I + b^I}{1 - (a^I - b^I)(c^I - d^I)} \tag{20}$$

$$y^I = \frac{(c^I - d^I)b^I + d^I}{1 - (c^I - d^I)(a^I - b^I)}. \tag{21}$$

We will first prove that if there is conditional uncertainty on both issues, then there must be unconditional uncertainty on both. Subtracting one half from both sides in Equations (20) and (21) yields, after some manipulation:

$$x^I - \frac{1}{2} = \frac{1}{2}\frac{(b^I - a^I)(1 - (c^I + d^I)) + (1 - (a^I + b^I))}{1 - (a^I - b^I)(c^I - d^I)} \tag{22}$$

$$y^I - \frac{1}{2} = \frac{1}{2}\frac{(d^I - c^I)(1 - (a^I + b^I)) + (1 - (c^I + d^I))}{1 - (c^I - d^I)(a^I - b^I)}. \tag{23}$$

By Lemma 10, conditional uncertainty on both issues means

$$\lim_{I\to\infty} \sqrt{I}\left|1 - (a^I + b^I)\right| < \infty$$

and

$$\lim_{I\to\infty} \sqrt{I}\left|1 - (c^I + d^I)\right| < \infty.$$

Since $1 - (a^I - b^I)(c^I - d^I)$ is uniformly bounded away from 0 and $|(a^I - b^I)|$ is bounded by 1, this suffices to show that $\lim_{I\to\infty} \sqrt{I}\left|x^I - \frac{1}{2}\right|$ and $\lim_{I\to\infty} \sqrt{I}\left|x^I - \frac{1}{2}\right|$ given the expressions in (22) and (23) are both finite. By Lemma 11, this implies unconditional uncertainty on both issues.

We finally show that unconditional uncertainty on both issues implies conditional uncertainty on both. Equations (18) and (19) imply:

$$
\begin{aligned}
x^I - y^I &= \left(a^I + b^I - 1\right) y^I + 2a^I\left(\tfrac{1}{2} - y^I\right)\\
y^I - x^I &= \left(c^I + d^I - 1\right) x^I + 2c^I\left(\tfrac{1}{2} - x^I\right).
\end{aligned}
$$

These can be rewritten as

$$\left(a^I + b^I - 1\right) y^I = \left(x^I - \tfrac{1}{2}\right) + 2\left(\tfrac{1}{2} - a^I\right)\left(\tfrac{1}{2} - y^I\right)$$
$$\left(c^I + d^I - 1\right) x^I = \left(y^I - \tfrac{1}{2}\right) + 2\left(\tfrac{1}{2} - c^I\right)\left(\tfrac{1}{2} - x^I\right).$$

By Lemma 11, unconditional uncertainty on both issues provides $\lim_{I\to\infty} \sqrt{I}\left|\frac{1}{2} - x^I\right|$ and $\lim_{I\to\infty} \sqrt{I}\left|\frac{1}{2} - y^I\right|$ are both finite. Since both $\left|\frac{1}{2} - a^I\right|$ and $\left|\frac{1}{2} - c^I\right|$ are bounded by $\frac{1}{2}$, this suffices to show that

$$\lim_{I\to\infty} \sqrt{I}\left|a^I + b^I - 1\right| y^I < \infty.$$

By Lemma 10, this implies issue 1 is conditionally uncertain. Similarly, issue 2 is also conditionally uncertain.

□

A.10 Proof of Proposition 9

We first record the following lemma, which we will also use in future proofs.

Lemma 13. *Suppose the support of μ is the set of supermodular types. Then for $x = 1, 2$:*

$$\mu(\theta_x \geq \theta_\emptyset) \leq \mu(x \in s^*(\theta)) \leq \mu(\theta_{x'} \geq \theta_{12}).$$

Proof. Consider the case where $x = 1$. By Lemma 1, there exist $\alpha \in [0,1]$ such that:

$$\begin{aligned} \mu(1 \in s^*(\theta)) &= \mu(\alpha\theta_{12} + (1-\alpha)\theta_1 \geq \alpha\theta_2 + (1-\alpha)\theta_\emptyset) \\ &= \mu(\alpha[\theta_{12} + \theta_\emptyset - \theta_1 - \theta_2] \geq \theta_\emptyset - \theta_1) \\ &= \mu((1-\alpha)[\theta_1 + \theta_2 - \theta_{12} - \theta_\emptyset] \geq \theta_2 - \theta_{12}). \end{aligned}$$

