Harmful Signaling in Matching Markets

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Abstract

A signaling mechanism has been proposed as a device to improve agent welfare in decentralized two-sided matching markets. An example of such an environment is the job market for new Ph.D. economists. We study a market game of incomplete information between firms and workers. Workers have almost aligned preferences over firms: each worker has "typical" commonly known preferences with probability close to one and "atypical" idiosyncratic preferences with the complementary probability close to zero. Firms have some commonly known preferences over workers. We show that the introduction of a signaling mechanism is harmful for this environment. Though signals transmit previously unavailable information, they also facilitate information asymmetry that leads to coordination failures. As a result, the introduction of a signaling mechanism lessens the expected total number of matches.

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1 Introduction

Signaling as an actual mechanism design instrument was first implemented by the Ad Hoc Committee[1] of American Economic Association (AEA) in December 2006 to facilitate match formation in the job market for new Ph.D. economists. This market begins in early Fall each year, when departments of economics departments advertise open faculty positions and graduate students nearing completion of their dissertations apply for these positions according to their preferences. Additionally, each student has an opportunity to send two signals to two departments prior to the market.[2] Each signal states only that the student has indicated her interest to a given department. The signals are private, only the faculty of the chosen department knows the student's signal. The main part of the market happens later, when departments invite students to interviews and finally select the candidates to whom they make job offers. However, each department can interview only a small portion of available students, which creates congestion in the market. The decision which candidates to invite for interviews is a strategic one. An average department probably does not want spend time interviewing candidates who are being interviewed by the elite departments.

The Ad Hoc Committee introduced the signals in order to alleviate congestion at the interviewing stage. Signaling is essentially a costless communication, or cheap talk. There is no penalty attached for lying, and claims do not directly affect payoffs[3]. Therefore, signals can only enlarge the set of equilibria. Crawford and Sobel (1982) show that cheap talk can be credible in an equilibrium, if parties have common interests. Moreover, costless communication leads to new equilibria that are Pareto-superior to the one without communication. Therefore, one may conjecture that cheap talk should be also beneficial for decentralized matching markets. Roth (2008b) suggests also that the limited number of signals can credibly transmit information about students' preference, which could help to reduce the coordination failures faced by the market participants and facilitate better match formation (see also "Signaling for Interviews in the Economics Job Market" AEA (2005)[4] for more discussion). Recently, Coles et al. (2009) obtain results that support this intuition. They consider many-to-many matching markets among firms and workers, whose preferences are ex-ante block-correlated. Specifically, there are several blocks of firms. Workers have the same and commonly known ranking of firms across the blocks and idiosyncratic

[1] The Ad Hoc Committee was established in 2005 in order to develop ways to facilitate the job market for new Ph.D. economists. Its members are Alvin E. Roth (chair), John Cawley, Philip Levine, Muriel Niederle, and John Siegfried.

[2] This mechanism is implemented via the AEA website: http://www.aeaweb.org/joe/signal/.

[3] Ration talk is a better name for signals in our setting. Though signals are costless, an agent can send only a limited number of signals.

[4] The document was created by the Ad Hoc Committee to provide advice to participants in the job market for new Ph.D. economists; http://www.aeaweb.org/joe/signal/signaling.pdf.

preferences over firms within blocks that are uniformly distributed (preferences are equally likely). Firms' preferences are uniformly distributed over the range of all possible preference order lists. Each worker can send several signals to firms. Coles et al. (2009) show that the introduction of signals increases the expected number of matches and the welfare of workers in equilibrium.

Another example of a market where signaling plays a significant role is the market for clinical psychologists, described by Roth and Xing (1997). They show that the ability of candidates to convey information about the likelihood to accept an offer is crucial in the market. Program directors for internships in clinical psychology have a tendency to hire applicants who explicitly express their readiness to accept an offer immediately, even if these applicants are of a low quality.

Preference signaling also plays an important role in match formation in the U.S. college admission market. More than a hundred colleges adopted some form of early admission program in the 1990s (Avery et al., 2003), and many schools fill a significant fraction of their entering class with early applicants. There are two types of early admission programs: *early action* programs, where students may apply early but without any commitment to enroll, and *early decision* programs, where students commit to enroll if accepted. Many schools also require that applicants not send early applications to other schools. Colleges view an early application as a signal of a student's enthusiasm for a particular school. Avery and Levin (2009) show that selective (or elite) schools benefit from adopting early action policy. At the same time, a lower ranked school, by adopting early decision policy, can attract some highly qualified but cautious students, drawing them away from highly ranked schools.

Our main concern in this paper is the job market for new Ph.D. economists. We consider a model similar to that of Coles et al. (2009) and show that signals impede match formation in some environments. Though signals transmit information about agents' preference truthfully, they also introduce information asymmetry. The information asymmetry facilitates coordination failures that decrease the expected number of matches and ambiguously affects the welfare of agents. The negative effect on agents' welfare in new cheap talk equilibria is in line with Farrell and Gibbons (1989)'s results, though it differs in its intuition.[5] Costless communication in their two-agent bargaining model gives the buyer an opportunity to pretend to have a lower value and the seller an opportunity to pretend to have a higher value (compared to the truthful information transmission in our model). This enhances their bargaining positions at the cost of the risk of no trade. New cheap talk equilibria are characterized by both less trade and a reduction in the expected gains from trade.

