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TARGETING IN ADVERTISING MARKETS: IMPLICATIONS FOR OFFLINE VS. ONLINE MEDIA

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Targeting in Advertising Markets: Implications for Offline vs. Online Media*

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Abstract

We develop a model with many advertisers (products) and many advertising markets (media). Each advertiser sells to a different segment of consumers, and each medium has a different ability to target advertising messages. We characterize the competitive equilibrium in the media markets and evaluate the implications of targeting in advertising markets.

An increase in the targeting ability leads to an increase in the total number of purchases (matches), and hence in the social value of advertising. Yet, an improved targeting ability also increases the concentration of firms advertising in each market. Surprisingly, we then find that the equilibrium price of advertisements is first increasing, then decreasing in the targeting ability.

We trace out the implications of targeting for competing media. We distinguish offline and online media by their targeting ability: low versus high. As consumers' relative exposure to online media increases, the revenues of offline media decrease, even though the price of advertising might increase.

KEYWORDS: Targeting, Advertising, Online Advertising, Sponsored Search, Media Markets.

JEL CLASSIFICATION: D44, D82, D83.

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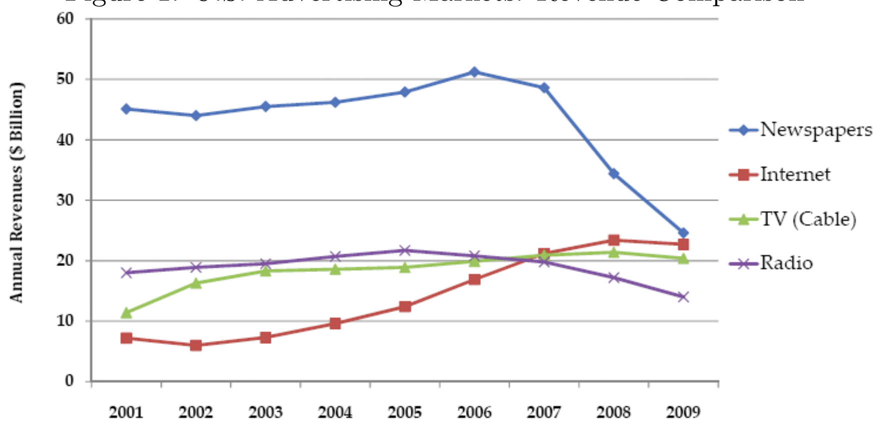
1 Introduction

Over the past decade the internet has become an increasingly important medium for advertising. The arrival of the internet has had important consequences on the market position of many traditional media, i.e. offline media such as print, audio and television. For some of these media, most notably the daily newspapers, the very business model is under the threat of extinction due to competition from the internet for the placement of advertising. The following chart shows the recent changes in aggregate spending for advertising on different media.[1]

Figure 1: U.S. Advertising Markets: Revenue Comparison



At the same time, through a variety of technological advances, the internet has allowed many advertisers to address a targeted audience beyond the reach of traditional media. In fact, it has been argued that the distinguishing feature of internet advertising is its ability to convey information to a targeted audience. In particular, targeting improves the quality of the match between the consumer and the advertisement message, and enables smaller businesses to access advertising markets from which they were previously excluded.[2] While this holds for display advertising, it is even more true for sponsored search, where the individual consumer declares her intent or preference directly, by initiating a query.

The objective of this paper is to develop a model of competition between offline (traditional) and online (new) media, in which the distinguishing feature of the online media is the ability to (better) target advertisement messages to their intended audience. We

[1] Source: Price Waterhouse Coopers annual reports for the Interactive Advertising Bureau.

[2] Anderson (2006) refers to this phenomenon as the “long tail of advertising.”

investigate the role of targeting in the determination of (a) the allocation of advertisements across different media, and (b) the equilibrium price for advertising. For this purpose, we first develop a framework to analyze the role of targeting, and then use this to model to analyze the interaction between offline and online advertising.

We present a model in which advertising creates awareness for a product. We consider an economy with a continuum of buyers and a continuum of products. Each product has a potential market size which describes the mass of consumers who are contemplating to purchase it. Each consumer is contemplating only one of the available products, and the role of the advertisement is to generate a match between product and consumer. The placement of an advertisement constitutes a message from the advertiser to a group of consumers. If the message happens to be received by a consumer with interest in the advertiser's product, then the potential customer turns into an actual customer and a sale is realized. A message received by a customer who is not in the market for the product in question is irretrievably lost and generates no tangible benefit for the advertiser. At the same time, a potential customer might be reached by multiple and hence redundant messages from the same advertiser. Consequently, the probability that a potential customer is turned into an actual customer is an increasing but concave function of the number of messages sent.

We begin the analysis with a single advertising market in which all consumers are present and can be reached by any advertiser. It is useful to think of the single advertising market as a national platform, such as the nationwide newspapers or the major television networks. We show that in this market structure only the largest firms, measured by the size of their potential market, purchase any advertising space. We also show that the concentration of consumers' interests (i.e. the degree of asymmetry in firms' potential market sizes) has an initially positive, but eventually negative effect on the equilibrium price of messages.

We then introduce the possibility of targeting by introducing a continuum of advertising markets. Each advertising market is characterized by the composition of the audience in terms of its preferences over products. While each consumer is at most present in one advertising market, the likelihood of her presence in a specific market is correlated with her potential interest. As each consumer segment becomes more concentrated in a few advertising markets, the probability of a match between consumers and advertisements increases. In consequence, the social welfare is increasing with the ability of the advertisers to reach their preferred audience. We then investigate the equilibrium advertising prices as the degree of targeting improves. While the marginal product of each message is increasing in the targeting ability, thus potentially increasing the prices for the advertisement, a second and more powerful effect appears. As consumers become more concentrated, the competition among different advertisers becomes weaker. In fact, each advertiser focuses his attention on a few important advertising markets and all but disappears from the other advertising markets. Therefore, the price of advertising is declining in the degree of targeting, even though the value of advertising is increasing. The number of participating advertisers shows a similarly puzzling behavior. While improved targeting increases the total number of advertisers participating across all markets – by allowing smaller advertisers to appear – it reduces the number of actively advertising firms in each specific advertising market.

In the second part of the paper we introduce competition among different media for the attention of the consumer. Thus, while each consumer is still only interested in one product, he can now receive a message from any advertiser through two different advertising media. A single message received in either one of the media is sufficient to create a sale. The "dual-homing" of the consumer across the two media markets may then lead to duplicative efforts by the advertisers, who therefore view messages in the two competing markets as substitutes. We first describe the advertising allocation when the competitors are both traditional media without any targeting ability. In this case, messages on the two media are perfect substitutes, and the equilibrium prices are equalized. Furthermore, the allocation of messages only depends on the total supply, not on its distribution across media.

The competition among two offline media markets presents a useful benchmark when we next consider competition between an offline and an online market. We analyze the interaction of offline media – such as newspapers or TV – with online media, such as display (banner) and sponsored search advertisements. Display advertisements allow for targeting through superior knowledge of the consumer's preferences (attribute targeting). Sponsored keyword search advertisements allow advertisers to infer the consumer's preferences from her actions (behavioral targeting). As expected, competition lowers the equilibrium revenues of the traditional medium. However, if entry by an online competitor reduces the available advertising space on the traditional media (for example, by reducing the time consumers spend on each channel), then the effect of competition on the equilibrium price of advertising is non monotonic. In particular, as the consumers shift their attention from traditional to new (targeted) media, the price on the traditional channels is first decreasing, then increasing.

This paper is related to several strands in the literature on advertising and media competition. Anderson and Coate (2005) provide the first model of competing broadcasters, with exclusive assignment of viewers to stations; their setup is extended by Ferrando, Gabszewicz, Laussel, and Sonnac (2004), and Ambrus and Reisinger (2006) to the case of non-exclusive assignments. However, the role of targeting for the structure of advertising markets has received scant attention in the literature. The most prominent exception is Iyer, Soberman, and Villas-Boas (2005), who analyze the strategic choice of advertising in an imperfectly competitive market with product differentiation. In their model, the consumers are segmented into different audiences that the firms can target with advertising messages. Yet, Iyer, Soberman, and Villas-Boas (2005) are mostly concerned with the equilibrium product prices that result from the competitive advertising strategies. This approach is also followed in the work by Esteban, Gil, and Hernandez (2001) and de Cornière (2010). In contrast, we take sales prices as given, and focus our attention on the equilibrium prices of advertising messages themselves. Our results on equilibrium advertising prices and competing media are in line with recent empirical work by Goldfarb and Tucker (2010), who exploit the variation in targeting ability generated by the legal framework across states. They show that prices for sponsored search advertising are higher when regulations limit the offline alternatives for targeted advertising. In another empirical study, Chandra (2009) relates the degree of segmentation (targeting) of a newspaper's subscriber base to the price it charges for the advertising space. The results imply a substantial benefit to advertisers and media firms from targeted advertising; see also Chandra and Kaiser (2010) for related evidence from the magazine markets.

In this paper, each advertising message generates a match between a product and a potential customer. The present interpretation of advertising as matching products and users is shared with recent papers, such as Athey and Ellison (2008) and Chen and He (2006). Yet, in these contributions, the primary focus is on the welfare implications of position auctions in a search model where consumers are uncertain about the quality of the match. Similarly, several recent papers, Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) among others, focus on the specific mechanisms used in practice to sell advertising messages online, such as auctions for sponsored links in keyword searches. In contrast, we model the market for advertisements as a competitive market and the allocation of advertising messages is determined by the competitive equilibrium price.

In closely related work, Athey and Gans (2010) analyze the impact of targeting on the supply and price of advertising in a model with local and general outlets. In their model, targeting improves the efficiency of the allocation of messages, and leads to an increase in demand. They observe that as long as advertising space can be freely expanded, the revenue effects of targeting can also be obtained by increasing the supply of (non targeted) messages, yielding an equivalence result. More generally, Athey and Gans (2010) show that supply-side effects mitigate the value of targeting. Finally, Levin and Milgrom (2010) discuss the trade-off between value creation and market thickness in the context of online advertising, and describe several instances of excessive targeting leading to lower revenues for publishers.

The remainder of the paper is organized as follows. Section 2 introduces the model and describes the targeting technology. Section 3 opens with the equilibrium analysis in a single advertising market. Section 4 investigates the general model with many advertising markets. Section 5 extends the analysis by allowing each consumer to be present in several media markets. Section 6 investigates the competition between offline and online media. The Appendix collects the formal proofs of all propositions in the main body of the text.

2 Model

2.1 Advertising and Product Markets

We consider a model with a continuum of products and a continuum of advertising markets. Each product x is offered by firm x with $x \in [0, \infty)$. The advertising markets are indexed by $a \in [0, \infty)$. There is a continuum of buyers with unit mass and each buyer is present in exactly one product market and one advertising market. The population of consumers is jointly distributed across advertising markets a and product markets x according to $S(a, x)$, with a density $s(a, x)$. For brevity of notation, we often denote the density by $s_{a,x}$. The market share of product x is given by the marginal distribution, integrating over all the advertising markets a:

$$s_x \triangleq \int_0^\infty s(a, x)\, da. \tag{1}$$

Similarly, the size of the advertising market a is given by the marginal distribution, integrating over all the products x:

$$s_a \triangleq \int_0^\infty s(a,x)\,dx. \tag{2}$$

Each buyer is only interested in one specific product x. A sale of product x occurs if and only if the buyer is interested in the product *and* she receives at least one message from firm x. A message by firm x is hence only effective if it is received by a buyer in segment x. In other words, we adopt the complementary view of advertising (see Bagwell (2007)), in which both the message and the right receiver are necessary to generate a purchase. Each sale generates a gross revenue of \$1, constant across all product markets.

