Verifiability and Group Formation in Markets*

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Abstract

We consider group formation with asymmetric information. Agents have unverifiable characteristics as well as the verifiable qualifications required for memberships in groups. The characteristics can be chosen, such as strategies in games, or can be learned, such as skills required for jobs. They can also be innate, such as intelligence. We assume that the unverifiable characteristics are observable ex post (after groups have formed) in the sense that they may affect the output and utility of other agents in the group. They are not verifiable ex ante, which means that prices for memberships cannot depend on them, and they cannot be used for screening members. The setup includes problems as diverse as moral hazard in teams, screening on ability, and mechanism design. Our analysis, including the definition of equilibrium and existence, revolves around the randomness in matching. We characterize the limits on efficiency in such a general equilibrium, and show that a sufficiently rich set of group types can ensure the existence of an efficient equilibrium.

Keywords: clubs, games, contracts, lotteries, general equilibrium
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1 Introduction

What determines the contracts, mechanisms, games, and other organizational forms that are used in an economy? What role does competition play in shaping incentives and institutional design? How does private information enter markets, and to what extent does competition mitigate or magnify the inefficiencies that arise from asymmetric information? This paper develops a model designed to address these questions.

Classical general equilibrium theory focuses on anonymous price-taking agents, typically ignoring any strategic effects or incentives. Modern theory of institutions, contracts, and mechanism design focuses on incentives and private information in isolation, typically ignoring market forces that might alter organizational design. As a consequence, neither can explain how incentives might influence markets or how competition might select among institutions.

To address such issues, this paper develops a model that melds key aspects of contract theory, mechanism design and game theory with general equilibrium theory. Agents interact strategically in small groups, taking into account incentives and the effects of their actions on group outcomes, but trade anonymously in markets, taking prices as given. This allows us to study the interplay between market forces, private information, the provision of incentives, and the structure of institutions, and to assess the role of markets in limiting inefficiencies that stem from asymmetric information.

We take as a starting point models of group formation in markets developed in club theory. In these models, agents choose memberships in finite groups ("clubs"), and also trade private goods. Agents act as price takers in markets for memberships and goods. Market clearing determines prices and the types of groups that emerge. These models extend general equilibrium theory to include a vast array of economic and social interactions that take place in groups. In particular, as emphasized by Ellickson, Grodal, Scotchmer and Zame (2005), club theory provides a natural model of firms. Prescott and Townsend (2006) and Zame (2007) expanded these ideas to incorporate more general contracting problems with private information.

Our model extends the group formation model of Ellickson, Scotchmer, Grodal and Zame (1999, 2005) (EGSZ below) to incorporate asymmetric information. Agents may have both verifiable and unverifiable characteristics. Unverifiable characteristics can be either hidden actions or hidden information. Thus they can be chosen, such as actions in games, learned, such as skills required for jobs, or innate, such as intelligence. We assume that unverifiable characteristics are observable ex post (after groups have formed) in the sense that they may affect the output and utility of other agents in the group. They are not verifiable ex ante. Thus, prices for memberships cannot depend on them, and they cannot be used for screening members. This framework includes problems as diverse as moral hazard in teams, signaling, screening, and mechanism design.

Because characteristics are unverifiable and groups form randomly, risk is a central feature of the model, both aggregate risk and idiosyncratic risk. Once agents have chosen memberships and strategies, a matching process determines who is matched with whom, and therefore determines the unverifiable characteristics or strategies played in each agent's groups. We model this matching process as random, and construct the associated stochastic processes so that the resulting distribution on possible matchings is uniform. Because the model has a continuum of agents, there are subtleties in making this precise. To do so, we adapt the construction of random matching in pairs in Duffie and Sun (2007) to the more general group setting. This construction has several important consequences. First, it leads to an exact law of large numbers. Second, it highlights the aggregate uncertainty that arises from matching: each possible matching is a random outcome that applies to the economy as a whole, and affects each agent's wealth and preferences for private goods.

In our model, aggregate risk is not ruled out by the law of large numbers. This contrasts with the approach of Prescott and Townsend (2006) and Zame (2007), who focus on purely idiosyncratic risk. For example, Zame (2007) argues that, due to the law of large numbers, aggregate consumption and production are deterministic, and as a consequence, private-goods prices are deterministic. This is not true in our model. Agents' outcomes in the random matching are independent by construction, but individual demands may be correlated by prices. The law of large numbers can be applied in aggregating individual demands only after first assuming that prices are constant. Instead of assuming this, we show that constant prices materialize in equilibrium if a certain kind of insurance is offered in the market. With insurance, constant prices emerge as a conclusion, rather than an assumption. Insurance also provides efficiency gains. Absent insurance, equilibrium prices need not be constant, and trades in private goods can be inefficient even if prices are constant.[1]

We use the matching process we construct to develop two equilibrium concepts, one in which agents are sophisticated enough to realize that their chosen groups might not form, and another in which they assume their demands for memberships are always met. The second equilibrium notion is close in spirit to that of Zame (2007), under the additional assumption that prices are constant across all matchings. We also develop a refinement that links the two equilibrium notions.

Our main results focus on the resulting efficiency in the trading of private goods and in the formation of groups. The mere fact that agents choose their groups is a force toward efficiency; that is probably the main message of club theory. On the other hand, most games permit inefficient outcomes, especially in the context of asymmetric information. Since these two lenses give contradictory intuitions, how much efficiency can we expect?

Our main result shows that efficiency can be achieved by introducing a sufficiently rich

[1]See example 6 below.

set of group types using reporting mechanisms and residual claimants in the spirit of Maskin (1999). Roughly, we show that if groups include appropriately designed mechanisms, there are equilibrium states that replicate those that would arise if all strategies were verifiable. These states are efficient, provided efficiency can be achieved in deterministic states of the economy.[2]

Over the past 25 years, there has been significant interest in embedding private information, particularly contracts, within the framework of markets and general equilibrium. A number of papers have considered related themes in the context of particular applications. Examples include Cole, Mailath and Postlewaite (2001), McAfee (1993), Peters (1997, 2001), Bulow and Levin (2006), Magill and Quinzii (2005), Acemoglu and Simsek (2010), and Legros and Newman (1996, 2008, 2009). In particular, Legros and Newman (1996) study a general equilibrium model of the determination of monitoring and incentive provision in firm formation. Using the specificity of their model, they determine a number of important relationships between the distribution of wealth and the pattern of organizational forms used in firms. Similarly to club theory, they view firms as finite groups of agents engaged in an activity. Their model differs from the clubs model of EGSZ (1999, 2005) and from our model in that they adopt a cooperative, core-based equilibrium concept.

A number of other papers focus instead on general competitive models incorporating asymmetric information. This work can be grouped around three broad themes: lotteries on consumption plans, clubs, and pooling. Our model touches on and extends each, but also diverges in important ways. We discuss each in turn below.

The pioneering work of Prescott and Townsend (1984) formulated the trading of contracts in general equilibrium by modeling incentive constraints as a restriction on contract trades. Due to the resulting nonconvexities, agents are modeled not as choosing a particular consumption plan, but rather a lottery that is a distribution over consumption plans. This is the framework adapted by Cole and Prescott (1997) to clubs, and by Prescott and Townsend (2006), who extend the clubs model to accommodate unverifiable effort in firms. In these models, a lottery is offered by an intermediary who serves a continuum of agents (for simplicity, the whole economy). Because firms must serve a continuum of agents, the model is no longer a foundation for competitive theory.[3] We show instead how lotteries can be introduced with finite group types.

We adapt the clubs framework of EGSZ (1999, 2005) instead of Cole and Prescott (1997), and therefore our model shares features with that of Zame (2007). We diverge by constructing the random group formation process and allowing for aggregate as well

[2]A subtlety is that, due to indivisibilities in consumption, efficiency may require randomization. We comment on this further below.

[3]In addition, Rustichini and Siconolfi (2010) show that equilibria may fail to exist when incentive compatibility is taken as a constraint on lotteries the firm can offer rather than a constraint on lotteries an agent can purchase.

as idiosyncratic uncertainty; in the basic equilibrium notion we adopt; and in focusing on efficiency and the role of additional markets in enhancing efficiency. In particular, we show that an insurance market can eliminate randomness in private-goods prices, that lotteries can be modeled as group memberships, and that a sufficiently rich set of groups embedding appropriately designed mechanisms can lead to efficient equilibria.

"Pooling" provides an alternative approach for incorporating contracts and asymmetric information in general equilibrium, as pioneered by Dubey, Geanakoplos, and Shubik (2005). See also Bisin et al (2001), Minelli and Polemarchakis (2000), and Dubey and Geanakoplos (2004). In these models, sellers deliver to a pool, and buyers buy from this pool. When the goods differ in quality, each buyer receives the average delivery or average quality from the pool. Due to pooling, the market for goods will clear if the market for contracts clears, and it is not necessary to match sellers with buyers. In contrast, our model allows trade with unknown quality in finite trading groups, in which some members deliver goods, and other members consume them. Membership prices establish payments from users to suppliers. Some sellers with high-quality goods will stay off the market, but beliefs in equilibrium will reflect the distribution of qualities that are supplied.

In section 2 we lay out the model. In section 3, we give two examples to illustrate the model, emphasizing the difference between verifiability and observability. In section 4, we formalize the notion of random group formation. In section 5 we define our basic equilibrium notion. In section 6 we define a second equilibrium notion with beliefs on membership characteristics, and explore the connection to our basic equilibrium notion by means of a refinement. In section 7, we introduce insurance markets that smooth the consumption of private goods, and establish a constrained version of the first welfare theorem. In section 8, we illuminate the role of residual claimants in achieving efficiency, arguing that group types with residual claimants will often drive out group types without residual claimants, and give our main efficiency theorem. In section 9, we show that randomization can be introduced as a choice variable through lotteries modeled as group types.

2 The Model

2.1 Private goods and Groups

There are $N \geq 1$ divisible, publicly traded private goods.

Groups are described by a finite, exogenous set of *group types,* $\mathbf{G}$. The group type embeds organizational characteristics such as games, production technologies, transfers, and many other aspects of the internal organization of a group; we elaborate below.[4]

[4] The notion of an exogenously given set of group types follows EGSZ (1999, 2001, 2005) who defined the group type by the characteristics of its members and organizational characteristics from a set. Our formulation is equivalent, although less descriptive.

A *group type* $g \in \mathbf{G}$ has associated to it a finite set $\mathbf{M}(g)$ designating memberships and, implicitly, the number of members. A *membership* in group type g is denoted $m \in \mathbf{M}(g)$. We write $\mathbf{M}$ for the set of memberships $\cup_{g \in \mathbf{G}} \mathbf{M}(g)$.

A *membership list* is an indicator function $\ell : \mathbf{M} \to \{0, 1\}$, with the interpretation that $\ell(m) = 1$ means the agent consumes a membership of type m. Let $\mathbf{Lists}(\mathbf{M})$ denote the set of lists. More generally, for any set C, we write $\mathbf{Lists}(C)$ for the set of indicator functions on C, and given $\ell \in \mathbf{Lists}(C)$ we write $|\ell|$ for the number of elements $c \in C$ such that $\ell(c) = 1$.

In addition to their verifiable membership characteristics, encoded in m, group members may have unverifiable characteristics or strategies. For each membership $m \in \mathbf{M}$, let S_m be the set of unverifiable characteristics that could be chosen in m. For example, in problems with moral hazard, S_m may include unverifiable effort, while in screening problems, S_m may include unverifiable personal characteristics that are nevertheless observable and affect the utility of others. In normal-form games where m is the membership corresponding to a particular player, the set S_m represents the set of actions available to that player. An agent's choice of an unverifiable characteristic in S_m may be constrained by the agent's consumption set; we formalize this below. For example, characteristics that are interpreted as innate cannot be different for a given agent in different memberships.

Given a group type $g \in \mathbf{G}$, let $S(g) := \prod_{m \in \mathbf{M}(g)} S_m$ denote the possible *strategy profiles* the members of g could adopt. Given a membership $m \in \mathbf{M}(g)$ and a strategy profile $s \in S(g)$, write $\tilde{m} = (m, s)$ for the resulting *augmented membership* in group type g. Let $\tilde{\mathbf{M}}(g) := \{\tilde{m} = (m, s) : m \in \mathbf{M}(g) \text{ and } s \in S(g)\}$ represent the set of all possible augmented memberships in a given group type g, and write $\tilde{\mathbf{M}} = \cup_{g \in \mathbf{G}} \tilde{\mathbf{M}}(g)$ for the set of all augmented memberships. An *augmented membership list* is an indicator function $\tilde{\ell} : \tilde{\mathbf{M}} \to \{0, 1\}$. Write $\mathbf{Lists}(\tilde{\mathbf{M}})$ for the set of augmented membership lists.

Corresponding to each group type g is then a set of possible augmented group types, depending on the strategies chosen by the agents who take memberships in the group. Given g and $s \in S(g)$, (g, s) is the corresponding *augmented group type*. Each augmented group type (g, s) thus has the same set of memberships $\mathbf{M}(g)$ and one particular strategy profile $s \in S(g)$.

Let $|\mathbf{M}(g)|$ denote the number of memberships in a group type g, or equivalently, in any augmented group type (g, s) derived from g.

Groups may engage in productive activities, summarized by an input-output vector which may depend on the unverifiable characteristics of group members. We capture this by associating to each augmented group type (g, s) an *input-output vector* $h(g, s) \in \mathbf{R}^N$, which is assumed to be verifiable. The input-output vector could arise, for example, from the equilibrium of a game played within the group, or could simply be a required input vector.

The input-output vector of a group will be shared among its members according to *transfer functions* $t_g : \mathbf{M}(g) \times \mathbf{R}^N \to \mathbf{R}^N$, for each $g \in \mathbf{G}$. The vector $t_g(m, y)$ is transfered to an agent holding membership $m \in \mathbf{M}(g)$ when the input-output vector produced by the group is y. The transfers must allocate the input-output vector among the members, that is,

$$\sum_{m \in \mathbf{M}(g)} t_g(m, y) = y \quad \text{for each } y \in \mathbf{R}^N$$

While the transfers cannot depend on unverifiable characteristics directly, they will depend on the unverifiable characteristics through the output of the group. In the augmented group type (g, s), the transfer received by an agent holding membership m is $t_g(m, h(g, s))$. The total transfer received by an agent consuming augmented list $\tilde{\ell}$ is then

$$\tilde{\ell}t := \sum_{g \in \mathbf{G}, m \in \mathbf{M}(g), s \in S(g)} \tilde{\ell}(m, s) t_g(m, h(g, s))$$

The net payment that an agent receives when consuming an augmented list $\tilde{\ell}$ depends both on these transfers, which are part of how the group type is defined, and on the membership prices discussed below, which are endogenous.

2.2 Agents

The set of agents is a nonatomic finite measure space $(A, \mathcal{F}, \lambda)$. That is, A is a set, $\mathcal{F}$ is a σ-algebra of subsets of A, and λ is a non-atomic measure on $\mathcal{F}$ with $\lambda(A) < \infty$.

A complete description of an agent $a \in A$ consists of a consumption set, endowments, and a utility function; we define each of these in turn.

Agents choose lists $\mu \in \mathbf{Lists}(\mathbf{M})$ and strategies $\sigma \in \Sigma$, where the strategy space Σ is defined by

$$\Sigma := \prod_{m \in \mathbf{M}} S_m$$

with generic element $\hat{\sigma} \in \Sigma$. To simplify notation, this formulation requires each agent to choose a strategy for each membership, even if he does not choose the membership.

Agents consume unverifiable augmented lists $\tilde{\mu} \in \mathbf{Lists}(\tilde{\mathbf{M}})$. Let

$$\mathcal{U} := \{\tilde{\mu} : A \to \mathbf{Lists}(\tilde{\mathbf{M}})\}$$

denote the set of all possible assignments of augmented lists to agents. The augmented lists that agents consume in equilibrium will be constrained by the memberships and strategies they choose, and also by the memberships and strategies chosen by others.

Agent a's *consumption set* $X_a \subset \mathbf{R}^N_+ \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$ specifies the triples (x_a, μ_a, σ_a) of private goods, lists of memberships, and strategies that the agent may choose. Each agent $a \in A$ has an *endowment* $(e_a, 0, \sigma_a^o) \in X_a$ and an *ex post utility function* $u_a : \mathbf{R}_+^N \times \mathbf{Lists}(\tilde{\mathbf{M}}) \to \mathbf{R}$.

A central feature of the model is the underlying randomness arising from group formation. Private-goods consumption and prices can both be contingent on the realized state in this model. Because the state space will be derived endogenously based on all agents' membership and strategy choices, as part of the random group formation model, we describe only the ex post utility here. Below we assume that agents have beliefs over the state space that arises, and choose contingent consumption bundles, memberships and strategies to maximize expected ex post utility. We assume that neither the agent's endowment nor his feasibility constraints on consumption of private goods depends on the resolution of the randomness.[5]

2.3 Economies

An *economy* $\mathcal{E}$ is a mapping $a \mapsto (X_a, e_a, u_a)$ for which:

- the consumption set mapping $a \mapsto X_a$ is a measurable correspondence such that
  - for each $a \in A$, $X_a \subset \mathbf{R}_+^N \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$
  - for each $a \in A$, if $(x_a, \mu_a, \sigma_a) \in X_a$ and $x'_a \geq x_a$ then $(x'_a, \mu_a, \sigma_a) \in X_a$
  - for each $a \in A$, if $(x_a, \mu_a, \sigma_a) \in X_a$ and $\mu'_a \leq \mu_a$ then $(x_a, \mu'_a, \sigma_a) \in X_a$
  - there exists $M > 0$ such that for each $a \in A$ and $(x_a, \mu_a, \sigma_a) \in X_a$,

$$\sum_{m \in \mathbf{M}} \mu_a(m) \leq M$$

- the endowment mapping $a \mapsto e_a$ is an integrable function
- the ex-post utility mapping $(a, x, \tilde{\ell}) \mapsto u_a(x, \tilde{\ell})$ is a jointly measurable function of its arguments, and for each a, u_a is strictly monotone and continuous in x.
- $\bar{e} := \int_A e_a \, d\lambda(a) \gg 0$

Restrictions on the consumption set can be used to model, among other things, settings in which some characteristics are innate. We assume that increased consumption of private goods is always possible, while there is a fixed bound on the number of memberships that each agent can choose. To handle disequilibrium states where some chosen memberships do not result in groups forming, we assume that if some memberships are dropped from

[5] In reality there may be settings where an agent's feasible consumption of private goods would depend on the characteristics that materialize in the agent's groups. For example, the agent might have to buy locks in order to protect against a roommate who turns out to be a kleptomaniac. For simplicity, we have chosen to put this type of requirement into preferences rather than the consumption set.

a feasible bundle, then the new bundle is still feasible. This is a restriction, but it makes the definition of equilibrium tractable. The restriction can be removed in several ways, for example, by defining group types that combine memberships that must be consumed together.

We follow EGSZ (1999) by defining consistency of choices in terms of aggregates. Define an *aggregate membership vector* to be an element $\bar{\mu} \in \mathbf{R}^{\mathbf{M}}$. An aggregate membership vector $\bar{\mu} = \int_A \mu_a d\lambda(a)$ is *consistent* if for every group type $g \in \mathbf{G}$, there is a real number $\alpha(g)$ such that

$$\bar{\mu}(m) = \alpha(g) \text{ if } m \in \mathbf{M}(g)$$

Given a measurable set $B \subset A$ and a measurable choice function $\mu : A \to \mathbf{Lists}(\mathbf{M})$, we say that μ is *consistent* if the aggregate membership vector $\int_A \mu_a \, d\lambda(a)$ is consistent.

3 Two examples

Before continuing, we give two examples to illustrate the model. The first example illustrates the difference between observability and verifiability. The second example shows how the standard principal-agent problem can be embedded in a group model, and shows how transfer payments can be used to solve the moral hazard problem.

Example 1: Observable but Unverifiable Characteristics

There is a single group type g with two memberships $\{m_1, m_2\} \in \mathbf{M}(g)$. A member can have one of two unverifiable characteristics, b or c. Thus $S_{m_1} = S_{m_2} = \{b, c\}$, and $\Sigma = \{(b,b), (b,c), (c,b), (c,c)\}$. The utility of each member depends on all the members' unverifiable characteristics, revealed after the group forms. These characteristics are observable after the group has formed, but not before. Thus, membership prices and choices cannot depend on them.

Let the set of agents be $A = [0,1]$. The characteristics b and c are understood to be innate, and we assume that there is a proportion $\rho \in (0,1)$ such that agents $a \in [0, \rho)$ have characteristic b, that is, are constrained by their consumption sets to choose strategy (b,b). Similarly, agents $a \in [\rho, 1]$ are constrained to choose strategy (c,c). We adopt the shorthand notations $\tilde{m}_{bb}, \tilde{m}_{cc}, \tilde{m}_{bc}$, and $\tilde{m}_{cb}$ for the augmented group types where both members have unverifiable characteristic b, both have characteristic c, or one member has each characteristic.

Agents are limited to a single membership, so $M = 1$, and there is a single private good of which each agent has an endowment $e \in \mathbf{R}_+$. Agents $a \in [0, \rho)$, who have characteristic

b, have ex-post utility function v_b given by

$$v_b(x, \tilde{\ell}) = \begin{cases} x - 1 & \text{if } \tilde{\ell} = 0 \\ 6 + x & \text{if } \tilde{\ell}(\tilde{m}_{bb}) = 1 \\ x & \text{if } \tilde{\ell}(\tilde{m}_{bc}) = 1 \text{ or } \tilde{\ell}(\tilde{m}_{cb}) = 1 \end{cases}$$

Agents $a \in [\rho, 1]$, who have characteristic c, have ex-post utility function v_c given by

$$v_c(x, \tilde{\ell}) = \begin{cases} x - 1 & \text{if } \tilde{\ell} = 0 \\ x & \text{if } \tilde{\ell}(\tilde{m}_{bc}) = 1 \text{ or } \tilde{\ell}(\tilde{m}_{cb}) = 1 \\ 2 + x & \text{if } \tilde{\ell}(\tilde{m}_{cc}) = 1 \end{cases}$$

Thus for any fixed consumption level x, agents get more utility if matched with like agents. If the characteristics were verifiable and could be encoded into the memberships, the efficient matching would result in as many homogeneous groups as possible.

In equilibrium, which we formalize in section 5, membership prices must sum to zero within each group. Since the memberships are indistinguishable in this example, each membership price will be zero in equilibrium. Thus, we can think of agents simply choosing with whom to match. The question is whether an equilibrium will result in efficient matches of like with like.

When an agent takes a membership, he cannot observe the unverifiable characteristic of the other member, but has beliefs about its distribution. For each $s_1 \in S_{m_1}$, let $f(s_1; m_2)$ denote the probability an agent holding membership $m_2 \in \mathbf{M}(g)$ assigns to being matched with an agent whose characteristic is s_1. Define $f(s_2; m_1)$ symmetrically for $s_2 \in S_{m_2}$.

We first show that there are beliefs that support an equilibrium in which half of the agents with each unverifiable characteristic take each membership. Suppose agents' common beliefs are $f(b; m) = \rho$ and $f(c; m) = 1 - \rho$ for each $m \in \mathbf{M}(g)$. Given these beliefs, all agents are indifferent among all memberships. Given this indifference, choices such that half the agents with each characteristic take each membership are optimal, and generate a distribution over augmented group types that agrees with f. This will be an equilibrium in our model. Nonetheless, it is not Pareto optimal, and is dominated by the matchings that would arise if characteristics were verifiable.

Now consider enlarging the set of group types. Let $\mathbf{G} = \{g_{bb}, g_{bc}, g_{cb}, g_{cc}\}$, where each group type has two memberships $\{m_1, m_2\}$ as before. These new group types are identical in every way except their labels, which can serve as a coordinating mechanism.

In the economy with the enlarged set of group types, there is still an equilibrium where no coordination takes place. Agents ignore the labels, have the beliefs given above for each group type, and are still matched randomly in groups of like or unlike agents according to the population distribution.