We analyze one-to-one a matching market between workers and firms in this paper. We examine an environment in which workers have almost aligned preferences. Each worker has

[5] We are thankful for Lones Smith who drew our attention to this comparison.

either "typical" commonly known preferences with a probability close to one or "atypical" preferences taken from some distribution with the complementary probability close to zero. The preferences of workers are ex-ante independently distributed.[6] Firms have some fixed and commonly known preferences over workers. We consider a decentralized matching game with three stages. First, each worker chooses a firm, to which she sends her signal. Each worker sends up to one signal; workers send signals simultaneously. Only firms that receive signals observe them. Second, firms make decisions about job offers by taking into account signals received at the first stage. Each firm can make only one offer. Finally, each worker chooses an offer to accept among the available offers. Each worker can accept at most one offer.

We show that if firms respond to signals in this environment, i.e. treat signals informatively, the introduction of signals decreases the expected number of matches. The effect of signals on the welfare of agents is ambiguous. Intuitively, signals help workers with atypical preferences to obtain better matches. This also increases the welfare of some firms. At the same time, signals deprive some agents of their matches. Overall, our analysis suggests that signals play two important roles: 1) they reduce coordination failures because they transmit previously unavailable information about workers' preferences, and 2) they introduce information asymmetry. They transmit information about the preferences of workers to a limited number of firms, leaving the other firms uninformed. This information asymmetry facilitates coordination failures.

Finally, we analyze how the welfare implications change if all agents observe the signals each firm receives, i.e. signals are public. Though the expected number of matches increases compared to the offer game with private signals, public signals still impede match formation for some environments. Public signals do not transmit enough information about worker preferences. This induces some firms to compete for the same workers, which creates mismatches.

A simple example

Let us illustrate why signals can facilitate coordination failures by a simple example with three firms and three workers. The firms rank the workers in the same way (w_1, w_2, w_3), i.e. they strictly prefer worker w_1 to worker w_2 to worker w_3. Each worker's preference is either typical (f_1, f_2, f_3) with probability $1 - \varepsilon$ or atypical with the complementary probability ε, where ε is small. The atypical preferences are uniformly distributed among all possible preference order lists. All workers are acceptable to all firms and vice versa.

If signals are not allowed, the only possible match in an equilibrium is the assortative

[6] We assume that typical workers rank firms according to some public ranking. For example, typical candidates in the job market for new Ph.D. economists rank departments of economics in their field according to the *U.S. News and World Report* ranking.

match, in which each firm is matched to the corresponding worker. If signals are allowed, we consider the following equilibrium strategies of agents.[7] Each worker with typical preferences sends her signal to the corresponding firm (worker w_i sends her signal to firm f_i). Each worker with atypical preferences sends her signal to the best firm worse or equal to the corresponding one (according to typical preferences). Each firm makes its offer to a worker better or equal to the corresponding one, only if it receives a signal from her. Each firm ignores all signals from workers worse than the corresponding one. If a firm receives no signals, it makes an offer to the best worker worse than the corresponding one.

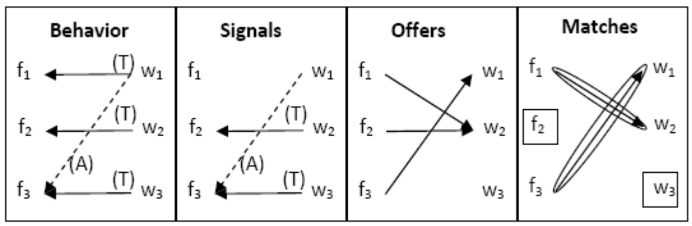

Figure I.

Let us consider the realization of preference profiles when only worker w_1 is atypical and firm f_3 is her favorite firm. Worker w_2 and worker w_3 are typical. Figure I illustrates the equilibrium behavior. Worker w_1 sends her signal to firm f_3. Worker w_2 and worker w_3 send their signals to firm f_2 and firm f_3 correspondingly. Firm f_3 makes an offer to worker w_1, and firm f_1 anticipates that worker w_1 is atypical and makes an offer to worker w_2. However, the coordination failure arises because firm f_2 has no information about worker w_1's type and cannot anticipate firm f_1's behavior. Firm f_2 also makes its offer to worker w_2; however, it eventually ends up unmatched because worker w_2 accepts firm $f_1's$ offer. Thus, the number of matches for some realization of preferences is smaller than the number of matches when the signals are not allowed. Therefore, the expected number of matches is also smaller.