The advertising policy of firm x determines the number of messages $m_{a,x}$ it distributes in advertising market a. Each message of advertiser x reaches a random consumer in advertising market a with uniform probability. Given the size of the advertising market s_a and the message volume $m_{a,x}$, the probability that a given consumer in market a is aware of product x is then given by:

$$f(m_{a,x}, s_a) \triangleq 1 - e^{-m_{a,x}/s_a}. \tag{3}$$

We refer to $f(m_{a,x}, s_a)$ as the awareness level for product x in advertising market a. The exponential form of the matching probability (3) is a result of the uniform random matching process in the presence of a large number of messages. In detail, suppose a large number of messages, denoted by m, is distributed with uniform probability across a large number of agents, denoted by s. Now, the exact probability that a representative agent has received *none* out of the m messages is given by:

$$(1 - 1/s)^m.$$

If we now let $r = m/s$, then, by the definition of the exponential function, we have that:

$$\lim_{m\to\infty}(1 - r/m)^m = e^{-r},$$

and the complementary probability is given by (3).

Finally, the supply of messages M_a in every advertising market a is proportional to the size s_a of the advertising market and given by

$$M_a \triangleq s_a \cdot M,$$

for some constant $M > 0$. The constant M can be interpreted as the attention or time that the representative consumer allocates to receiving messages from the advertising outlet.

Firms can purchase advertisement messages at a unit price p_a in each market a. The total profits of firm x are given by:

$$\pi_x \triangleq \int_0^\infty \left[s_{a,x} f\left(m_{a,x}, s_a\right) - p_a m_{a,x}\right] da. \tag{4}$$

The awareness function $f(\cdot)$ described above applies literally to the case of display advertising online, where each impression corresponds to a message. However, it can be easily amended for the analysis of broadcast media. In particular, let $m_{a,x}$ be the product of an advertisement's air time $t_{a,x}$ times the audience size s_a of channel a. The awareness level generated by an ad of firm x placed in market a is then given by $1 - \exp\left(-t_{a,x}\right)$. We can define the cost of advertising in market a as $p_a \cdot m_{a,x}$, so that firms are charged proportionally to time and audience, and hence we again obtain the expression (4) as the profit function of advertiser x.

2.2 Exponential Model

In order to efficiently capture the role of product market concentration and advertising market targeting, the allocation of buyers across product and advertising markets is assumed to be governed by an exponential distribution. Firms are ranked, without loss of generality, in decreasing order of market share, so s_x is decreasing in x. In particular, the market share of product x is given by:

$$s_x \triangleq \lambda e^{-\lambda x}. \tag{5}$$

The parameter $\lambda \geq 0$ measures the concentration in the product market, and a large value of λ represents a more concentrated product market. In turn, the conditional distribution of consumers in product segment x over advertising markets a is given by a (truncated) exponential distribution:

$$\frac{s_{a,x}}{s_x} \triangleq \begin{cases} \gamma e^{-\gamma(x-a)}, & \text{if} \quad 0 < a \leq x, \\ 0, & \text{if} \quad a > x, \end{cases} \tag{6}$$

with a mass point

$$\frac{s_{a,x}}{s_a} \triangleq e^{-\gamma x} \text{ if } a = 0.$$

In other words, we model market $a = 0$ as a large advertising market, in which all advertisers are potentially interested (as $s_{x,0} > 0$ for all x), such as the Yahoo! front page, or a national newspaper.[3] The parameter $\gamma \geq 0$ measures the concentration of the consumers in the advertising markets. A larger value of γ represents a heavier concentration of more consumers in fewer advertising markets. The corresponding unconditional market shares are given by:

$$s_{a,x} \triangleq \begin{cases} \lambda e^{-(\lambda+\gamma)x}, & \text{if} \quad a = 0, \\ \lambda\gamma e^{-(\lambda+\gamma)x} e^{\gamma a}, & \text{if} \quad a \leq x, \\ 0, & \text{if} \quad a > x. \end{cases}$$

For $\gamma > 0$, the distribution of consumers over product and advertising markets has a triangular structure. The consumers who are interested in product x are present in all advertising markets $a \leq x$, but are not present in the advertising markets $a > x$. The distribution of consumers across a one-dimensional product space and a one-dimensional advertising space has a natural interpretation in terms of specialization of preferences and audiences. In this interpretation, a product with a larger index x represents a more specialized product with a smaller market. Correspondingly, an advertising market with a larger index a represents an outlet with a narrower audience. To give a precise example, consider the market for bicycles. Here, products naturally range from mass-produced comfort bikes, to quality-produced fitness bikes, to high-end racing bikes with successively smaller market shares. Similarly, there is a natural range of advertising markets, from daily newspapers with a large audience, to monthly magazine with well-defined audience such as "Sports Illustrated," to narrowly focused publications, such as "Velonews". Now, the triangular structure of the joint distribution implies that the consumer with an interest in racing bikes

[3] As we will show, despite the introduction of mass point at $a = 0$, the equilibrium in this market has the same properties as all other markets $a > 0$.

may read either one of the publications, but that a consumer with interest in fitness bikes does *not* read "Velonews," and by extension that a consumer with an interest in comfort bikes does *not* read "Velonews" nor "Sports Illustrated". In other words, the triangular structure represents a positive but less than perfect correlation of the preference and the audience characteristics of a consumer. The specific feature of the triangular structure, namely the unidirectional diffusion of the consumer x across advertising markets $a \leq x$, is not essential for the qualitative character of our results, but allow us to represent the strength of the targeting in a single variable, namely the parameter γ of the exponential distribution.

As we vary the targeting measure γ from 0 to ∞, we change the distribution and the concentration in each advertising market. The limit values of γ, namely $\gamma = 0$ and $\gamma = \infty$, represent two special market structures. If $\gamma = 0$, then all consumers are present in advertising market 0 and hence there is a single advertising market. If, on the other hand, $\gamma \to \infty$, then all consumers of product x are present in advertising market x, and hence we have advertising markets with perfect targeting. More generally, as we increase γ, an increasing fraction of consumers of product x move away from the large advertising markets (near $a = 0$) to the smaller advertising markets (near $a = x$). Figure 2 illustrates the cross section, represented by the conditional distribution $s_{a,x}/s_x$, of how the consumer segments of two different advertisers are distributed across the advertising markets (for a low and high degree of targeting in the left and right panel respectively). The mass points indicate the number of consumers of each firm that are present in advertising market 0.

Figure 2: Conditional Distribution of Consumers across Advertising Markets

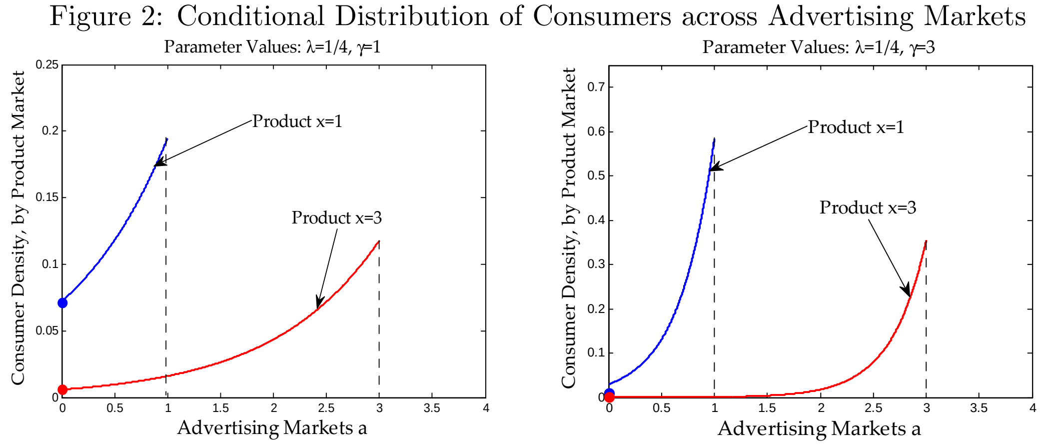
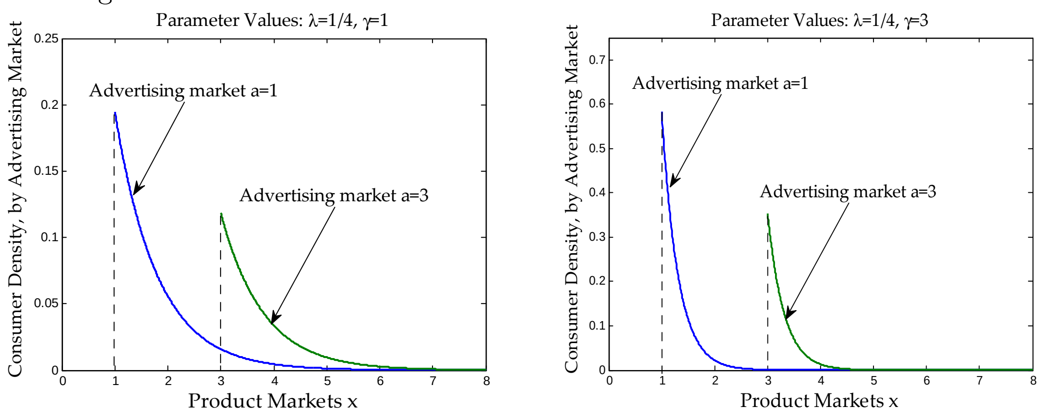


Figure 3 shows how two advertising media host consumers across different product categories, represented by the conditional distribution $s_{a,x}/s_a$, for a low and high degree of targeting, respectively.

Figure 3: Conditional Distribution of Consumers across Product Markets



3 Single Advertising Market

We begin the equilibrium analysis with the benchmark of a single advertising market. In other words, consumers of all product market segments are present in a single advertising market. In terms of the distribution of the consumers over the advertising markets, this corresponds to setting $\gamma = 0$. Each firm x can now reach its consumers by placing messages in the single advertising market $a = 0$. Consequently, in this section we drop the subscript a in the notation without loss of generality. The objective of each firm x is to maximize the profit given the unit price for advertising p. The profit π_x is given by:

$$\pi_x = \max_{m_x} \left[s_x f\left(m_x\right) - p m_x \right].$$

An advertising policy m_x generates a gross revenue $s_x \cdot f\left(m_x\right)$. The information technology $f\left(m_x\right)$, given by (3), determines the probability that a representative consumer is aware of product x, and s_x is the market share of product x. The cost of an advertising policy m_x is given by $p \cdot m_x$. The demand function of messages by firm x, as determined by the first order conditions, is given by:

$$m_x = \begin{cases} \ln\left(s_x / p\right) & \text{if} \quad s_x \geq p, \\ 0 & \text{if} \quad s_x < p. \end{cases}$$

It is an implication of the above optimality conditions that firms with a larger market share s_x choose to send more messages to the consumers. In consequence, at the equilibrium price, the firms with the largest market share choose to advertise. Let $[0, X]$ be the set of participating firms, where X is the marginal firm, and let M be the total supply of messages. The equilibrium price p for messages is then determined by the market clearing condition:

$$\int_0^X m_x dx = M.$$

Using the optimal demand of firm x and the formula for product market shares (5), we obtain

$$\int_0^X \left(\ln\left(\lambda / p\right) - \lambda x\right) dx = M. \tag{7}$$

The equilibrium price and participation are determined by imposing $m_X = 0$ and the market clearing condition in (7). The competitive equilibrium is then characterized by (p^*, X^*) with:

$$p^* = \lambda e^{-\sqrt{2\lambda M}}, \tag{8}$$

$$X^* = \sqrt{2M / \lambda}. \tag{9}$$

By inserting these formulas into the demand functions of the advertisers, we obtain the competitive equilibrium allocation of messages for a single advertising market with a given capacity M:

$$m_x^* = \begin{cases} \sqrt{2\lambda M} - \lambda x, & \text{if} \quad x \leq X^*, \\ 0, & \text{if} \quad x > X^*. \end{cases} \tag{10}$$

Thus, in the competitive equilibrium the X^* largest firms enter the advertising market and the remaining smaller firms stay out of the advertising market. With the exponential distribution of consumers across products, the number of messages sent by an active firm is linear in its rank x in the market.