There is a second equilibrium, however, in which the expanded set of group types coordinates the agents' beliefs, hence choices. In this equilibrium, agents $a \in [0, \rho)$, those with strategy (b, b), take memberships in $\mathbf{M}(g_{bb})$, with half taking each membership. Agents $a \in [\rho, 1]$, those with strategy (c, c), take memberships in $\mathbf{M}(g_{cc})$, again with half taking each membership. No one takes memberships in the mixed groups g_{bc} or g_{cb}. Membership prices are still zero. This equilibrium, which results in an allocation that Pareto dominates the previous one, can be sustained by the following beliefs:

$$
\begin{aligned}
f(b; m) &= 1 \text{ if } m \in \mathbf{M}(g_{bb}) \\
f(c; m) &= 1 \text{ if } m \in \mathbf{M}(g_{cc}) \\
f(c; m) &= f(b; m) = \frac{1}{2} \text{ if } m \in \mathbf{M}(g_{bc}) \text{ or } m \in \mathbf{M}(g_{cb})
\end{aligned}
$$

It may not always possible to find an equilibrium that screens with respect to the unverifiable characteristics. Suppose, for example, that the utility received by agents with strategy (c, c) is reversed for homogeneous groups and mixed groups, that is, suppose that $v_c(x, \tilde{\ell}) = 2 + x$ if $\tilde{\ell}(m_{bc}) = 1$ or $\tilde{\ell}(\tilde{m}_{cb}) = 1$ and $v_c(x, \tilde{\ell}) = x$ if $\tilde{\ell}(\tilde{m}_{cc}) = 1$. Then optimal screening cannot be achieved by introducing new group types. In addition, a screening equilibrium is not always superior; see example 11. ◊

Example 2: A Standard Principal-Agent Problem

There is one type of firm with two memberships, a worker w and a principal p. The principal has a "null" strategy s_o while the worker's action can be either low or high effort, $\{e_\ell, e_h\}$. If the worker works hard, the output is higher:

$$h(g, (s_o, e_h)) = y_h > y_\ell = h(g, (s_o, e_\ell))$$

The internal transfers give all the output to the worker:

$$t_g(w, y) = y, t_g(p, y) = 0 \text{ for } y = y_\ell, y_h$$

Let $A = [0, 5]$, and suppose that agents $a \in [0, 3)$ are constrained to take memberships as workers, that is, they cannot take principal memberships, while agents $a \in [3, 5]$ are constrained to take memberships as principals.

We assume that all reservation utilities are zero. Principals have utility equal to their consumptions of the private good, and workers have the following utility functions

$$
\begin{aligned}
v_a(x, e_\ell) &= v_a(x - 1, e_h) \text{ if } a \in [0, 1) \\
v_a(x, e_\ell) &= v_a(x - 3, e_h) \text{ if } a \in [1, 3)
\end{aligned}
$$

where v_a is increasing in the first component. Thus, all agents dislike effort, but effort is costlier for those in $[1, 3)$.

Assume that $1 < y_h - y_\ell < 3$. Then it is efficient for low type workers, those in $[1, 3)$, to exert low effort e_ℓ, while high type workers, those in $[0, 1)$, exert high effort e_h. We argue that this is what will happen in equilibrium.

For any membership price $q(w)$, an agent in $[0, 1)$ will choose high effort because $v_a(-q(w) + t_g(w, y_\ell), e_\ell) < v_a(-q(w) + t_g(w, y_h) - 1, e_h)$. An analogous calculation shows that an agent in $[1, 3)$ will choose low effort. Membership fees in each group sum to zero, so the worker's loss is the principal's gain: $q(p) = -q(w)$.

We claim there is an equilibrium with membership prices $q(p) = -y_\ell$ and $q(w) = y_\ell$. At this equilibrium, all potential principals are in firms, all high-type workers are in firms, and half the low-type workers are in firms. Principals get utility equal to y_ℓ, which exceeds their reservation payoff since they are in short supply. Low-type workers get zero consumption, since they are in excess supply, and high-type workers get rents equal to $y_h - y_\ell - 1$, since they are in short supply among agents who will be matched. This is an equilibrium because no principal or worker can improve utility by choosing to shed or add memberships, and no worker can improve utility by choosing a different effort level. Equilibrium is first-best efficient. $\Diamond$

4 Random Matching

A key aspect of our model is the matching process that underlies group formation. We imagine that once agents have made membership and strategy choices, groups form that are consistent with those choices. Since each agent's utility and income may depend on the outcome of matching, the agent's expected utility (hence membership and strategy choices) depend on the probabilities of different matchings.[6]

Loosely, we assume that matching is random and uniform, so that every matching consistent with agents' choices is equally likely. There are mathematical subtleties in defining such a process precisely, due to the well-known issues stemming from a continuum of agents, and a continuum of random variables. The matching process we use for groups is adapted from the construction of Duffie and Sun (2007) for matching in pairs. This gives a precise meaning to random and uniform matching in a continuum economy, and leads to a natural law of large numbers.

To make this precise, let $\mathbb{M}$ be a finite index set and let $\{A_m \subset A | m \in \mathbb{M}\}$ be measurable sets of agents such that $A_m \cap A_{m'} = \emptyset$ for $m \neq m' \in \mathbb{M}$. In our model, $\mathbb{M}$ represents memberships in a given group type.[7] Write $A^{\mathbb{M}} = \prod_{m \in \mathbb{M}} A_m$ and $A^{\mathbb{M}}_{-m} = \prod_{m' \neq m} A_{m'}$, so $a_{-m} \in A^{\mathbb{M}}_{-m}$ is a list of $|\mathbb{M}| - 1$ agents.

[6]This is a major difference between the model here and EGSZ (1999), where the matching does not matter, provided the matching is consistent as to verifiable characteristics. Even there, though, the matching could matter in the sense of "sunspots" for coordinating on different private-goods prices.

[7]If a given agent has two memberships in a given group type, then he appears in two sets A_m and $A_{m'}$. Implicitly, we imagine the copies of the agent to be distinct agents when defining the correpsonding group matching. When matching is random and uniform, any given agent will be matched with himself in a group with probability zero, so we can ignore such groups.

Definition 1 *A* group matching *is a function* $\Psi : A^{\mathbb{M}} \to \{0,1\}$ *such that for every* $m \in \mathbb{M}$ *and for every* $b \in A_m$*, there exists at most one* $a \in A^{\mathbb{M}}$ *such that* $\Psi(a) = 1$ *and* $a_m = b$*.*

If $\Psi(a) = 0$ *for all* $a \in A^{\mathbb{M}}$ *such that* $a_m = b$*, then* b *is* unmatched.

If $\Psi(a) = 1$ *then* $a \in A^{\mathbb{M}}$ *is a* match.

Given a group matching Ψ, for each $m \in \mathbb{M}$, let the function $g_m : A_m \to A^{\mathbb{M}}_{-m} \cup \emptyset$ describe the matches for the agents in A_m. Then $g_m(b) = \emptyset$ if $b \in A_m$ is unmatched, and if b is matched, so $g_m(b) \neq \emptyset$, he is matched with $g_m(b) \in A^{\mathbb{M}}_{-m}$.

If the measures of the sets $\{A_m \subset A | m \in \mathbb{M}\}$ are different, then not all agents will be matched. The measure of the subset of agents in A_m who are matched will be $\underline{\zeta} := \min_{m \in \mathbb{M}} \lambda(A_m)$. For each $m \in \mathbb{M}$, set

$$\zeta(m) := \begin{cases} 1 & \text{if } \underline{\zeta} = 0 \\ \frac{\lambda(A_m) - \underline{\zeta}}{\lambda(A_m)} & \text{otherwise} \end{cases}$$

The values $\{\zeta(m) | m \in \mathbb{M}\}$ are the *no-match probabilities* associated with the collection $\{A_m \subset A | m \in \mathbb{M}\}$.

In our model, the sets $\{A_m \subset A | m \in \mathbb{M}\}$ will represent the agents who have chosen the various memberships in a given group type $g \in \mathbf{G}$. The characteristics of these agents will be defined by their strategy choices. In this section, we simply imagine that agents have characteristics specified by functions $\alpha_m : A_m \to S_m$, each $m \in \mathbb{M}$, where the sets $\{S_m : m \in \mathbb{M}\}$ represent characteristics that could be attached to the membership m. We use α_{-m} to refer to $\{\alpha_{m'} : m' \in \mathbb{M}, m' \neq m\}$.

For each $s_m \in S_m$, let $A_m(s_m) := \{a \in A_m \mid \alpha_m(a) = s_m\}$. We define p_m to be the relative frequency of strategies in the set A_m, thus for each $m \in \mathbb{M}$,

$$p_m(s_m) := \frac{\lambda(A_m(s_m))}{\lambda(A_m)} \text{ if } \lambda(A_m) > 0$$

Similarly, we define p_{-m} to be the relative frequencies of strategies in matches, excluding the member from A_m. To account for the possibility that an agent is not matched, we add the "null" characteristic $\emptyset$. Let $S_{-m} := \prod_{m' \in \mathbb{M} \setminus m} S_{m'}$. For $s \in S_{-m} \cup \emptyset$ let

$$p_{-m}(s) := \begin{cases} (1 - \zeta(m)) \prod_{\substack{m' \in \mathbb{M} \setminus m \\ \lambda(A_{m'}) > 0}} p_{m'}(s_{m'}) & \text{if} \quad s = \{s_{m'}\}_{m' \in \mathbb{M} \setminus m} \in S_{-m} \\ \zeta(m) & \text{if} \quad s = \emptyset \end{cases}$$

These definitions describe matching and relative frequencies of characteristics, but do not describe what it means to match randomly. Intuitively, we assume matching is random and uniform; thus we will want p_{-m} to be the probability distribution on characteristics in a match, from the perspective of the m^{th} member, for each membership m. Part of the contribution of this section is to show that a state space and random variables describing matchings can be constructed such that this is the case.

To formalize this, we start by letting V denote a state space and $(V, \mathcal{V}, \nu)$ an associated probability space. For now we take these as given, so that we can define the notation needed to describe random matching. In the construction of random group formation models, this probability space will be determined endogenously, as a function of membership and strategy choices of the agents.

For each $b \in A_m$, $m \in \mathbb{M}$, let $g_m(b, \cdot) : V \to A_{-m}^{\mathbb{M}} \cup \emptyset$ be a random variable that gives the match for agent b, and let $\omega(b, \cdot) : V \to S_{-m} \cup \emptyset$ be the corresponding random variable that describes the characteristics in agent b's random match. Thus

$$\omega(b, v) = \begin{cases} \alpha_{-m} \circ g_m(b, v) & \text{if} \quad g_m(b, v) \neq \emptyset \\ \emptyset & \text{if} \quad g_m(b, v) = \emptyset \end{cases}$$

Then $\omega(b, \cdot) = \emptyset$ if and only if agent b is not matched. If b is matched, then $g_m(b, v)$ specifies the names of the agents in his match, and $\alpha_{-m} \circ g_m(b, v)$ specifies their characteristics.

Definition 2 *Let* $\{A_m \subset A | m \in \mathbb{M}\}$ *be measurable subsets of agents, and* $\{\zeta(m) | m \in \mathbb{M}\}$ *be the associated no-match probabilities. Let* $(V, \mathcal{V}, \nu)$ *be a probability space. A* random group matching *is a function* $\Psi : A^{\mathbb{M}} \times V \to \{0, 1\}$ *such that:*

(i) for every $v \in V$, $\Psi(\cdot, v)$ *is a group matching*

(ii) for almost every $v \in V$,

$$\lambda(\{a \in A_m \mid a \text{ is unmatched in } \Psi(\cdot, v)\}) = \zeta(m) \text{ for each } m \in \mathbb{M}$$

(iii) for each $m \in \mathbb{M}$ *and almost every* $b \in A_m$, p_{-m} *is the distribution of* $\omega(b, \cdot)$

(iv) for each $m \in \mathbb{M}$ *and almost every* $b, b' \in A_m$, $\omega(b, \cdot)$,$\omega(b', \cdot)$ *are independent.*

To use these notions in our model, we imagine that the list and strategy choices (μ, σ) are given. The list choices determine the sets of agents who might be matched in any given group type, and the strategy choices determine the corresponding distribution of unverifiable characteristics. This naturally leads to the notion of random matchings that are consistent with population choices (μ, σ).

Definition 3 *For* $g \in \mathbf{G}$, *a random group matching* $\Psi_g : A^{\mathbf{M}(g)} \times V \to \{0, 1\}$ *is* consistent with (μ, σ) *if*

(i) for each $m \in \mathbf{M}(g)$, $A_m = \{a \in A \mid \mu_a(m) = 1\}$;

(ii) for each $m \in \mathbf{M}(g)$ and $a \in A_m$, $\alpha_m(a) = s_m$ iff $\sigma_{a,m} = s_m$.

Definition 4 *A* random group formation model consistent with (μ, σ) *is a probability space $(V, \mathcal{V}, \nu)$ and a collection of maps $\{\Psi_g : g \in \mathbf{G}\}$ such that*

(i) $\{\Psi_g : g \in \mathbf{G}\}$ are random group matching functions consistent with (μ, σ).

(ii) for every $m \in \mathbb{M}_g, m' \in \mathbb{M}_{g'}$ and for almost every $b \in A_m$, $b' \in A_{m'}$, the random variables $\omega(b, \cdot)$ and $\omega(b', \cdot)$ are independent.

Henceforth, we write $\mathcal{R}(\mu, \sigma)$ for a random group formation model consistent with (μ, σ), and write $(V, \mathcal{V}, \mathcal{P}(\mu, \sigma))$ for the associated probability space.

For a standard pairwise matching problem, or equivalently, a setting with a single group type with two memberships, Duffie and Sun (2007) show that such a random group formation model exists. Our notion of random group formation is the natural extension of this construction to multiple group types with an arbitrary finite number of members in each group type.

Theorem 4.1 *For every (μ, σ) there exists a random group formation model consistent with (μ, σ).*

We omit the proof, which mimics the proof in Duffie and Sun (2007, Theorem 2.6) for the case of pairwise matching.

Assignments $\tilde{\mu} \in \mathcal{U}$ are random variables on the probability space $(V, \mathcal{V}, \mathcal{P}(\mu, \sigma))$. Each $v \in V$ induces a random group matching $\{\Psi_g(\cdot, v) : g \in \mathbf{G}\}$, and each random matching generates an assignment of augmented lists $\tilde{\mu} \in \mathcal{U}$. However, the assignment $\tilde{\mu}$ contains less information, since the assignment of augmented lists $\tilde{\mu}$ is preserved if two agents with the same augmented lists, $\tilde{\mu}_a = \tilde{\mu}_{a'}$, trade places in all groups. Thus different random matchings can lead to the same assignment of augmented lists $\tilde{\mu}$, but not vice versa. Nevertheless, in the following we will often use the word "matching" interchangeably for the assignment $\tilde{\mu}$.

We let $\tilde{\mu}^r(v)$ denote the random assignment $\tilde{\mu}$ that is realized at a state v. Thus agent a's assignment in the state v is denoted $\tilde{\mu}^r_a(v)$. The probability of a random assignment $\tilde{\mu}$ is $\mathcal{P}(\mu, \sigma)(V(\tilde{\mu}))$ where $V(\tilde{\mu}) := \{v \in V : \tilde{\mu}^r(v) = \tilde{\mu}\}$. The probability of a given $\tilde{\mu}_a \in \mathbf{Lists}(\tilde{\mathbf{M}})$ is understood analogously.

An important consequence of the construction of random group formation models is that it leads to an exact law of large numbers governing the distribution of lists $\tilde{\mu}_a$ for every $a \in A$.

To make this precise, using the notation set out in definition 3, the implied no-match probabilities for each given $g \in \mathbf{G}$ and $m \in \mathbf{M}(g)$ are

$$\zeta(m;\mu) = \begin{cases} 1 - \frac{\min_{m' \in \mathbf{M}(g)} \lambda(A_{m'})}{\lambda(A_m)} & \text{if } \lambda(A_m) > 0 \\ 1 & \text{if } \lambda(A_m) = 0 \end{cases}$$

When an agent with membership m and characteristic s_m is matched, the distribution of other members' characteristics, $s_{-m} \in S_{-m}$, is independent of the member's own characteristic s_m. We can thus write the probability that the agent ends up in the augmented membership $(m, s) = (m, (s_m, s_{-m}))$, conditional on being matched, as

$$\bar{\phi}_{(\mu,\sigma)}(s_{-m}; m) := \prod_{m' \in \mathbf{M}(g) \setminus m} \frac{\lambda\left(\left\{a \in A : \sigma_{a,m'} = s_{m'}, \mu_a(m') = 1\right\}\right)}{\lambda(A_{m'})} \tag{1}$$

Thus, when the population choices are (μ, σ), all agents who choose the list and strategy $(\ell, \hat{\sigma}) \in \mathbf{Lists}(\mathbf{M}) \times \Sigma$ face the probability distribution on augmented lists $\tilde{\ell}$ given by:[8]

$$\begin{aligned} \eta_{(\mu,\sigma)}(\tilde{\ell}; \ell, \hat{\sigma}) &= \prod_{g \in \mathbf{G}} \prod_{\{m \in \mathbf{M}(g), s_{-m} \in S_{-m}(g) : \tilde{\ell}(m,(s_{-m},\hat{\sigma}_m)) = 1\}} \ell(m)(1 - \zeta(m;\mu))\bar{\phi}_{(\mu,\sigma)}(s_{-m}; m) \\ &\quad \times \prod_{\{m \in \mathbf{M}(g) : \ell(m) = 1, \sum_{s_{-m} \in S_{-m}(g)} \tilde{\ell}(m,(s_{-m},\hat{\sigma}_m)) = 0\}} \zeta(m;\mu) \end{aligned} \tag{2}$$

For memberships m such that $\ell(m) = 1$, the second line in (2) gives the probability of not matching. The first line gives the probability of the particular match that is made (where $\tilde{\ell}(m, (s_{-m}, \hat{\sigma}_m)) = 1$).

Conditional on being matched in each membership in ℓ, these probabilities become

$$\bar{\eta}_{(\mu,\sigma)}(\tilde{\ell}; \ell, \hat{\sigma}) = \prod_{g \in \mathbf{G}} \prod_{\{m \in \mathbf{M}(g), s_{-m} \in S_{-m}(g) : \tilde{\ell}(m,(s_{-m},\hat{\sigma}_m)) = 1\}} \ell(m)\bar{\phi}_{(\mu,\sigma)}(s_{-m}; m)$$

We refer to $\bar{\phi}_{(\mu,\sigma)}$, $\eta_{(\mu,\sigma)}$ and $\bar{\eta}_{(\mu,\sigma)}$ as the *empirical distributions*, since they are derived from the choices made by agents in the population. That these empirical distributions are generated by the random matching model is an important consequence of the exact law of large numbers, recorded below.[9]

Theorem 4.2 *Let $(V, \mathcal{V}, \mathcal{P}(\mu, \sigma))$ be the probability space associated with the random group formation model $\mathcal{R}(\mu, \sigma)$. For almost every pair of agents $a, b \in A$, the random variables $\tilde{\mu}^r_a(\cdot)$ and $\tilde{\mu}^r_b(\cdot)$ are pairwise independent and identically distributed, with distribution $\eta_{(\mu,\sigma)}$ on $\mathbf{Lists}(\tilde{\mathbf{M}})$ given by (2).*

[8] We take the product over the empty set to be 1 in this expression.

[9] This is analogous to Duffie and Sun (2007, Theorem 2.6).

5 Group Equilibrium

We assume that agents are price-takers in membership and private goods markets, and choose actions or characteristics strategically. Agents' choices depend both on membership and private goods prices, and on the membership and strategy choices of other agents. In particular, agents understand the random group formation model $\mathcal{R}(\mu, \sigma)$. Agents' membership and strategy choices are then a best response to the choices of other agents, given their knowledge of the matching process. Although this is a familiar idea in game theory, it creates a tension with the general equilibrium idea that agents' demands do not depend on choices of other agents or whether their demands can be satisfied. In section 6, we assume instead that agents choose memberships on the assumption (perhaps incorrect) that their demands for memberships will always be met. We develop a refinement below that connects these two equilibrium concepts.

Let $(\mathbf{R}_{+}^{N})^{V}$ be endowed with the product topology.

A *state* is a measurable mapping $(x, \mu, \sigma) : A \to (\mathbf{R}_{+}^{N})^{V} \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$, together with a random group formation model $\mathcal{R}(\mu, \sigma)$.

A state $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$ is *feasible* if for almost every $a \in A$, $(x_a(v), \mu_a, \sigma_a) \in X_a$ for $\mathcal{P}(\mu, \sigma)$-almost all v, $\int_A \mu_a \, d\lambda(a)$ is consistent for A, and *material balance* holds, that is,

$$\int_A x_a(v)\, d\lambda(a) \le \int_A e_a \, d\lambda(a) + \int_A \left[\sum_{g \in \mathbf{G}} \sum_{\substack{m \in \mathbf{M}(g) \\ s \in S(g)}} \tilde{\mu}_a^r(v)\,(m, s)\, \frac{h(g, s)}{|\mathbf{M}(g)|} \right] d\lambda(a)$$

for $\mathcal{P}(\mu, \sigma)$-almost all v.

Given (μ, σ) and an associated random group formation model $\mathcal{R}(\mu, \sigma)$ with probability space $(V, \mathcal{V}, \mathcal{P}(\mu, \sigma))$, we assume that agents hold beliefs $\{P_a, a \in A\}$ on $(V, \mathcal{V})$. We also assume that each agent evaluates combinations of state-contingent private goods, membership and strategy choices by expected ex post utility, given P_a. When evaluating deviations from membership and strategy choices, we assume that each agent takes membership and strategy choices of other agents as given, as well as the random group formation model $\mathcal{R}(\mu, \sigma)$. We assume that each agent has beliefs over the characteristics that will materialize in groups, as a function of his membership and strategy choices. We let $\tilde{\ell}_a(\ell, \hat{\sigma})$ denote the corresponding random variable on $\mathbf{Lists}(\tilde{\mathbf{M}})$ for each a, and let $n_a(\cdot; \ell, \hat{\sigma}) \in \Delta(\mathbf{Lists}(\tilde{\mathbf{M}}))$ denote the beliefs of agent a given his membership and strategy choices $(\ell, \hat{\sigma})$. We require these beliefs to coincide in equilibrium with the empirical frequencies generated in the random group formation model.

To allow for the possibility that not all chosen memberships result in matches, we let

$\ell(\tilde{\ell})$ denote the memberships represented in a given augmented list $\tilde{\ell}$, that is,

$$\ell(\tilde{\ell})(m) := \sum_{\substack{s \in S(g) \\ g \in \mathbf{G}}} \tilde{\ell}(m,s) \quad \text{for each } m \in \mathbf{M}$$

Thus $\ell(\tilde{\ell})(m) = 1$ if and only if there is some s such that $\tilde{\ell}(m,s) = 1$, that is, if the membership m is part of some augmented membership (m,s) in the support of $\tilde{\ell}$.

Definition 5 *A* group equilibrium *consists of a feasible state* $(x,\mu,\sigma), \mathcal{R}(\mu,\sigma)$*, beliefs* $\{P_a, n_a, a \in A\}$*, private goods prices* $p \in \left(\mathbf{R}_+^N\right)^V$ *with* $p \neq 0$*, where* p *is measurable, and membership prices* $q \in \mathbf{R}^{\mathbf{M}}$ *such that (E1)-(E3) hold:*

(E1) ***Budget balance for group types:*** *For each* $g \in \mathbf{G}$,

$$\sum_{m \in \mathbf{M}(g)} q(m) = 0$$

(E2) ***Optimization by agents:*** *For almost every* $a \in A$*, if* $(x_a'(v), \mu_a', \sigma_a') \in X_a$ *for* P_a*-almost every* v *and*

$$\int_V \sum_{\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})} n_a(\tilde{\ell}; \mu_a', \sigma_a') u_a(x_a'(v), \tilde{\ell}) dP_a(v) > \int_V u_a(x_a(v), \tilde{\mu}_a^r(v)) dP_a(v)$$

then there exists $V' \subset V$ *with* $P_a(V') > 0$ *and* $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$ *with* $n_a(\tilde{\ell}; \mu_a', \sigma_a') > 0$ *such that*

$$p(v) \cdot x_a'(v) + q \cdot \ell(\tilde{\ell}) > p(v) \cdot e_a + p(v) \cdot (\tilde{\mu}_r^v(v)\, t) \text{ for every } v \in V'$$

(E3) ***Beliefs are correct:*** *For almost every* $a \in A$*,* $P_a = \mathcal{P}(\mu,\sigma)$ *and* $n_a(\cdot; \ell, \hat{\sigma}) = \eta_{(\mu,\sigma)}(\cdot; \ell, \hat{\sigma})$ *for each* $(\ell, \hat{\sigma})$.