Related literature

A substantial part of the literature on two-sided matching markets focuses on centralized markets that employ the deferred acceptance algorithm proposed by Gale and Shapley (1962). The outcome of this algorithm is a "stable" matching, in which no agent is matched to an unacceptable agent on the other side of the market, and no pair of agents is unmatched if it prefers to be matched. Centralized clearinghouses organized around the deferred acceptance algorithm can deliver thickness to the market, help to deal with the congestion, and make it safe to participate (Roth, 2008b). These desirable properties have allowed some previously failed markets to be successfully reorganized. Roth (2008a) and Roth and Sotomayor

[7] See Theorem 1 for the proof that these strategies constitute a sequential equilibrium.

(1990) present an excellent overview of the main theoretical accomplishments in this area. As an illustration that preference signaling can be useful in centralized matching markets, Abdulkadiroglu et al. (2008) show that the introduction of signaling technology can improve the ex-ante efficiency of the deferred acceptance algorithm in case of weak preferences.

Still, many labor markets are decentralized or at least preceded by decentralized opportunities for participants to match. Therefore, analysis of decentralized matching markets outcomes and the devices that facilitate match formation for them are an important issue. Good examples of papers that study models of decentralized matching markets with non-transferable utility are Pais (2006) and Niederle and Yariv (2009). The former one models decentralized matching markets by means of a sequential game where firms are randomly given the opportunity to make job offers. It shows that every stable match can be reached as the outcome of an equilibrium play of the game. The latter ones study decentralized matching markets under incomplete information. They show that strong assumptions are required for the existence of equilibrium strategies that yield a stable outcome in the presence of uncertainty and frictions.

Shimer (2005) and Konishi and Sapozhnikov (2008) study models of decentralized matching markets with transferable utility. The former paper studies the assignment of heterogeneous workers to heterogeneous jobs in a game with transferable utility. It shows that in an equilibrium when workers use anonymous strategies, i.e. agents' strategies depend on agent's type, not on a particular identity of an agent, a worker's wage is increasing in her job's productivity and a firm's profit is increasing in its employees' productivity. The latter one studies an assignment game of Shapley-Shubik, where each firm makes a take-it-or-leave-it salary offer to one applicant. They show that applicants' (firms') equilibrium salary vectors are bounded above (below) by the minimal competitive salary vector; this suggests that adopting the minimum competitive salary vector as the equilibrium outcome for decentralized markets does not have a strong justification.

Finally, we want to note that this paper does not analyze search for matches. Agents usually need to perform costly search to locate a better partner in decentralized matching markets. Contrary to search literature (see Chade and Smith, 2006; Lee and Schwarz, 2007; Kircher, 2008), we assume that agents perfectly know the payoffs from their matches.

The paper proceeds as follows. Section 2 outlines our general model and introduces some notations. Equilibrium analysis is presented in Section 3. Section 4 analyzes the welfare of agents in the model with and without signals. Section 5 compares these welfare implications with the results in the previous literature and discusses two controversial roles of signals in matching markets. The case of public signals is considered in Section 6. Finally, Section 7 discusses some assumptions of our model and concludes.

2 Model

We consider a two-sided matching model with W workers and F firms, $W \geq F$. The set of workers and the set of firms are denoted as W and F correspondingly. Both W and $\mathcal{F}$ include the empty set. Each worker w orders firms according to some strict preference list θ_w. Similarly, each firm f orders workers according to some preference list θ_f. $\Theta_{\mathcal{W}}$ and $\Theta_{\mathcal{F}}$ together comprise the set of all possible workers' and firms' preference lists.

Each agent a has cardinal utility compatible with her/its preference list θ_a[8]. If worker w with preferences θ_w is matched with firm f, she receives cardinal utility $u_w(f, \theta_w)$. Similarly, if firm f with preferences θ_f is matched with worker w, it receives cardinal utility $u_f(w, \theta_f)$. We assume that agent utility depends only on the rank of an agent with which it is matched. Specifically, the utility of an agent from being matched with an agent on the kth position in her/its preference list equals $u_a(k)$. We assume that agents have the same utility function; i.e. for any agent a, $u_a(k) = u(k)$. Our results do not depend on the last assumption; however, this assumption simplifies the exposition.

Additionally, agent's cardinal utility from being unmatched is normalized to zero. We also assume that there is no worker whom firms do not want to hire, and there is no worker who prefers being unemployed to being matched with some firm; i.e. for any k, $u(k) > 0$.

Each agent knows only her/its preferences and has some ex-ante common beliefs about the other agents' preferences. We consider an environment where each firm f has some fixed publicly known preference list θ_f. Each worker is one of two types: "typical" or "atypical". A "typical" worker w is denoted as $w(T)$. All workers of typical type have the same commonly known preference list θ_0. An "atypical" worker w is denoted as $w(A)$. The preferences of atypical workers are identically and independently distributed according to some distribution $A(\Theta_{\mathcal{W}})$. Each worker is ex-ante typical with probability $1 - \varepsilon$ and atypical with probability ε, for some $\varepsilon \in (0, 1)$. Our main analysis considers the case when ε is small.[9] We also assume that the distribution of atypical preferences, $A(\Theta_{\mathcal{W}})$, has a full support, i.e. each firm can be the top firm of an atypical worker with positive probability[10].