We note that in the current environment, the advertising firms face only a pecuniary, or indirect, congestion effect, as messages sent by competing firms do not directly reduce the effectiveness of an advertising campaign. Rather, as other firms demand a larger number of messages, the market clearing price is driven upwards, reducing the demands of each firm x. In other words, there are no direct externalities in our model. In consequence, the competitive equilibrium implements the socially efficient allocation of advertisement messages (given λ). An easy way to see this is that with a uniform unit price of messages, the marginal returns to the messages bought by different firms are equalized. A natural question is how does the social value of advertising depend on the product market concentration. Consider holding the allocation m_x^* fixed, and increasing λ. Now the total market share of the *advertising firms* has increased, and thus fewer messages are wasted and more matches are formed. At the new equilibrium (and socially optimal allocation), welfare will be even higher, as the allocation is adjusted towards the new market shares of the advertising firms.

Proposition 1 (Single Market, Efficiency)
The social value of advertising is increasing in λ.

We next determine how the equilibrium allocation depends on the primitives of the advertising market, namely λ and M.

Proposition 2 (Single Market, Comparative Statics)

1. *The equilibrium demand of messages m_x^* is increasing in λ for $x \leq X^*/2$, and decreasing for $x > X^*/2$.*

2. *The number of advertising firms X^* is increasing in M and decreasing in λ.*

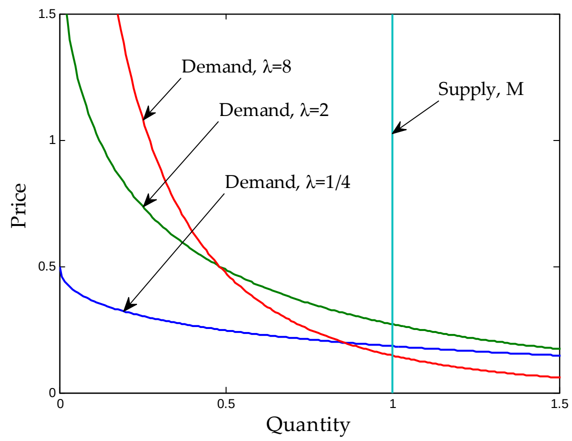
3. *The equilibrium price p^* is decreasing in M for all λ.*

4. *The equilibrium price p^* is increasing in λ iff $\lambda < 2/M$.*

5. *The price per consumer reached is increasing in x. It is decreasing in λ for $x \leq X^*/2$.*

The equilibrium price responds to the concentration measure λ in a subtle way. If the product market is diffuse, given the fixed supply, an increase in the concentration measure increases the market share (and hence the returns from advertising) of all the active firms. This drives up market demand and causes the equilibrium price to increase. Conversely, if the concentration in the product market is already large, then a further increase in the concentration weakens the marginal firm's willingness to pay for advertising. In other words, the demand of the inframarginal firms (whose market share increases) has a positive effect on the price, which is contrasted by the falling demand of the smaller, marginal firms. But as the market share of the large firms is already substantial, the increase in their demand is not sufficient to pick up the decrease in demand coming from the marginal firms, and consequently the equilibrium price falls. The additional demand of the large firms is weak because of decreasing marginal returns: an increase in the already large advertising volume leads to many more redundant messages, which generate few additional sales. Figure 4 shows the market demand and supply for different values of the concentration measure λ.

Figure 4: Equilibrium Demand for Different Concentration Measures



We can view the dichotomy in the comparative statics as driven by the determination of the marginal demand for advertising. For high enough λ, the source of the marginal demand is the marginal firm, and the price goes down with an increase in λ. Likewise for low λ, the marginal demand is driven by the inframarginal firms, and then the advertising price is increasing with λ. In this sense, the non monotonic behavior of prices is not specific to the exponential distribution of firms' market shares. On the contrary, it is a consequence of the natural tension between competition and concentration.

It is useful to recast the equilibrium of the model in hedonic terms. In this respect, Proposition 2 shows that larger firms pay a decreasing amount *per consumer reached* as λ increases. This result is driven by the concentration of the equilibrium messages in the hands of a few firms, who make large profit levels on the inframarginal units. Conversely, the price per consumer reached is increasing in λ for firms smaller than the median advertising firm. For these firms, the price per consumer reached increases until it attains a value of one (which is the marginal return to the first message $f'(0)$). In particular, for all λ, the marginal firm $X^*(\lambda)$, which pays a price per consumer reached equal to one.

One may wonder how relaxing the assumption of perfectly inelastic supply affects the comparative statics result in Proposition 2. For the case of constant supply elasticity $q = Mp^{\varepsilon}$, we can show that the equilibrium price retains the same comparative static properties: it is first increasing, then decreasing in λ. Moreover, as M becomes larger, the equilibrium price will be increasing in λ over a larger range. In particular, when the product market is very concentrated, so that the willingness to pay of the marginal firm is low, a more elastic supply reduces the number of active firms in the market. A further increase in concentration may then increase the demand of the active firms, and therefore also the price. But for high values of λ, it continues to hold that the demand falls off fast enough that the equilibrium price decreases. In particular, as λ goes to infinity, both the price and the quantity traded go to zero. However, since an increase in λ causes a drop in the quantity sold, the welfare result with respect to an increase in the concentration measure λ now becomes ambiguous.

Finally, we observe that we assumed that the value of a match is constant across product markets. The introduction of product specific profit margins – which may be thought of as the value of a match – would affects the equilibrium price and the distribution of messages. In the case of exponentially declining profit levels, the rate of decrease of profits plays a role similar to that of the concentration parameter λ. Intuitively, faster declining profits imply a more skewed equilibrium allocation of messages. As the profit margins are declining faster, the competitive equilibrium displays a decline in the number of participating firms.

4 Many Advertising Markets

We are now in a position to analyze the general model with a continuum of advertising markets. We described the distribution of consumers over different advertising markets by a (truncated) exponential distribution, and we now allow the targeting parameter γ to be strictly positive. We recall that the share of consumers who are active in product category x, and located in advertising market a is given by (6), or:

$$\frac{s_{a,x}}{s_x} \triangleq \begin{cases} e^{-\gamma x}, & \text{if} \quad a = 0, \\ \gamma e^{-\gamma(x-a)}, & \text{if} \quad a \leq x, \\ 0, & \text{if} \quad a > x. \end{cases}$$

The model with a single advertising market was described by $\gamma = 0$, while the case of perfect targeting is described by $\gamma = \infty$. The population size in advertising market $a > 0$ is given by the integral over the population shares:

$$s_{a>0} \triangleq \int_a^\infty \lambda\gamma e^{-(\lambda+\gamma)x} e^{\gamma a} dx = \frac{\gamma\lambda}{\gamma+\lambda} e^{-\lambda a}. \tag{11}$$

The share of consumers active in product market x and located in advertising market $a = 0$ is given by the residual probability of the product market segment x. As a result, the population size in advertising market $a = 0$ is:

$$s_{a=0} \triangleq \int_0^\infty \lambda e^{-(\lambda+\gamma)x} dx = \frac{\lambda}{\gamma+\lambda}. \tag{12}$$

An important implication of the exponential distribution across advertising and product markets is a certain stationarity in the composition over the consumers across the advertising markets. In particular, the relative shares of the product markets are constant across advertising markets:

$$\frac{s_{a,x}}{s_a} = (\lambda+\gamma)\, e^{-(\lambda+\gamma)(x-a)} = \frac{s_{a+n,x+n}}{s_{a+n}},$$

for all $x \geq a$ and all $n \geq 0$. Thus, while the exact composition of each advertising market is changing, the size distribution of the competing advertisers remains constant across advertising markets. This stationarity property allows us to transfer many of the insights of the single advertising market to the world with many advertising markets.

Now we consider the optimization problem of firm x in market a,

$$m_{a,x} = \arg\max_m [s_{a,x}(1 - e^{-m/s_a}) - p_a m].$$

The demand function of firm x in market a is then by:

$$m_{a,x} = s_a \ln\left(s_{a,x} / p_a s_a\right). \tag{13}$$

The equilibrium in each market a is determined through the demand functions (13), the market clearing condition, and the definition of the marginal firm in market a, given by

$m_{a,X_a^*} = 0$. In particular, we have:

$$\int_a^{X_a^*} m_{a,x} dx = s_a M,$$

and

$$s_{a,X_a^*}/s_a = p_a.$$

We can now characterize the equilibrium prices p_a^*, the number of active firms $X_a^* - a$, and the allocation $m_{a,x}^*$ of messages. The price and the number of active firms are stationary in the index a of the advertising market, that is:

$$\begin{aligned} p_a^* &= (\gamma + \lambda)\, e^{-\sqrt{2M(\gamma+\lambda)}}, & (14) \\ X_a^* - a &= \sqrt{2M/(\gamma + \lambda)}, & (15) \end{aligned}$$

for all $a \geq 0$. Observe that the stationarity of the equilibrium prices implies that the marginal utility of an additional message is equalized across markets. We also know that the competitive equilibrium allocation of the advertising space M_a in each market is efficient. Therefore, the efficient allocation of a fixed advertising space M is proportional to the size of the advertising market: $M_a = s_a \cdot M$. In other words, if the social planner had the opportunity to rearrange the supply of messages across markets, she would not find it optimal to do so. Finally, the allocation of messages is given by

$$m_{a,x}^* = \begin{cases} \gamma\lambda e^{-\lambda a}(\sqrt{2M/(\gamma+\lambda)} - (x-a)), & \text{if} \quad a > 0, \\ \lambda(\sqrt{2M/(\gamma+\lambda)} - x), & \text{if} \quad a = 0. \end{cases} \qquad (16)$$

Clearly, the larger firms $x \geq a$ receive a higher fraction of the message supply. If in particular we consider firm $x = a$, then the number of messages it receives is also increasing in the targeting ability. We can now turn to comparative statics results as we vary the targeting technology.

Proposition 3 (Many Markets, Efficiency)
The social value of advertising is strictly increasing in the targeting ability γ.

To understand the implications of targeting on social welfare, consider the relative size of consumer segment x in advertising market $a = x$:

$$\frac{s_{x,x}}{s_{a=x}} = \gamma + \lambda.$$

We observe that better targeting increases the value that firm x assigns to a message in the advertising market $a = x$. Now let us consider holding the allocation of messages $m_{a,x}$ constant, and increasing the degree of targeting γ. The volume of matched consumers and firms is increasing because of the shift in the relative sizes of advertising markets. Since we know that the competitive allocation of messages is Pareto efficient, the equilibrium (for the new γ) has unambiguously improved the social value of advertising.