Supermodularity implies that

$$\theta_{12} + \theta_\emptyset - \theta_1 - \theta_2 \geq 0. \tag{24}$$

This provides the following inequality:

$$\begin{aligned} \mu(1 \in s^*(\theta)) &= \mu(\alpha[\theta_{12} + \theta_\emptyset - \theta_1 - \theta_2] \geq \theta_\emptyset - \theta_1) \\ &\geq \mu(0 \geq \theta_\emptyset - \theta_1) \\ &= \mu(\theta_1 \geq \theta_\emptyset). \end{aligned}$$

Inequality (24) also provides the following inequality:

$$\begin{aligned} \mu(1 \in s^*(\theta)) &= \mu((1-\alpha)[\theta_1 + \theta_2 - \theta_{12} - \theta_\emptyset] \geq \theta_2 - \theta_{12}) \\ &\leq \mu(\theta_{12} \geq \theta_2) \\ &= \mu(\theta_{12} \geq \theta_2). \end{aligned}$$

□

Then part (i) of the proposition follows from the first part of the inequality in Lemma 13: if

$$\mu(x \in s^*(\theta)) \geq \mu(\theta_x \geq \theta_\emptyset) > \frac{1}{2},$$

then the Strong Law of Large Numbers for triangular arrays implies that issue x is unconditionally certain to pass. Similarly, part (ii) follows from the second part of the inequality in Lemma 13.

A.11 Proof of Proposition 10

We begin by proving a useful implication of conditional certainty.

Lemma 14. *If A is conditionally certain, then*

$$\mu\left(\theta_A \geq \theta_{A'}\right) \mu\left(\theta_A \geq \theta_{A''}\right) \geq \mu\left(\theta_{A'} \geq \theta_A\right) \mu\left(\theta_{A''} \geq \theta_A\right).$$

Proof. Suppose that A is conditionally certain. By Proposition 7 and Lemma 2, we know that the following hold:

$$\mu\left(\theta_A \geq \theta_{A'} \,|\, \theta_{A''} \geq \theta_A\right) \geq \mu\left(\theta_{A'} \geq \theta_A \,|\, \theta_A \geq \theta_{A''}\right) \tag{25}$$

$$\mu\left(\theta_A \geq \theta_{A''} \,|\, \theta_{A'} \geq \theta_A\right) \geq \mu\left(\theta_{A''} \geq \theta_A \,|\, \theta_A \geq \theta_{A'}\right). \tag{26}$$

Observe that:

$$\mu\left(\theta_A \geq \theta_{A'} \mid \theta_{A''} \geq \theta_A\right) = \mu\left(\theta_{A''} \geq \theta_A \mid \theta_A \geq \theta_{A'}\right) \mu\left(\theta_A \geq \theta_{A'}\right) \mu\left(\theta_A \geq \theta_{A''}\right)$$

$$\mu\left(\theta_{A'} \geq \theta_A \mid \theta_A \geq \theta_{A''}\right) = \mu\left(\theta_A \geq \theta_{A''} \mid \theta_{A'} \geq \theta_A\right) \mu\left(\theta_{A'} \geq \theta_A\right) \mu\left(\theta_{A''} \geq \theta_A\right)$$

We can then rewrite (25) as:

$$\begin{aligned}
&\mu\left(\theta_{A''} \geq \theta_A \mid \theta_A \geq \theta_{A'}\right) \mu\left(\theta_A \geq \theta_{A'}\right) \mu\left(\theta_A \geq \theta_{A''}\right) \\
&\quad \geq \mu\left(\theta_A \geq \theta_{A''} \mid \theta_{A'} \geq \theta_A\right) \mu\left(\theta_{A'} \geq \theta_A\right) \mu\left(\theta_{A''} \geq \theta_A\right)
\end{aligned}$$

This is equivalent to:

$$\frac{\mu(\theta_{A'} \geq \theta_A)\mu(\theta_{A''} \geq \theta_A)}{\mu(\theta_A \geq \theta_{A'})\mu(\theta_A \geq \theta_{A''})} \leq \frac{\mu(\theta_{A''} \geq \theta_A \mid \theta_A \geq \theta_{A'})}{\mu(\theta_A \geq \theta_{A''} \mid \theta_{A'} \geq \theta_A)} \tag{27}$$