In group equilibrium, prices, budget sets and demand for private goods depend on the state v. An agent's budget set is affected not only by verifiable memberships, but also by the unverifiable characteristics of other members, which are random. We denote an agent's budget set by

$$B(a,p,q;P_a,n_a) := \{(x_a, \ell, \hat{\sigma}) \ : \ (x_a(v), \ell, \hat{\sigma}) \in X_a, \ p(v) \cdot x_a(v) + q \cdot \ell(\tilde{\ell}) \leq p(v) \cdot e_a + p(v) \cdot (\tilde{\ell} t)$$
$$\text{for } P_a\text{-almost all } v \in V \text{ and every } \tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}}) \text{ such that } n_a(\tilde{\ell}; \ell, \hat{\sigma}) > 0\}$$

Denote the optimizing choices of private goods, memberships, and strategies by

$$d_a(p,q;P_a,n_a) \ := \ \arg \max_{(x_a, \mu_a, \sigma_a)} \int_V \sum_{\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})} n_a(\tilde{\ell}; \mu_a, \sigma_a) u_a(x_a(v), \tilde{\ell}) dP_a(v)$$
$$\text{s.t. } (x_a, \mu_a, \sigma_a) \in B(a,p,q;P_a,n_a)$$

Of this optimizing triple, let $\xi_a(p, q; P_a, n_a)$ denote the demand for private goods. Then aggregate demand for private goods at the state v is given by

$$\xi(p, q)(v) := \int_A \xi_a(p, q; P_a, n_a)(v) d\lambda(a)$$

These definitions make clear that agents can be thought of as making their choices in two steps, first choosing their memberships and strategies at the prices q, while having rational expectations regarding p, and then choosing their consumptions of private goods after the state v is realized and the prices $p(v)$ are known. Equivalently, agents have contingent consumption plans for private goods, contingent on the realizations of v and $\tilde{\ell}$. Thus the state and matching affect the choices of private goods both directly through agents' preferences and indirectly through their budget sets.

A group equilibrium trivially exists, namely, one in which no groups form, since our assumptions are strong enough to guarantee that there is an equilibrium in the exchange economy with no groups. In that equilibrium, no agent can improve utility by choosing a membership, because no agent believes the membership would result in formation of a group. Typically there will be equilibria, or at least quasi-equilibria, with groups as well.[10]

To focus on non-trivial equilibria, we develop a refinement that uses expanded economies in which at least a small mass of every type of group always forms. The limit of the expanded economies corresponds to the real economy, and the refinement selects equilibria that can be approximated arbitrarily closely by equilibria in expanded economies. In the limit, some of the group types may vanish. For an equilibrium with no groups of some types to survive this refinement, agents must hold common beliefs on strategies of other members such that they do not wish to join the types of groups that have vanished.

To formalize, let $\mathcal{E}$ be a group economy. Fix $\varepsilon > 0$, and let $A^\varepsilon_m \subset \mathbf{R}$ be disjoint intervals of length ε for each $m \in \mathbf{M}$. Set

$$A^\varepsilon = A \cup \bigcup_{m \in \mathbf{M}} A^\varepsilon_m$$

The agent space for the ε-expansion $\mathcal{E}^\varepsilon$ is then $(A^\varepsilon, \mathcal{F}^\varepsilon, \lambda^\varepsilon)$, where $\mathcal{F}^\varepsilon$ is the σ-algebra generated jointly by $\mathcal{F}$ and the Lebesgue measurable subsets of $\cup_{m \in \mathbf{M}} A^\varepsilon_m$, and λ^ε is λ on A and Lebesgue measure on $\cup_{m \in \mathbf{M}} A^\varepsilon_m$.

We will say that $\mathcal{E}^\varepsilon$ is an *ε-expanded group economy* if consumption sets, endowments and utility functions of agents in A are the same in the expanded economy $\mathcal{E}^\varepsilon$ as in the original economy $\mathcal{E}$, and the measurable map $a \mapsto (\sigma^\varepsilon_a, u^\varepsilon_a, e^\varepsilon_a)$ on $\cup_{m \in \mathbf{M}} A^\varepsilon_m$ satisfies:

[10] In group-formation models, inputs required for groups can exhaust the endowments of members, who may end up in the zero-wealth position. Guaranteeing that a quasi-equilibrium is an equilibrium therefore requires more assumptions than in an exchange economy. We return to this issue below.

- for each $m \in \mathbf{M}$, agents $a \in A_m^\varepsilon$ have consumption sets

$$X_a = \mathbf{R}_+^N \times \{\ell \in \mathbf{Lists}(\mathbf{M}) : \ell\,(m) = 1 \text{ and } |\ell| = 1\} \times \{\sigma_a^\varepsilon\},$$

- the map $a \mapsto e_a^\varepsilon$ is integrable
- the ex-post utility mapping $(a, x, \tilde{\ell}) \mapsto u_a^\varepsilon(x, \tilde{\ell})$ is a jointly measurable function of its arguments, and for each a, u_a^ε is monotone and continuous in x

In this economy, agents $a \in A_m^\varepsilon$ are "endowed" with membership m and strategy σ_a^ε. Thus in an ε-expanded group economy, a mass of each group type of at least ε will always form, with some distribution of characteristics influenced by the fixed map σ^ε. This gives each agent an empirical basis for forming beliefs over matchings. Choices of memberships and strategies will then be based on these beliefs in an equilibrium.

Our objective is to study a class of equilibria that can be represented as limits of equilibria in these expansions as $\varepsilon \to 0$. To ensure that equilibrium consumptions and prices are comparable across different expansions, we focus on a subclass of equilibria in the economies $\mathcal{E}^\varepsilon$ that are invariant to these expansions.

For each $\tilde{\mu} \in \mathcal{U}$ and $\varepsilon > 0$, let

$$\begin{aligned}
\mathcal{U}^\varepsilon\,(\tilde{\mu}) &:= \{\tilde{\mu}^\varepsilon : A^\varepsilon \to \mathbf{Lists}(\tilde{\mathbf{M}}), \tilde{\mu}_a^\varepsilon = \tilde{\mu}_a \text{ for each } a \in A\} \\
V(\tilde{\mu}) &:= \{v \in V : \tilde{\mu}_a^r\,(v) = \tilde{\mu}_a \text{ for each } a \in A\} \\
V^\varepsilon(\tilde{\mu}) &:= \{v^\varepsilon \in V^\varepsilon : \tilde{\mu}_a^r\,(v^\varepsilon) = \tilde{\mu}_a \text{ for each } a \in A\}
\end{aligned}$$

If $v' \in V^{\varepsilon'}(\tilde{\mu})$ and $v'' \in V^{\varepsilon''}(\tilde{\mu})$, the assignments $\tilde{\mu}^r(v')$ and $\tilde{\mu}^r(v'')$ are indistinguishable for agents in the original economy.

Say that $x^\downarrow \in (\mathbf{R}_+^N)^V$ is a *reduction* of $x \in (\mathbf{R}_+^N)^{V^\varepsilon}$ if $x^\downarrow(v) = x(v^\varepsilon)$ whenever $v^\varepsilon \in V^\varepsilon(\tilde{\mu})$ and $v \in V(\tilde{\mu})$, for each $\tilde{\mu} \in \mathcal{U}$.

Given $\varepsilon > 0$, say that the equilibrium $(x^\varepsilon, \mu^\varepsilon, \sigma^\varepsilon), \mathcal{R}(\mu^\varepsilon, \sigma^\varepsilon), (p^\varepsilon, q^\varepsilon), \{P_a^\varepsilon, n_a^\varepsilon, a \in A\}$ in $\mathcal{E}^\varepsilon$ is *expansion-invariant* if p^ε and x_a^ε for each $a \in A$ have reductions. Expansion invariance restricts attention to equilibria that are equivalent for agents in the original economy (that is, agents in A) whenever the random matching gives them the same augmented lists. With expansion invariance, agents' consumption bundles, as well as the prices they face, depend only on $\tilde{\mu}$, the assignment to agents in A.[11]

Definition 6 *A group equilibrium $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q), \{P_a, n_a, a \in A\}$ in $\mathcal{E}$ is* group perfect *if there exist ε-expansions $\mathcal{E}^\varepsilon$ of the economy $\mathcal{E}$ such that*

$$(p, q) = \lim_{\varepsilon \to 0}\,(p^{\varepsilon\downarrow}, q^\varepsilon)$$

[11] This raises the question whether such equilibria exist. Lemma 6.1 in section 6 shows that the restriction to constant prices on any set of matchings with positive measure is possible by the law of large numbers. It follows from Theorem 6.1 that such an equilibrium exists for each expanded economy $\mathcal{E}^\varepsilon$.

and for almost every $a \in A$,

$$(x_a, \mu_a, \sigma_a) = \lim_{\varepsilon \to 0} (x_a^{\varepsilon \downarrow}, \mu_a^{\varepsilon}, \sigma_a^{\varepsilon})$$

where $(x^\varepsilon, \mu^\varepsilon, \sigma^\varepsilon), \mathcal{R}(\mu^\varepsilon, \sigma^\varepsilon), (p^\varepsilon, q^\varepsilon), \{P_a^\varepsilon, n_a^\varepsilon, a \in A\}$ is an expansion-invariant group equilibrium in $\mathcal{E}^\varepsilon$ for each ε.

Group perfect equilibria exist under our assumptions, but we defer the proof to the following section, where we relate group perfectness to the second equilibrium notion we study.

We close this section by giving two examples to illustrate the effects that matchings may have on markets and prices. Different matchings of agents into groups can create different market conditions. Example 3 illustrates how, as a consequence, prices may be different at different matchings. Example 4 shows that variation in prices can be a source of inefficiency from an ex ante perspective, even though trades in private goods are always efficient ex post.

Example 3: Matching and a continuum of prices

Suppose there are two private goods. Every agent's endowment is $e_a = (1, 1)$. It is convenient to normalize and write the prices as $p = (1, p_2)$. A single group type g has two memberships $m_1, m_2 \in \mathbf{M}(g)$. In each of these memberships agents can take the unverifiable characteristic b or c, so $S_{m_1} = S_{m_2} = \{b, c\}$.

Each agent's consumption set allows a single membership. We assume, as in example 1, that characteristics are innate, with agents $a \in [1, 1/2)$ constrained to choose b in every membership, while agents $a \in [1/2, 1]$ are constrained to choose c. Utilities are given by

$$u_a(x, \tilde{\ell}) = \begin{cases} x_1^a x_2 & \text{if } \tilde{\ell}(m_1, bb) = \tilde{\ell}(m_2, bb) = 0 \\ x_1^{2a} x_2 & \text{if } \tilde{\ell}(m_1, bb) = 1 \text{ or } \tilde{\ell}(m_2, bb) = 1 \end{cases}$$

By this specification, the agents with unverifiable characteristic c, $a \in [1/2, 1]$, have the utility function $x_1^a x_2$. Agents $a \in [0, 1/2)$ with characteristic b have utility for private goods that depends on whether or not they are matched with a like agent. When matched with another b agent, these agents have an agent-specific increase in marginal utility for good 1. Different matchings $\tilde{\mu}$ therefore lead to different demands for private goods, which in turn lead to different equilibrium prices.

We focus on matchings $\tilde{\mu}$ that generate a measure 1/8 of augmented groups (g, bb), since the total measure of groups is 1/2 if everyone joins a group, and of those, 1/4 will be (g, bb). More specifically, for a fixed $\beta \leq 1/4$, consider matchings $\tilde{\mu}$ such that agents $a \in A_\beta := (\beta, \beta + 1/4)$ form the groups of augmented type (g, bb).

Because every agent is indifferent between the two memberships and groups make zero profit, the equilibrium membership prices must be $q = 0$. To calculate equilibrium private-goods prices, we first calculate demand. Demands for good 1 are given by

$$x_{a1}(p_2) = \begin{cases} (p_2+1)\left(\frac{2a}{1+2a}\right) & \text{if } a \in A_\beta \\ (p_2+1)\left(\frac{a}{1+a}\right) & \text{if } a \in A \backslash A_\beta \end{cases}$$

Integrating to get the aggregate demand and setting aggregate demand equal to aggregate supply yields

$$\int_0^\beta (p_2+1)\left(\frac{a}{1+a}\right) d\lambda(a) + \int_\beta^{\beta+1/4} (p_2+1)\left(\frac{2a}{1+2a}\right) d\lambda(a) + \int_{\beta+1/4}^1 (p_2+1)\left(\frac{a}{1+a}\right) d\lambda(a) = 1$$

This equation defines a continuous implicit function $p_2(\beta)$ that is decreasing in β, as the derivative of the left hand side with respect to β is positive. Thus, there are a continuum of equilibrium prices indexed by β. $\Diamond$

Example 4: Inefficient trading due to random prices

This example illustrates the inefficiencies that can arise when prices vary with matchings. We start with an ordinary exchange economy with two types of agents having strictly concave utility functions $u^0, u^1 : \mathbf{R}_+^N \to \mathbf{R}$ and endowments $e_0, e_1 \in \mathbf{R}_+^N$ respectively. Suppose that this exchange economy has three equilibria, (p^*, x^*), $(p^\dagger, x^\dagger)$, and $(p^\ddagger, x^\ddagger)$, satisfying

$$\begin{aligned} u^0(x_0^\dagger) &< u^0(x_0^*) < u^0(x_0^\ddagger) \\ u^1(x_1^\dagger) &> u^1(x_1^*) > u^1(x_1^\ddagger) \end{aligned}$$

There is a single group type $g \in \mathbf{G}$ with two memberships $m_1, m_2 \in \mathbf{M}(g)$, and a single (null) characteristic $S_{m_1} = S_{m_2} = \{s^o\}$. Ex-post utility is independent of membership characteristics, and given by

$$u_a(x, \tilde{\ell}) = \begin{cases} u^0(x) & \text{if } a \in [0, 1/2) \\ u^1(x) & \text{if } a \in [1/2, 1] \end{cases}$$

Membership prices will be zero in any equilibrium, and agents will always be indifferent over memberships. We focus on equilibria in which all agents choose memberships; let (μ, σ) represent equilibrium choices. Fix $\alpha \in (0,1)$ and let $\{V', V''\}$ be a partition of V such that $\mathcal{P}(\mu,\sigma)(V') = \alpha$ and $\mathcal{P}(\mu,\sigma)(V'') = 1-\alpha$. An equilibrium with variation in private goods prices can be constructed in which private goods prices and consumptions (p^α, x^α) satisfy

$$(p^\alpha(v), x^\alpha(v)) = \begin{cases} (p^*, x^*) & \text{for each } v \in V' \\ (p^\dagger, x^\dagger) & \text{for each } v \in V'' \end{cases}$$

To see that the resulting allocation is inefficient, let y_0, y_1 be the average consumptions:

$$\begin{aligned} y_0 &= \alpha x_0^\dagger + (1-\alpha) x_0^* \\ y_1 &= \alpha x_1^\dagger + (1-\alpha) x_1^* \end{aligned}$$

The consumptions (y_0, y_1) are feasible, since they integrate to the aggregate endowment. Moreover, by strict concavity,

$$\begin{aligned} u^0(y_0) &> \alpha u^0(x_0^\dagger) + (1-\alpha)\, u^0(x_0^*) \\ u^1(y_1) &> \alpha u^1(x_1^\dagger) + (1-\alpha)\, u^1(x_1^*) \end{aligned}$$

Thus the feasible allocation (y_0, y_1) Pareto dominates the equilibrium allocation x^α.

As this example illustrates, private goods can be inefficiently distributed in an equilibrium in which prices vary with v. Moreover, even though equilibria with random prices may be inefficient, there is not necessarily an equilibrium with constant prices that is Pareto superior. In this example, none of the three possible equilibria with constant prices is Pareto superior to x^α. ◊

6 Equilibrium with Beliefs on Membership Characteristics

The existence of a group equilibrium is trivial because there is always an equilibrium with no groups in which "no one goes there because no one goes there." In this section, we consider a second equilibrium concept, in which agents assume (perhaps incorrectly) that their chosen memberships can always be accommodated. We show that equilibria of this type also exist and, with constant prices, are equivalent to group-perfect equilibria. As a corollary, this yields the existence of group perfect equilibria as well.

As before, we require that beliefs on membership characteristics must agree in equilibrium with the conditional distribution on characteristics generated by the random group formation model, for group types that form. For groups that do not form in equilibrium, beliefs on membership characteristics cannot be derived from the random group formation model. For such groups, we simply require that agents hold common beliefs over membership characteristics that rationalize their choices not to join these groups.[12] When agents hold beliefs on membership characteristics, they are only partially sophisticated. On one hand, they are assumed to know the probability distribution on the characteristics that will materialize in their groups, conditional on the groups forming, but on the other hand, do not understand that the groups might not form.

To formalize this, let $\Delta(S_{-m}(g))$ be the set of probability distributions on $S_{-m}(g)$. Let

$$\mathbf{F} := \prod_{g \in \mathbf{G}} \prod_{m \in \mathbf{M}(g)} \Delta(S_{-m}(g))$$

Then *beliefs on membership characteristics* are an element $f \in \mathbf{F}$, where $f(s_{-m}; m)$ denotes the probability that each agent assigns to ending up in augmented membership (m, s) when

[12]The restriction to common beliefs is not necessary, and is done simply to save notation. We show that there is always an equilibrium even with this more restrictive assumption.

he chooses (and is matched in) a membership $m \in \mathbf{M}(g)$ and plays strategy s_m. Given f, for each $(\ell, \hat{\sigma}) \in \mathbf{Lists}(\mathbf{M}) \times \Sigma$, let $n(\cdot; \ell, \hat{\sigma}, f) \in \Delta(\mathbf{Lists}(\tilde{\mathbf{M}}))$ be the corresponding distribution on augmented lists, defined by

$$n(\tilde{\ell}; \ell, \hat{\sigma}, f) = \prod_{g \in \mathbf{G}} \prod_{\substack{\{m \in \mathbf{M}(g), s_{-m} \in S_{-m}(g): \\ \tilde{\ell}(m,(s_{-m}, \hat{\sigma}_m))=1\}}} \ell(m) f(s_{-m}; m)$$

The distributions f and n are arbitrary for the moment, although in equilibrium we will require that they coincide with the empirical distributions $\bar{\phi}_{(\mu,\sigma)}$ and $\bar{\eta}_{(\mu,\sigma)}$ for all memberships having positive match probabilities.

Definition 7 *A* group equilibrium with beliefs on membership characteristics *consists of a feasible state $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$, private goods prices $p \in (\mathbf{R}_+^N)^V$ with $p \neq 0$ such that p is measurable, membership prices $q \in \mathbf{R}^{\mathbf{M}}$, beliefs $\{P_a, a \in A\}$ and beliefs on membership characteristics f such that (E1), (E4) and (E5) hold, where:*

(E4) ***Optimization by agents:*** *For almost all $a \in A$, if $(x'_a(v), \mu'_a, \sigma'_a) \in X_a$ for P_a-almost all v and*

$$\int_V \sum_{\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})} n(\tilde{\ell}; \mu'_a, \sigma'_a, f) u_a(x'_a(v), \tilde{\ell}) \, dP_a(v) > \int_V u_a(x_a(v), \tilde{\mu}^r_a(v)) \, dP_a(v)$$

then there exists $V' \subset V$ with $P_a(V') > 0$ and $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$ with $n(\tilde{\ell}; \mu'_a, \sigma'_a, f) > 0$ such that

$$p(v) \cdot x'_a(v) + q \cdot \ell(\tilde{\ell}) > p(v) \cdot e_a + p(v) \cdot (\tilde{\mu}^r_a(v) t) \text{ for every } v \in V'$$

(E5) ***Beliefs are correct:*** *For almost every $a \in A$, $P_a = \mathcal{P}(\mu, \sigma)$ and $f(\cdot; m) = \bar{\phi}_{(\mu,\sigma)}(\cdot; m)$ for each $m \in \mathbf{M}$ such that $\zeta(m; \mu) < 1$.*

In equilibrium, μ is consistent, which implies that $\zeta(m; \mu) \in \{0, 1\}$ for every membership m. If $\zeta(m; \mu) = 1$, groups with membership m form with probability zero, so beliefs on membership characteristics $f(\cdot; m)$ are not constrained by the choices (μ, σ) and the random group formation model $\mathcal{R}(\mu, \sigma)$. When $\zeta(m; \mu) = 0$, the corresponding group forms with probability one, in which case $f(\cdot; m)$ must agree with the empirical distribution derived from the random group formation model.

Nothing in our definition of equilibrium implies that private-goods prices are the same at every matching. Nonetheless, if prices are state-independent, then aggregate demand is constant almost everywhere by the law of large numbers, which provides a tractable way to prove existence of equilibrium. Modeling the matching process gives us a foundation for a precise version of an exact law of large numbers, but also illuminates the fact that constant prices would be an assumption in our model. This assumption is, in effect, maintained in Prescott and Townsend (2006) and Zame (2007).

We say that $\bar{p} \in (\mathbf{R}_+^N)^V$ is a *constant price* if $\bar{p}(v) = p$ for some $p \in \mathbf{R}_+^N$ and for almost all $v \in V$. When p is a constant price, agents face idiosyncratic uncertainty regarding their groups, but no price uncertainty. As a consequence, an agent's private-goods demand set depends only on his own augmented list, but not on the entire matching. Because agents' augmented lists are independent random variables, their demands are independent. The idiosyncratic randomness faced by each agent vanishes in aggregate by a law of large numbers, as we show below.

Given a state space V, in order to describe demand define, for each $a \in A$, and $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$,

$$V_a(\tilde{\ell}) := \{v \in V : \tilde{\mu}_a^r(v) = \tilde{\ell}\}$$

The aggregate output of groups and the resulting transfers are random, because they depend on the random matching. The expected output and transfers are given by

$$\begin{aligned} H(\mu,\sigma) &:= \int_A \int_V \left[\sum_{g\in\mathbf{G}} \sum_{\substack{m\in\mathbf{M}(g)\\ s\in S(g)}} \tilde{\mu}_a^r(v)(m,s)\,\frac{h(g,s)}{|\mathbf{M}(g)|} \right] d\mathcal{P}(\mu,\sigma)(v) d\lambda(a) \\ T(\mu,\sigma) &:= \int_A \int_V \tilde{\mu}_a^r(v) t \; d\mathcal{P}(\mu,\sigma)(v)\; d\lambda(a) \end{aligned}$$

$H(\mu,\sigma)$ and $T(\mu,\sigma)$ are equal if μ is consistent. Moreover, each expectation is equal to the corresponding value for almost all v by the law of large numbers. The following lemma formalizes these results.

Lemma 6.1 *Let $(V, \mathcal{V}, \mathcal{P}(\mu,\sigma))$ be the probability space associated with the random group formation model $\mathcal{R}(\mu,\sigma)$. Let $\bar{p}$ be a constant price and $q \in \mathbf{R}^{\mathbf{M}}$.*

(a) For $\mathcal{P}(\mu,\sigma)$-almost all $v \in V$,

$$\begin{aligned} H(\mu,\sigma) &= \int_A \left[\sum_{g\in\mathbf{G}} \sum_{\substack{m\in\mathbf{M}(g)\\ s\in S(g)}} \tilde{\mu}_a^r(v)(m,s)\,\frac{h(g,s)}{|\mathbf{M}(g)|} \right] d\lambda(a) \\ T(\mu,\sigma) &= \int_A \tilde{\mu}_a^r(v) t \; d\lambda(a) \end{aligned}$$

(b) If μ is consistent, then $H(\mu,\sigma) = T(\mu,\sigma)$.

(c) For each $a \in A$, $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$, and $\mathcal{P}(\mu,\sigma)$-almost all $v, v' \in V_a(\tilde{\ell})$,

$$\xi_a(\bar{p}, q; P_a, n_a)(v) = \xi_a(\bar{p}, q; P_a, n_a)(v').$$

(d) For $\mathcal{P}(\mu, \sigma)$-almost all $v \in V$

$$\int_A \xi_a(\bar{p}, q; P_a, n_a)(v)\ d\lambda(a) = \int_A \int_V \xi_a(\bar{p}, q; P_a, n_a)(v) d\mathcal{P}(\mu, \sigma)(v)\ d\lambda(a)$$

Proof Agents' transfers $\tilde{\mu}_a t$ are pairwise independent as a consequence of Theorem 4.2. Part (a) follows from the law of large numbers (Corollary 2.10 of Sun (2006)).

Part (b) holds by definition of t when μ is consistent.

Part (c) follows because with a constant price, v affects agent a's preferences and budget set only through his own augmented list $\tilde{\mu}_a$.