To model the influence of signals on congested markets, we consider a model with three periods:

1. *Agents' preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. Signals are observed only by firms who have received them.*

2. *Each firm makes an offer to at most one worker; offers are made simultaneously*[11].

[8] We employ cardinal utilities compatible with ordinal ranking similar to Bogomolnaia and Moulin (2001).

[9] The exact bound on ε depends on the parameters of distribution $A(\Theta_{\mathcal{W}})$. However, for each distribution $A(\Theta_{\mathcal{W}})$, one could find an upper bound of ε. We provide a more detailed discussion in Section 7.

[10] Formally, for any $f \in \mathcal{F}$ and any $w \in \mathcal{W}$ $\Pr(f = \max_{\theta_w}(f' : f' \in \mathcal{F})) > 0$.

[11] In practice, some firms should rationally make several offers, anticipating that some workers probably

3. *Each worker may accept at most one offer from the set of offers she receives.*

We restrict our analysis to pure strategies.[12] A strategy of worker w is a duple $s_w = (s_w^1, s_w^2)$ that represents her decisions at the first and third stages. A strategy of a worker at the first stage is to choose a firm she sends her signal to, $s_w^1 : \Theta_{\mathcal{W}} \to \mathcal{F}$. A strategy of a worker on the last stage is to choose an offer among those available to her, $s_w^2 : \Theta_{\mathcal{W}} \times 2^F \to \mathcal{F}$, where $2^F = \{h : h \subset \mathcal{F}\}$. A strategy of firm f is its decision at the second stage. Firm f chooses the worker to whom it makes an offer based on a set of signals it receives, $s_f : 2^W \to$W, where $2^W = \{h : h \subset \mathcal{W}\}$. The dependence of firm strategy on preferences is omitted, because we assume that each firm has some fixed preferences.

For a given strategy profile of agents $s = (s_w, s_f)$ and realized agents' types $\theta \in (\Theta_{\mathcal{W}})^W \times (\Theta_{\mathcal{F}})^F$ one can determine the final matching and agents' utilities. We denote the utility of agent a given a strategy profile s and a profile of types θ as $\pi_a(s, \theta)$. The interim expected payoff of worker w with preferences θ_w from strategy s_w when the other agents follow a strategy profile s_{-w} equals

$$u_w(s_w|s_{-w}, \theta_w) = \sum\nolimits_{\theta_{-w}} t(\theta_{-w}) \pi_w((s_w, s_{-w}), (\theta_w, \theta_{-w})),$$

where $t(\theta_{-w})$ denotes the joint distribution of all agents except worker w preferences. The interim expected payoff of firm f given a subset of received signals $h \subset \mathcal{W}$, beliefs $\mu_f(\cdot|h)$, and other agents' strategy profile s_{-f} is

$$u_f(s_f|s_{-f}, h) = \sum\nolimits_{\theta} \mu_f(\theta|h) \pi_f(s_f, s_{-f}, \theta).$$

We employ the concept of sequential equilibrium for multi-stage games with observed actions and incomplete information in order to solve the game (see Fudenberg and Tirole, 1991).

reject their offers. We do not model these strategic decisions.

[12] The analysis of the offer game in which agents can use mixed strategies does not give additional intuition to our main result that signals could impede match formation for some environments. However, this analysis is available upon request.

Definition 1 *A strategy profile* (s_w, s_f) *and posterior beliefs* $\mu_f(\cdot|h)$ *for each firm* f *and each subset of workers* $h \subset \mathcal{W}$ *is a sequential equilibrium if*

- *for any* $w \in \mathcal{W}$, $\theta_w \in \Theta_{\mathcal{W}} : s_w^1(\theta_w) \in \arg\max_{\alpha \in \mathcal{F}} u_w(\alpha|s_{-w}, \theta_w)$,
- *for any* $f \in \mathcal{F}$, $h \subset \mathcal{W} : s_f(h) \in \arg\max_{\beta \in \mathcal{W}} u_f(\beta|s_{-f}, h)$, *and*
- *for any* $w \in \mathcal{W}$, $\theta_w \in \Theta_{\mathcal{W}}$, $h' \subset \mathcal{F} : s_w^2(h', \theta_w) \in \arg\max_{\gamma \in h'} u_w(\gamma, \theta_w)$,

where beliefs are defined using Bayes' rule.[13]

Now we introduce some notations that will be useful in our further discussion. Though worker strategy is a duple $s_w = (s_w^1, s_w^2)$, we will talk mainly about worker strategies at the first stage. The reason is that each worker has a strictly dominant strategy at the last stage−accept the best offer available−since she knows her preferences and the preferences are strict. To simplify notation, we omit the upper index and write $s_w(\theta_w)$ instead of $s_w^1(\theta_w)$.