The comparative statics results with respect to the concentration measure λ and message volume M do not differ qualitatively from the case of a single advertising market. More importantly, the effect of targeting ability γ and product market concentration λ on the equilibrium allocation is remarkably similar. In particular, prices are increasing in λ if and only if both the concentration and the targeting parameter are low enough. We now focus on the comparative statics with respect to γ, where a higher γ means more precise targeting. We define the equilibrium advertising revenues on each advertising market a as $R_a^* \triangleq s_a p_a^*$.

Proposition 4 (Role of Targeting)

1. *The number of messages per capita* $m_{a,x}^*/s_a$ *is increasing in* γ *for* $x \leq (a + X_a^*)/2$, *and decreasing for* $x > (a + X_a^*)/2$.

2. *The number of participating firms* $X_a^* - a$ *is decreasing in* γ.

3. *The equilibrium price* p_a^* *is increasing in* γ *if and only if* $\lambda + \gamma < 2/M$.

4. *The equilibrium revenue* R_0^* *is decreasing in* γ. *The revenues* $R_{a>0}^*$ *are increasing in* γ *if and only if* $\gamma < (1 + \sqrt{1 + 2M\lambda})/M$.

The equilibrium number of messages $m_{a,x}^*$ is increasing in γ for the participating firms larger than the median firm active on each market a. Furthermore, more precise targeting implies a lower number of active firms. Notice that the relationship between targeting ability and equilibrium price is generally hump-shaped. However, if either M or λ are large, then p_a^* is decreasing in γ for all values of γ. In other words, despite the increased social value of advertising, the equilibrium price of advertising is decreasing in the targeting ability over a large range of parameter values. In terms of revenues, it is immediate to see from equations (11) and (12) that an increase in γ leads to an increase in the size of markets $a > 0$ and to a decrease in the size of market 0. Since prices are constant, revenues in market 0 are decreasing in γ. Finally, targeting has the same qualitative effect on the equilibrium revenues in all markets $a > 0$.

We now come back to the similar effects of concentration and targeting. In particular, as with product market concentration, an increase in targeting γ reduces the demand of the marginal firm on each advertising market a. At the same time, better targeting increases the demand of the inframarginal firms. The underlying tension is the one between identifying a consumer segment precisely, and finding several advertisers who are interested in it. The resulting trade-off between competition and inframarginal willingness to pay applies to a number of contexts, such as generic vs. specific keyword searches, and more or less precise attributes targeting on social networks.

For example, Goldfarb and Tucker (2010) analyze bidding data for "personal injury" Google keywords, and report the prices paid by advertisers (law firms) in several locations. The variation in prices across locations is considerable, ranging from over $50 per click to nearly zero. We can reinterpret these results in light of our comparative statics results. In particular, fix γ at a high value, reflecting the precise targeting ability of a specific Google keyword. The different markets (zip codes) considered by Goldfarb and Tucker (2010) differ by product market concentration (λ), measured by the number of personal injury lawyers, and presumably also by the average exposure (M) of consumers to online advertising. In our Proposition 4.3, we show how these market conditions affect the profitability of a high level of targeting. In particular, the effect of targeting on the equilibrium price is positive if both λ and M are low enough. Therefore, variation in concentration and supply across different advertising markets can lead to a wide dispersion in prices for precisely targeted advertisement messages.

To conclude this section, we should point out that the exponential distributions over advertising and product markets provide particularly tractable expressions. The insights about the non monotonic behavior of the equilibrium price of advertising extend to more general production and distribution functions. In the appendix, we present a set of sufficient conditions for the comparative static to remain true beyond the exponential model presented here. The general conditions are stated in terms of the matching function and distribution over advertising and product markets: (i) the production function $f(m)$ is strictly concave, the marginal returns to the first message are bounded, and marginal returns vanish as the number of messages grows; (ii) changes in the targeting technology induce a rotation of the density function $s_{a,x}$, resulting in higher market shares for the largest firms; (iii) the targeting technology modifies (via rotations) the distribution of consumers, ranging from a (limit) diffuse distribution to complete concentration. Under these conditions, the market clearing price for a fixed finite supply is initially increasing in the targeting ability. As targeting becomes extreme, the largest firm's market share grows, as does the number of messages this firm purchases. If we impose the additional condition (iv) requiring marginal returns to decline fast enough, then the equilibrium price is eventually decreasing in the targeting parameter.

As a special case, we have derived the main results under the alternative assumption of Pareto-distributed consumers over product and advertising markets. The key difference with the exponential distribution lies in the fat tails (and hence decreasing hazard rate) of the Pareto distribution. In the product markets, this means two niche (high x) products have more similar market shares, compared to two mass (low x) products. Analogously, consumers in smaller advertising markets are relatively more dispersed than in larger advertising markets. It follows that, in small advertising markets, the marginal and inframarginal firms have more similar message demands under the Pareto than under the exponential distribution. The number of active firms in each advertising market is then no longer a constant, but rather it is increasing in a. In consequence, the willingness to pay of the marginal firm is decreasing in a, and therefore so are the equilibrium prices p_a.

5 Media Competition

In this section, we deploy our model of general and targeted advertising markets as a framework to provide insights into the effects of competition between new and established media. For this reason, we shall weaken the single-homing assumption to allow each consumer to be present in multiple markets. A first effect of competition is then to multiply the opportunities for matching an advertiser with a customer. At the same time, we maintain all the assumptions of the previous sections, namely that each buyer is only interested in one product, and that one message is sufficient to generate a sale.

We initially consider competition between traditional media, i.e. sellers of non-targeted messages, where each medium is described by a single advertising market. For example, this may represent the competition between nation-wide TV broadcasting and nation-wide newspaper publishers. We initially abstract away from the role of targeting, in order to trace out the implications of (a) the number of consumers present on each market, and (b) the distribution of consumer characteristics in each market. The analysis of competition between traditional advertising markets can shed light on the interaction of new and established (offline and online) media along at least two dimensions.

First, new media are likely to have an initially smaller user base. As a consequence, advertisement messages have a more narrow reach, though a smaller market makes it easier to reach a large fraction of the audience. Our results show that only the largest advertisers buy a positive number of messages in both markets. Furthermore, these firms purchase a constant number of advertising messages in the (new) smaller market. Therefore, media competition allows medium-sized firms to have a relatively larger presence on the new advertising market, compared to the case of a single medium.

Second, the main feature of a targeted, online advertising market is a higher concentration of consumers of a particular product, compared to a traditional market. Therefore, the degree of product market concentration, which we focus on here, plays a similar role to the degree of advertising market targeting of Section 4. In particular, differences in market concentration lead firms to sort into those markets where their messages have a higher probability of forming a match with the desired customer segment.

5.1 Competition by Symmetric Offline Media

We begin the analysis with a model of competition between two traditional media. The two media have the same distribution of consumer characteristics in their respective advertising markets. This model provides a useful benchmark to understand the effects of different user bases and consumer distributions. Therefore, we consider two media, A and B, competing for advertisers. Let $m_{a,x}$ denote the number of messages bought by firm x on advertising market $a \in \{A, B\}$, and denote by M_a the exogenous supply of advertising space on each market. We can also interpret M_a as the constant amount of time each consumer spends on market a.

As in our baseline model, the fraction of consumers reached by firm x on advertising market a is given by

$$f_{a,x} \triangleq 1 - e^{-m_{a,x}}.$$

The main novel feature of media competition is that each firm x views messages displayed in advertising markets A and B as (perfect) substitutes. We can therefore define the total awareness level generated by firm x as

$$f\left(m_{A,x}, m_{B,x}\right) \triangleq f_{A,x} + f_{B,x} - f_{A,x} f_{B,x} = 1 - e^{-m_{A,x} - m_{B,x}}.$$

As each consumer is dual-homing, there is a loss in the frequency of productive matches generated by messages in market A because the consumer may have received a duplicate message in market B (and conversely). The distribution of consumers in product markets (x) is common to the two media (a), and it is given by $s_x = \lambda \exp\left(-\lambda x\right)$. Each firm then maximizes the following profit function:

$$\pi_x \triangleq s_x f\left(m_{A,x}, m_{B,x}\right) - \sum_{a \in \{A,B\}} p_a m_{a,x}.$$

It follows that the demand function of firm x in market a is given by

$$m_{a,x} = \ln\left(\lambda / p_a\right) - m_{x,-a} - \lambda x.$$

This expression differs from the demand function in a single advertising market only because of the perfect substitutability of messages across markets. Intuitively, each firm advertises in market a until the critical level at which the value of advertising in a falls below p_a. This level depends on the amount of advertising in the other market. We denote by $m_x \triangleq \sum_a m_{a,x}$ the total number of messages demanded by firm x, and we describe the equilibrium allocation in the following proposition.

Proposition 5 (Offline Media)
The equilibrium with two competing offline media is described by:

$$\begin{aligned} p_A^* &= p_B^* = \lambda e^{-\sqrt{2\lambda(M_A+M_B)}}, \\ m_x^* &= \sqrt{2\lambda\left(M_A+M_B\right)}-\lambda x, \text{ for } x \leq X^*, \\ X^* &= \sqrt{2\left(M_A+M_B\right)/\lambda}. \end{aligned}$$

Since the messages on the two markets are perfect substitutes, it is intuitive that the equilibrium prices must also be identical. The number of active firms X^* in equilibrium reflects the increase in the total supply of messages (M_A+M_B), but it is otherwise analogous to the case of a single advertising market.

In this symmetric model, the equilibrium allocation of messages is not characterized in terms of each $m_{a,x}$. This is because perfect substitutability of messages across the two media leads to an indeterminacy in the division of message purchases across the two media. In particular, both media specialization – in which each firm $x \leq X^*$ buys messages exclusively on one market – and proportional representation of advertisers on each market, may occur in equilibrium.

Finally, notice that the equilibrium revenues of market a are non monotonic in the supply level M_a and decreasing in M_{-a}. Therefore, if we considered advertising space M_a as a strategic variable – such as a capacity choice – then market interaction would be analogous to quantity competition between the two media.

5.2 Media Markets of Different Size

We now turn to the effects of introducing a new advertising medium with a smaller user base, which is visited only by a subset of the consumers. To capture this asymmetry between the new and the established medium in a simple way, let the number of consumers present on (new) market B be given by $\delta \leq 1$. Furthermore, all consumers who visit the new medium B also visit the established medium A. For example, one may think of the early days of online advertising, or more recently about new online advertising channels (such as social networks).

We normalize the supply of messages *per capita* to M_a in each market a. Since each firm x can reach a subset of its customers on the new market B, the profit function is given by

$$\pi_x = \lambda e^{-\lambda x}((1 - e^{-m_{A,x}}) + e^{-m_{A,x}}\delta(1 - e^{-m_{B,x}/\delta})) - \sum_{a\in\{A,B\}} p_a m_{a,x}.$$

Whenever firm x buys a positive number of messages on both media, the first order conditions imply the following demand functions:

$$m_{A,x} = \ln \frac{\lambda(1-\delta)}{p_A - \delta p_B} - \lambda x, \qquad m_{B,x} = \delta \ln \frac{p_A - \delta p_B}{p_B(1-\delta)}.$$

In particular, for those firms buying on both markets, $m_{A,x}$ is decreasing in x, while $m_{B,x}$ is constant in x. In other words, the largest firms enter the new market with a constant number of messages. Intuitively, larger firms stand more to lose by shifting messages to market B and reaching fewer potential customers. More formally, suppose (as is the case) that larger firms buy a larger number of messages on the established market (A). Given the substitutability of messages across markets, this increases the demand by smaller firms in the new market. In equilibrium, this effect exactly offsets the differences in demand due to firm size, and the resulting allocation of messages on market B is flat for all dual-homing firms. Compared to the single market case, the new advertising market is then characterized by a strong presence of "medium-size" firms, and by a longer tail of relatively smaller firms.