For (d), we use the fact that agent a's demand set $\xi_a(\bar{p}, q; P_a, n_a)(v)$ is the same for all $v \in V_a(\tilde{\ell})$, for each $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$. For each $a \in A$ and $v \in V$, let $\bar{\xi}_a(\bar{p}, q; P_a, n_a)(v)$ be a selection from the demand set. By Theorem 4.2, for different agents these selections define pairwise independent random variables on V. Then by the law of large numbers (Corollary 2.10 of Sun (2006)), for $\mathcal{P}(\mu, \sigma)$-almost every $v \in V$

$$\int_A \bar{\xi}_a(\bar{p}, q; P_a, n_a)(v) d\lambda(a) = \int_A \left[\int_V \bar{\xi}_a(\bar{p}, q; P_a, n_a)(v) d\mathcal{P}(\mu, \sigma)(v) \right] d\lambda(a)$$

Since the righthand side does not depend on v, the selection generates the same aggregate demand with probability one. Since every element of aggregate demand in (d) is defined by some selection, the result follows. ■

Restricting to constant prices allows us to recast the existence problem in finite dimensions. If prices are constant, then consumptions can be restricted to be elements of $(\mathbf{R}^N)^{\mathbf{Lists}(\tilde{\mathbf{M}})}$ instead of $(\mathbf{R}^N_+)^V$ without loss of generality. With this restriction, say the choices $(x, \mu, \sigma) : A \to (\mathbf{R}^N_+)^{\mathbf{Lists}(\tilde{\mathbf{M}})} \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$ are *feasible* if the aggregate membership vector $\int_A \mu_a\, d\lambda(a)$ is consistent, and for $\mathcal{P}(\mu, \sigma)$-almost every $v \in V$,

$$\int_A x_a(v)\, d\lambda(a) \leq \int_A e_a\, d\lambda(a) + \int_A \left[\sum_{g \in \mathbf{G}} \sum_{\substack{m \in \mathbf{M}(g) \\ s \in S(g)}} \tilde{\mu}^r_a(v)(m, s) \frac{h(g, s)}{|\mathbf{M}(g)|} \right] d\lambda(a) \tag{3}$$

Let $W_a(\cdot; f) : \hat{X}_a \to \mathbf{R}$ represent agent a's expected utility, defined as

$$W_a(x_a, \mu_a, \sigma_a; f) := \sum_{\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})} n(\tilde{\ell}; \mu_a, \sigma_a, f) u_a(x_a(\tilde{\ell}), \tilde{\ell})$$

where

$$\hat{X}_a = \{(x, \ell, \hat{\sigma}) \in (\mathbf{R}^N_+)^{\mathbf{Lists}(\tilde{\mathbf{M}})} \times \mathbf{Lists}(\mathbf{M}) \times \Sigma \ : \ (x(\tilde{\ell}), \ell, \hat{\sigma}) \in X_a \text{ for each } \tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}}) \text{ and } (\ell, \hat{\sigma}) \in \mathbf{Lists}(\mathbf{M}) \times \Sigma\}$$

We use this reformulation in the appendix to show that an equilibrium with beliefs on membership characteristics exists.[13]

A basic problem encountered in club models is that group formation can deplete members' resources entirely, so they end up in the zero-wealth situation. We modify assumptions used in EGSZ (1999, 2005) to avoid this problem, and to restore the equivalence between quasi-equilibrium and equilibrium. First, we say that *endowments are desirable* if $u_a(e_a, 0) > u_a(0, \tilde{\ell})$ for all $(\tilde{\ell}, \hat{\sigma}) \in \mathbf{Lists}(\tilde{\mathbf{M}}) \times \Sigma$ such that $(0, \tilde{\ell}, \hat{\sigma}) \in X_a$. Next, let $\mathcal{E}$ be a group economy and let (x, μ, σ) be a feasible state. Let $I \subset \{1, \ldots, N\}$ be a non-empty set of private goods. Say that the feasible state (x, μ, σ) is a *minimum consumption configuration for good* i if for almost all agents $a \in A$ there does not exist a bundle x'_a of private goods such that $x'_a \leq x_a, x'_{ai} < x_{ai}$ and $(x'_a, \mu_a, \sigma_a) \in X_a$. (If $(0, \mu_a, \sigma_a) \in X_a$ then a feasible state is a minimum consumption configuration for good i only if the entire social endowment of i is used in group formation.) Say that (x, μ, σ) is *group linked* if for every partition $I \cup J = \{1, \ldots, N\}$ of the set of consumption goods for which (x, μ, σ) is a minimum expenditure configuration for each good $i \in I$, then for almost every $a \in A$ there is a real number $r \in \mathbf{R}$ and an index $j \in J$ such that

$$u_a(e_a + r\delta_j, 0) > u_a(x_a, \tilde{\mu}_a)$$

for each $\tilde{\mu}_a \in \{\tilde{\mu}_a \in \mathbf{Lists}(\tilde{\mathbf{M}}) | \tilde{\mu}_a(m, s) = 1 \Rightarrow \mu_a(m) = 1 \text{ and } \sigma_{a,m} = s_m\}$, where δ_j is the j^{th} unit vector. We say that $\mathcal{E}$ is *group irreducible* if every feasible state is group linked. That is, if the entire social endowment of the private goods in I is used up in production, then for almost all agents a, there is some good $j \notin I$ and some sufficiently large level of consumption of good j such that agent a would prefer consuming his endowment together with this large level of good j, and belong to no groups, rather than consume the bundle x_a in the augmented group memberships $\tilde{\mu}_a$.

Theorem 6.1 *If endowments are desirable and the economy is group irreducible, then a group equilibrium with beliefs on membership characteristics exists.*

In the proof of this theorem, given in the appendix, we show that the argument of EGSZ (1999) can be extended to account for the introduction of unverifiable characteristics, the dependence of choices on beliefs over membership characteristics, and to secure correct beliefs in equilibrium. In fact, the proof actually establishes the stronger result that there exists a constant-price group equilibrium with beliefs on membership characteristics.

If we restrict to constant prices, then equilibria with beliefs on membership characteristics coincide with group perfect equilibria, as the following theorem shows.

[13]To avoid confusions that might arise from the change in commodity space, we define the restricted notion of equilibrium formally in the appendix.

Theorem 6.2 *Suppose that endowments are desirable and the economy is group irreducibile.*

(i) Every group perfect equilibrium with constant prices is a group equilibrium with beliefs on membership characteristics.

(ii) Every group equilibrium with beliefs on membership characteristics and constant prices is a group perfect equilibrium.

Proof For (i), let $\varepsilon > 0$ be given, and let $(x^\varepsilon, \mu^\varepsilon, \sigma^\varepsilon), \mathcal{R}(\mu^\varepsilon, \sigma^\varepsilon), (p^\varepsilon, q^\varepsilon)$ be a group equilibrium in $\mathcal{E}^\varepsilon$. We first show that there exist beliefs f^ε on membership characteristics such that $(x^\varepsilon, \mu^\varepsilon, \sigma^\varepsilon), \mathcal{R}(\mu^\varepsilon, \sigma^\varepsilon), (p^\varepsilon, q^\varepsilon), f^\varepsilon$ is an equilibrium with beliefs on membership characteristics in the economy $\mathcal{E}^\varepsilon$.

Let f^ε be defined by

$$f^\varepsilon(s_{-m}; m) := \bar{\phi}^\varepsilon_{(\mu^\varepsilon, \sigma^\varepsilon)}(s_{-m}; m) \text{ for each } s \text{ and } m$$

Then (*E5*) (correct beliefs) is satisfied by definition.

Since every group type forms in a group equilibrium in the economy $\mathcal{E}^\varepsilon$, each agent believes with probability one that his chosen groups will form if he deviates. That is, almost every realized list is equal to the chosen list. The optimization condition (*E4*) is therefore equivalent to the optimization condition (*E2*). Thus the group equilibrium is also an equilibrium with beliefs on membership characteristics, with beliefs f^ε.

By passing to a subsequence if necessary, define f by

$$f(s_{-m}; m) = \lim_{\varepsilon \to 0} \bar{\phi}^\varepsilon_{(\mu^\varepsilon, \sigma^\varepsilon)}(s_{-m}; m) \tag{4}$$

Take $(x, \mu, \sigma), (p, q)$ to be the limit of $(x^{\varepsilon\downarrow}, \mu^\varepsilon, \sigma^\varepsilon), (p^{\varepsilon\downarrow}, q^\varepsilon)$ as $\varepsilon \to 0$. We show that $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q), f$ is an equilibrium with beliefs on membership characteristics and constant prices.

For each $(y, \ell, \hat{\sigma}) \in (\mathbf{R}^N_+)^{\mathbf{Lists}(\tilde{\mathbf{M}})} \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$ and $a \in A$,

$$\lim_{\varepsilon \to 0} W_a(y, \ell, \hat{\sigma}; \bar{\phi}^\varepsilon_{(\mu^\varepsilon, \sigma^\varepsilon)}) = W_a(y, \ell, \hat{\sigma}; f)$$

Next we show that (*E4*) holds, that is, that (x, μ, σ) is optimal given (p, q) for almost all agents. To that end, suppose not. Then by Lusin's Theorem, there exist $\delta_0 > 0$, $A_0 \subset A$ with $\lambda(A_0) > 0$, and $(x', \mu', \sigma') : A_0 \to (\mathbf{R}^N_+)^{\mathbf{Lists}(\tilde{\mathbf{M}})} \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$ such that for all $a \in A_0$

$$W_a(x'_a, \mu'_a, \sigma'_a; f) > W_a(x_a, \mu_a, \sigma_a; f) + \delta_0 \tag{5}$$

and

$$p \cdot x'_a(\tilde{\ell}) + q \cdot \mu'_a \leq p \cdot e_a + p \cdot (\tilde{\ell}t) \text{ if } n(\tilde{\ell}; \mu'_a, \sigma'_a, f) > 0$$

We show that this is impossible because it must imply that $(x^\varepsilon, \mu^\varepsilon, \sigma^\varepsilon), \mathcal{R}(\mu^\varepsilon, \sigma^\varepsilon), (p^\varepsilon, q^\varepsilon), f^\varepsilon$ is not an equilibrium for ε sufficiently close to 0.

To that end, let $\tilde{\ell} : A_0 \to \textbf{Lists}(\tilde{\mathbf{M}})$ be a selection of augmented lists such that for each $a \in A_0$, $n(\tilde{\ell}_a; \mu'_a, \sigma'_a, f) > 0$ and $x'_{ai}(\tilde{\ell}_a) > 0$ for some commodity i. The existence of these lists follows from the assumption that endowments are desirable. Further, since $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q)$ is a group equilibrium, it follows from group irreducibility and desirability of endowments that $p \in \mathbf{R}^N_{++}$ (see Proposition 3.3 of EGSZ 1999), so there exists $p_{\min}$ such that $p_i \geq p_{\min} > 0$ for each i.

Next, choose $\delta_1 > 0$ and $z : A_0 \to \left(\mathbf{R}^N_+\right)^{\textbf{Lists}(\tilde{\mathbf{M}})}$ such that

(a) $z_a \leq x'_a$

(b) $W_a(x'_a, \mu'_a, \sigma'_a; f) - W_a(z_a, \mu'_a, \sigma'_a; f) < \delta_0/3$ for each $a \in A_0$

(c) for each $a \in A_0$ and some commodity i, $x'_{ai}(\tilde{\ell}_a) > z_{ai}(\tilde{\ell}_a) + \delta_1 > 0$

Choose $\delta_2 > 0$ such that

(d) $\delta_2 < \delta_1 p_{\min}$

Now take a sequence $\varepsilon^n \to 0$. By Egoroff's Theorem, there exists $\bar{n}$ and a set $A_1 \subset A_0$ with $\lambda(A_1) > 0$ such that for each $a \in A_1$ and $n \geq \bar{n}$,

(e) $\left| W_a(z_a, \mu'_a, \sigma'_a; f) - W_a(z_a, \mu'_a, \sigma'_a; \bar{\phi}^{\varepsilon^n}_{(\mu^{\varepsilon^n}, \sigma^{\varepsilon^n})}) \right| < \delta_0/3$

(f) $\left| W_a(x_a, \mu_a, \sigma_a; f) - W_a(x^{\varepsilon^n}_a, \mu^{\varepsilon^n}_a, \sigma^{\varepsilon^n}_a; \bar{\phi}^{\varepsilon^n}_{(\mu^{\varepsilon^n}, \sigma^{\varepsilon^n})}) \right| < \delta_0/3$

(g) $p^{\varepsilon^n} \cdot x'_a(\tilde{\ell}_a) + q^{\varepsilon^n} \cdot \mu'_a < p^{\varepsilon^n} \cdot e_a + p^{\varepsilon^n} \cdot (\tilde{\ell}_a t) + \delta_2$

We conclude the proof by arguing that for $n \geq \bar{n}$ and $a \in A_1$:

$$W_a(z_a, \mu'_a, \sigma'_a; \bar{\phi}^{\varepsilon^n}_{(\mu^{\varepsilon^n}, \sigma^{\varepsilon^n})}) > W_a(x^{\varepsilon^n}_a, \mu^{\varepsilon^n}_a, \sigma^{\varepsilon^n}_a; \bar{\phi}^{\varepsilon^n}_{(\mu^{\varepsilon^n}, \sigma^{\varepsilon^n})}) \tag{6}$$

$$p^{\varepsilon^n} \cdot z_a(\tilde{\ell}_a) + q^{\varepsilon^n} \cdot \mu'_a < p^{\varepsilon^n} \cdot e_a + p^{\varepsilon^n} \cdot (\tilde{\ell}_a t) \tag{7}$$

(6) follows from (e), (b), (5), and (f). For (7), notice first that (a) and (c) imply $p^{\varepsilon^n} \cdot x'_a(\tilde{\ell}_a) > p^{\varepsilon^n} \cdot z_a(\tilde{\ell}_a) + \delta_1 p_{\min}$. Adding (d) and (g) yields

$$\begin{aligned} p^{\varepsilon^n} \cdot z_a(\tilde{\ell}_a) + q^{\varepsilon^n} \cdot \mu'_a &< p^{\varepsilon^n} \cdot x'_a(\tilde{\ell}_a) + q^{\varepsilon^n} \cdot \mu'_a - \delta_1 p_{\min} \\ &< p^{\varepsilon^n} \cdot e_a + p^{\varepsilon^n} \cdot (\tilde{\ell}_a t) + \delta_2 - \delta_1 p_{\min} < p^{\varepsilon^n} \cdot e_a + p^{\varepsilon^n} \cdot (\tilde{\ell}_a t) \end{aligned}$$

from which (7) follows.

Together (6) and (7) contradict that $(x^{\varepsilon^n}, \mu^{\varepsilon^n}, \sigma^{\varepsilon^n}), \mathcal{R}\left(\mu^{\varepsilon^n}, \sigma^{\varepsilon^n}\right), (p^{\varepsilon^n}, q^{\varepsilon^n}), f^{\varepsilon^n}$ is an equilibrium with beliefs on memberships for each $\mathcal{E}^{\varepsilon^n}$.

For the converse, part (ii), let $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q), f$ be a group equilibrium with beliefs on membership characteristics and constant prices. We construct an economy $\mathcal{E}^\varepsilon$ for each $\varepsilon > 0$. Choose a mapping $\sigma^\varepsilon : A^\varepsilon \to \Sigma$ such that for each g, each $m \in \mathbf{M}(g)$, and each $s_{-m} \in S_{-m}(g)$,

$$\frac{\prod_{m' \in \mathbf{M}(g) \setminus m} \lambda^\varepsilon \{a \in A^\varepsilon_{m'} : \sigma^\varepsilon_{a,m'} = s_{m'}\}}{\lambda^\varepsilon(A^\varepsilon_{-m})} = f(s_{-m}; m)$$

That is, for each membership m, choose a distribution of strategies among the agents $a \in A^\varepsilon_m$ such that the induced joint distribution on membership characteristics matches that given by f.

Choose preferences for each $a \in \cup_m A^\varepsilon_m$ by setting $u_a(x, \tilde{\ell}) = p \cdot x$.

Next, construct the equilibrium consumption bundles and endowments for each added agent in the economy $\mathcal{E}^\varepsilon$. Each agent in A^ε_m is constrained to choose the list μ_a that includes a single membership, m, and the strategy given above. Correspondingly, agent a's augmented lists $\tilde{\mu}_a$ are constrained to $\mathbf{Lists}(\tilde{\mathbf{M}})(\mu_a) := \{\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}}) : \tilde{\ell}(m, s) = 1 \Rightarrow \mu_a(m) = 1\}$. Choose endowments e^ε_a so that for almost every $a \in A^\varepsilon_m$, $e^\varepsilon_a + t\tilde{\mu}_a \gg 0$ for all $\tilde{\mu}_a \in \mathbf{Lists}(\tilde{\mathbf{M}})(\mu_a)$. Then for each $a \in \cup_m A^\varepsilon_m$, set $x^\varepsilon_a(v) = e^\varepsilon_a + t\tilde{\mu}^r_a(v)$ for each v.

For each $a \in A$, set $(\mu^\varepsilon_a, \sigma^\varepsilon_a) = (\mu_a, \sigma_a)$, and set x^ε_a so that $x^\varepsilon_a(v^\varepsilon) = x_a(v)$ for every $v^\varepsilon \in V^\varepsilon(\tilde{\mu})$ and every $v \in V(\tilde{\mu})$, each $\tilde{\mu} \in \mathcal{U}$. Similarly, set $q^\varepsilon = q$ and p^ε so that $p^\varepsilon(v^\varepsilon) = p(v)$ for every every $v^\varepsilon \in V^\varepsilon(\tilde{\mu})$ and every $v \in V(\tilde{\mu})$, each $\tilde{\mu} \in \mathcal{U}$. By construction, p^ε and each x^ε_a are expansion invariant, with $p^{\varepsilon\downarrow} = p$ and $x^{\varepsilon\downarrow}_a = x_a$ for each $a \in A$.

By construction, $(x^\varepsilon, \mu^\epsilon, \sigma^\varepsilon), \mathcal{R}(\mu^\varepsilon, \sigma^\varepsilon), (p^\varepsilon, q^\varepsilon)$ is a group equilibium in $\mathcal{E}^\varepsilon$ for each ε. Furthermore,

$$(p, q) = \lim_{\varepsilon \to 0} (p^{\varepsilon\downarrow}, q^\varepsilon)$$

and for almost every $a \in A$,

$$(x_a, \mu_a, \sigma_a) = \lim_{\varepsilon \to 0} (x^{\varepsilon\downarrow}_a, \mu^\varepsilon_a, \sigma^\varepsilon_a)$$

For the original economy, to show that $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q)$ is a group perfect equilibrium, it remains to show that it is a group equilibrium. But this follows because $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q), f$ is an equilibrium with beliefs on membership characteristics. Any utility improvement available to an agent who knows correctly the group formation model is also available if the agent also thinks his groups will form with probability one. This implies that almost all agents are optimizing under both equilibrium concepts. ■

This equivalence result, coupled with the existence of equilibrium with beliefs on membership characteristics and constant prices, establishes the existence of group perfect equilibria as well.

Theorem 6.3 *If endowments are desirable and the economy is group irreducible, then a group perfect equilibrium exists.*

Proof This follows from Theorem 6.1 and 6.2. ■

For group types that do not form in equilibrium, there is no empirical basis for the beliefs on membership characteristics. The next example illustrates the importance of beliefs regarding group types that do not form in equilibrium. Example 5 shows that beliefs can support an inefficient state with no group formation at all, even if agents believe that their chosen groups will be filled.

As example 4 showed, variation in prices can be a source of inefficiency. Restricting to constant prices evidently eliminates this source of inefficiency, but nevertheless does not ensure efficient outcomes. This is not surprising, since the basic inefficiencies of game theory, such as coordination problems, remain. More strikingly, though, example 6 demonstrates that equilibria with constant prices can be Pareto ranked, even when the choices (μ, σ) are fixed and the equilibria entail the same distribution on matchings and beliefs.

Example 5: Inefficient equilibrium with no groups

Suppose there is a single group type g with two memberships $m_1, m_2 \in \mathbf{M}(g)$. As in our previous examples, suppose agents can take one of two unverifiable characteristics in each membership, so $S_{m_1} = S_{m_2} = \{b, c\}$. Each agent can choose at most one membership. There is a single private good, of which every agent is endowed with $e = 3$ units. Agents $a \in [0, 2/3)$ are constrained to choose strategy b in each membership, and have utility function

$$u_a(x, \tilde{\ell}) = \begin{cases} x & \text{if } \tilde{\ell}(m, bb) = 1 \text{ for } m \in \mathbf{M}(g) \\ x - 1 & \text{if } \tilde{\ell}(m_1, bc) = 1 \text{ or } \tilde{\ell}(m_2, cb) = 1 \\ x & \text{if } \tilde{\ell} = 0 \end{cases}$$

Agents $a \in [2/3, 1]$ are constrained to choose c in each membership, and have utility function

$$u_a(x, \tilde{\ell}) = \begin{cases} x - 4 & \text{if } \tilde{\ell}(m, cc) = 1 \text{ for } m \in \mathbf{M}(g) \\ x + 1/2 & \text{if } \tilde{\ell}(m_1, cb) = 1 \text{ or } \tilde{\ell}(m_2, bc) = 1 \\ x - 1 & \text{if } \tilde{\ell} = 0 \end{cases}$$

One equilibrium with beliefs on membership characteristics in this example has half of the b agents taking m_1 memberships, and all of the c agents taking m_2 memberships. The remaining b agents take no memberships. These choices are supported by membership prices $q(m_1) = -1$ and $q(m_2) = 1$, and (correct) beliefs $f(c; m_1) = 1$, $f(b; m_2) = 1$.

This is not the only equilibrium with beliefs on membership characteristics, however. Suppose instead that all agents believe that the distribution of characteristics in their groups will match the population distribution. That is, suppose agents hold beliefs $f(c;m) = 1/3$, $f(b;m) = 2/3$ for each $m \in \mathbf{M}(g)$. Since the memberships are indistinguishable, membership prices are zero, and it is optimal to take no memberships. No groups form in this equilibrium, which is inefficient. $\diamondsuit$

Example 6: Constant-price equilibria may be Pareto ranked

As in example 4, start with an ordinary two-person exchange economy with strictly concave utility functions $v_1, v_2 : \mathbf{R}_+^N \to \mathbf{R}_+$, and endowments $e_1 = e_2 \in \mathbf{R}_+^N$. We suppose that this exchange economy has two equilibria $(p^\dagger, x^\dagger), (p^\ddagger, x^\ddagger)$, with corresponding equilibrium utilities that satisfy

$$\begin{aligned} v_1(x_1^\dagger) &> v_1(x_1^\ddagger) \\ v_2(x_2^\ddagger) &> v_2(x_2^\dagger) \\ \frac{1}{2}v_1(x_1^\dagger) + \frac{1}{2}v_2(x_2^\dagger) &> \frac{1}{2}v_1(x_1^\ddagger) + \frac{1}{2}v_2(x_2^\ddagger) \end{aligned}$$

We embed this exchange economy in a group formation model by imagining that each agent's ex-post utility for private goods depends on the augmented group type into which he is randomly matched, and may be either v_1 or v_2. There is a single group type g with two memberships $\mathbf{M}(g) = \{m_1, m_2\}$, and two unverifiable characteristics, $S_{m_1} = S_{m_2} = \{b, c\}$. Each agent can choose one membership. Agents $a \in [0, 1/2)$ are constrained to choose strategy b in each membership, and agents $a \in [1/2, 1]$ must choose strategy c.

The utility function for each agent $a \in A$ is:

$$u_a(x, \tilde{\ell}) = \begin{cases} v_1(x) & \text{if } \tilde{\ell}(m, bc) = 1 \text{ for } m \in \mathbf{M}(g) \\ v_2(x) & \text{if } \tilde{\ell}(m, bb) = 1 \text{ or } \tilde{\ell}(m, cc) = 1 \text{ for } m \in \mathbf{M}(g) \\ \frac{1}{2}\min(v_1(x) - 1, v_2(x) - 1) & \text{if } \tilde{\ell} = 0 \end{cases}$$

There is a constant price equilibrium with private goods prices $p^\ddagger$ and membership prices $q = 0$ in which half the agents of each type take each membership. Half of the agents will be matched with like agents, and half will be in a mixed group. The agents in mixed groups (g, bc) or (g, cb) consume $x_1^\ddagger$ while the agents in homogeneous groups (g, bb) or (g, cc) consume $x_2^\ddagger$. The expected utility of every agent in this equilibrium is

$$\frac{1}{2}v_1(x_1^\ddagger) + \frac{1}{2}v_2(x_2^\ddagger)$$

By the same argument, there is a second constant price equilibrium with private goods prices $p^\dagger$ and membership prices $q = 0$ in which the agents in mixed groups (g, bc) or (g, cb) consume $x_1^\dagger$ and the agents in homogeneous groups (g, bb) or (g, cc) consume $x_2^\dagger$. By construction, this equilibrium Pareto dominates the previous one, since

$$\frac{1}{2}v_1(x_1^\dagger) + \frac{1}{2}v_2(x_2^\dagger) > \frac{1}{2}v_1(x_1^\ddagger) + \frac{1}{2}v_2(x_2^\ddagger)$$

$\Diamond$

7 Efficiency: Private goods and Insurance

The trading of private goods must be efficient from an ex post point of view, after the state v has been realized. Because the consumption of private goods depends on the state and is therefore random, this does not imply that trades are efficient from an ex ante point of view, even keeping the memberships and strategies (μ, σ) fixed. Example 4 illustrates that a Pareto improvement can be achieved by averaging over the consumptions supported ex post by different prices. Example 6 illustrates that constant-price equilibria can sometimes be Pareto ranked. An agent might be willing to trade lower utility at some states for higher utility at another state, both predicated on the same memberships and strategies (μ, σ).