For convenience, we name firms according to the typical preference list $\theta_0 = (f_1, ..., f_F)$; i.e. f_1 is the best firm, f_2 is the second best, etc. Similarly, we name workers in the following way: worker w_1 is the best worker among all workers $\mathcal{W}$ according to firm f_1's preferences, $w_1 = \max_{\theta_{f_1}}(w|w \in \mathcal{W})$; worker w_2 is the best worker among $\mathcal{W}\backslash \{w_1\}$ according to firm f_2's preferences, $w_2 = \max_{\theta_{f_2}}(w|w \in \mathcal{W}\backslash\{w_1\})$; and so on. Generally, worker $w_i = \max_{\theta_{f_i}}(w|w \in \mathcal{W}\backslash\{w_1, ..., w_{i-1}\})$ if $i \leq F$. The other workers $\mathcal{W}\backslash\{w_1, ..., w_F\}$ are named according to some prespecified order.[14]

We say a subset of workers $h \subset \mathcal{W}$ is *reached for firm* f *when workers follow strategy profile* $s_{\mathcal{W}}$ if ex-ante probability that only workers from set h send their signals to firm f strictly more than zero.

Definition 2 *A subset of workers* $h \subset \mathcal{W}$ *is reached for firm* f *when workers follow strategy profile* $s_{\mathcal{W}}$ *if*

$$\Pr(h_f = h) = \sum\nolimits_{\theta} t(\theta) \prod\nolimits_{w \in h} I_{s_w(\theta_w)=f} \prod\nolimits_{w' \notin h} (1 - I_{s_{w'}(\theta_{w'})=f}) > 0,$$

where $I_{s_w(\theta_w)=f} = \begin{cases} 1 \text{ if } s_w(\theta_w) = f \\ 0 \text{ otherwise} \end{cases}$ *and* $t(\theta)$ *denotes the joint distribution of all agents' preferences.*

We also say that firm f responds to worker w's signal, when workers follow strategy profile $s_{\mathcal{W}}$, if her signal changes the strategy of firm f with positive probability.

[13] Off-equilibrium beliefs are defined by considering the limits of completely mixed strategies.

[14] For instance, if all firms have the same preferences θ^*, workers are named according to this preference list $\theta^* = \{w_1, ..., w_W\}$.

Definition 3 *Firm f responds to worker w's signal, when workers follow strategy profile $s_{\mathcal{W}}$, if there exists a subset of workers h, $w \notin h$, such that both h and $h \cup w$ are reached for firm f, and $s_f(h) \neq s_f(h \cup w)$.*

We proceed with equilibrium analysis in the next section.

3 Equilibrium analysis

As a benchmark, we first consider an environment in which workers cannot send signals. Then, the model outlined above is a static game of incomplete information. Therefore, the notion of sequential equilibrium coincides with the notion of Bayesian equilibrium and agents' beliefs are irrelevant. There is a unique equilibrium match in this case.

If signals are not allowed and ε is small, the only optimal strategy of firm f_1 is to make an offer to its best worker $w_1 = \max_{\theta_{f_1}}(w|w \in \mathcal{W})$. The second top firm anticipates that worker w_1 is likely to accept firm f_1's offer. Hence, the only optimal strategy of firm f_2 is to make an offer to its best worker among $\mathcal{W}\backslash\{w_1\}$, $w_2 = \max_{\theta_{f_2}}(w|w \in \mathcal{W}\backslash\{w_1\})$ and so on. Workers accept the best available offer. Overall, there is the maximum number of matches, F (since $F \leq W$), in the equilibrium when signals are not allowed.

Proposition 1 (No signaling equilibrium) *For sufficiently small ε, there is a unique equilibrium when signals are not allowed: firm f_j, $j = 1, ..., F$, makes an offer to worker w_j; worker $w_i, i = 1, ..., F$, accepts the best available offer.*

We further call the match in our benchmark model as "no signaling" match.

Now, we analyze the set of equilibria in the matching market with signals. Though signals are voluntary in our model, they could still play a negative role and draw away firm offers. In order to eliminate such equilibria, we assume that if firm f makes an offer to worker w when it does not receive her signal, firm f makes an offer to worker w when it receives her signal.[15]

Assumption PRS (Positive Role of Signals). *For any firm $f \in \mathcal{F}$ and any worker $w \in \mathcal{W}$ and any $h \subset \mathcal{W}$, $w \notin h$, if $s_f(h) = w$ then $s_f(h \cup w) = w$.*

We further distinguish three types of equilibria in the matching model with signals.

[15] See Example A1 in Appendix for an example of an equilibrium in which Assumption PRS is violated.

Definition 4

- *An equilibrium is "babbling" if no firm responds to any signal.*
- *An equilibrium is "informative", if at least one firm responds to some worker's signal.*
- *An equilibrium is "very informative", if each firm responds to all signals from workers better or equal to its no signaling match.*

The set of the first and second type equilibria, i.e. babbling and informative, exhaust the set of all possible equilibria in our model. The set of equilibria of the last type is a subset of the set of informative equilibria.

A babbling equilibrium always exists in our model because signals are costless. If firms do not respond to signals, signals play no role in equilibria. Hence, the only possible match in a babbling equilibrium is no signaling match.