In order to complete the description of the equilibrium allocation, we identify two thresholds, X and Z, such that firms $x \in [0, X]$ buy messages on both markets, while firms $x \in [X, Z]$ only buy a positive number of messages on market B. We summarize our findings in the following proposition.

Proposition 6 (New Advertising Medium)

1. *The equilibrium allocation of messages in the established market A is*

$$m^*_{A,x} = \sqrt{2\lambda M_A} - \lambda x, \quad \text{for } x \leq \sqrt{2M_A/\lambda}.$$

2. *The equilibrium allocation of messages in the new market B is given by*

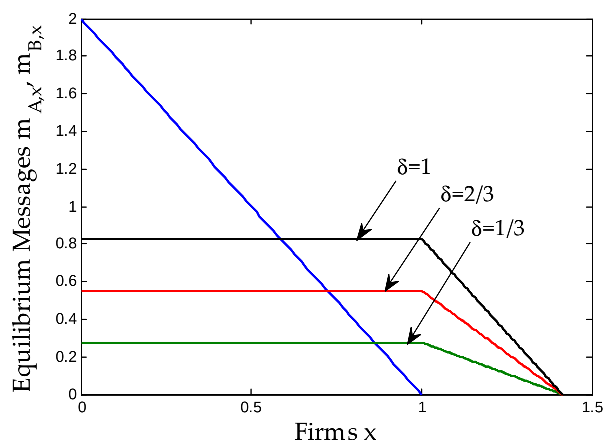
$$m^*_{B,x} = \begin{cases} \delta(\sqrt{2(M_A+M_B)\lambda} - \sqrt{2M_A\lambda}), & \text{for } \quad x \leq \sqrt{2M_A/\lambda}, \\ \delta(\sqrt{2(M_A+M_B)\lambda} - \lambda x), & \text{for } \quad \sqrt{2M_A/\lambda} < x \leq \sqrt{2(M_A+M_B)/\lambda}. \end{cases}$$

3. *The equilibrium prices are given by*

$$\begin{aligned} p^*_A &= \delta\lambda e^{-\sqrt{2(M_A+M_B)\lambda}} + (1-\delta)\lambda e^{-\sqrt{2M_A\lambda}}, \\ p^*_B &= \lambda e^{-\sqrt{2(M_A+M_B)\lambda}}. \end{aligned}$$

Figure 5 illustrates the allocation for $M_A = M_B = 1$, $\lambda = 2$, and several values of δ. When $\delta = 1$, we return to the case of symmetric advertising markets, and the specific allocation displayed below is just one of the possible equilibrium allocations. The displayed allocation for $\delta = 1$ is however the unique limit for the equilibrium allocations as $\delta \to 1$.

Figure 5: Equilibrium Demand for Different Market Sizes



Proposition 6 shows that the number of active firms in market A is determined by the single market threshold, when supply is equal to M_A. The total number of active firms is instead determined by the symmetric competition threshold, when supply is equal to

$M_A + M_B$. Finally, the equilibrium price on the larger market p_A is decreasing in the size of the smaller market δ, while the price on the smaller market p_B is independent of δ. Both results can be traced back to changes in the supply of messages in the new market. Indeed as δ increases, demand by the larger advertisers also increases. This would drive the price up and reduce the number of active firms, but this effect is offset by a proportional increase in supply.

5.3 Media Markets with Different Distributions

As we saw in Section 4, the key advantage of more targeted advertising markets is to allow fewer firms to deliver messages to a more concentrated consumer population. We now shift our attention to the role of the distribution of consumer characteristics for the competition between different media markets.

We consider two advertising markets, $a \in \{A, B\}$ and let the distribution of consumers in market a be given by $s_{a,x} \triangleq \lambda_a \exp(-\lambda_a x)$. We assume that the advertising market A has a more concentrated distribution over consumer characteristics than advertising market B, or $\lambda_A > \lambda_B$. As the distribution of consumers across advertising markets is now assumed to be different, it follows that not all consumer will be dual-homing. In particular, if a firm x has a larger presence in market A, then all its potential customers are present in market A, but only a subset of them is present in market B. Given that $\lambda_A > \lambda_B$, this is the case for the larger firms, for which $s_{A,x} > s_{B,x}$. The converse holds for the smaller firms, which have more consumers in market B. If we denote the matching technology by $f(m)$, we can write the objective function of a large firm x (for which $s_{A,x} > s_{B,x}$) as:

$$\pi_x = s_{A,x} f(m_{A,x}) + (1 - f(m_{A,x})) s_{B,x} f(m_{B,x}) - \sum_{a \in \{A,B\}} p_a m_{a,x}.$$

In other words, firm x perceives market B as a lower-quality substitute, analogous to a market with a smaller user base. Market A plays a similar role for smaller firms, for which $s_{A,x} < s_{B,x}$. It follows that larger firms have an incentive to focus on medium A and to disregard medium B.

The equilibrium allocation is now characterized by three threshold firms, $X < Y < Z$, and in particular:

1. The largest firms $x \in [0, X]$ only buy on market A.

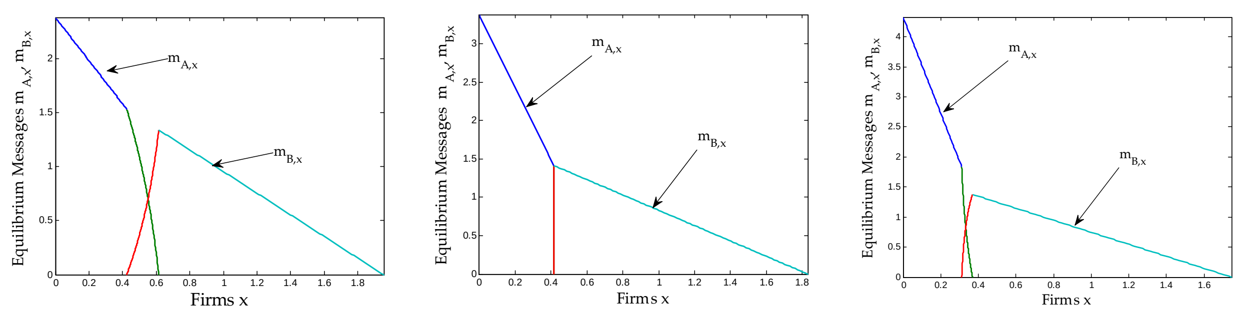
2. A set of "medium-sized" firms $x \in [X, Y]$ buy on both markets. These firms divide their purchases in varying proportions. In particular, the demand for messages in market A is decreasing in x, while the demand on market B is increasing in x. The total demands are decreasing in x.

3. The relatively smaller firms $x \in [Y, Z]$ only buy on market B.

In equilibrium, the more concentrated market attracts the largest, most valuable, firms. In particular, large firms advertise exclusively on the more concentrated market, while a subset of medium-sized firms advertise on both, and relatively smaller firms only advertise on the more diffuse market, where they can reach a larger fraction of their consumers.

The cutoff values X, Y and Z solve the market clearing conditions given the demand functions. The equilibrium market shares do not allow for an explicit expression in the case of different concentration levels, and the details of the equilibrium construction are presented in the appendix. In Figure 6, we show the allocations of messages $m_{A,x}$ and $m_{B,x}$ as a function of λ_A. The remaining parameter values are $\lambda_B = 1$, and $M_A = M_B = 1$.

Figure 6: Equilibrium Demand for Different Concentration Measures



$\lambda_A = 2$ $\qquad$ $\lambda_A = \lambda_A^* \approx 4.73$ $\qquad$ $\lambda_A = 8$

For large differences in the concentration levels λ_a, all dual homing firms $x \in [X, Y]$ satisfy $s_{A,x} < s_{B,x}$, which means they are located to the right of the crossing point of the two density functions. For small differences in the concentration levels, all $x \in [X, Y]$ satisfy $s_{A,x} > s_{B,x}$. For a given choice of the parameters (λ_B, M_A, M_B), the number of dual-homing firms $(Y - X)$ is non monotonic in λ_A, and it is equal to zero for a single value $\lambda_A = \lambda_A^*$.

When this is the case, the marginal firm $X = Y$ has an identical share of consumers in each of the two distributions.

The results in this section provide two kinds of insights into the interaction of online and offline advertising markets. Indeed, we can view each online advertising market as a separate medium with a higher concentration of consumers. With this interpretation, the prediction of the model is that Internet advertising induces the largest, most profitable advertisers to switch away from the offline medium, and to advertise only on the more concentrated online markets.

In this sense, competition by a more concentrated (targeted) market is very different from an (identical) emerging market with a smaller user base. In the former case, the established media lose the most valuable firms, as these firms find a more profitable market where to reach their customers. In the latter case, the established media share the largest buyers with the new media, and actually hold a relatively favorable position (in terms of the allocation of messages purchased by the largest firms).

In an alternative interpretation, we can view market B as the newer medium, such as the Internet, with a relatively larger presence of consumers of small (long tail) firms. Competition with a more concentrated (established) market then causes the demand for messages by smaller firms to completely crowd out the demand of larger firms, and to partially offset the demand of medium-size firms. In this sense, online advertising increases the number of firms that have access to messages in equilibrium, and allows for a more significant participation of smaller firms.

6 Offline vs. Online Media

The internet has introduced at least two technological innovations in advertising, namely (a) the ability to relate payments and performance (e.g. pay per click), and (b) an improved ability to target advertisement messages to users. We focus on the latter aspect, and in particular on the equilibrium allocation of advertising when both traditional and targeted media are present.

In our model, the targeted markets represent specialized websites, and messages can be thought of as display advertisements. We therefore refer to the traditional medium as "offline," and to the many targeted markets as "online." We then consider a population of dual-homing consumers, who spend a total time of M_A on the offline medium, and M_B on a single market a in the online, targeted, medium. More specifically, $s_a M_B$ denotes the supply of messages on each targeted market.

Because of the risk of duplication, messages sent online and offline are viewed as substitutes by each firm. This is not the case for messages sent on two different online markets, since each consumer only visits one website (in addition to the offline market). Therefore, if firm x sends a total of m_x non targeted messages and $m_{a,x}$ messages on each online market a, its profit function is given by

$$\pi_x = \int_0^x (s_{a,x}(1 - e^{-m_x - m_{a,x}/s_a}) - p_a m_{a,x}) da - p m_x.$$

The analysis of firms' advertising choices between offline and online media is intricate. In general, each firm x will want to advertise on a subset of the online markets $a \leq x$ where its consumers are located (see Figure 2), and some firms will also advertise offline. Both for tractability concerns, and to focus on the revenue implications of competition and targeting, we assume that the online medium allows to perfectly target messages to consumers. We then ask what is the equilibrium unit price of advertisement messages, and how it is affected by each firm's demands offline.