Moreover, (26) implies that

$$\frac{\mu(\theta_{A''} \geq \theta_A \mid \theta_A \geq \theta_{A'})}{\mu(\theta_A \geq \theta_{A''} \mid \theta_{A'} \geq \theta_A)} \leq 1 \tag{28}$$

Together, (27) and (28) imply

$$\frac{\mu(\theta_{A'} \geq \theta_A)\mu(\theta_{A''} \geq \theta_A)}{\mu(\theta_A \geq \theta_{A'})\mu(\theta_A \geq \theta_{A''})} \leq 1$$

which is the desired conclusion. □

Without loss of generality consider the case where the Condorcet winner is $\{1,2\}$. We first prove sufficiency. So, suppose that at least one issue agrees with $\{1,2\}$. There are five cases to consider:

Case 1: $\{1,2\}$ is conditionally certain. Since issue 2 is conditionally certain to pass, by Proposition 5, $1 \in s^*(\theta)$ whenever $\theta_{12} \geq \theta_2$. But since $\{1,2\}$ is a local Condorcet winner, $\mu(\theta_{12} \geq \theta_2) > \frac{1}{2}$, i.e. $\mu(1 \in s^*(\theta)) > \frac{1}{2}$. Then, by the Strong Law of Large Numbers for triangular arrays, issue 1 is unconditionally certain to pass. A similar argument establishes that issue 2 is also unconditionally certain to pass. Thus $\{1,2\}$ is unconditionally certain.

Case 2: The bundle $\{1\}$ is conditionally certain. By Lemma 14, we have

$$\mu(\theta_1 \geq \theta_{12})\mu(\theta_1 \geq \theta_\emptyset) \geq \mu(\theta_{12} \geq \theta_1)\mu(\theta_\emptyset \geq \theta_1). \tag{29}$$

Since $\{1, 2\}$ is a local Condorcet winner, we know $\mu(\theta_{12} \geq \theta_1) > \frac{1}{2}$ so:

$$\mu(\theta_1 \geq \theta_{12}) < \mu(\theta_{12} \geq \theta_1). \tag{30}$$

The in order to maintain the inequality (29), it must be that

$$\mu(\theta_1 \geq \theta_\emptyset) > \mu(\theta_\emptyset \geq \theta_1). \tag{31}$$

However, (30) and Proposition 5 imply that $\mu(1 \in s^*(\theta)) > \frac{1}{2}$. By the Strong Law of Large Numbers for triangular arrays, issue 1 is unconditionally certain to pass. Similarly, (31) and Proposition 5 imply that issue 2 is also unconditionally certain to pass.

Case 3: The bundle $\{2\}$ is conditionally certain. This case can be argued similarly to Case 2.

Case 4: Issue 1 is conditionally certain to pass and issue 2 is conditionally uncertain. Recall (22) from the proof of Lemma 12:

$$\sqrt{I}\mu(1 \in s_I^*(\theta)) - \frac{1}{2} = \frac{1}{2}\frac{\sqrt{I}(b^I - a^I)(1 - (c^I + d^I)) + \sqrt{I}(1 - (a^I + b^I))}{1 - (a^I - b^I)(c^I - d^I)}$$

where

$$\begin{aligned}
a^I &= \mu(1 \in s_I^*(\theta) \mid 2 \in s_I^*(\theta)) \\
b^I &= \mu(1 \in s_I^*(\theta) \mid 2 \notin s_I^*(\theta)) \\
c^I &= \mu(2 \in s_I^*(\theta) \mid 1 \in s_I^*(\theta)) \\
d^I &= \mu(2 \in s_I^*(\theta) \mid 1 \notin s_I^*(\theta)).
\end{aligned}$$

By Lemma 10, we have $\lim \sqrt{I}(1 - (c^I + d^I)) < \infty$ since issue 2 is conditionally uncertain. Similarly, $\lim \sqrt{I}(1 - (a^I + b^I)) = \infty$. Since $1 - (a^I - b^I)(c^I - d^I)$ is uniformly bounded away from 0, this suffices to prove $\sqrt{I}\mu(1 \in s_I^*(\theta)) - \frac{1}{2} \to \infty$. Then by Lemma 11, we conclude that issue 1 is unconditionally certain to pass. The argument that issue 2 is also unconditionally certain to pass is symmetric.