Proposition 2 *For sufficiently small ε, the only possible match in a babbling equilibrium is no signaling match.*

If some firms respond to signals, then signals transmit information about workers' preferences in an equilibrium, which changes the overall matching outcome. However, there is a great multiplicity of informative equilibria. One may suggest to use refinements proposed by (Cho and Kreps, 1987) and (Banks and Sobel, 1987).[16] However, these criteria are very powerful in the case of one sender and one receiver. The situation with many senders and receivers is more difficult. Though these criteria significantly reduce the number of equilibria, they do not guarantee uniqueness.

However, it is sufficient to restrict ourselves to the case in which each firm responds to all signals from workers better or equal to its no signaling match, i.e. very informative equilibria, in order to guarantee uniqueness. This equilibrium consists of the following strategies. Worker w_i sends her signal to the best firm among the firms that prefer worker w_i to their no signaling match $\Delta(w_i) = (f_j \in \mathcal{F} : w_i \succeq_{f_j} w_j)$. If firm f_j receives at least one signal from the set of workers $\Delta(f_j) = (w \in \mathcal{W} : w \succeq_{f_j} w_j)$, i.e. workers better or equal to worker w_j, it makes its offer to the best such worker; otherwise, it makes an offer to its best worker among $\mathcal{W} \backslash \{w_1, ..., w_j\}$.

[16] Cho and Kreps (1987) analyze *never a weak responce, intuitive criterion, D1,* and *D2* refinements. Banks and Sobel (1987) analyze *divinity* and *universal divinity* refinements.

Theorem 1 *For a sufficiently small ε, under Assumption PRS the set of strategies,*

- $s_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}}(f \in \Delta(w_i))$,
- $s_{f_j}(h) = \begin{cases} \max_{\theta_{f_j}}(w : w \in h) & \text{if } h \cap \Delta(f_j) \neq \varnothing \\ \max_{\theta_{f_j}}(w : w \in \mathcal{W}\backslash\{w_1, ..., w_j\}) & \text{if } h \cap \Delta(f_j) = \varnothing \end{cases}$,

and the set of firms' beliefs consistent with agents' strategies constitute a unique very informative equilibrium.[17]

The above theorem is remarkable because it shows that the equilibrium of the model is unique, if we restrict our attention to the case in which firms use signals most extensively. However, we should point out that we do not intend to eliminate all other equilibria. First, the theorem illustrates typical agents' behavior in an informative equilibrium. Workers do not just send signals to the best firms. They send their signals to the best firms that respond to these signals, which is in line with AEA advice to participants in the job market for new Ph.D. economists (see AEA, 2005). Similarly, firms do not respond to all signals. Instead they respond to the signals from workers better than those they could secure in the no signaling equilibrium. Second, our results of welfare comparison do hold for most other sequential equilibria.

4 Welfare properties of equilibria

We evaluate the effect of signals on the matching market from an ex-ante perspective. We mainly use the following quantitative characteristics: the expected number of matches, the expected total welfare of firms, and the expected total welfare of workers.

Let us denote the ex-post number of matches for the profile of preferences $\theta \in \Theta_{\mathcal{W}} \times \Theta_{\mathcal{F}}$, when agents follow the profile of strategies s as $m(s, \theta)$. Then, the expected number of matches can be expressed as

$$E[M(s)] = \textstyle\sum_{\theta} t(\theta) m(s(\theta), \theta),$$

where $t(\theta)$ denotes the joint distribution of all agents' preferences. Similarly, the expected total welfare of workers and firms can be expressed as

$$\begin{aligned} E[W_{\text{firm}}(s)] &= \textstyle\sum_{f}\sum_{\theta} t(\theta)\pi_f(s(\theta), \theta), \text{ and} \\ E[W_{\text{worker}}(s)] &= \textstyle\sum_{w}\sum_{\theta} t(\theta)\pi_w(s(\theta), \theta) \end{aligned}$$

[17] We should point out that there is a multiplicity of beliefs that could support this equilibrium on off-equilibrium path.

correspondingly.

Proposition 1 shows that the expected number of matches in any no signaling equilibrium is the maximum one. Hence, it is impossible that the expected number of matches in any informative equilibrium exceeds the expected number of matches in any "no signaling" equilibrium. Example 1 and Example 2 demonstrate the case of strict inequality and equality for this welfare criterion.

Example 1 is presented in the introduction and considers the very informative equilibrium with three firms and three workers. To avoid a repetition, we do not discuss it here.

Example 1 (Fewer expected number of matches) *There are three firms and three workers .Firms have the same ranking over workers* (w_1, w_2, w_3). *The typical worker preference list is* $\theta_0 = (f_1, f_2, f_3)$. *Atypical worker preferences are uniformly distributed. Firm* f_j, $j = 1, 2, 3$, *and worker* w_i, $i = 1, 2, 3$, *equilibrium strategies are*

- $s_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}}(f \in \Delta(w_i))$,
- $s_{f_j}(h) = \begin{cases} \max_{\theta_{f_j}}(w : w \in h) & \text{if } h \cap \Delta(f_j) \neq \varnothing \\ \max_{\theta_{f_j}}(w : w \in \mathcal{W}\backslash\{w_1, ..., w_j\}) & \text{if } h \cap \Delta(f_j) = \varnothing \end{cases}$,

and the set of firms' beliefs consistent with agents' strategies.