6.1 Perfect Targeting

With perfect targeting, each advertising market a is only visited by consumers of product a. Since the size of market a is identical to the market share of firm $x = a$, we immediately obtain the allocation and prices online from the individual firms' demands:

$$m_{x,x} = \lambda e^{-\lambda x} M_B, \quad (17)$$

$$p_{a=x} = e^{-M_B} e^{-m_x}. \quad (18)$$

Equation (17) implies that in equilibrium, given the supply of messages on each market, each firm reaches a constant fraction $1 - \exp(-M_B)$ of its customers.[4] Equation (18) shows

[4] Strictly speaking, we should interpret this as the limit of a model with a discrete number of product and advertising markets. In the discrete model, all consumers of product x are located in the advertising market $a = x$. Each firm x only advertises in the online market $a = x$, supply is proportional to the number of consumers in the market, and as a consequence, the probability of a match is constant across firms. These results hold independently of the number of products and markets, and carry over to our continuous model.

that the more firm x advertises offline, the lower the price on the corresponding online market $a = x$. This is again a consequence of the substitutability of messages across media.

We now turn to the message demands offline. Since each firm reaches a constant fraction $1 - \exp(-M_B)$ of its customers online, the supply of messages online simply acts as a scaling factor for each firm's demand function offline. Intuitively, each firm now has $s_x \exp(-M_B)$ potential customers offline. The equilibrium allocation is then given by

$$X^* = \sqrt{2M_A/\lambda}, \tag{19}$$

$$m_x^* = \sqrt{2\lambda M_A} - \lambda x. \tag{20}$$

Equations (19) and (20) show that the equilibrium distribution of offline messages across the participating firms, as well as the number of active firms, are both identical to the single market case. However, competition has a clear effect on the equilibrium prices and revenues, as we show in the next proposition.

Proposition 7 (Equilibrium Prices)

1. *The equilibrium price on the offline medium is given by*

$$p^* = \lambda \exp(-M_B - \sqrt{2\lambda M_A}).$$

2. *The equilibrium prices on the online markets are given by*

$$p_a^* = \begin{cases} \exp(\lambda a - M_B - \sqrt{2\lambda M_A}), & \text{for} \quad a \leq X^*, \\ \exp(-M_B), & \text{for} \quad a > X^*. \end{cases}$$

Consistent with intuition, the offline price p^* is decreasing in M_B. This reflects the decline in each firm's willingness to pay for regular advertisements when an alternative, better targeted market is present. In other words, a targeted online market does not modify the composition of the offline market, but lowers the equilibrium profits. The prices in the online markets are initially increasing in a, and then constant. This reflects the allocation of messages offline, where relatively smaller firms buy a lower number of messages, and are willing to pay more for M_B messages per capita online. Furthermore, the prices paid online are constant for all the firms that do not participate in the offline market. In other words, "niche" online markets, where customers of long tail firms are likely to be present, are not affected at all by media competition. In this sense, as emphasized by Anderson (2006), online advertising allows to reach new segments of the consumer population, which are distinct from the intended audience of the firms that actively advertise offline.

Up to this point, we have imposed no restrictions on the total supply of advertising space. We now seek to assess the implications of the consumer's relative exposure to online and offline advertisements. For this reason, we interpret the supply as the outcome of the consumer's time allocation decision. In particular, we assume each consumer spends a fraction β of her time M in the online medium. We then have $M_A = (1-\beta)\,M$ and $M_B = \beta M$, and we obtain the following comparative statics result.

Proposition 8 (Online Exposure)

1. *The equilibrium price in online markets $a > X^*$ is decreasing in β.*

2. *The equilibrium prices in online markets $a \leq X^*$ and in the offline market are decreasing in β if and only if $\beta \leq 1 - \lambda/2M$.*

3. *The equilibrium revenues in the offline market are decreasing in β if and only if*

$$\beta \geq \beta^* = \begin{cases} \max\left\{1 - \frac{\sqrt{2}(\sqrt{\lambda}+\sqrt{\lambda-8})}{4\sqrt{M}}, 0\right\} & \text{if} \quad \lambda \geq 8, \\ 0 & \text{if} \quad \lambda < 8. \end{cases}$$

In other words, the equilibrium price of advertisements offline does not vary monotonically with consumers' exposure to online media. When online exposure is low, the greater efficiency of online targeted messages reduces the marginal willingness to pay for advertisements offline. This reduction more than offsets the price increase resulting from a lower supply of offline messages. In particular, recall that (Proposition 5) under competition by symmetric offline media, the equilibrium price depends on the total supply M. In this sense, for low β, the growth of online advertising markets is more detrimental for the price of an offline medium than the loss of market share to new traditional competitors. However, as the online exposure β increases further, an important additional effect appears: a decrease in supply on the offline medium changes the identity – and hence the willingness to pay – of the marginal firm. As offline supply decreases, the largest, most valuable customers buy most of the advertising space. This effect, which is due to a reduction in supply, keeps the marginal returns high, and hence drives up the equilibrium price. This is radically different from the increase in concentration with a fixed supply we analyzed in Section 3.

In addition, as the price offline increases, so do the online prices on the online markets $a \leq X^*$. In fact, the decline in the number of matches offline more than compensates for the increase in advertising space on these online markets. Notice that, for higher degrees of concentration λ, the change in the composition of the offline demand occurs faster, leading prices to increase in β. In particular, for $\lambda > 2M$, the equilibrium price is increasing in β everywhere.

Finally, we turn to the equilibrium revenues. As consumers' exposure to online advertising β goes to 1, the equilibrium revenues offline must vanish, as the medium runs out of advertising space. More generally, the offline media's revenues are decreasing in β as long as the concentration level λ is not too extreme. However, as shown in Proposition 8, when the concentration level and the total supply are very high, and β is low, an increase in online exposure can actually improve offline revenues. This occurs through the demand composition effect, and the resulting higher price of advertising, in spite of the reduction in the quantity of messages sold offline.

6.2 Imperfect Targeting

When we consider imperfect targeting levels, our predictions are similar to those of the model with different degrees of concentration. In particular, the online market $a = 0$ is a close substitute for the offline medium, as all consumer types are present (though with different intensities). Consider the hierarchical structure of the advertising markets on a targeted medium. If firm x advertises on all markets $a \leq x$, then it can reach all of its customers both offline and online. This means the two media are perfect substitutes, and the prices would have to make the firm indifferent, in order to justify dual homing. Therefore, the price offline must be to equal a weighted average of the prices on the online markets firm x is active in:

$$p = \int_0^x p_a s_a da.$$

Clearly, this condition cannot hold for more than one firm. In equilibrium, it must then be the case that firms $[x_L, x_H]$ advertise offline, while no firm $x > x_L$ advertises on online market $a = 0$. The message is similar to the model with different concentrations. Indeed, the two models are very close, as the concentration parameter of the distribution of consumers on market 0 is equal to $\gamma + \lambda$. As a result, the largest firms leave the offline medium and advertise exclusively online, in the largest markets a, leading to a decrease in the price of the offline medium.

This effect is somewhat mitigated if the online market has a smaller user base. As in the case of competition between offline media of different sizes, we can show that all firms larger than a critical x^* advertise both offline and on all the available online markets (*i.e.*, each firm x buys messages on markets $a \in [0, x]$). In terms of comparative statics, better targeting reduces the demand for online messages by "long tail" firms, and induces a higher concentration in the offline medium. At the same time, larger firms are also able to reach a larger fraction of their customers online, and this reduces the overall profitability of the offline medium. As a consequence, the offline price p^* and the number of firms participating offline x^* are decreasing in γ, while the distribution of offline messages is more concentrated as γ increases.

7 Concluding Remarks

In this paper, we developed a novel model to understand and evaluate the implications of targeting in advertising markets. The model provided a framework for the systematic analysis of the trade-offs that arise due to changes in the targeting technology. We adopted a hierarchical framework to rank products and advertising markets of different sizes. We explored in particular the tension between competition and value extraction that appears as the targeting ability of the various media improve.

We discussed earlier the robustness of our findings to alternative matching structures and targeting technologies. Our analysis identifies conditions that extend our results beyond the exponential framework adopted here. As these conditions are not specific to the case of display advertising, or broadcasting, it follows that our model can provide insight into the effects of detailed users information in the hands of social networks and on the profitability of IP address tracking.

The analysis we have presented is the outcome of a number of modeling choices which constrain the scope of our results in some directions. We now conclude by discussing several directions for future research. The price of advertising was determined in a competitive equilibrium model. While the competitive equilibrium is the natural benchmark, it is of interest to consider the pricing of advertising in strategic environments. In an earlier version, (Bergemann and Bonatti (2010)), we investigate the equilibrium pricing when each advertising market is owned by a single or a few publishers, each one maximizing his revenues. For the case of a single publisher, we analyze the optimal nonlinear pricing tariff in order to discriminate across advertising firms of different segment size. We then extend the model to a small number of publishers in which each publisher determines the supply of messages in his outlet. In particular, we establish that for a sufficiently large, but finite number of publishers, the Nash equilibrium yields the competitive equilibrium outcome analyzed here. Clearly, extending our model to incorporate the auctions for keywords in the sponsored search environment, or the emerging ad exchange model, might offer valuable additional insights in this respect.

In our model, the advertisers were competing for messages but they were not competing for consumers. In other words, competition among firms for advertising messages did not interact with their competition in the product market. A natural next step therefore might be to enrich the current model with advertisers that are directly competing in the product markets. The equilibrium price for advertising, in particular in highly targeted markets, may then interact with the intensity of competition in the product market.

Finally, in the current work, the distribution of the consumers across the advertising markets was assumed to be given exogenously. A natural next step would be to extend the model to consumers whose location choice with respect to the advertising outlets reflects an optimization decision. Along the lines of Anderson and Coate (2005), each medium provides content and advertising for the consumer. While content has positive value to the consumers, advertising has a negative value. In the spirit of the current model, the disutility of advertising could be increasing in the distance of the advertisement message from the interest of the reader. In such a framework one could investigate the competition between a general interest traditional medium, such as the New York Times or the Wall Street Journal, and a general interest portal, such as Google or Yahoo!, that can personalize the distribution of advertising message by conditioning on the characteristics about the reader.