Case 5: Issue 2 is conditionally certain to pass and issue 1 is conditionally uncertain. This case can be argued similarly to Case 4.

Now to prove necessity, we will show that the nonexistence of any issue satisfying (i) or (ii) implies that $\{1, 2\}$ cannot be conditionally certain. So, suppose either both issues are conditionally uncertain or that one issue is conditionally certain to fail.

Case 6: Both issues are conditionally uncertain. Then, by Lemma 12, both issues are also unconditionally uncertain. So $\{1, 2\}$ is not unconditionally certain.

Case 7: Issue 1 is conditionally certain to fail and issue 2 is conditionally uncertain. This case can be argued the same way that Case 4 but noting that Lemma 10 gives that $\lim \sqrt{I}\left(1 - \left(a^I + b^I\right)\right) = -\infty$ when issue 1 is conditionally certain to fail so (22) gives that $\lim \sqrt{I}\left(\mu\left(1 \in s_I^*(\theta)\right) - \frac{1}{2}\right) \to -\infty$. Then by Lemma 11, issue 1 is unconditionally certain to fail.

Case 8: Issue 2 is conditionally certain to fail and issue 1 is conditionally uncertain. This can be argued similarly to Case 7.

A.12 Proof of Proposition 11

Without loss of generality, assume $A = \emptyset$ is the local Condorcet loser. First, observe that if both issues are conditionally uncertain, Lemma 12 implies there is no unconditionally certain bundle. In particular, $\emptyset$ cannot be unconditionally certain.

We next argue that if either issue is conditionally certain to fail, then $\emptyset$ cannot be unconditionally certain. So, suppose issue 1 is conditionally certain to fail, Proposition 5 implies $2 \in s^*(\theta)$ if $\theta_2 > \theta_\emptyset$. However, since $\emptyset$ is a local Condorcet loser, $\mu(\theta_2 > \theta_\emptyset) > \frac{1}{2}$. By the Strong Law of Large Numbers for triangular arrays, this means issue 2 is unconditionally certain to pass. A symmetric argument holds if issue 2 is conditionally certain to fail. So, we can now assume without loss of generality that there is at least one issue that is conditionally certain to pass, and that the other issue is either conditionally certain to pass or is conditionally uncertain. Consider the case where issue 1 is conditionally certain to pass; the argument for issue 2 is identical.

Case 1: Issue 2 is conditionally certain to pass. Then $\{1, 2\}$ is conditionally certain. Now, by way of contradiction, suppose $\emptyset$ is unconditionally certain. So $\mu(1 \in s^*(\theta)) \leq \frac{1}{2}$. By Proposition 5, $\mu(\theta_1 \geq \theta_{12}) \geq \frac{1}{2} \geq \mu(\theta_{12} \geq \theta_1)$ because issue 2 is conditionally certain to pass. Symmetrically, we can also conclude $\mu(\theta_2 \geq \theta_{12}) \geq \frac{1}{2} \geq \mu(\theta_{12} \geq \theta_2)$. Then

$$\mu(\theta_1 \geq \theta_{12})\mu(\theta_2 \geq \theta_{12}) \geq \mu(\theta_{12} \geq \theta_1)\mu(\theta_{12} \geq \theta_2).$$

At the same time, the fact $\{1, 2\}$ is conditionally certain also implies, through Lemma 14 in the proof of Proposition 10, that

$$\mu(\theta_1 \geq \theta_{12})\mu(\theta_2 \geq \theta_{12}) \leq \mu(\theta_{12} \geq \theta_1)\mu(\theta_{12} \geq \theta_2).$$

The only way to maintain the prior two inequalities is for $\mu(\theta_1 \geq \theta_{12}) = \frac{1}{2}$ and $\mu(\theta_1 \geq \theta_{12}) = \frac{1}{2}$. But then the Condorcet ranking $\succ_{\mathcal{C}}$ is incomplete, contradicting the hypothesis that $\succ_{\mathcal{C}}$ is complete.

Case 2: Issue 2 is conditionally uncertain. Since issue 1 is conditionally certain to pass, we can replicate the argument for Case 4 in the proof of Proposition 10 verbatim, and conclude that issue 1 is unconditionally certain to pass. Then $\emptyset$ cannot be unconditionally certain.