In this section, we investigate whether insurance can allow efficient trades across states, and the effect this has on resulting equilibrium prices. The insurance we describe is feasible provided augmented membership lists are are not only observable ex post, but also verifiable ex post.

We begin with an example to illustrate the ideas.

Example 7: Efficient trading with insurance

There is a single private good, all agents have the same endowment, $e = 0$, and the ex-post utility function of every agent $a \in A$ is given by $u_a(x, \tilde{\ell}) = \sqrt{x}$. There is a single group type g with two memberships, $\mathbf{M}(g) = \{m_1, m_2\}$, that either agent can take. Since nothing verifiable distinguishes memberships, $q = 0$ in any equilibrium.

The unverifiable characteristics in the two memberships are $S_{m_1} = S_{m_2} = \{b, c\}$. Agents $a \in [0, 1/2)$ are constrained to play strategy b in every membership and agents $a \in [1/2, 1]$ are constrained to play strategy c. The output in each augmented group is

$$\begin{aligned} h(g, bc) &= h(g, cb) = 4 \\ h(g, cc) &= h(g, bb) = 0 \end{aligned}$$

The internal transfers t give the same payment to each member, which is half the output y.

Consider the equilibrium of the economy in which half the agents of each type choose each membership. If an agent is matched in an augmented group (g, bb) or (g, cc), he consumes 0. If matched in an augmented group (g, bc) or (g, cb) he consumes 2. Thus, consumption is risky, and expected utility is $(1/2)\sqrt{2} = \sqrt{1/2}$. Expected utility can be improved if the lucky agents in augmented groups (g, bc) or (g, cb) transfer consumption to the unlucky agents in augmented groups (g, bb) and (g, cc), so that all agents consume 1 regardless of the matching. The riskiness in consumption can be eliminated by insurance. $\Diamond$

We model insurance by modifying the agents' budget constraints. Each agent faces a single budget constraint that holds in expectation, rather than a separate budget constraint at each state. Implicitly, this allows the agent to transfer income between states.

Definition 8 *A* group equilibrium with insurance *consists of a feasible state* $(x,\mu,\sigma), \mathcal{R}(\mu,\sigma)$, *private goods prices* $p \in (\mathbf{R}^N_+)^V$ *with* $p \neq 0$, *where* p *is measurable, membership prices* $q \in \mathbf{R}^{\mathbf{M}}$, *and beliefs* $\{P_a, n_a, a \in A\}$ *such that (E1), (E3), and (E6) hold, where*

(E6) ***Optimization by agents:*** *For almost all* $a \in A$, *if* $(x'_a(v), \mu'_a, \sigma'_a) \in X_a$ *for* P_a-*almost all* v *and*

$$\int_V \sum_{\tilde{\ell}\in\mathbf{Lists}(\tilde{\mathbf{M}})} n_a(\tilde{\ell};\mu'_a,\sigma'_a)u_a(x'_a(v),\tilde{\ell})dP_a(v) > \int_V u_a(x_a,\tilde{\mu}^r_a(v))dP_a(v)$$

then

$$\int_V \sum_{\tilde{\ell}\in\mathbf{Lists}(\tilde{\mathbf{M}})} n_a(\tilde{\ell};\mu'_a,\sigma'_a)\left[(p(v)\cdot x'_a(v) + q\cdot \ell(\tilde{\ell}) - p(v)\cdot(e_a+\tilde{\mu}^r_a(v)t)\right]\ dP_a(v) > 0$$

This model of insurance is similar to that of Malinvaud (1973), in which agents are understood to be insured at actuarially fair prices when their consumption choices maximize utility subject to an expected budget constraint. A natural question is how to implement such insurance, and in particular, whether such insurance can be achieved by trading Arrow securities or other assets. Cass, Chichilnisky and Wu (1996) consider this question in a model like Malinvaud's, with a finite number of types of consumers, and with finitely many collective states arising from the independent risks faced by the individuals. They show that insurance in the Malinvaud sense can be achieved if agents trade $H(S-1)T$ insurance contracts against individual risks, together with T Arrow securities against collective risks, where H is the number of consumer types, S is the number of individual states and T is the number of collective states.

In our model, due to the continuum and to the law of large numbers, there is no collective risk on supply of commodities, although there is collective risk on prices. A natural interpretation of Arrow securities would be that claims depend on states v, and the claims trade at actuarially fair prices relative to the probability distribution $\mathcal{P}(\mu,\sigma)$. No further insurance instruments would be necessary, as the probability distribution describes the social risks and, through the induced probability distribution on $\tilde{\mu}$, the individual risks. A conjecture in the spirit of Cass, Chichilnisky and Wu (1996) would be that it is enough to trade Arrow securities with claims linked to a reduced set of states indexed by private-goods prices, and in addition, agents insure individually against variation in their individual augmented lists.[14]

[14]We note that the insurance we have modeled cannot be implemented by an insurance firm that involves a finite collection of agents, so we cannot replicate these results by introducing insurance group types.

Introducing insurance in this sense yields an efficient allocation of private goods, conditional on the memberships and strategies, (μ, σ). A state $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$ is *Pareto dominated* if there exists a feasible state $(x', \mu', \sigma'), \mathcal{R}(\mu', \sigma')$ such that

$$\int_{V'} u_a\left(x'_a(v), \tilde{\mu}^r_a(v')\right)\, d\mathcal{P}(\mu', \sigma')(v') \geq \int_V u_a\left(x_a(v), \tilde{\mu}^r_a(v)\right)\, d\mathcal{P}(\mu, \sigma)(v)$$

for almost every $a \in A$, with strict inequality for a set of agents $A' \subset A$ with positive measure. A feasible state $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$ is *Pareto optimal* if it is not Pareto dominated.

Theorem 7.1 *Suppose $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q), \{P_a, n_a, a \in A\}$ is a group equilibrium with insurance. Then no feasible state $(x', \mu, \sigma), \mathcal{R}(\mu, \sigma)$ Pareto dominates $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$.*

Proof Let $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma), (p, q), \{P_a, n_a, a \in A\}$ be an equilibrium with insurance, and suppose that $(x', \mu, \sigma), \mathcal{R}(\mu, \sigma)$ is a feasible state that Pareto dominates $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$. Then there is a set $A' \subset A$ of positive measure such that

$$\int_V u_a\left(x'_a(v), \tilde{\mu}^r_a(v)\right)\, d\mathcal{P}(\mu, \sigma)(v) > \int_V u_a\left(x_a(v), \tilde{\mu}^r_a(v)\right)\, d\mathcal{P}(\mu, \sigma)(v)$$

for all $a' \in A'$, with

$$\int_V u_a\left(x'_a(v), \tilde{\mu}^r_a(v)\right)\, d\mathcal{P}(\mu, \sigma)(v) \geq \int_V u_a\left(x_a(v), \tilde{\mu}^r_a(v)\right)\, d\mathcal{P}(\mu, \sigma)(v)$$

for almost all $a \in A \setminus A'$. For $a \in A'$,

$$q \cdot \mu_a + \int_V p(v) \cdot \left[x'_a(v) - \tilde{\mu}^r_a(v)t - e_a\right] d\mathcal{P}(\mu, \sigma)(v) > 0$$

while for $a \in A \setminus A'$, strict monotonicity implies

$$q \cdot \mu_a + \int_V p(v) \cdot \left[x'_a(v) - \tilde{\mu}^r_a(v)t - e_a\right] d\mathcal{P}(\mu, \sigma)(v) \geq 0$$

Integrating over A and using the fact that $\lambda(A') > 0$ yields

$$q \cdot \int_A \mu_a d\lambda(a) + \int_A \int_V p(v) \cdot \left[x'_a(v) - \tilde{\mu}^r_a(v)t - e_a\right] d\mathcal{P}(\mu, \sigma)(v) d\lambda(a) > 0$$

Since μ is consistent, $q \cdot \int_A \mu_a d\lambda(a) = 0$. Thus

$$\int_A \int_V p(v) \cdot \left[x'_a(v) - \tilde{\mu}^r_a(v)t - e_a\right] d\mathcal{P}(\mu, \sigma)(v) d\lambda(a) > 0 \tag{8}$$

However, using Lemma 6.1(a), and the feasibility of (x', μ, σ), for $\mathcal{P}(\mu, \sigma)$-almost all $v \in V$,

$$\int_A [x'_a(v) - \sum_{g \in \mathbf{G}} \sum_{\substack{m \in \mathbf{M}(g) \\ s \in S(g)}} \tilde{\mu}^r_a(v)(m, s) \frac{h(g, s)}{|\mathbf{M}(g)|} - e_a] d\lambda(a) \leq 0 \tag{9}$$

Because μ is consistent, again by Lemma 6.1, for $\mathcal{P}(\mu,\sigma)$-almost all $v \in V$,

$$\begin{aligned}\int_A \sum_{g\in\mathbf{G}} \sum_{\substack{m\in\mathbf{M}(g)\\ s\in S(g)}} \tilde{\mu}_a^r(v)(m,s)\frac{h(g,s)}{|\mathbf{M}(g)|}d\lambda(a) &= \int_A \sum_{g\in\mathbf{G}} \sum_{\substack{m\in\mathbf{M}(g)\\ s\in S(g)}} \tilde{\mu}_a^r(v)(m,s)\,t_g(m,h(g,s))\,d\lambda(a)\\ &= \int_A \tilde{\mu}_a^r(v)t\;d\lambda(a)\end{aligned}$$

Substituting in (9) yields

$$\int_A \left[x'_a(v) - \tilde{\mu}_a^r(v)t - e_a\right] d\lambda(a) \leq 0 \text{ for } \mathcal{P}(\mu,\sigma)\text{-almost all } v \in V$$

This violates feasibility of x'. ■

Theorem 7.1 is a constrained version of the first welfare theorem, since the comparison is only among feasible states that share the same membership and strategy choices (μ,σ). In example 7, the equilibrium with insurance is efficient conditional on membership choices, but there is a Pareto-superior equilibrium with complete sorting in which agents of type b choose m_1 and agents of type c choose m_2. The Pareto superior equilibrium is possible despite Theorem 7.1 because the two equilibria involve different membership choices.

On the other hand, suppose that (μ,σ) is "efficient" in the sense that (x,μ,σ) is an efficient state for some x. An implication of examples 4 and 6 is that, absent insurance, an equilibrium state (x',μ,σ) might *not* be efficient.

The insurance scheme described in (*E6*) implicitly allows the agent to insure against both sources of randomness, the randomness due to variation in prices and the randomness in matching. However, the next theorem shows that, with insurance, one of these sources of randomness disappears. Equilibrium prices are constant, provided utility for private goods consumption is suitably concave and differentiable. If utility is concave, insurance leads to constant consumption of private goods. Insurance also leads to constant prices, provided there is a unique price vector that supports the given consumption of private goods.

Theorem 7.2 *Suppose that* $(x,\mu,\sigma), \mathcal{R}(\mu,\sigma), (p,q)$ *is a group equilibrium with insurance in which* p *is strictly positive and* x_a *is strictly positive for almost all* $a \in A$. *Suppose for almost all* $a \in A$, $u_a(\cdot,\tilde{\ell})$ *is* C^2, *strictly concave, and* $D_x u_a(x,\tilde{\ell}) \gg 0$ *for each* $x \in \mathbf{R}^N_{++}$ *and* $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$. *Then the private-goods prices* p *are constant and the consumption* x *satisfies*

$$x_a(v) = x_a(v') \text{ for a.e. } a \in A,\ \text{ a.e. } v, v' \in V_a(\tilde{\ell}),\ \text{ for each } \tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}}) \tag{10}$$

Proof We first show (10). We show that if x does not satisfy (10), then there is a feasible state $(x',\mu,\sigma), \mathcal{R}(\mu,\sigma)$ that Pareto dominates the equilibrium state $(x,\mu,\sigma), \mathcal{R}(\mu,\sigma)$, in contradiction to Theorem 7.1.

For each $a \in A$ and $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$, let $\bar{x}_a(\tilde{\ell})$ be the expected consumption of a at $\tilde{\ell}$:

$$\bar{x}_a(\tilde{\ell}) = \begin{cases} \frac{1}{\mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell}))} \int_{V_a(\tilde{\ell})} x_a(v) d\mathcal{P}(\mu,\sigma)(v) & \text{if } \mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell})) > 0 \\ 0 & \text{if } \mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell})) = 0 \end{cases}$$

Let $x'_a(v) = \bar{x}_a(\tilde{\ell})$ for each $a \in A$, $v \in V_a(\tilde{\ell})$, and $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$. Using strict concavity of the utility functions, the state (x', μ, σ) is preferred to the state (x, μ, σ) by every agent. Since x does not satisfy (10) and x' does satisfy (10), there is a set of agents A' of positive measure for whom $x'_a(v) \neq x_a(v)$ on a set of positive measure. For these agents, the preference is strict. Further, the constructed state (x', μ, σ) is feasible by the law of large numbers. This contradicts Theorem 7.1, thus (10) holds.

To see why p must be constant, we use the fact that agents' augmented lists are independent, from Theorem 4.2. Choose $A' \subset A$, with $\lambda(A \setminus A') = 0$, such that for each pair $a, b \in A'$, $\tilde{\mu}^r_a$ and $\tilde{\mu}^r_b$ are independent. Using (10) and the additional assumptions on preferences, A' can also be chosen so that for each $a \in A'$ and $\tilde{\ell}_a$ such that $\mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell}_a)) > 0$, p is constant on $V_a(\tilde{\ell}_a)$. Then if p is not constant on V, there exist sets $V', V'' \subset V$ of $\mathcal{P}(\mu,\sigma)-$positive measure such that

(i) $p(v') \neq p(v'')$ for all $v' \in V', v'' \in V''$

Further, there are agents $a, b \in A'$ and augmented lists $\tilde{\ell}_a, \tilde{\ell}_b$ such that

(ii) $\mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell}_a) \cap V') > 0$, $\mathcal{P}(\mu,\sigma)(V_b(\tilde{\ell}_b) \cap V'') > 0$

¿From (i) and (ii), and because p is constant on each of $V_a(\tilde{\ell}_a)$ and $V_b(\tilde{\ell}_b)$, we conclude that

$$\mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell}_a) \cap V_b(\tilde{\ell}_b)) = 0$$

In particular,

$$\mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell}_a) | \tilde{\mu}^r_b = \tilde{\ell}_b) = 0 \neq \mathcal{P}(\mu,\sigma)(V_a(\tilde{\ell}_a))$$

which contradicts the independence of $\tilde{\mu}^r_a$ and $\tilde{\mu}^r_b$. ■

8 Efficiency and Residual Claimants

So far we have illustrated two broad classes of inefficiency that can arise in our model, belief-driven coordination problems, and missing insurance markets. These do not exhaust the inefficiencies that may arise, however, such as those due to screening or moral hazard. An important question is which inefficiencies are irremediable, and which can be remedied with a sufficiently rich set of group types, in particular, with groups incorporating appropriately designed mechanisms. Since agents are allowed to choose the groups they join, we might expect them to choose groups with mechanisms that support efficient outcomes. That is the point of this section.

We show that when characteristics or actions are observable to all group members (but not verifiable), efficiency can be achieved if group types incorporate reporting mechanisms and residual claimants in the spirit of Maskin (1999). Roughly, by incorporating appropriately designed mechanisms, some equilibrium states replicate those in a model in which all characteristics and strategies are verifiable. In these equilibria, the randomness that comes from the unverifiability of agents' actions or characteristics is eliminated. If the elimination of randomness leads to efficiency, the resulting equilibria are efficient. These equilibria are akin to the efficient equilibria described by EGSZ (1999, 2005). However, the qualification has bite. As we discuss below, efficiency can sometimes be improved by introducing randomness, although not necessarily the randomness that arises naturally through the unverifiability of strategies.

We begin the section with three examples that illustrate the role of residual claimaints. Residual claimants can enable screening, can solve moral hazard problems, and can allow agents to choose efficient group types. In the remainder of the section, we show how these ideas can be extended and generalized.

8.1 Three Examples

Example 8 shows that a verifiable signal of unverifiable characteristics can be used to screen members. A residual claimant administers punishments by collecting the profit when screening fails. Example 9 illustrates how a residual claimant can prevent the moral hazard in teams that arises from budget balance (Holmstrom 1984). Example 10 shows how direct revelation mechanisms can be embedded in general equilibrium, and illustrates how a residual claimant can be used to elicit correlated information that no one observes until the group has formed.

Example 8: Verifiable Signals of Unverifiable Characteristics

In this example, the group's output of the private good is a verifiable signal of the unverifiable characteristics. In this case screening may be possible by punishing workers if they do not produce the intended output. The punishment is to give all the output to a residual claimant, called a supervisor.

There are three group types $\mathbf{G} = \{g_{bb}, g_{bc}, g_{cc}\}$. The labels on the group types are intended to be used as a coordinating device. There are three memberships in each group type, denoted $\mathbf{M}(g) = \{sp, w_1, w_2\}$ for each $g \in \mathbf{G}$. For each $g \in \mathbf{G}$, the supervisor sp has a single null characteristic $S_{sp}(g) = \{sp\}$, while workers can be of two types $S_w(g) = \{b, c\}$ for $w = w_1, w_2$. There are two private goods. The production technology in each group type is the same, but output varies with the unverifiable characteristics of members. In particular, for each $g \in \mathbf{G}$,

$$\begin{aligned} h(g,(sp,b,b)) &= (6,0) \\ h(g,(sp,b,c)) &= h(g,(sp,c,b)) = (5,0) \\ h(g,(sp,c,c)) &= (0,2) \end{aligned}$$

The set of agents is $A = [0, 2]$. Each $a \in [0, 1]$ is constrained to be a worker, and each $a \in (1, 2]$ is constrained to be a supervisor. Each agent can join only one group. Workers $a \in [0, \rho)$ are constrained to choose the action b in each membership, and workers $a \in [\rho, 1]$ are constrained to choose the action c in each membership. Each agent has the ex-post utility function given by $u_a(x, \tilde{\ell}) = x_1 + x_2$.

The transfer payments that enable screening are the following, for $i = 1, 2$.

$$t_{bb}(w_i, y) = \begin{cases} (3,0) & \text{if } y = (6,0) \\ (0,0) & \text{if } y \neq (6,0) \end{cases} \text{ and } t_{bb}(sp, y) = \begin{cases} (0,0) & \text{if } y = (6,0) \\ (6,0) & \text{if } y \neq (6,0) \end{cases}$$

$$\begin{aligned} t_{cc}(w_i, y) &= \begin{cases} (0,1) & \text{if } y = (0,2) \\ (0,0) & \text{if } y \neq (0,2) \end{cases} \text{ and } t_{cc}(sp, y) = \begin{cases} (0,0) & \text{if } y = (0,2) \\ (0,2) & \text{if } y \neq (0,2) \end{cases} \\ t_{bc}(w_1, y) &= \begin{cases} (3,0) & \text{if } y = (5,0) \\ (0,0) & \text{if } y \neq (5,0) \end{cases} t_{bc}(w_2, y) = \begin{cases} (2,0) & \text{if } y = (5,0) \\ (0,0) & \text{if } y \neq (5,0) \end{cases} \end{aligned}$$

and

$$t_{bc}(sp, y) = \begin{cases} (0,0) & \text{if } y = (5,0) \\ (5,0) & \text{if } y \neq (5,0) \end{cases}$$

If $\rho < 1/2$, the transfers t support an equilibrium in which private goods prices are $p = (1, 1)$, all workers $a \in [0, \rho)$ choose memberships $w_1 \in \mathbf{M}(g_{bc})$, while a proportion ρ of workers $a \in [\rho, 1]$ choose memberships $w_2 \in \mathbf{M}(g_{bc})$, with the remaining measure $1 - 2\rho$ of agents in $[\rho, 1]$ divided equally between memberships w_1 and w_2 in $\mathbf{M}(g_{cc})$. The membership prices for all groups are zero. The common beliefs that support this equilibrium satisfy

$$\begin{aligned} f(c; w_1) &= 1 \text{ for } w_1 \in \mathbf{M}(g_{bc}) \\ f(b; w_2) &= 1 \text{ for } w_2 \in \mathbf{M}(g_{bc}) \\ f(b; w_1) &= f(b; w_2) = 1 \text{ for } w_1, w_2 \in \mathbf{M}(g_{bb}) \\ f(c; w_1) &= f(c; w_2) = 1 \text{ for } w_1, w_2 \in \mathbf{M}(g_{cc}) \end{aligned}$$

◊

Example 9: Moral hazard and budget balance

As discussed by Holmstrom (1984), team production will be inefficient because of budget balance. Since the team members share the output, not every team member can be rewarded with his marginal product, and effort will be suboptimal. Prescott and Townsend (2006) assumed that this problem can be solved by merely having a supervisor present. We show instead that the problem can be solved by designating the supervisor as a residual claimant. Provided there is no cost to engaging a residual claimant, firms with residual claimants will drive out teams with no residual claimants.

More specifically, let $\mathbf{G} = \{g_t, g_f\}$, with $\mathbf{M}(g_f) = \{sp, w_1, w_2\}$ and $\mathbf{M}(g_t) = \{w_1, w_2\}$. The team g_t and the firm g_f have the same production technology, each with two workers. In addition, the firm has a supervisor, who acts as a residual claimant.

In each group, workers can take low effort or high effort, and the supervisor has a null strategy, *sp*. That is, $S_w(g) = \{e_\ell, e_h\}$ for $w = w_1, w_2$ and $S_{sp}(g) = \{sp\}$ for $g \in \mathbf{G}$.

There is a single private good, produced by the team or firm according to the production function

$$h(g, s) = \begin{cases} y_\ell & \text{if } (s_{w_1}, s_{w_2}) = (e_\ell, e_\ell) \\ y_m & \text{if } (s_{w_1}, s_{w_2}) = (e_h, e_\ell) \text{ or } (e_\ell, e_h) \\ y_h & \text{if } (s_{w_1}, s_{w_2}) = (e_h, e_h) \end{cases}$$

for $g \in \mathbf{G}$, where $y_\ell < y_m < y_h$.

The transfers in a team divide the output between the workers. The transfers in the supervised firm divide the output between the workers if the output is high (which means the workers took high effort), but otherwise give the output to the supervisor.

$$t_{g_t}(w_1, y) = t_{g_t}(w_2, y) = \frac{1}{2}y \qquad \text{for } y \in \mathbf{R}$$

$$t_{g_f}(w_1, y) = t_{g_f}(w_2, y) = \begin{cases} \frac{1}{2}y & \text{if } y = y_h \\ 0 & \text{if } y \neq y_h \end{cases}$$

$$t_{g_f}(sp, y) = \begin{cases} 0 & \text{if } y = y_h \\ y & \text{if } y \neq y_h \end{cases}$$

We suppose that each agent can take a single membership. Agents $a \in [0, \rho)$ are equipped to be workers, while agents $a \in [\rho, 1]$ are equipped to be supervisors. More agents are equipped to be supervisors than workers, that is, $\rho < 1/2$.

Preferences are as follows. The supervisor cares only about income. Each worker's utility can be written $x + v(e_\ell)$, where $x \in \mathbf{R}_+$, and in particular, the outputs and disutility of effort are such that

$$(1/2)\,y_h + v(e_h) = 6 \qquad (1/2)\,y_\ell + v(e_\ell) = 3$$
$$(1/2)\,y_m + v(e_\ell) = 7 \qquad (1/2)\,y_m + v(e_h) = 2$$

First consider the strategies chosen by the workers. The games played by workers in the team g_t and the supervised firm g_f are respectively:

	e_ℓ	e_h
e_ℓ	$(3,3)$	$(7,2)$
e_h	$(2,7)$	$(6,6)$

	e_ℓ	e_h
e_ℓ	$(0,0)$	$(0,0)$
e_h	$(0,0)$	$(6,6)$

Low effort levels (e_ℓ, e_ℓ) are the equilibrium strategies in the team and high effort levels (e_h, e_h) are the equilibrium strategies in the supervised firm, as is efficient.