Example 2 shows that some informative equilibria could have the maximum expected number of matches. Intuitively, it is possible that if some firm f_j secures a better match with some atypical worker w_i, firm f_i always makes its offer to firm f_j's no signaling match, worker w_j, in an equilibrium. Therefore, firms exchange their matches and it does not decrease the number of matches.

Example 2 (Equal expected number of matches) *Let us consider three firms and three workers. All firms have the same preferences* $\theta_{f_j} = \{w_1, w_2, w_3\}$. *Let us consider the following equilibrium strategies:*

- $s_{w_1}(\theta_{w_1}) = \max_{\theta_{w_1}}(f : f \in \{f_1, f_2\})$ *and* $s_{w_i}(\theta_{w_i}) = f_i$, $i = 2, 3$;
- $s_{f_j}(h) = \begin{cases} \max_{\theta_{f_j}}(w : w \in h) & \text{if } h \cap \Delta(f_j) \neq \varnothing \\ \max_{\theta_{f_j}}(w : w \in \mathcal{W}\backslash\{w_1, ..., w_j\}) & \text{otherwise} \end{cases}$, *for* $j = 1, 2$;
- $s_{f_3}(h) = \begin{cases} \max_{\theta_{f_3}}(w : w \in h) & \text{if } h \cap \Delta(f_3) \neq \varnothing \\ w_3 & \text{otherwise} \end{cases}$.

The set of equilibrium beliefs is such that if firm f_1 or f_2 receives a signal from worker w_1, it believes that it is worker w_1's top firm. If firm f_3 receives a signal from worker w_1, its belief coincides with her prior, i.e. worker w_1 is typical with probability $1-\varepsilon$ and atypical with probability ε. Similarly, if any firm f_j receives a signal from worker w_2 or w_3, its belief coincides with its prior. To put it briefly, only firm f_1 and firm f_2 respond to worker w_1's signal. All other signals are ignored. One may check that the described strategies constitute an informative equilibrium.

The results about the expected total welfare of firms and the expected total welfare of workers are not so straightforward and depend on the relative magnitudes of $u(k)$. The intuition is that signals in an informative equilibrium play two roles. On the one hand, signals help to secure "better" matches between some atypical workers and firms. On the other hand, the introduction of signals leaves some workers and firms unmatched. Example 3 illustrates that the introduction of signals is beneficial for a matching market according to egalitarian welfare criterion if and only if the decrease in the number of matches is offset by better matches of atypical workers. A similar example can show that the total welfare of firms changes ambiguously.

Example 3 (Welfare of firms and workers) *Let us again consider the game of Example 1. Workers' cardinal utilities from being matched to first, second, and third choice are $\delta+\lambda, \delta$, and $\delta-\lambda$ correspondingly. The expected total welfare of workers in no signaling match*

$$E[W_{worker}^{nosignals}] = \sum_{i=1}^{3} \left[(1-\varepsilon)\, u(i) + \varepsilon \tfrac{1}{3} \sum_{l=1}^{3} u(l) \right] = 3\delta.$$

One may check that the expected total welfare of workers in very informative equilibrium is[18]

$$E[W_{worker}^{signals}] = 3\delta + \left(-\tfrac{1}{3}\delta + \tfrac{19}{6}\lambda\right)\varepsilon$$

Hence, the expected total welfare of workers increases, if and only if the difference in utilities between adjacent firms is large enough, $\lambda > \frac{2}{19}\delta$.

The theorem below summarizes the results derived above.

Theorem 2 *For a sufficiently small ε :*

- *the expected number of matches in any informative equilibrium is weakly fewer than in any no signaling equilibrium;*
- *the effect of signals on the expected total welfare of firms and the expected total welfare of workers is ambiguous.*

[18] Terms of the order of ε^2 and ε^3 are ignored.

We have compared above the properties of any informative and no signaling equilibrium. However, more strict result holds for very informative equilibrium. Under the assumption that there are at least three workers and there exists a worker w, such that there are at least three firms that weakly prefer worker w to their no signaling matches, $|\{f_j \in \mathcal{F} : w \succeq_{f_j} w_j\}| \geq 3$, the expected total number of matches is strictly fewer in very informative equilibrium than in the corresponding no signaling match. The intuition for this result is similar to the one for Example 1. If firms respond to signals, some of the realized matches differ from no signaling match. Moreover, if at least three firms respond to some worker's signal the exchange of matches−the situation presented in Example 2−is impossible for each realization of preferences. Therefore, the expected number of matches is smaller than the maximum one in this case.