A.13 Proof of Proposition 12

We first record a straightforward but useful implication of quasi-separability.

Lemma 15. *Suppose $\succ_{\mathcal{C}}$ is quasi-separable. If A is a Condorcet winner, then its complement $X \setminus A$ is a Condorcet loser.*

Proof. Without loss of generality, suppose $\{1, 2\}$ is a Condorcet winner. Then $\{1, 2\} \succ_{\mathcal{C}} \{1\}$. By quasi-separability of $\succ_{\mathcal{C}}$, we have $\{2\} \succ_{\mathcal{C}} \emptyset$. Similarly, $\{1, 2\} \succ_{\mathcal{C}} \{2\}$ implies $\{1\} \succ_{\mathcal{C}} \emptyset$. Moreover, since $\{1, 2\}$ is a Condorcet winner, we have $\{1, 2\} \succ_{\mathcal{C}} \emptyset$. Therefore $\emptyset$ is a Condorcet loser. □

To prove the Proposition, without loss of generality consider the case where $\{1, 2\}$ is the Condorcet winner. Since $\emptyset$ is the Condorcet loser, then it cannot be conditionally certain. To see this, observe that the necessary inequalities in part (iii) of Lemma 2 are impossible because both $\mu(\theta_\emptyset \geq \theta_1)$ and $\mu(\theta_\emptyset \geq \theta_2)$ are strictly less than one half, while one of the ratios on the right hand sides of the inequality must be weakly greater than one half. But since there is conditional certainty on both issues and $\emptyset$ is not conditionally certain, one of the issues must be conditionally certain to pass. Then by Proposition 10, this implies that $\{1, 2\}$ is conditionally certain.

A.14 Proof of Proposition 13

We will prove the case when types are supermodular; the argument for the submodular case then follows by relabeling "up" to "down" on the second issue.

We first record the following implications of Lemma 13

$$\mu(1 \in s^*(\theta)) \geq \mu(\theta_1 \geq \theta_\emptyset) \quad (32)$$

$$\mu(2 \in s^*(\theta)) \geq \mu(\theta_2 \geq \theta_\emptyset) \quad (33)$$

$$\mu(1 \in s^*(\theta)) \leq \mu(\theta_2 \geq \theta_{12}) \quad (34)$$

$$\mu(2 \in s^*(\theta)) \leq \mu(\theta_1 \geq \theta_{12}) \quad (35)$$

We now argue by cases that the Condorcet winning bundle A is a limit outcome of the election and that it is the unique limit outcome.

Case 1: $A = \{1,2\}$**.** Since $\{1,2\}$ is the Condorcet winner, we have $\{1,2\} \succ_{\mathcal{C}} \{2\}$. By quasi-separability, $\{1\} \succ_{\mathcal{C}} \emptyset$. Recalling the definition of the Condorcet order, we have $\mu(\theta_1 \geq \theta_\emptyset) > \frac{1}{2}$. But, using (32), this implies $\mu(1 \in s^*(\theta) > \frac{1}{2}$. Appealing to the Strong Law of Large Numbers for triangular arrays, this implies issue 1 is unconditionally certain to pass. Quasi-separability of $\succ_{\mathcal{C}}$ similarly implies $\{2\} \succ_{\mathcal{C}} \emptyset$, i.e. $\mu(\theta_2 \geq \theta_\emptyset) > \frac{1}{2}$. Using (33), this similarly implies issue 2 is unconditionally certain to pass. Thus $\{1,2\}$ is the only limit outcome.

Case 2: $A = \{1\}$**.** Then quasi-separability of $\succ_{\mathcal{C}}$ implies $\mu(\theta_1 \geq \theta_\emptyset) > \frac{1}{2}$. By (32), we have that issue 1 is unconditionally certain to pass. Also, quasi-separability implies $\mu(\theta_\emptyset \geq \theta_2) > \frac{1}{2}$, that is $\mu(\theta_2 \geq \theta_0) < \frac{1}{2}$. Then by (35), it must be that issue 2 is unconditionally certain to fail.

Case 3: $A = \{2\}$**.** This argument is nearly identical to Case 2, using (34) and (33).

Case 4: $A = \emptyset$**.** This argument is nearly identical to Case 1, using (34) and (35).

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