Since supervisors are in excess supply, they will get zero payoff in equilibrium. Thus, the equilibrium membership prices are

$$\begin{aligned} q_{g_t}(w_1) &= q_{g_t}(w_2) = 0 \\ q_{g_f}(w_1) &= q_{g_f}(w_2) = q_{g_f}(sp) = 0 \end{aligned}$$

Workers who choose teams get utility 3 through the internal transfers, and workers who choose supervised firms get utility 6. Clearly, workers will choose supervised firms instead of teams, since supervised firms support the efficient level of effort, and all the proceeds go to the workers. $\Diamond$

Example 10: Direct Revelation and Bayesian Mechanisms

This example illustrates how a group type can accommodate implementation by Bayesian equilibrium in a direct-revelation game. The mechanism reveals information that is not observable to anyone before the group has formed, namely a patient's medical condition. Screening is not possible because the patient does not know the diagnosis, and the physicians only observe it after seeing the patient. The direct revelation game will reveal the patient's condition by using the patient as a residual claimant.[15]

There are three types of medical clinics $\mathbf{G} = (g_o, g_r, g_m)$, each with two doctors and a patient with an injured knee, thus $\mathbf{M}(g) = \{p, d_1, d_2\}$. After the clinic has formed, the doctors receive private, correlated signals regarding the correct treatment, and private, uncorrelated information about their own costs of treating the patient.

The medical clinic g_o is aggressive in the sense that it always treats the knee by operating, while the clinic g_r is conservative in the sense that it always treats the knee with RICE (rest, ice, compression and elevation). The third clinic g_m implements a mechanism-design approach to discover which is the better treatment. In the clinic g_m, two problems must be solved: to discover the correct treatment, and, if an operation is required, to discover the lower-cost doctor. The patient has no signal of which treatment is correct, and will not be able to distinguish ex-post whether he got the right treatment.

The clinic plays a direct-revelation game to reveal the best treatment, and, if necessary, to find the lower-cost physician. The patient acts as a residual claimant in the resulting information-revelation game, and can thus avoid the impasse that would arise from budget

[15] Alternatively, a shareholder could be the residual claimant.

balance if the doctors could only make payments between themselves. In the absence of a residual claimant, there might not be a mechanism that elicits their true information about the patient's condition, as we will see.

After examining the patient, each doctor has a true diagnosis about the best treatment, $\theta_1, \theta_2 \in \{r, o\}$ (RICE or operate). The mechanism in the clinic will implement the best treatment as a function of the doctors' diagnoses, $\tau(\theta_1, \theta_2)$, which is assumed to satisfy:

$$\begin{aligned} \tau(o,o) &= o \\ \tau(o,r) &= \tau(r,o) = \tau(r,r) = r \end{aligned}$$

The doctors' costs of operating are $c_1, c_2 \in \{c^\ell, c^h\}$.

We assume that for each doctor, the prior probability of each diagnosis is $\pi(r) = \pi(o) = 1/2$, and that the doctors agree with probability 2/3. That is, the conditional probabilities satisfy

$$\begin{aligned} \pi(r|r) = \pi(o|o) = 2/3 \\ \pi(r|o) = \pi(o|r) = 1/3 \end{aligned}$$

We define the mechanism of the clinic $\gamma = (\gamma_1, \gamma_2)$ in two stages. The first stage is given by $\gamma_1 = (t, \tau)$, where t defines transfers in a direct-revelation game in which the doctors report their diagnoses, and τ is the efficient treatment. In γ_1, the transfers t, which are payments from the patient to the doctors, are symmetric and independent of the cost reported in the second stage. Let $\hat{\theta}_1, \hat{\theta}_2 \in \{r, o\}$ be the reported diagnoses of the two doctors in the first stage. The transfers to the doctors are denoted $t(\hat{\theta}_1, \hat{\theta}_2, d_1)$, $t(\hat{\theta}_1, \hat{\theta}_2, d_2)$.

If $\hat{\theta}_1 = \hat{\theta}_2 = o$, the patient will receive an operation, and the second stage of the mechanism is reached. This stage, γ_2, is a mechanism to choose the lower-cost doctor. To shorten the discussion, we will not specify the mechanism γ_2, but summarize the relevant aspects in the information rents $r(c^\ell)$ or $r(c^h)$, with $r(c^\ell) > r(c^h)$. Because there are information rents in the second stage of the mechanism, the doctors have an incentive to reach that stage, and would not report their diagnoses truthfully if merely asked. The corresponding incentive compatability constraints for doctor d_1 in the first mechanism γ_1 are the following (and symmetrically for doctor d_2).

$$\begin{aligned} \frac{2}{3}r(c_1) + \frac{2}{3}t(o,o,d_1) + \frac{1}{3}t(o,r,d_1) &\geq \frac{2}{3}t(r,o,d_1) + \frac{1}{3}t(r,r,d_1) \\ \frac{2}{3}t(r,r,d_1) + \frac{1}{3}t(r,o,d_1) &\geq \frac{2}{3}t(o,r,d_1) + \frac{1}{3}t(o,o,d_1) + \frac{1}{3}r(c_1) \end{aligned} \tag{11}$$

There may be no balanced-budget revelation game between the doctors that elicits the true diagnosis when the true diagnosis is r. By symmetry, budget balance would imply $t(o,o,d_i) = t(r,r,d_i) = 0$ for $d_i = d_1, d_2$, and $-t(o,r,d_1) = t(o,r,d_2) = t(r,o,d_1) =$

$-t(r, o, d_2)$, hence, (11) would imply

$$\begin{aligned} \frac{2}{3}r(c_1) &\geq t(o, r, d_1) \geq \frac{1}{3}r(c_1) \\ \frac{2}{3}r(c_2) &\geq t(r, o, d_2) = t(o, r, d_1) \geq \frac{1}{3}r(c_2) \end{aligned}$$

If $r(c_2) > 2r(c_1)$ (as may occur when $c_2 = c^\ell, c_1 = c^h$), these two inequalities are inconsistent, so there is no balanced-budget incentive-compatible mechanism. However, without budget balance, there are many mechanisms that support truth-telling, for example, $t(o, o, d_1) = t(r, r, d_1) = 0$, $t(o, r, d_1) = -\frac{2}{3}r(c^\ell)$, $t(r, o, d_1) = -\frac{1}{3}r(c^\ell)$, analogously for doctor 2.[16]

If the doctors only care about expected income, the membership prices for doctors (their wages) must be the same in expectation in the three types of clinics, provided that all three are used in equilibrium. Of course there will be variance in income in the g_m clinic, due to uncertainty regarding the doctors' diagnoses. If the patient is risk neutral with respect to income, and weakly prefers the better treatment, he will always use the clinic g_m. If he wants to avoid variation in income, and if he is reasonably certain what the correct diagnosis will be, he will use either g_o or g_r, depending on which treatment he believes is correct. $\Diamond$

8.2 Residual Claimant Economies

Examples 8, 9 and 10 illustrate ways in which residual claimants can increase efficiency by providing an enforcement mechanism. We now elaborate on this idea by defining a class of economies in which every group type includes a residual claimant. We show that, provided efficiency can be achieved in a deterministic state of the economy, as defined below, the introduction of residual claimants can result in an efficient equilibrium.

In the mechanism described below, we eliminate randomness in strategies by labeling each group type with target strategies, and punishing members for not playing the target strategies. This is done with the help of a residual claimant.[17] If the strategies are verifiable ex post (as well as observable), then the punishments can be created directly by giving all the output to the residual claimant when the target strategies are not played. That is the spirit of the mechanism described below, but we address the more difficult case that strategies are never verifiable. This is why we require reporting mechanisms.

As in our basic model, there are $N \geq 1$ divisible, publicly traded private goods, and we begin with a finite, exogenous set of *primitive group types*, **G**. As above, associated to

[16] A complication in g_m is that truth-telling is not the only equilibrium of the game. There may also be equilibrium strategies in which each doctor lies; this outcome is inefficient, since it leads to the wrong treatment, but cannot be ruled out in our framework for the same reasons it cannot be ruled out in standard mechanism design.

[17] This cannot be accomplished directly, for example, by appealing to a court or other enforcer to punish a member who deviates from the target, or by requiring a certain strategy as a condition of a membership, because by assumption no enforcer can observe the strategy.

group type g is a finite set of *primitive memberships* $\mathbf{M}(g)$. For each primitive membership $m \in \mathbf{M}(g)$, let $S_m(g)$ be the set of unverifiable characteristics that might be chosen by the member m. Let $S(g) := \prod_{m \in \mathbf{M}(g)} S_m(g)$ denote the *characteristics profiles* for g.

Let $\mathbf{G}^c = \{g_s : g \in \mathbf{G}, s \in S(g)\}$ be the set of *group types*. That is, we create a copy of the primitive group type g for each $s \in S(g)$ and label it g_s. Each such group type will have the same set of memberships as the underlying primitive group type g, with an additional distinguished member c_{g_s} who will be the *residual claimant*. As we formalize below, the index s represents the target characteristics of the mechanism to be played in the group type g_s.

Formalizing, for each group type $g_s \in \mathbf{G}^c$ the set of memberships is

$$\mathbf{M}^c(g_s) = \{m_s : m \in \mathbf{M}(g)\} \cup c_{g_s}$$

Let $\mathbf{M}^c = \cup_{g_s \in \mathbf{G}^c} \mathbf{M}^c(g_s)$.

To each membership $m_s \in \mathbf{M}^c$ is associated a set of strategies for that membership. The strategy set is the product of, first, the set of unverifiable characteristics $S_m(g)$ associated with the primitive membership m, and, second, a set of reporting strategies $R_g := \{r : S(g) \to S(g)\}$. The strategy set associated to each membership c_{g_s} is a singleton null strategy $\{(s_{g_s}, r_{g_s})\}$. We let $r_{g_s} \in R_g$, so all members have the same set of reporting strategies.

Each agent chooses a *strategy* $\sigma \in \Sigma$, where

$$\Sigma := \prod_{g_s \in \mathbf{G}^c} \Sigma\,(g_s)$$

and

$$\Sigma(g_s) := \{(s_{m_s}, r_{m_s}) \in S_m(g) \times R_g : m_s \in \mathbf{M}^c(g_s)\}$$

An element $\theta \in \Sigma\,(g_s)$ represents the *strategy profile* chosen by a group of type g_s, that is, the strategies of the different members. The strategy θ has two parts, the characteristics chosen by the members of the group, and the reporting strategies chosen by members of the group. Each reporting strategy $r \in R_g$ is a function that operates on the chosen characteristics $s \in S\,(g)$. The strategies generate *reports*, $r_\theta \in S\,(g)^{\mathbf{M}^c(g_s)\backslash c_{g_s}}$. We use the notation $(s_\theta, r_\theta) \in S\,(g) \times S\,(g)^{\mathbf{M}^c(g_s)\backslash c_{g_s}}$ to represent the characteristics chosen by, and the reports delivered by, members of the group other than the residual claimant when the members choose $\theta \in \Sigma\,(g_s)$.

We will focus on equilibria in which agents' strategies are honest in two ways: the unverifiable characteristics chosen by the members are the target characteristics, and the members make honest reports to the residual claimants. For each group type g_s, the *honest reporting strategy* $r \in R_g$ satisfies $r\,(s) = s$ for each $s \in S\,(g)$. An agent's strategy $\sigma \in \Sigma$ is *honest* if for each $g_s \in \mathbf{G}^c$ and $m_s \in \mathbf{M}^c(g_s)$, $\sigma_{m_s} = (s_m, r)$ where $(s_m, s_{-m}) = s$, and r is the honest reporting strategy.

For each $g_s \in \mathbf{G}^c$ and $\theta \in \Sigma(g_s)$, (g_s, θ) is an *augmented group type*. Again abusing notation a bit, let $\tilde{\mathbf{M}}^c(g_s)$ be the set of memberships in the augmented group types derived from g_s, namely,

$$\tilde{\mathbf{M}}^c(g_s) = \{(m_s, \theta) \in \mathbf{M}^c(g_s) \times \Sigma(g_s)\}$$

Group types are also defined by input-output vectors $h(g_s, s_\theta)$, as before, and by transfer functions $t_{g_s} : \mathbf{M}^c(g_s) \times \mathbf{R}^N \to \mathbf{R}^N$ that divide up the input-output vectors. We assume, as is natural, that for each $g \in \mathbf{G}$ and $s, s' \in S(g)$, $h(g_s, s_\theta) = h(g_{s'}, s_{\theta'})$ if $s_\theta = s_{\theta'}$. That is, the input-output vector depends only on the characteristics that are actually chosen, not on the label.

To each residual claimant group type g_s is also associated a *reporting transfer function* $t^W_{g_s} : \mathbf{M}^c(g_s) \times S(g)^{\mathbf{M}^c(g_s) \setminus c_{g_s}} \to \mathbf{R}^N$. The transfer function is parameterized by a penalty $W > 0$. Transfers are paid based on reports as follows, where $\mathbf{1} = (1, 1, \ldots, 1)$.

$$\begin{aligned} t^W_{g_s}(m_s, r) &= \begin{cases} 0 & \text{if } r = s \times s \times \cdots \times s \\ -W\mathbf{1} & \text{otherwise} \end{cases} \\ t^W_{g_s}(c_{g_s}, r) &= \begin{cases} 0 & \text{if } r = s \times s \times \cdots \times s \\ \sum_{m_s \in \mathbf{M}^c(g_s) \setminus c_{g_s}} W\mathbf{1} & \text{otherwise} \end{cases} \end{aligned}$$

As we show below, there is an equilibrium in which all members report strategies truthfully, provided the penalty W is sufficiently large.

Agent a's *consumption set* $X_a \subset \mathbf{R}^N_+ \times \mathbf{Lists}(\mathbf{M}^c) \times \Sigma$ specifies the triples (x_a, μ_a, σ_a) of private goods, lists of memberships, and strategies that the agent may choose. We assume that each agent $a \in A$ has an endowment $(e_a, 0, (s_a, r_a)) \in X_a$. As above, we assume agents' endowments of private goods are state-independent.

A *residual-claimant economy* is an economy in which the set of group types, memberships, and strategies are as defined above, and in which the mapping $a \mapsto (X_a, e_a, u_a)$ satisfies:

- the consumption set mapping $a \mapsto X_a$ is a measurable correspondence such that for each $a \in A$, if $(x_a, \mu_a, \sigma_a) \in X_a$, then $(x_a, \mu_a, \sigma'_a) \in X_a$ if σ_a and σ'_a entail the same choices of unverifiable characteristics in all memberships, but different reporting strategies
- for each a, $u_a(x, \tilde{\ell}) = u_a(x, \tilde{\ell}')$ whenever for each $g \in \mathbf{G}$ and $\hat{s} \in S(g)$,

$$\sum_{\substack{s \in S(g) \\ \theta \in \Sigma(g_s) : \hat{s} = s_\theta}} \tilde{\ell}(m_s, \theta) = \sum_{\substack{s \in S(g) \\ \theta \in \Sigma(g_s) : \hat{s} = s_\theta}} \tilde{\ell}'(m_s, \theta)$$

The first condition says that agents can choose any possible reports. The second condition stipulates that if there is a mismatch between target characteristics and chosen characteristics, it is only the chosen characteristics that determine utility.

Let $\Sigma_{-m_s}(g_s)$ denote the set of strategies of the members in $\mathbf{M}^c(g_s)$ except m_s and the residual claimant c_{g_s}, and $\Delta(\Sigma_{-m_s}(g_s))$ be the set of probability distributions on $\Sigma_{-m_s}(g_s)$. Let

$$\mathbf{F} := \prod_{\substack{g_s \in \mathbf{G}^c \\ m_s \in \mathbf{M}^c(g_s) \setminus c_{g_s}}} \Delta(\Sigma_{-m_s}(g_s))$$

Then *beliefs on membership characteristics* are an element $f \in \mathbf{F}$. The value $f(\theta_{-m_s}; m_s)$ is the probability that members of a group of type g_s other than m_s choose $\theta_{-m_s} \in \Sigma_{-m_s}(g_s)$. We say that *beliefs are on honest strategies* if $f(\theta_{-m_s}; m_s) = 1$ for each m_s when each element of θ is honest.

The transfer received by an agent consuming the augmented list $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}}^c)$ is

$$\tilde{\ell}(t^W + t) := \sum_{\substack{g_s \in \mathbf{G}^c \\ (m_s, \theta) \in \mathbf{M}^c(g_s) \times \Sigma(g_s)}} \tilde{\ell}(m_s, \theta) \quad \left[t^W_{g_s}(m_s, r_\theta) + t_{g_s}\left(m_s, h\left(g_s, s_\theta\right)\right)\right]$$

8.3 Efficiency in the Residual Claimant Economy

We have constructed the residual claimant economy to ensure that there is an equilibrium in which agents report honestly on the characteristics chosen within their groups, and that the characteristics chosen in equilibrium match the target characteristics of the group label. Such an equilibrium eliminates the randomness that can otherwise result from the unverifiability of characteristics, and the inefficiency that results from this randomness.

Say that a feasible state $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$ is *deterministic* if for almost every $a \in A$, x_a is a constant bundle, that is, $x_a(v)$ has the same value for almost all v, and

$$\mu_a(m_s) = 1 \Rightarrow \sigma_{a,m_s} = (s_m, r) \quad \text{for some } r \in R_g$$

For a deterministic feasible state, we will use x_a interchangeably to mean $x_a \in (\mathbf{R}^N_+)^V$ and $x_a \in \mathbf{R}^N_+$. In a deterministic state, agents in a group g_s choose the characteristics that match the characteristics profile s.[18]

Our objective is to find conditions under which an efficient equilibrium exists. We do this by first finding an equilibrium that is deterministic. Then say that *nonrandomness is efficient* if every deterministic feasible state that is not Pareto dominated by another deterministic feasible state is also Pareto optimal in the residual claimant economy.

[18] We could define the deterministic state more generally, such that the agents choose a different characteristics profile, say $\hat{s}(g_s)$, but that would be cumbersome without adding anything. Due to our assumption that members of a group care only about the characteristics profile, and not about the label of the group, the relabeling would have no effect on utility.

An equilibrium with honest strategies will be deterministic. To ensure that there is an equilibrium with honest strategies, the punishment $W\mathbf{1} \in \mathbf{R}_+^N$ in the reporting transfer functions must be large enough that paying it would either be infeasible with an agent's budget or make the agent worse off than playing the honest strategy. To this end, say that t^W *induces honesty* if $\forall a \in A$, $\forall x \in \mathbf{R}_+^N$, $\forall \tilde{\ell}, \tilde{\ell}'$,

$$u_a(x + W\mathbf{1}, \tilde{\ell}) > u_a(x, \tilde{\ell}') \tag{12}$$

In order that there exists $W > 0$ such that t^W induces honesty, it is enough that each agent has a bounded willingness to pay for his most preferred augmented list, as compared to his least preferred augmented list. To this end, say that *willingness to pay for strategies is bounded* if $\exists W > 0$ such that $\forall a \in A$, $\forall x \in \mathbf{R}_+^N$, $\forall \tilde{\ell}, \tilde{\ell}'$, $u_a(x + W\mathbf{1}, \tilde{\ell}) > u_a(x, \tilde{\ell}')$.

If all agents play honest strategies and choose the target characteristics for their group types in the deterministic state (x, μ, σ), then the resulting distribution on $\tilde{\mu}$ is degenerate.

Theorem 8.1 *Suppose willingness to pay for strategies is bounded and that t^W induces honesty. If nonrandomness is efficient, then there is an equilibrium state $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$ that is Pareto optimal.*

Proof We will prove the stronger result that there is such an equilibrium with constant prices p. In this equilibrium the random group formation model plays no role, and we will supress the notation for it for simplicity.

We first show that there is an equilibrium of the residual-claimant economy $(x, \mu, \sigma), (p, q), f$ in which σ is honest, beliefs f are on honest strategies, and prices p are constant.

Consider an artificial economy derived from the residual claimant economy in which all memberships in $\mathbf{M}^c$ are verifiable. This can be modeled by constraining σ to be honest. Notice that with this restriction, $|\mathbf{M}^c| = |\tilde{\mathbf{M}}^c|$. Theorem 6.1 establishes that in the artificial economy, there is an equilibrium with beliefs on membership characteristics, $(x, \mu, \sigma), (p, q), f$, in which x is constant, p is constant, and beliefs f are, correctly, on honest strategies. In every random matching, almost every agent receives his chosen list, and because characteristics are verifiable, they match the target strategies stipulated as part of the group type.

Now consider the true residual-claimant economy. The equilibrium of the artificial economy is also an equilibrium of the true residual-claimant economy, together with honest strategies and beliefs on honest strategies. No agent can improve on playing the honest strategy. If any agent deviates from the honest strategy, either by choosing a characteristic other than the target characteristic or by misreporting the strategies of others, he is punished by paying $W\mathbf{1}$. This makes him worse off because of (12). In particular, the support of $\eta_{(\mu,\sigma)}$ is the subset of $\mathbf{Lists}(\tilde{\mathbf{M}}^c)$ such that $\tilde{\ell}\,(m_s, \theta) = 1$ if and only if $\ell\,(m_s) = 1$ and $s_\theta = s$.

To establish that there is no deterministic state that Pareto dominates (x, μ, σ), suppose to the contrary that the state (x', μ', σ'), is deterministic and Pareto dominates. Then, because the distributions on $\tilde{\mu}$ and $\tilde{\mu}'$ are degenerate,

$$u_a\left(x'_a, \tilde{\mu}'_a\right) \geq u_a\left(x_a, \tilde{\mu}_a\right)$$

for almost every $a \in A$, with strict inequality for a set of agents $A' \subset A$ with positive measure.

Since (x, μ, σ) is an equilibrium state of the economy and (x', μ', σ') and (x, μ, σ) entail honest strategies, $\tilde{\mu}_a t^W = \tilde{\mu}'_a t^W = 0$ for almost all $a \in A$. Therefore, since $(x, \mu, \sigma), (p, q), f$ is an equilibrium,

$$p \cdot x'_a + q \cdot \mu'_a \geq p \cdot e_a + p \cdot \mu_a t$$

and $\forall a \in A'$,

$$p \cdot x'_a + q \cdot \mu'_a > p \cdot e_a + p \cdot \mu_a t$$

Integrating yields

$$\int_A p \cdot x'_a d\lambda(a) + \int_A q \cdot \mu'_a d\lambda(a) > \int_A p \cdot e_a d\lambda(a) + \int_A p \cdot \mu_a t d\lambda(a) \tag{13}$$

By consistency, and because each agent consumes a single augmented list with probability one,

$$\int_A p \cdot \mu_a t d\lambda(a) = \int_A \sum_{g_s \in \mathbf{G}^c, m_s \in \mathbf{M}^c(g_s) \setminus c_{g_s}} \tilde{\mu}_a\left(m_s, s\right) \frac{h\left(g_s, s\right)}{\left|\mathbf{M}^c\left(g\right)\right|} d\lambda(a)$$

Therefore consistency and (13) imply

$$\begin{aligned}
\int_A p \cdot x'_a d\lambda(a) &> \int_A p \cdot e_a d\lambda(a) + \int_A p \cdot \mu_a t d\lambda(a) \\
&= p \cdot \left[\int_A e_a d\lambda(a) + \int_A \sum_{g_s \in \mathbf{G}^c, m_s \in \mathbf{M}^c(g_s) \setminus c_{g_s}} \tilde{\mu}_a\left(m_s, s\right) \frac{h\left(g_s, s\right)}{\left|\mathbf{M}^c\left(g\right)\right|} d\lambda(a)\right]
\end{aligned}$$

which contradicts feasibility. The result now follows. ■

Theorem 8.1 implies that there is an efficient equilibrium, provided efficiency can be achieved in a deterministic state. The following example illustrates a limitation of this result: a deterministic equilibrium can be Pareto dominated by an equilibrium with randomness.

Example 11: An equilibrium with randomness can Pareto-dominate a deterministic equilibrium

In the primitive economy there is a single group type g with two memberships, $\mathbf{M}(g) = \{B, G\}$, where B denotes blue and G denotes green (both verifiable). For each membership, there are two unverifiable strategies, $S_B = \left\{B^T, B^N\right\}$ and $S_G = \left\{G^L, G^F\right\}$. These can be interpreted as meaning that each color comes in two unverifiable shades: blue can be either turquoise or navy, and green can be either lime green or forest green. The expanded set of group types in the residual-claimaint economy is $\mathbf{G}^c = \{g_{G^L B^T}, g_{G^L B^N}, g_{G^F B^T}, g_{G^F B^N}\}$.