Appendix A

Proof of Proposition 1. The average probability of a match, which is equal to the total fraction of consumers reached, is given by

$$W(\lambda, M) = \int_0^{X^*} s_x(1 - e^{-m_x^*})dx = 1 - \frac{1 + \sqrt{2M\lambda}}{e^{\sqrt{2M\lambda}}},$$

which is increasing in λ. ■

Proof of Proposition 2. (1.)–(4.) The comparative statics results can be derived directly by differentiating expressions (8), (9), and (10) in the text.
(5.) The total expenditure of firm $x \leq X^*$ is given by

$$p^* m_x^* = \lambda e^{-\sqrt{2\lambda M}}(\sqrt{2\lambda M} - \lambda x),$$

and the total number of consumers reached is

$$s_x(1 - e^{-m_x^*}) = \lambda e^{-\lambda x}(1 - e^{\lambda x - \sqrt{2\lambda M}}).$$

Therefore, the price paid by firm x per consumer reached is given by

$$\frac{p^* m_x^*}{s_x\left(1 - e^{-m_x^*}\right)} = \frac{\sqrt{2\lambda M} - \lambda x}{e^{\sqrt{2\lambda M} - \lambda x} - 1} = \frac{z}{e^z - 1},$$

which is decreasing in z (with $z = \sqrt{2\lambda M} - \lambda x$), and therefore increasing in x. It is also decreasing in λ if $x < \sqrt{M/2\lambda}$ (which represents the median active firm). ■

Proof of Proposition 3. The average probability of a match now takes into account the fraction of consumers reached in the exterior market as well as in the interior markets. It is given by,

$$W(\lambda, \gamma, M) = \int_0^{\infty} \int_a^{X_a^*} s_{a,x}(1 - e^{-m_{a,x}/s_a})dadx + \int_0^{X_0^*} s_{x,0}(1 - e^{-m_{x,0}/s_0})dx,$$

where $m_{a,x}^*$ is given by (16) in the text. Therefore, we obtain

$$W(\lambda, \gamma, M) = 1 - \frac{1 + \sqrt{2M(\lambda + \gamma)}}{e^{\sqrt{2M(\lambda+\gamma)}}},$$

which is increasing in λ and γ. ■

Proof of Proposition 4. (1.)–(4.) These statements follow from differentiation of expressions (14), (15), and (16) in the text. ■

Proof of Proposition 5. From the first order conditions for firm x, we obtain

$$\begin{aligned} 1 - f_{A,x} &= e^{-m_{A,x}} = e^{\lambda x} \frac{p_A}{\lambda \left(1 - f_{B,x}\right)}, \\ 1 - f_{B,x} &= e^{-m_{B,x}} = e^{\lambda x} \frac{p_B}{\lambda \left(1 - f_{A,x}\right)}. \end{aligned}$$

It follows that in equilibrium we must have $p_A = p_B = p$, and that the sum of the demands is given by

$$m_{A,x} + m_{B,x} = \ln \frac{\lambda}{p} - \lambda x.$$

Consider the market clearing condition for A and B combined,

$$\int_0^X \left(m_{A,x} + m_{B,x}\right) dx = M_A + M_B,$$

and the results follow as in the single-homing case. ■

Proof of Proposition 6. The first order conditions may be written as

$$\begin{aligned} \lambda_A e^{-\lambda_A x} \left(1 - \delta \left(1 - e^{-m_{B,x}/\delta}\right)\right) e^{-m_{A,x}} - p_A &= 0, \\ \lambda_A e^{-\lambda_A x} e^{-m_{A,x}} \left(1 - e^{-m_{B,x}/\delta}\right) - p_B &= 0. \end{aligned}$$

Solving for $m_{A,x}$ and $m_{B,x}$, and simplifying, we obtain

$$\begin{aligned} m_{A,x} &= \ln \frac{\lambda \left(1 - \delta\right)}{p_A - \delta p_B} - \lambda x, \\ m_{B,x} &= m_B = \delta \ln \frac{p_A - \delta p_B}{p_B \left(1 - \delta\right)}, \text{ for } x \in [0, X]. \end{aligned}$$

For all firms $x \in [X, Z]$, we have $m_{A,x} = 0$ and $m_{B,x} = \delta \left(\ln \lambda / p_B - \lambda x\right)$ as in the single-homing case. Since by construction, the marginal firm X satisfies $m_{A,X} = 0$, we have $(1 - \delta) \lambda \exp(-\lambda X) = p_A - \delta p_B$. Similarly, we have $m_{B,Z} = 0$, and so $\lambda \exp(-\lambda Z) = p_B$.

We can now write the market clearing conditions as follows:

$$\begin{aligned} \int_0^X m_{A,x} dx &= \int_0^X \lambda (X - x)\, dx = M_A, \\ X m_{B,x} + \int_X^Z m_{B,x} dx &= X\delta\lambda (Z - X) + \int_X^Z \delta\lambda (Z - x)\, dx = \delta M_B. \end{aligned}$$

Therefore

$$X = \sqrt{\frac{2M_A}{\lambda}}, \quad Z = \sqrt{\frac{2(M_A + M_B)}{\lambda}},$$

which implies that

$$\begin{aligned} p_A &= \delta\lambda e^{-\sqrt{2(M_A+M_B)\lambda}} + (1-\delta)\,\lambda e^{-\sqrt{2M_A\lambda}} \\ p_B &= \lambda e^{-\sqrt{2(M_A+M_B)\lambda}}, \end{aligned}$$

which completes the proof. ■

Proof of Proposition 7. The price offline is equal to $\lambda \exp(-\lambda X^*)$, where X^* is the marginal firm characterized in (19). The prices offline follow from substitution of (20) into (17) and (18). ■

Proof of Proposition 8. (1.) The equilibrium price p_a for $a > X^*$ is decreasing in M_B and hence in β.

(2.) The sign of the derivative of p and p_a for $a \leq X^*$ depends on the term

$$\beta M + \sqrt{2\lambda (1-\beta) M},$$

which is decreasing in β everywhere if $\lambda > 2M$, or increasing in β for $\beta \leq 1 - \lambda/2M$.

(3.) The sign of the derivative of the offline revenues

$$\frac{\partial}{\partial p}\left[M(1-\beta)\exp\left(-\beta M - \sqrt{2\lambda(1-\beta)M}\right)\right],$$

depends on the sign of the following term:

$$\sqrt{M\lambda(1-\beta)} - \sqrt{2}M(1-\beta) - \sqrt{2}.$$

This expression is always negative for $\lambda < 8$. If $\lambda \geq 8$ then the relevant root is given by

$$\max\left\{1 - \frac{\sqrt{\lambda} + \sqrt{\lambda - 8}}{2\sqrt{2M}}, 0\right\},$$

which concludes the proof. ■

Appendix B

This Appendix contains the construction of the competitive equilibrium with competing media markets and different consumer concentrations across media markets.

The case of $s_{A,X} > s_{B,X}$ We begin with the case of $s_{A,X} > s_{B,X}$. Consider the objective function of any firm x such that $\lambda_A e^{-\lambda_A x} > \lambda_B e^{-\lambda_B x}$:

$$\pi_x = \lambda_A e^{-\lambda_A x}\left(1-e^{-m_{A,x}}\right) + e^{-m_{A,x}}\lambda_B e^{-\lambda_B x}\left(1-e^{-m_{B,x}}\right) - p_A m_{A,x} - p_B m_{B,x}.$$

The first order conditions are given by

$$\begin{aligned} e^{-m_{A,x}}\left(\lambda_A e^{-\lambda_A x} - \lambda_B e^{-\lambda_B x}\left(1-e^{-m_{B,x}}\right)\right) - p_A &= 0, \\ e^{-m_{B,x}} e^{-m_{A,x}} \lambda_B e^{-\lambda_B x} - p_B &= 0. \end{aligned}$$

By construction, the marginal firm satisfies $m_{A,X} > 0$ and $m_{B,X} = 0$. The indifference condition for the marginal firm requires that

$$e^{-m_{A,x}}\left(\lambda_A e^{-\lambda_A X} - \lambda_B e^{-\lambda_B X}\right) = p_A - p_B,$$

and therefore, from the first order conditions, $m_{A,X} = \ln\left(s_{A,X}/p_A\right)$, or equivalently:

$$X = \frac{1}{\lambda_A - \lambda_B} \ln \frac{\lambda_A/\lambda_B}{p_A/p_B}.$$

Similarly, the marginal firm Y has $m_{A,Y} = 0$ and $m_{B,Y} > 0$. Therefore, indifference requires

$$\lambda_A e^{-\lambda_A Y} - \lambda_B e^{-\lambda_B Y} = p_A - p_B.$$

Since we know from the marginal firm X that $p_A > p_B$, we also know $\lambda_A e^{-\lambda_A Y} > \lambda_B e^{-\lambda_B Y}$, that is, all firms active on both media will lie before the crossing point of the two density functions. Furthermore, for all the dual-homing firms $x \in [X, Y]$, we know both first order conditions hold with equality, and so

$$\begin{aligned} e^{-m_{A,x}} &= \frac{p_A - p_B}{\lambda_A e^{-x\lambda_A} - \lambda_B e^{-x\lambda_B}}, \\ e^{-m_{B,x}} &= \frac{p_B}{\lambda_B e^{-\lambda_B x}} e^{m_{A,x}}. \end{aligned}$$

Finally, exploiting the two conditions $m_{A,Y} = 0$ and $m_{B,X} = 0$, we obtain the equilibrium prices as a function of the cutoff firms:

$$\begin{aligned} p_A &= \lambda_A e^{-X\lambda_A} \frac{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}, \\ p_B &= \lambda_B e^{-X\lambda_B} \frac{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}. \end{aligned}$$

Now we consider the market clearing conditions:

$$\int_0^X \left(\ln \frac{\lambda_A}{p_A} - \lambda_A x \right) dx + \int_X^Y \ln \frac{\lambda_A e^{-x\lambda_A} - \lambda_B e^{-x\lambda_B}}{p_A - p_B} dx = M_A, \tag{21}$$

$$-\int_X^Y \ln \frac{\lambda_A e^{-x\lambda_A} - \lambda_B e^{-x\lambda_B}}{p_A - p_B} dx + \int_X^Z \left(\ln \frac{\lambda_B}{p_B} - \lambda_B x \right) dx = M_B, \tag{22}$$

where the marginal firm Z satisfies:

$$Z = \frac{1}{\lambda_B} \ln \frac{\lambda_B}{p_B} = X + \frac{1}{\lambda_B} \ln \frac{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}.$$

We can then substitute the expressions for p_A, p_B, and Z into (21) and (22), solve them numerically for the two unknowns X and Y, and verify that $X < Y < h(\lambda_A, \lambda_B)$, where

$$h(\lambda_A, \lambda_B) = (\ln \lambda_A - \ln \lambda_B) / (\lambda_A - \lambda_B)$$

represents the crossing point of the two distributions.

The case of $s_{A,X} < s_{B,X}$ Next we consider the case of $s_{A,X} < s_{B,X}$. Consider the objective function of any firm x such that $\lambda_A e^{-\lambda_A x} < \lambda_B e^{-\lambda_B x}$:

$$\pi_x = \lambda_A e^{-\lambda_A x} e^{-m_{B,x}} \left(1 - e^{-m_{A,x}}\right) + \lambda_B e^{-\lambda_B x} \left(1 - e^{-m_{B,x}}\right) - p_A m_{A,x} - p_B m_{B,x}.$$

The first order conditions are given by

$$\begin{aligned}\lambda_A e^{-\lambda_A x} e^{-m_{B,x}} e^{-m_{A,x}} - p_A &= 0,\\ e^{-m_{B,x}}\left(\lambda_B e^{-\lambda_B x} - \lambda_A e^{-\lambda_A x}\left(1 - e^{-m_{A,x}}\right)\right) - p_B &= 0.\end{aligned}$$

By construction, $m_{A,X} > 0$ and $m_{B,X} = 0$. Again, indifference requires that

$$\lambda_A e^{-\lambda_A X} - \lambda_B e^{-\lambda_B X} = p_A - p_B.$$

Similarly, the marginal firm Y has $m_{A,Y} = 0$ and $m_{B,Y} > 0$. Therefore, indifference also requires

$$\frac{\lambda_A e^{-\lambda_A Y}}{\lambda_B e^{-\lambda_B Y}} = \frac{p_A}{p_B},$$

or equivalently

$$Y = \frac{1}{\lambda_B - \lambda_A} \ln \frac{\lambda_A/\lambda_B}{p_A/p_B}.$$

For all the dual-homing firms $x \in [X, Y]$, we know both first order conditions must hold with equality, and so we have

$$\begin{aligned}e^{m_{A,x}} &= \frac{\lambda_A e^{-\lambda_A x}}{p_A} e^{-m_{B,x}}\\ e^{m_{B,x}} &= \frac{\lambda_B e^{-x\lambda_B} - \lambda_A e^{-x\lambda_A}}{p_B - p_A}.\end{aligned}$$

We can then solve for the prices,

$$\begin{aligned}p_A &= \lambda_A e^{-Y\lambda_A} \frac{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}},\\ p_B &= \lambda_B e^{-Y\lambda_B} \frac{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}.\end{aligned}$$

Now we have the following market clearing conditions:

$$\begin{aligned}\int_0^X \left(\ln \frac{\lambda_A}{p_A} - \lambda_A x\right) dx + \int_X^Y \left(\ln \frac{\lambda_A}{p_A} - \lambda_A x - m_{B,x}\right) dx &= M_A,\\ \int_X^Y \ln \frac{\lambda_B e^{-x\lambda_B} - \lambda_A e^{-x\lambda_A}}{p_B - p_A} dx + \int_Y^Z \left(\ln \frac{\lambda_B}{p_B} - \lambda_B x\right) dx &= M_B,\end{aligned}$$

where the marginal firm Z satisfies:

$$\lambda_B e^{-\lambda_B Z} = p_B,$$

or

$$Z = Y + \frac{1}{\lambda_B} \ln \frac{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}.$$

Again, we can use the expressions for p_A, p_B, and Z, solve the market clearing conditions numerically, and verify that $h(\lambda_A, \lambda_B) < X < Y$.