All agents have the same endowment of a single private good, $e > 1$. Agents $a \in [0, 1/4)$ can take no membership or a B membership, with agents $a \in [0, 1/8)$ having characteristic B^T while agents $a \in [1/8, 1/4)$ are B^N. Agents $a \in [1/4, 1/2)$ can take no memberhsip or a G membership, with agents $a \in [1/4, 3/8)$ having characteristic G^L while agents $a \in [3/8, 1/2)$ are G^F. Agents $a \in [1/2, 1]$ can take no membership or the residual claimant membership. The measure of groups that will form is at most $1/4$.

The utility of an agent who does not take a membership or takes a residual claimant membership is equal to the consumption of the private good. For agents in B or G memberships, utility is given by

$$u_a(x, \ell) = \begin{cases} u(x) & \text{for } B^T\text{–agents, } a \in [0, 1/8), \text{ if } \ell\left(B^T, (B^T, G^L)\right) = 1 \\ u(x) + 1 & \text{for } B^T\text{–agents, } a \in [0, 1/8), \text{ if } \ell\left(B^T, (B^T, G^F)\right) = 1 \\ u(x) + 1 & \text{for } B^N\text{–agents, } a \in [1/8, 1/4), \text{ if } \ell\left(B^N, (B^N, G^L)\right) = 1 \\ u(x) & \text{for } B^N\text{–agents, } a \in [1/8, 1/4), \text{ if } \ell\left(B^N, (B^N, G^F)\right) = 1 \\ \end{cases}$$

and

$$u_a(x, \ell) = \begin{cases} u(x) + 1 & \text{for } G^L\text{–agents, } a \in [1/4, 3/8), \text{ if } \ell\left(G^L, (B^T, G^L)\right) = 1 \\ u(x) & \text{for } G^L\text{–agents, } a \in [1/4, 3/8), \text{ if } \ell\left(G^L, (B^N, G^L)\right) = 1 \\ u(x) & \text{for } G^F\text{–agents, } a \in [3/8, 1/2), \text{ if } \ell\left(G^F, (B^T, G^F)\right) = 1 \\ u(x) + 1 & \text{for } G^F\text{–agents, } a \in [3/8, 1/2), \text{ if } \ell\left(G^F, (B^N, G^F)\right) = 1 \\ \end{cases}$$

where private goods consumption is $x \in \mathbf{R}_+$ and $u : \mathbf{R}_+ \to \mathbf{R}_+$ is strictly concave, with $u(e) > e$. Notice agents prefer to join groups if the price is zero.

We describe two equilibria. First, there is a deterministic equilibrium with honest strategies and beliefs on honest strategies. In this equilibrium, the measure of each type of group in $\mathbf{G}^c$ is $1/16$. All B agents take B memberships, and all G agents take G memberships. A measure $1/4$ of agents in $[1/2, 1]$ take residual claimant memberships. Prices for memberships will make all agents indifferent among all memberships. Specifically, let $b > 0$ satisfy

$$u(e - b) + 1 = u(e + b)$$

Equilibrium prices are:

$$\begin{array}{llll} q_{G^L B^T}\left(G^L\right) = b & q_{G^L B^N}\left(G^L\right) = -b & q_{G^F B^T}\left(G^F\right) = -b & q_{G^F B^N}\left(G^F\right) = b \\ q_{G^L B^T}\left(B^T\right) = -b & q_{G^L B^N}\left(B^N\right) = b & q_{G^F B^T}\left(B^T\right) = b & q_{G^F B^N}\left(B^N\right) = -b \end{array}$$

In this equilibrium it does not matter how the agents are matched, because each agent gets utility $\bar{u} := u(e - b) + 1 = u(e + b)$.

In the second equilibrium, strategies are not honest, and agents do not believe in honest strategies. In each group type g_s, each agent chooses to report the target characteristics profile s regardless of the unverifiable characteristics that actually materialize. Because all agents report the target characteristics, there are no internal transfers. There is no sorting in the choice of group memberships. Each agent with a membership G has the belief $f(B^T;G) = f(B^N;G) = 1/2$, and each agent in membership B has the belief $f(G^L;B) = f(G^F;B) = 1/2$. In equilibrium, the agents with each unverifiable characteristic are divided equally between the memberships available to them, and every agent is indifferent among the memberships available to him. Thus, these beliefs are correct in equilibrium.

Membership prices satisfy $q \equiv 0$. Thus, each agent receives expected utility

$$\frac{1}{2}u(e) + \frac{1}{2}(u(e)+1) = u(e) + \frac{1}{2}$$

Agents are better off in the second equilibrium, with randomness, than in the first equilibrium, as:

$$\begin{aligned} u(e) + \frac{1}{2} &> \left[\frac{1}{2}u(e+b) + \frac{1}{2}u(e-b)\right] + \frac{1}{2} \\ &= \left[\frac{1}{2}\bar{u} + \frac{1}{2}(\bar{u}-1)\right] + \frac{1}{2} = \bar{u} \end{aligned}$$

◊

9 Lotteries and Efficiency

We showed that a residual-claimant economy has an efficient equilibrium, provided the efficient state does not involve randomness. However, example 11 shows that randomness might improve efficiency in some settings. Although randomness arises naturally in example 11 from the unverifiability of characteristics, verifiability is not the root issue. Cole and Prescott (1997) showed that randomness can improve efficiency even where all the characteristics are verifiable. The root issue is that consumption choices (group memberships) are indivisible.

In this section we introduce lotteries into the model by defining a class of lottery group types. Our lotteries differ from those of Cole and Prescott (1997) and Prescott and Townsend (2006) (denoted CPPT below) in several ways. First, the lotteries of CPPT are played in the population as a whole. Instead, our lotteries are played in finite groups. Second, our lotteries are on lists of memberships rather than on memberships, so that agents can correlate membership outcomes. The CPPT model does not need to address this problem, because agents are constrained to consume a single membership. Third, the CPPT lotteries are supplied by a profit-maximizing intermediary serving a continuum of agents. By selling to a continuum of agents, the intermediary can clear markets and balance its budget, even though the outcomes of the agents' randomizations are independent. Instead we incorporate randomization in finite lottery group types. The outcomes in our lottery group types are not independent (although outcomes are independent across lottery groups), and each finite lottery group balances its budget. This has the advantage that we do not need a distinguished type of firm that serves the whole economy (or a continuum within the economy). At the same time, it limits the efficiency gains of allowing for lotteries.

We define a lottery group type such that the random outcome of the lottery generates a consistent set of memberships. We start with a set of lists $\mathbf{L}$, each list in $\mathbf{Lists}(\mathbf{M})$. The set $\mathbf{L}$ may contain duplicate copies of some lists. Say that $\mathbf{L}$ is *consistent* if there are nonnegative integers $\{\alpha(g) : g \in \mathbf{G}\}$ such that $\sum_{\ell \in \mathbf{L}} \ell(m) = \alpha(g)$ for each $m \in \mathbf{M}(g)$ and $g \in \mathbf{G}$. A *lottery membership* is a function $l : \mathbf{L} \to \{0, 1\}$, where $l(\ell) = 1$ is interpreted to mean that the lottery member l would accept the list $\ell \in \mathbf{L}$. Thus, the membership l designates a collection of lists, each of which would be acceptable to the member. Given a consistent set of lists $\mathbf{L}$, let $\mathbf{M_L}$ be a set of lottery memberships.

A *lottery* is a pair $(\mathbf{L}, \mathbf{M_L})$ such that

1. $\mathbf{L}$ is consistent
2. for every $\ell \in \mathbf{L}$, $l(\ell) = 1$ for at least one $l \in \mathbf{M_L}$
3. $|\mathbf{L}| = |\mathbf{M_L}|$
4. $0 \notin \mathbf{M_L}$

It is understood that the lottery group type will assign members to lists randomly, with an equal probability on each assignment that is consistent with the memberships. More specifically, a *lottery assignment* for the lottery $(\mathbf{L}, \mathbf{M_L})$ is a one-to-one map $\gamma : \mathbf{M_L} \to \mathbf{L}$ such that $\gamma(l) = \ell$ only if $l(\ell) = 1$. Because γ is a one-to-one map, every list in $\mathbf{L}$ is assigned to some member. Write $\Gamma_{(\mathbf{L}, \mathbf{M_L})}$ for the set of all lottery assignments, and write $|\Gamma_{(\mathbf{L}, \mathbf{M_L})}|$ for the cardinality.

Write $\Gamma(l, \ell; \mathbf{L}, \mathbf{M_L})$ for the set of lottery assignments in which the member $l \in \mathbf{M_L}$ is assigned to $\ell \in \mathbf{Lists}(\mathbf{M})$, and write $|\Gamma(l, \ell; \mathbf{L}, \mathbf{M_L})|$ for the cardinality. Then the probability that an agent with membership $l \in \mathbf{M_L}$ is assigned to $\ell \in \mathbf{Lists}(\mathbf{M})$ is the fraction of assignments where that happens, namely, $|\Gamma(l, \ell; \mathbf{L}, \mathbf{M_L})|/|\Gamma_{(\mathbf{L}, \mathbf{M_L})}|$.

To illustrate, consider a lottery in which every member would be willing to take every list, that is, $l(\ell) = 1$ for each member $l \in \mathbf{M_L}$ and each list $\ell \in \mathbf{L}$. The probability that a given member l is assigned to a given list ℓ is calculated as follows. If $|\mathbf{M_L}| = K$, the number of lottery assignments is the number of permutations of members, $K!$. The number of lottery assignments where l is assigned to ℓ is $(K-1)!$. Thus, the probability that l is assigned to ℓ is $1/K = |\Gamma(l, \ell; \mathbf{L}, \mathbf{M_L})|/|\Gamma_{(\mathbf{L}, \mathbf{M_L})}| = (K-1)!/K!$.

Consider another lottery in which $\mathbf{L} = \{\ell_a, \ell_b, \ell_c\}$ and $\mathbf{M_L} = \{l_1, l_2, l_3\}$, where $l_1(\ell) = 1$ for $\ell = \ell_b, \ell_c$, $l_2(\ell) = 1$ only for $\ell = \ell_c$, and $l_3(\ell) = 1$ for $\ell = \ell_a, \ell_c$. There is a single lottery assignment, which must then occur with probability one. The probability that l_1 is assigned to ℓ_b is 1, even though this member is willing to accept ℓ_c as well, since no other member is willing to accept ℓ_b. Lottery membership l_2 must be assigned to ℓ_c, and that leaves only ℓ_a as a feasible assignment for l_3.

A natural conjecture is that with sufficiently large lottery group types, equilibria might be approximately efficient. We leave this for future work.

Appendix

To prove Theorem 6.1, we adapt the proof of existence of group equilibrium in EGSZ (1999, 2005) to account for randomness in augmented groups and beliefs on memberships.

We begin with a formal definition of constant-price equilibrium, making use of the finite-dimensional reformulation of the agent's problem from section 6.

Definition 9 *A* constant-price group equilibrium with beliefs on membership characteristics *consists of a feasible state $(x, \mu, \sigma), \mathcal{R}(\mu, \sigma)$, constant private goods prices $p \in \mathbf{R}^N_+$ with $p \neq 0$, membership prices $q \in \mathbf{R}^{\mathbf{M}}$, and beliefs on membership characteristics f such that (E1), (E7), and (E8) hold, where:*

(E7) ***Optimization by agents**: For almost all $a \in A$, if $(x'_a, \mu'_a, \sigma'_a) \in \hat{X}_a$ and $W_a(x'_a, \mu'_a, \sigma'_a; f) > W_a(x_a, \mu_a, \sigma_a; f)$, then there exists $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$ with $n(\tilde{\ell}; \mu_a, \sigma_a, f) > 0$ such that*

$$p \cdot x'_a(\tilde{\ell}) + q \cdot \mu_a > p \cdot e_a + p \cdot (\tilde{\ell} t)$$

(E8) ***Beliefs are correct**: $f(\cdot; m) = \bar{\phi}_{(\mu,\sigma)}(\cdot; m)$ for each $m \in \mathbf{M}$ such that $\zeta(m; \mu) < 1$.*

Proof of Theorem 6.1: We will show that a constant-price equilibrium with beliefs on membership characteristics exists.

For each $a \in A$, define a budget set $\hat{B}(a, p, q; f) \in (\mathbf{R}^N_+)^{\mathbf{Lists}(\tilde{\mathbf{M}})} \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$ and demand set $d_a(p, q; f) \in (\mathbf{R}^N_+)^{\mathbf{Lists}(\tilde{\mathbf{M}})} \times \mathbf{Lists}(\mathbf{M}) \times \Sigma$ as follows.

$$\begin{aligned} \hat{B}(a, p, q; f) := \{(x_a, \mu_a, \sigma_a) \in \hat{X}_a \quad : \quad & p \cdot x'_a(\tilde{\ell}) + q \cdot \mu_a \leq p \cdot e_a + p \cdot (\tilde{\ell} t) \\ & \forall\ \tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}}) \text{ such that } n(\tilde{\ell}; \ell, \hat{\sigma}, f) > 0\} \end{aligned}$$

$$\begin{aligned} d_a(p, q; f) \quad = \quad & \arg \max_{(x_a, \mu_a, \sigma_a)} W_a(x_a, \mu_a, \sigma_a; f) \\ & \text{s.t. } (x_a, \mu_a, \sigma_a) \in \hat{B}(a, p, q) \end{aligned}$$

Following EGSZ (2005), write

$$\mathbf{Cons} = \{\bar{\mu} \in \mathbf{R}^{\mathbf{M}} : \bar{\mu} \text{ is consistent } \}$$

$$\mathbf{Trans} = \{q \in \mathbf{R}^{\mathbf{M}} : q \cdot \mu = 0 \text{ for each } \mu \in \mathbf{Cons}\}$$

Assume without loss that $\lambda(A) = 1$. By assumption, aggregate endowment $\bar{e}$ is strictly positive and individual endowments are uniformly bounded above; say that $\bar{e} \geq w\mathbf{1} >> 0$ and that $e_a \leq W_0\mathbf{1}$ for all $a \in A$. Write $W_e = \max\{W_0, 1\}$. Recall that

$$\tilde{\ell}t = \sum_{g \in \mathbf{G}} \sum_{\substack{m \in \mathbf{M}(g) \\ s \in S(g)}} \tilde{\ell}(m, s) t_g(m, h(g, s))$$

There is a bound, say W_t, such that $|\tilde{\ell}t| \leq W_t\mathbf{1}$ for all $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$. Let $W = W_e + W_t$.

The technique of this proof is to define a sequence of economies $k = 1, 2, \ldots$, each with an augmented set of agents, such that an equilibrium exists and equilibrium prices are strictly positive for each economy in the sequence. We then argue that the limit of these equilibria is a quasi-equilibrium of the true economy, and that equilibrium can be assured by using the assumption of irreducibility.

Step 1 We construct the k^{th} economy $\mathcal{E}^k$. Fix an integer $k > 0$. Choose a family $\{A^k_m : m \in \mathbf{M}\}$ of pairwise disjoint intervals in $\mathbf{R}$, each of length $1/k$. Set

$$A^k = A \cup \bigcup_{m \in \mathbf{M}} A^k_m$$

The agent space for the perturbed economy $\mathcal{E}^k$ is $(A^k, \mathcal{F}^k, \lambda)$, where $\mathcal{F}^k$ is the σ-algebra generated by $\mathcal{F}$ and the Lebesgue measurable subsets of $\cup_{m \in \mathbf{M}} A^k_m$, λ^k is λ on A and Lebesgue measure on $\cup_{m \in \mathbf{M}} A^k_m$. Note that $\lambda^k(A^k) = 1 + \frac{|\mathbf{M}|}{k}$. External characteristics, consumption sets, endowments and utility functions of agents in A are just as in the original group economy $\mathcal{E}$. For agents $a \in A^k_m$, we define:

$$\begin{aligned} X_a &= \mathbf{R}^N_+ \times \{\ell \in \mathbf{Lists}(\mathbf{M}) : \ell(m) = 1, |\ell| = 1\} \times \Sigma \\ e_a &= W\mathbf{1} \\ u_a(x, \tilde{\ell}) &= |x| \text{ for all } (x, \tilde{\ell}) \end{aligned}$$

Step 2 We have enlarged the economy in such a way that market-clearing private-goods prices must be bounded away from zero, and market-clearing membership prices must be bounded. This is because the demand functions of the added agents are such that, for commodity prices near the boundary of the simplex and for membership prices that are large in absolute value, aggregate excess demand for commodities will be impossibly large.

Write $M^* = \max\{|\mathbf{M}(g)| : g \in \mathbf{G}\}$. Choose a real number $\varepsilon > 0$ so small that

$$[1 - (N-1)\varepsilon]\left[\frac{W}{kN\varepsilon} - W(1 + \frac{|\mathbf{M}|}{k})\right] - \varepsilon(N-1)W(1 + \frac{|\mathbf{M}|}{k}) > 0$$

Having chosen ε, choose a real number $R > 0$ so big that $R > 2|\tilde{\ell}t|$ for all $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$ and

$$[1 - (N-1)\varepsilon]\left[\frac{R}{2kNM^*} - W(1 + \frac{|\mathbf{M}|}{k})\right] - \varepsilon(N-1)W(1 + \frac{|\mathbf{M}|}{k}) > 0$$

Of course ε and R depend on k, although we supress the notation. Define a price simplex for private goods and a bounded price set for group memberships:

$$\begin{aligned}\Delta_\varepsilon &= \{p \in \mathbf{R}^N_+ : \sum_{n \in N} p_n = 1 \text{ and } p_n \geq \varepsilon \text{ for each } n\} \\ Q_R &= \{q \in \mathbf{Trans} : |q_m| \leq R \text{ for all } m \in \mathbf{M}\}\end{aligned}$$

Step 3 We define an aggregate excess demand correspondence together with a belief correspondence. The aggregate choices of private goods, memberships and strategies depend on both prices and beliefs, and the belief correspondence maps agents' membership and strategy choices into beliefs.

For each $m \in \mathbf{M}(g)$ and $(\mu, \sigma) : A \to \mathbf{Lists}(\mathbf{M}) \times \Sigma$, define the belief correspondence

$$\Phi(m; \mu, \sigma) := \zeta(m; \mu)\Delta(S_{-m}(g)) + [1 - \zeta(m; \mu)]\, \bar{\phi}_{(\mu,\sigma)}(\cdot; m)$$

At a fixed point of the mapping we construct below, μ must be consistent, so that, for each m, $\zeta(m; \mu)$ is either 0 or 1, where $\zeta(m; \mu)$ is the probability of not matching into a group. At such a fixed point, for any membership m such that $\zeta(m; \mu) = 1$, beliefs can be any selection from $\Delta(S_{-m}(g))$, and for memberships m with $\zeta(m; \mu) = 0$, beliefs are given by the empirical distribution.

Define $\bar{z} : \Delta_\varepsilon \times Q_R \times \mathbf{F} \to \mathbf{R}^N_+ \times \mathbf{R}^{\mathbf{M}} \times \mathbf{F}$ by

$$\begin{aligned}\bar{z}(p, q, f) := \{(z, \bar{\mu}, \pi) :\; & \\ & z = \int_{A^k} \left(\sum_{\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})} \left[x_a(\tilde{\ell}) - \tilde{\ell} t\right] n(\tilde{\ell}; \mu_a, \sigma_a, f) - e_a \right) d\lambda(a), \\ & \bar{\mu} = \int_{A^k} \mu_a d\lambda(a), \\ & \pi(\cdot; m) \in \Phi(m; \mu, \sigma) \text{ for each } m \in \mathbf{M}(g), \\ & \text{where } (x_a, \mu_a, \sigma_a) \in d_a(p, q; f) \text{ for all } a \in A\}\end{aligned}$$

The quantity z is aggregate expected excess demand. We argue in step 6 that z also equals the aggregate excess demand.

We claim that $\bar{z}$ is upper hemicontinuous. To that end, observe that endowments are bounded, group inputs and outputs are bounded (there are a finite number of input/output vectors $h(g, s)$), private good prices are bounded away from 0 and group membership prices are bounded above and below; hence the individual excess demand functions

$$\begin{aligned}(a, p, q, f) &\mapsto \sum_{\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})} \left[x_a(\tilde{\ell}) - \tilde{\ell} t\right] n(\tilde{\ell}; \mu_a, \sigma_a, f) - e_a \\ (a, p, q, f) &\mapsto \mu_a\end{aligned}$$

are uniformly bounded. This correspondence is also measurable and upper hemi-continuous since endowments are assumed to be desirable.

Let $(p^n, q^n, f^n) \to (p, q, f)$, and $(z^n, \bar{\mu}^n, \pi^n) \in \bar{z}(p^n, q^n, f^n)$ for each n such that $(z^n, \bar{\mu}^n, \pi^n) \to (z, \bar{\mu}, \pi)$. We must show that $(z, \bar{\mu}, \pi) \in \bar{z}(p, q, f)$.

By definition, for each n there exists $(x_a^n, \mu_a^n, \sigma_a^n) \in d_a(p^n, q^n, f^n)$ such that

$$\begin{aligned} z^n &= \int_{A^k} \left[\sum \left(x_a^n(\tilde{\ell}) - \tilde{\ell}t \right) n_a(\tilde{\ell}; \mu_a^n, \sigma_a^n, f^n) - e_a \right] d\lambda(a) \\ \bar{\mu}^n &= \int_{A^k} \mu_a^n d\lambda(a) \\ \pi^n(\cdot; m) &\in \Phi(m; \mu^n, \sigma^n) \ \ \forall m \end{aligned}$$

Because demands are uniformly bounded and upper hemicontinuous, for each a there exists $(x_a, \mu_a, \sigma_a) \in d_a(p, q, f)$ such that $(x_a^n, \mu_a^n, \sigma_a^n) \to (x_a, \mu_a, \sigma_a)$ for each a and

$$\begin{aligned} z &= \int_{A^k} \left[\sum \left(x_a(\tilde{\ell}) - \tilde{\ell}t \right) n_a(\tilde{\ell}; \mu_a, \sigma_a, f) - e_a \right] d\lambda(a) \\ \bar{\mu} &= \int_{A^k} \mu_a d\lambda(a) \end{aligned}$$

Now it suffices to show that $\pi(\cdot; m) \in \Phi(m; \mu, \sigma)$ for each m. To see this, fix $m \in \mathbf{M}(g)$. If $\zeta(m; \mu) = 1$, $\Phi(m; \mu, \sigma) = \Delta(S_{-m}(g)) \ni \pi(\cdot; m)$. Thus suppose $\zeta(m; \mu) < 1$. In this case, for each m', $\lambda(A_{m'}) > 0$. Since $\mu^n \to \mu$, without loss of generality take $\lambda(A^n_{m'}) > 0$ for each n and $m' \in \mathbf{M}(g)$, where $A^n_{m'} := \{a \in A : \mu_a^n(m') = 1\}$. Using a version of Fatou's lemma,

$$\begin{aligned} \zeta(m; \mu^n) &= 1 - \frac{\min_{m' \in \mathbf{M}(g)} \lambda(A^n_{m'})}{\lambda(A^n_m)} \\ &\to 1 - \frac{\min_{m' \in \mathbf{M}(g)} \lambda(A_{m'})}{\lambda(A_m)} = \zeta(m; \mu) \end{aligned}$$

and for each $s \in S(g)$,

$$\begin{aligned} \bar{\phi}_{(\mu^n, \sigma^n)}(s; m) &= \prod_{\hat{m} \in \mathbf{M}(g) \setminus m} \frac{\lambda\left(\left\{a \in A : \sigma^n_{a,\hat{m}} = s_{\hat{m}}, \mu_a^n(\hat{m}) = 1\right\}\right)}{\lambda(A^n_{\hat{m}})} \\ &\to \prod_{\hat{m} \in \mathbf{M}(g) \setminus m} \frac{\lambda\left(\left\{a \in A : \sigma^n_{a,\hat{m}} = s_{\hat{m}}, \mu_a^n(\hat{m}) = 1\right\}\right)}{\lambda(A^n_{\hat{m}})} = \bar{\phi}_{(\mu, \sigma)}(s; m) \end{aligned}$$

¿From this and the definition of Φ, we conclude that $\pi(\cdot; m) \in \Phi(m; \mu, \sigma)$ as desired.

Aggregate excess demand lies in a compact set. Individual income comes from selling endowments, possibly from receiving subsidies for group memberships, and from transfers within groups. The value of each individual's endowment is bounded by W_e and the value of transfers in the groups he joins is bounded by W_t. Thus, he can spend no more than

$W = W_e + W_t$ on consumption. Because group membership prices lie in the interval $[-R, +R]$ and individuals can choose no more than M group memberships, subsidies for group memberships are bounded by MR. Because private good prices are bounded below by ε, individual demand for each private good is bounded above by $\frac{1}{\varepsilon}(W + RM)$, and individual excess demand for each private good lies between $-W$ and $\frac{1}{\varepsilon}(W + RM)$. Hence aggregate excess demand for private goods lies in the compact set

$$X = \{x \in \mathbf{R}^N : -\lambda(A^k)W \leq x_n \leq \lambda(A^k)\frac{1}{\varepsilon}(W + RM) \text{ for each } n\}$$

Because individuals are constrained to demand at most M group memberships, aggregate demands for memberships lie in the set

$$C = \{\bar{\mu} \in \mathbf{R}^{\mathbf{M}}_+ : \sum_{m \in \mathbf{M}} \bar{\mu}(m) \leq \lambda(A^k)M\}$$

Step 4 We complete the construction of a correspondence for which fixed points are equilibria by defining the correspondence $\Upsilon : \Delta_\varepsilon \times Q_R \times X \times C \times \mathbf{F} \to \Delta_\varepsilon \times Q_R \times X \times C \times \mathbf{F}$ by

$$\begin{aligned} &\Upsilon(p, q, f, z, \bar{\mu}) \\ &\quad = \quad [\arg\max\{(p^*, q^*) \cdot (z, \bar{\mu}) : (p^*, q^*) \in \Delta_\varepsilon \times Q_R\}] \times \bar{z}(p, q, f) \end{aligned}$$

Υ is an upper hemi-continuous with compact convex values. Hence Kakutani's fixed point theorem guarantees that Υ has a fixed point.

A fixed point gives a price pair $(p^k, q^k) \in \Delta_\varepsilon \times Q_R$, beliefs $f^k \in \mathbf{F}$, and quantities $(z^k, \bar{\mu}^k, f^k) \in \bar{z}(p^k, q^k, f^k)$ such that

$$(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = \max\left[(p^*, q^*) \cdot (z^k, \bar{\mu}^k) \;:\; (p^*, q^*) \in \Delta_\varepsilon \times Q_R\right]$$

Walras's law implies that $(p^k, q^k) \cdot (z^k, \bar{\mu}^k) = 0$. Notice also that, provided $\bar{\mu}$ is consistent, beliefs will be correct at a fixed point for all groups that form in equilibrium.

The fixed point ensures that the beliefs on memberships f are correct; that is, $f(\cdot; m) = \bar{\phi}_{(\mu,\sigma)}(\cdot; m)$ for memberships that are taken by a group of agents with positive measure.

Step 5 We show in several steps that $z^k = 0$ and $\bar{\mu}^k \in$ **Cons**, that is, the state of the economy at a fixed point is feasible.

Step 5.1 We show first that $q^k \cdot \bar{\mu}^k = 0$. Suppose not. We obtain a contradiction by looking at excess demands at prices p^k, q^k of agents in $A^k \setminus A$. Because $0 \cdot \bar{\mu}^k = 0$, maximality and the definition of Υ imply $q^k \cdot \bar{\mu}^k \geq 0$. Maximality entails that $q^k \in \text{bdy}\, Q_R$ so that $|q^k_m| = R$ for some $m \in \mathbf{M}$. Budget balance for group types means that if some price has large magnitude and is positive then some other price must have large magnitude and be negative. Thus there is a membership m^* such that $q^k_{m^*} \leq -R/M^*$. An agent $b \in A^k_{m^*}$ could obtain a subsidy of R/M^* by choosing the membership m^* and no other. Such an agent, finding all private goods to be perfect substitutes and deriving no utility from group memberships, will consume only the least expensive private good(s) and group memberships with non-positive prices. Because $R > 2|\tilde{\ell}t|$ for each $\tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$, the wealth used on inputs to groups is less than $\frac{R}{2}$. Thus b's demand for the least expensive private good — which we may as well suppose is good 1 — is at least

$$x^k_{b1}(\tilde{\ell}) \geq \frac{R}{2NM^*} \text{ for each } \tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$$

Because $\lambda(A^k_\omega) = 1/k$ and individual excess demands are bounded below by $-W\mathbf{1}$, aggregate excess commodity demand z^k satisfies

$$\begin{aligned} z^k_1 &\geq \frac{1}{k}\frac{R}{2NM^*} - W(1 + \frac{|\mathbf{M}|}{k}) \\ z^k_n &\geq -W(1 + \frac{|\mathbf{M}|}{k}) \qquad \text{if } n > 1 \end{aligned}$$

Define $p \in \Delta_\varepsilon$ by:

$$p_n = \begin{cases} 1 - (N-1)\varepsilon & \text{if } n = 1 \\ \varepsilon & \text{if } n > 1 \end{cases} \tag{14}$$

It follows that

$$p \cdot z^k \geq [1 - (N-1)\varepsilon]\left[\frac{R}{2kNM^*} - W(1 + \frac{|\mathbf{M}|}{k})\right] - \varepsilon(N-1)W(1 + \frac{|\mathbf{M}|}{k})$$

Our choices of R, ε guarantee that the right side is strictly positive, so

$$(p, 0) \cdot (z^k, \bar{\mu}^k) > 0 = (p^k, q^k) \cdot (z^k, \bar{\mu}^k)$$

which contradicts maximality. We conclude that $q^k \cdot \bar{\mu}^k = 0$, as desired.

Step 5.2 We show next that $\bar{\mu}^k \in \mathbf{Cons}$. If not, we could find a pure transfer $q^* \in \mathbf{Trans}$ such that $q^* \cdot \bar{\mu}^k > 0$ and hence could find $q^{**} \in Q_R$ such that $q^{**} \cdot \bar{\mu}^k > 0$, contradicting maximality.

Step 5.3 We claim that $p^k_n > \varepsilon$ for each n. Suppose not. We once again obtain a contradiction by considering the excess demand of agents $b \in A^k \setminus A$. As before, we note that each such agent will consume only the least expensive private good(s) and group memberships with non-positive prices. It follows that b's demand for the least expensive private good — which we may as well suppose is good 1 — is at least

$$x_{b1}(\tilde{\ell}) \geq \frac{W}{N\varepsilon} \text{ for all } \tilde{\ell} \in \mathbf{Lists}(\tilde{\mathbf{M}})$$

As before, this means that aggregate excess commodity demand z^k satisfies

$$\begin{aligned} z^k_1 &\geq \frac{1}{k}\frac{W}{N\varepsilon} - W(1 + \frac{|\mathbf{M}|}{k}) \\ z^k_n &\geq -W(1 + \frac{|\mathbf{M}|}{k}) \qquad \text{if } n > 1 \end{aligned}$$

Defining p as in (14), it follows that

$$p \cdot z^k \geq [1-(N-1)\varepsilon]\left[\frac{W}{kN\varepsilon} - W(1+\frac{|\mathbf{M}|}{k})\right] - \varepsilon(N-1)W(1+\frac{|\mathbf{M}|}{k})$$

Our choice of ε guarantees that the right side is strictly positive so

$$(p,0)\cdot(z^k,\bar{\mu}^k) > 0 = (p^k,q^k)\cdot(z^k,\bar{\mu}^k)$$

which again contradicts maximality. We conclude that $p_n^k > \varepsilon$ for each n.

Step 5.4 We show that $z^k = 0$. Notice that $(p^k,q^k)\cdot(z^k,\bar{\mu}^k) = 0$ and $q^k \cdot \bar{\mu}^k = 0$ so $p^k \cdot z^k = 0$. Hence, if $z^k \neq 0$ there are indices i,j such that $z_i^k < 0$ and $z_j^k > 0$. Define $\hat{p}$ by

$$\begin{aligned} \hat{p}_i &= p_i^k - \frac{1}{2}(p_i^k - \varepsilon) \\ \hat{p}_j &= p_j^k + \frac{1}{2}(p_i^k - \varepsilon) \\ \hat{p}_n &= p_n^k \qquad \text{for } n \neq i,j \end{aligned}$$

Because $p_i^k > \varepsilon$, it follows that $\hat{p} \in \Delta_\varepsilon$. Because $p^k \cdot z^k = 0$, it follows that $\hat{p}\cdot z^k > 0$, a contradiction to maximality. We conclude that $z^k = 0$.

Step 6 We now show that a fixed point constitutes an equilibrium. By definition, there are selections $(x_a^k, \mu_a^k, \sigma_a^k)$ from the individual demand sets $d_a\,(p,q;f)$ such that

$$z^k = \int_{A^k}\left(\sum_{\tilde{\ell}\in\mathbf{Lists}(\tilde{\mathbf{M}})}\left[x_a^k(\tilde{\ell}) - \tilde{\ell}t\right] n(\tilde{\ell};\mu_a,\sigma_a,f) - e_a\right) d\lambda\,(a) = 0$$

Since μ is consistent, almost every agent's chosen memberships result in matches. Further, $f\,(\cdot;m) = \bar{\phi}_{(\mu,\sigma)}\,(\cdot;m)$ for every membership chosen by a set of agents of positive measure. It follows that, for almost every agent $a \in A$, $n(\tilde{\ell};\mu_a,\sigma_a,f) = \bar{\eta}_{(\mu,\sigma)}(\tilde{\ell};\mu_a,\sigma_a) = \eta_{(\mu,\sigma)}(\tilde{\ell};\mu_a,\sigma_a)$. As a consequence,

$$z^k = \int_{A^k}\left(\sum_{\tilde{\ell}\in\mathbf{Lists}(\tilde{\mathbf{M}})}\left[x_a^k(\tilde{\ell}) - \tilde{\ell}t\right] \bar{\eta}_{(\mu,\sigma)}(\tilde{\ell};\mu_a,\sigma_a) - e_a\right) d\lambda\,(a) = 0$$

At the selections $(x_a^k, \mu_a^k, \sigma_a^k)$, agents are optimizing. Since the fixed point ensures consistency, for feasibility it only remains to show that material balance holds when $z^k = 0$. The argument above is not quite enough, since it shows only that aggregate expected demand is zero – not that aggregate demand is zero. Setting $x_a^k(v) = x_a^k(\tilde{\ell})$ for each $v \in V_a(\tilde{\ell})$, this expression can be rewritten as

$$z^k = \int_{A^k}\left(\left[\int_V \left[x_a^k(v) - \tilde{\mu}_a^r(v)t\right] d\mathcal{P}\,(\mu,\sigma)\,(v)\right] - e_a\right) d\lambda\,(a) = 0$$

By Corollary 2.10 of Sun (2006), for $\mathcal{P}(\mu,\sigma)$-almost all $v \in V$ the aggregate demand at v is equal to the aggregate expected demand. Thus for $\mathcal{P}(\mu,\sigma)$-almost all $v \in V$,

$$z^k = \int_{A^k} \left(x_a^k(v) - \tilde{\mu}_a^r(v)t - e_a \right) d\lambda(a) = 0$$

When $\mu^k \in \mathbf{Cons}$,

$$\int_{A^k} \tilde{\mu}_a^r(v)t \; d\lambda(a) = \int_{A^k} \left[\sum_{g\in\mathbf{G}} \sum_{\substack{m\in\mathbf{M}(g) \\ s\in S(g)}} \tilde{\mu}_a^r(v)(m,s) \frac{h(g,s)}{|\mathbf{M}(g)|} \right] d\lambda(a)$$

Together the previous equalities yield (3).

Step 7 To argue that the limit of the equilibria as $k \to \infty$ is a quasi-equilibrium of the original economy, we must argue that the membership prices q^k stay bounded. They are bounded by R, but R depends on k. We now replace the sequence (q^k) by a bounded sequence $(\bar{q}^k)$ that leads to the same demands.

Passing to a subsequence if necessary, we may assume without loss that for each $\ell \in \mathbf{Lists}(\mathbf{M})$ the sequence $(q^k \cdot \ell)$ converges to a limit G_ℓ, which may be finite or infinite. Write:

$$\begin{aligned} L &= \{\ell \in \mathbf{Lists}(\mathbf{M}) : q^k \cdot \ell \to G_\ell \in \mathbf{R}\} \\ L_+ &= \{\ell \in \mathbf{Lists}(\mathbf{M}) : q^k \cdot \ell \to +\infty\} \\ L_- &= \{\ell \in \mathbf{Lists}(\mathbf{M}) : q^k \cdot \ell \to -\infty\} \end{aligned}$$

Choose $\bar{G} \in \mathbf{R}$ so large that $|q^k \cdot \ell| \le \bar{G}$ for each k, each $\ell \in L$.

Define the linear transformation $T : \mathbf{Trans} \to \mathbf{R}^L$ by $T(q)_\ell = q \cdot \ell$. Write ran $T = T(\mathbf{Trans}) \subset \mathbf{R}^L$ for the range of T and $\ker T = T^{-1}(0) \subset \mathbf{Trans}$ for the kernel (null space) of T. The fundamental theorem of linear algebra implies that we can choose a subspace $H \subset \mathbf{Trans}$ so that $H \cap \ker T = \{0\}$ and $H + \ker T = \mathbf{Trans}$. Write $T_{|H}$ for the restriction of T to H. Note that $T_{|H} : H \to$ ran T is a one-to-one and onto linear transformation, so it has an inverse $S :$ ran $T \to H$. Because S is a linear transformation, it is continuous, so there is a constant K such that $|S(x)| \le K|x|$ for each $x \in$ ran T.

Using Lemma 7.2 of EGSZ (1999), there exists a constant R^* and k_0 so large that $k \ge k_0$ implies

$$\begin{aligned} q^k \cdot \ell &> +2K\bar{G}M + W \qquad \text{if } \ell \in L_+ \\ q^k \cdot \ell &< -2K\bar{G}M - \frac{W}{R^*} \quad \text{if } \ell \in L_- \end{aligned}$$

Write ST for the composition of S with T. For each $k \ge k_0$ set

$$\bar{q}^k = ST(q^k) - ST(q^{k_0}) + q^{k_0} \in \mathbf{Trans}$$

Because $S, T_{|H}$ are inverses, the composition TS is the identity, so

$$T(\bar{q}^k) = TST(q^k) - TST(q^{k_0}) + T(q^{k_0}) = T(q^k)$$

We assert that for $k > k_0$, $\bar{\mu}^k_a \notin L_- \cup L_+$ for any $a \in A^k$. If $a \in A^k$ then $q^k \cdot \bar{\mu}^k_a \leq W$ (because the expenditures are bounded by W) so $\bar{\mu}^k_a \notin L_+$, by construction of L_+. Since $\{\bar{\mu}^k_a\}$ are strictly balanced and $q^k \in \mathbf{Trans}$, it follows from Lemma 7.2 in EGSZ (1999) that $\min_{a \in A^k}\{q^k \cdot \bar{\mu}^k_a\} \geq -\frac{1}{R^*} \max_{a \in A^k}\{q^k \cdot \bar{\mu}^k_a\} \geq -\frac{W}{R^*}$, and hence $\bar{\mu}^k_a \notin L_-$ by the construction of L_-.

Choose $k_1 \geq k_0$ so that $q^k \cdot \ell < q^{k_0} \cdot \ell - 2KGM$ for all $\ell \in L_-$ and all $k > k_1$. We claim that for $k > k_1$, $(x^k, \bar{\mu}^k, \sigma^k), (p^k, \bar{q}^k), f^k$ is an equilibrium for $\mathcal{E}^k$. Because $(x^k, \bar{\mu}^k, \sigma^k), (p^k, q^k), f^k$ is an equilibrium, it suffices to show that, for almost all $a \in A^k$ the choice $(x^k_a, \bar{\mu}^k_a, \sigma^k)$ is budget feasible and optimal at $(p^k, \bar{q}^k, f^k)$. We have shown above that $\bar{\mu}^k_a \in L$ for almost all a; by construction $\bar{q}^k \cdot \ell = q^k \cdot \ell$ for all $\ell \in L$ because $T(\bar{q}^k) = T(q^k)$. Hence choices are budget feasible. Suppose then that (y, ν, s) is budget feasible for a at $(p^k, \bar{q}^k, f^k)$ and preferred to $(x^k_a, \bar{\mu}^k_a, \sigma^k_a)$. Budget feasibility of (y, ν, s) at $(p^k, \bar{q}^k, f^k)$ implies that $\bar{q}^k \cdot \nu \leq W$ and hence $q^{k_0} \cdot \nu \leq W + 2K\bar{G}M$ because $|ST(q^k)| \leq K\bar{G}$ and $|ST(q^{k_0})| \leq K\bar{G}$. Thus $\nu \notin L_+$. For $\ell \in L_-$ and $k > k_1$, we similarly obtain $\bar{q}^k \cdot \ell > q^{k_0} \cdot \ell - 2K\bar{G}M > q^k \ell$. Thus, $\bar{q}^k \cdot \ell \geq q^k \cdot \ell$ for $\ell \in L_-$. Hence, $\bar{q}^k \cdot \ell \geq q^k \cdot \ell$ for $\ell \in L_- \cup L$. Thus, budget feasibility of (y, ν, s) at $(p^k, \bar{q}^k, f^k)$ implies budget feasibility of (y, ν, s) at (p^k, q^k, f^k), so $(x^k_a, \bar{\mu}^k_a, \sigma^k_a)$ is not optimal at (p^k, q^k, f^k). Thus $(x^k, \bar{\mu}^k, \sigma^k), (p^k, \bar{q}^k), f^k$ must be an equilibrium in $\mathcal{E}^k$.

Step 8 Finally we argue that the limit of equilibria is a quasi-equilibrium of the original economy, and also an equilibrium. By construction, $|\bar{q}^k \cdot \ell| \leq 2K\bar{G}M + |q^{k_0} \cdot \ell|$ for $k > k_0$ and all lists ℓ, so the prices of lists are bounded. Because singleton memberships are themselves lists, it follows that $(\bar{q}^k)$ is also a bounded sequence in $\mathbf{Trans}$, and f^k is bounded. We thus have bounded sequences (p^k), $(\bar{q}^k)$, $(f^k), (\bar{\mu}^k)$. Passing to a subsequence if necessary, we may assume that $p^k \to p^* \in \Delta$, $\bar{q}^k \to q^* \in \mathbf{Trans}$, $f^k \to f^* \in \mathbf{F}$, $\bar{\mu}^k \to \bar{\mu}^* \in \mathbf{Cons}$. The sequences $(\bar{\mu}^k)$ and (f^k) are uniformly bounded, hence uniformly integrable, so Schmeidler's version of Fatou's lemma (see Hildenbrand (1974, p. 225)) provides a measurable mapping $(x^*, \mu^*, \sigma^*) : A \to (\mathbf{R}^N_+)^{\mathbf{Lists}(\tilde{\mathbf{M}})} \times \mathbf{R}^{\mathbf{M}} \times \Sigma$ such that (i) for almost all $a \in A$: $(x^*_a, \mu^*_a, \sigma^*_a) \in B(a, p^*, q^*, f^*)$; (ii) for almost all $a \in A$: $(x^*_a, \mu^*_a, \sigma^*_a)$ belongs to agent a's quasi-demand set; that is, there does not exist a strictly preferred $(x', \ell', \sigma') \in \hat{X}_a$ that is budget feasible at (p^*, q^*, f^*) and strictly cheaper; (iii) $\int_A [x^*_a - \tilde{\mu}^*_a t]\, d\lambda \leq \bar{e}$; (iv) $\int_A \mu^*_a \, d\lambda = \bar{\mu}^*$. Conditions (i) and (ii) together imply that for almost all a, $(p^*, q^*) \cdot (x^*_a, \mu^*_a) - p^* \cdot \tilde{\mu}^*_a t = p^* \cdot e_a$. That is, left over goods (if any) are free. Distributing these free goods arbitrarily yields a quasi-equilibrium $(x^{**}_a, \mu^*_a, \sigma^*_a), (p^*, q^*)$ for $\mathcal{E}$. Group irreducibility implies that $(x^{**}_a, \mu^*_a, \sigma^*_a), (p^*, q^*), f^*$ is an equilibrium for $\mathcal{E}$, so the proof is complete.[19] ■

[19]Because utility functions are strictly monotone in private goods, no goods are free at equilibrium, so in fact there are no leftover goods to distribute.

References

[1] Acemoglu, D. and A. Simsek. 2010. "Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading." Working Paper, MIT, Department of Economics.

[2] Bisin, A., J. D. Geanakoplos, P. Gottardi, E. Minelli, H. Polemarchakis. 2001. "Markets and Contracts." Working Paper, Brown University, Department of Economics.

[3] Buchanan, J. 1965. "An Economic Theory of Clubs," *Economica* 33:1-14.

[4] Bulow, J. and J. Levin 2006. "Matching and Price Competition," *The American Economic Review*, 96(3): 652-668

[5] Cole, Harold L. and Edward S. Prescott. 1997. "Valuation Equilibrium with Clubs." *Journal of Economic Theory* 74:19-39.

[6] Cole, Harold L., G. Mailath, and A. Postlewaite. 2001. "Efficient Non-Contractible Investments in Large Economies." *Journal of Economic Theory* 101: 333-373.

[7] Dubey, P., J. Geanakoplos, and M. Shubik. 2005. "Default and Punishment in General Equilibrium." *Econometrica* 73: 1-37.

[8] Dubey, P. and J. Geanakoplos. 2004. "Signaling and Default: Rothschild-Stiglitz Reconsidered." *Quarterly Journal of Economics*

[9] Duffie, D. and Y. Sun. 2007. "Existence of Independent Random Matching." *Annals of Applied Probability* 17(1):386-419.

[10] Ellickson, B., B. Grodal, S. Scotchmer, W. Zame. 1999. "Clubs and the Market." *Econometrica* 67:1185-1217.

[11] Ellickson, B., B. Grodal, S. Scotchmer, W. Zame. 2005. "The Organization of Consumption, Production and Learning, Institute of Economics." In K. Vind, ed. *The Birgit Grodal Symposium.* Berlin: Springer-Verlag.

[12] Geanakoplos, J. D., and H. M. Polemarchakis. 1986. "Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete." In W. P. Heller, R. M. Starr and D. A. Starrett, eds. *Uncertainty, Information and Communication: Essays in Honor of K. J. Arrow.* Vol III, 66-95. Cambridge, UK: Cambridge University Press.

[13] Holmstrom, B. 1984. "Moral hazard and observability." *Bell Journal of Economics* 10:74-91.

[14] Legros, P. and A. Newman. 1996. "Wealth Effects, Distribution, and the Theory of Organization," *Journal of Economic Theory*, 70: 312-341.

[15] Legros, P. and A. Newman. 2008. "Competing for Ownership," *Journal of the European Economic Association*, 6(6): 1-30.

[16] Legros, P. and A. Newman. 2009. "A Price Theory of Vertical and Lateral Integration," working paper, February 2009.

[17] Magill, M. and M. Quinzii. 2008. "Normative Properties of Stock Market Equilibrium with Moral Hazard ." *Journal of Mathematical Economics* 44(7-8):785-806.

[18] Malinvaud, E. 1973. "Markets for an Exchange Economy with Individual Risks." *Econometrica* 41:383-410.

[19] Maskin, E. 1999. "Nash Equilibrium and Welfare Optimality." *The Review of Economic Studies* 66(1): 23-38.

[20] McAfee, P. 1993. "Mechanism Design by Competing Sellers," *Econometrica*, vol. 61(6): 1281-1312.

[21] Minelli, E. and H. M. Polemarchakis. 2000. "Nash-Walras equilibria of a large economy." *Proceedings of the National Academy of Sciences* 97: 5675-5678.

[22] Peters, M. 1997. "A Competitive Distribution of Auctions," *Review of Economic Studies* 64(1): 97-123.

[23] Peters, M. 2001. "Surplus Extraction and Competition," *Review of Economic Studies*, 68(3): 613-631.

[24] Prescott, Edward C. and Robert M. Townsend. 1984. "Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard." *Econometrica* 52: 21-45.

[25] Prescott, Edward S. and Robert M. Townsend. 2006. "Firms as Clubs in Walrasian Markets with Private Information." *Journal of Political Economy*, August 2006: 644-671.

[26] Rustichini, A. and P. Siconolfi. 2010. "Economies with Observable Types." Working Paper.

[27] Sun, Yeneng. 2006. "The Exact Law of Large Numbers via Fubini Extension and Characterization of Insurable Risks," *Journal of Economic Theory* 126(1): 31-69

[28] Zame, W. 2007. "Contracts and Markets – A General Equilibrium Theory of Firms." *Econometrica* 75(5): 1453-1500.