The case of $s_{A,X} = s_{B,X}$ We conclude the construction of the equilibrium by considering $s_{A,X} = s_{B,X}$ For the marginal firms X and Y to be both equal to $h(\lambda_A, \lambda_B) = (\ln \lambda_A - \ln \lambda_B) / (\lambda_A - \lambda_B)$, it must be that $p_A = p_B = p$ and that $X = Y$. Thus

$$X = \frac{1}{\lambda_A - \lambda_B} \ln \frac{\lambda_A/\lambda_B}{p_A/p_B} = \frac{1}{\lambda_A - \lambda_B} \ln(\lambda_A/\lambda_B). \tag{23}$$

Market clearing on $a = A$,

$$\begin{aligned} \int_0^X \left(\ln \frac{\lambda_A}{p} - \lambda_A x\right) dx &= X \ln \frac{\lambda_A}{p} - \lambda_A X^2/2 = M_A, \\ \lambda_A \exp\left(-\frac{M_A}{X} - \frac{\lambda_A X}{2}\right) &= p, \end{aligned}$$

and then on $a = B$:

$$\int_X^{\frac{1}{\lambda_B}\left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2}\right)} \left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2} - \lambda_B x\right) dx = M_B.$$

Substituting the definition of X from (23), we obtain the condition that λ_A must satisfy:

$$\begin{aligned} \int_X^{\frac{1}{\lambda_B}\left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2}\right)} \left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2} - \lambda_B x\right) dx &= M_B, \\ -\ln(\lambda_A/\lambda_B) \frac{1}{2} \frac{\lambda_A}{\lambda_A - \lambda_B} + \frac{(\lambda_A - \lambda_B) M_A}{\ln(\lambda_A/\lambda_B)} &= \sqrt{2\lambda_B M_B}. \end{aligned}$$

Appendix C

We now present general conditions beyond the exponential model of Section 2.2 that extend the comparative statics results on the equilibrium price of advertising. It suffices here to state the result for a single advertising market, as the generalization to the case of a representative targeted market a is straightforward.

Let consumers be distributed across product markets x according to a density function $s(x, \lambda)$. The parameter $\lambda \geq 0$ measures the concentration of the consumer distribution. Each firm x buys messages m in order to maximize

$$V(x) = \max_{m} \left[s(x, \lambda) \cdot f(m) - pm \right].$$

We now introduce the following assumption on the production function.

Assumption 1 (Production Function)

1. *$f(m)$ is strictly increasing and strictly concave.*

2. *$f'(0)$ is bounded (and here we normalize it to 1).*

3. $\lim_{m\to\infty} f'(m) = 0$.

We also introduce the following assumption on the distribution of consumers across product markets.

Assumption 2 (Consumer Distribution)

1. *The density $s(x, \lambda)$ is strictly decreasing on $[0, \infty)$ and $s(0, \lambda)$ is bounded.*

2. *The family of densities $s(\cdot, \lambda)$ is ordered by rotations. In particular, there exists a unique $\hat{x}(\lambda)$ such that $s_\lambda(x, \lambda) > 0$ if and only if $x < \hat{x}(\lambda)$.*

3. *The densities $s(\cdot, \lambda)$ range from diffuse to completely concentrated. In particular, $\lim_{\lambda\to 0} \int_0^x s(t, \lambda)\, dt = 0$ for all $x > 0$ and $\lim_{\lambda\to\infty} \int_0^x s(t, \lambda)\, dt = 1$ for all $x > 0$.*

4. *The rotation point $\hat{x}(\lambda)$ is continuous and strictly decreasing in λ. Furthermore, $\lim_{\lambda\to 0} \hat{x}(\lambda) = \infty$ and $\lim_{\lambda\to\infty} \hat{x}(\lambda) = 0$.*

Assumption 2.2 requires the concentration parameter λ to rotate the *density* of consumer tastes. This implies the distributions of consumers are ranked by first-order stochastic dominance. This concept is clearly related to the rotation order introduced by Johnson and Myatt (2006). The two notions are slightly different, because Johnson and Myatt (2006) focus on rotations of the *distribution* of consumer valuations. Their rotation order is therefore much closer to the notion of second-order stochastic dominance. However, in our model, a rotation of the density of consumer tastes induces a rotation in the firms' demand curve for advertising. In Johnson and Myatt (2006), the demand curve is derived from consumers' valuations directly, and rotations in the distribution translate into rotations of the market demand curve. In this sense, the two notions are very similar: in our model, a higher degree of concentration increases demands of high-valuation consumers, and decreases those of low-valuation consumers. This is analogous to the effect of releasing private-value information to consumers in Johnson and Myatt (2006), which typically induces a mean-preserving spread in the distribution of valuations.

The first order condition for the maximization problem of firm x is given by

$$s(x,\lambda)\, f'(m) = p.$$

Firm x then demands $m(x)$ messages, where:

$$m(x) = g\left(s(x,\lambda)/p\right).$$

The individual firm's demand function $g(y) \triangleq (f')^{-1}(1/y)$ is increasing, strictly concave and (by Assumption 1.2) satisfies $g(1) = 0$.

The market clearing condition as a function of the marginal firm $X(\lambda,p)$ may be written as:

$$\int_0^{X(\lambda,p)} g\left(s(x,\lambda)/p\right) dx = M.$$

We now rewrite this condition in terms of firms' sizes. Let

$$\theta \triangleq s(x,\lambda),$$

and since $s(x,\lambda)$ is strictly decreasing in x we can use the inverse function

$$x = h(\theta, \lambda),$$

which denotes the *pdf* of the distribution of sizes $h(\theta,\lambda)$. Firms' sizes θ take values between 0 and $\bar{\theta}(\lambda) \triangleq s(0,\lambda)$. For example, for the exponential distribution, $h = (1/\lambda)\ln(\lambda/\theta)$ with $\theta \in [0,\lambda]$.

By the change of variable $x \to \theta$ (so that $dx = h_\theta(\theta,\lambda)\,d\theta$) we can rewrite the market clearing condition as

$$\int_p^{\bar{\theta}(\lambda)} -g(\theta/p)\,h_\theta(\theta,\lambda)\,d\theta = M.$$

Integration by parts (using the fact that $h\left(\bar{\theta}(\lambda),\lambda\right) = 0$) yields

$$\int_p^{\bar{\theta}(\lambda)} \frac{1}{p} g'(\theta/p)\,h(\theta,\lambda)\,d\theta = M. \tag{24}$$

It is immediate to verify that the market demand is decreasing in p. Therefore, the sign of $dp/d\lambda$ is determined by

$$sign\left[p'(\lambda)\right] = sign\left[\int_p^{\bar{\theta}(\lambda)} \frac{1}{p} g'(\theta/p)\,h_\lambda(\theta,\lambda)\,d\theta\right]. \tag{25}$$

Finally, define the analogue to the rotation point $\hat{x}(\lambda)$ as $\hat{\theta}(\lambda)$. Assumption 2 implies

$$\begin{aligned} h_\lambda(\theta,\lambda) &> 0 \text{ iff } \theta > \hat{\theta}(\lambda), \\ \hat{\theta}'(\lambda) &> 0. \end{aligned}$$

Notice from (25) that if $p > \hat{\theta}(\lambda)$ then $p'(\lambda) > 0$. Similarly, if $g(\cdot)$ were linear, since $\int_0^{\bar{\theta}(\lambda)} h_\lambda(\theta,\lambda)\,d\theta = 0$, we would have $p'(\lambda) \propto \int_p^{\bar{\theta}(\lambda)} h_\lambda(\theta,\lambda)\,d\theta > 0$. Intuitively, the higher the curvature of $g(\cdot)$ the lower is the relative weight assigned to the high θ (with a positive h_λ) in (25). We can make this intuition formal through Assumption 3.

Assumption 3 (Production and Distribution Functions)
The production and distribution functions satisfy

$$\lim_{\lambda\to\infty}\left[s(0,\lambda) f'\left(\frac{M}{\hat{x}(\lambda)}\right)\right] < \infty.$$

With this additional condition, we can now prove the following Lemma.

Lemma 1 *Under Assumptions 1–3, $p < \hat{\theta}(\lambda)$ for λ high enough.*

Proof. Notice first that since demands are decreasing in x and f is strictly concave,

$$p(\lambda) = s(0,\lambda) f'(m(0)) < s(0,\lambda) f'\left(\frac{M}{X(\lambda)}\right).$$

Suppose now $p \geq \hat{\theta}(\lambda)$. Then the marginal sized firm (p) is larger than the rotation size $\hat{\theta}$, implying $X(\lambda) < \hat{x}(\lambda)$. But then Assumption 3 implies p is bounded since

$$p(\lambda) < s(0,\lambda) f'\left(\frac{M}{X(\lambda)}\right) < s(0,\lambda) f'\left(\frac{M}{\hat{x}(\lambda)}\right).$$

Since $\hat{\theta}(\lambda) \to \infty$ by Assumption 2.3, we derived a contradiction. ■

For high levels of concentration λ we can then show the following.

Proposition 9 (Equilibrium Price)
Suppose the production and distribution functions satisfy Assumptions $1-3$*. Then for λ high enough $p'(\lambda) < 0$, and for λ low enough $p'(\lambda) > 0$.*

Proof. We first rewrite the right hand side of equation (25) as (p^{-1} times)

$$\int_{p}^{\hat{\theta}(\lambda)} g'\left(\frac{\theta}{p}\right) h_\lambda(\theta,\lambda)\, d\theta + \int_{\hat{\theta}(\lambda)}^{\bar{\theta}(\lambda)} g'\left(\frac{\theta}{p}\right) h_\lambda(\theta,\lambda)\, d\theta.$$

By the Lemma, for λ high enough, we have $p < \hat{\theta}(\lambda)$. By Assumption 2 we have $\lim_{\lambda\to\infty}\bar{\theta}(\lambda) = \infty$ and $\lim_{\lambda\to\infty}\hat{\theta}(\lambda) = \infty$. Therefore, as $\lambda \to \infty$, the weights $g'(\theta/p)$ in the second term are vanishing, and the sign of $p'(\lambda)$ is determined by the first term, for which $h_\lambda < 0$.

Finally, it follows from Assumption 2.3 that $p(0) = 0$. But then the price $p(\lambda)$ cannot be monotone in λ, and must be increasing for λ low enough. ■

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