Theorem 3 *For sufficiently small ε, if there are at least three workers and for some worker w, $|\Delta(w)| \geq 3$, the expected number of matches is strictly smaller in the very informative equilibrium than in the corresponding no signaling equilibrium.*

Theorem 2 proves that the expected total welfare of workers changes ambiguously with the introduction of signals. However, the following proposition shows that signals are harmful to workers only because they deprive them of matches. Workers receive weakly better offers conditional on the fact that they receive any offer.

Proposition 3 *If a worker receives an offer in any informative equilibrium, this offer is from a firm weakly better than her no signaling match.*

It is easy to see that the above statement is not true for firms, because some firms may have to make offers to workers worse than their no signaling match if she is atypical.

5 Role of signals in matching markets

Coles et al. (2009) show that the introduction of signals increases the expected number of matches and the welfare of workers. However, they consider an environment where agents' preferences are block-uniform. Specifically, there exists a partition $\mathcal{F}_1, \ldots, \mathcal{F}_B$ of the firms into *blocks* and

1. For any $b < b'$, where $b, b' \in \{1, \ldots, B\}$, each worker prefers every firm in block $\mathcal{F}_b$ to any firm in block $\mathcal{F}_{b'}$;

2. Each worker's preferences within block $\mathcal{F}_b$ are uniform and uncorrelated, for all b;

3. Firm preferences over workers are uniform and uncorrelated.

This paper shows that Coles et al. (2009) results rely on the assumption that preferences are block-uniform. If the preferences of workers are almost aligned and the preferences of firms are fixed and commonly known, the introduction of signals decreases the expected number of matches. The effect of signals on the expected total welfare of agents is ambiguous. Overall, Table I presents the effects from the introduction of the signals for the two different environments: almost complete (this paper) and block-uniform distribution of preferences.

Preferences	No signals	$Matches$	$E[W_{\text{worker}}]$	$E[W_{\text{firm}}]$
Almost complete	0	$-$	$\pm$	$\pm$
Block-uniform	0	$+$	$+$	$\pm$

Table I. Almost complete VS Block-uniform distribution preferences.

A natural question is why signals influence matching markets in different ways. We argue that the signals play two different roles: transmit information and facilitate information asymmetry. On the one hand, the introduction of signals helps atypical workers to transmit information about their preferences and locate a better match. On the other hand, signals transmit information only to some firms, thus facilitating information asymmetry. This information asymmetry leads to coordination failures that decrease the number of matches.

When there is ex-ante small amount of information about agents' preferences, information transmission plays a more important role in match formation. This happens when agents' preferences are ex-ante block-uniform, as in Coles et al. (2009). However, when there is almost complete information about agents' preferences—as in the model of this paper—the introduction of signals leads to coordination failures. Table II presents the comparison.

Preferences	Transmit information	Facilitate information asymmetry
Almost complete	Small	**Large**
Block-uniform	**Large**	Small

Table II. The roles of signals

Overall, the signals play controversial roles in the match formation process. This could make them a less powerful tool than it was previously anticipated.

6 Public signals

One could conjecture that should signals be public, they would always benefit match formation. Public signals introduce no asymmetry of information among firms. Firms have the same beliefs about the distribution of workers' preferences and the same beliefs about the strategies other firms use. Therefore, firms should be able to make offers that are more likely to be accepted. Unfortunately, this intuition is incorrect. This section illustrates that the expected number of matches in an equilibrium of the offer game with public signals could be smaller than the expected number of matches in the offer game without signals.

We consider a market with three firms and three workers. The distribution of agents' preferences is the same as in Section 2. Each worker can send at most one signal and accept at most one offer. Each firm has only one vacant position and can make at most one offer. The timing of the game is as follows:

1. Agents' preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. All agents observe what signals each firm receives.

2. Each firm makes an offer to at most one worker; offers are made simultaneously.

3. Each worker chooses an offer to accept from the set of offers she receives.

The only difference from the game we considered previously is that all agents observe the signals each firm receives. The strategies of workers are the same as in Section 2. However, a strategy of firm f now depends on the set of signals each firm receives, $s_f : \mathcal{F}^W \to \mathcal{W}$.[19]

As previously, the only equilibrium outcome of the offer game with signals is a full match. However, the expected number of matches could be smaller than three if we allow workers to send public signals. Intuitively, public signals do not transmit enough information about workers' preferences. This could introduce a considerable amount of uncertainty about workers' preferences. Therefore, some firms can optimally engage in a competitive behavior for some workers; i.e. firms make their offers to the same worker in an equilibrium. This produces mismatches.

Example 4 *There are three firms and three workers. Firms have the same ranking over workers, which we denote as* (w_1, w_2, w_3). *The typical worker preference list is* (f_1, f_2, f_3). *The atypical worker preferences are uniformly distributed. We assume that all firms have the same cardinal utility and their utility from being matched to second top worker, i.e.* $u(2)$, *is at least twice as great as the cardinal utility from being matched to the third top worker, i.e.* $u(3)$.

[19] Note that we again omit the dependence of strategies on firms' preferences, as we assume that each firm has fixed commonly known preferences.

We consider the following strategies of agents in the offer game with public signals. Worker w_i sends her signal to the best firm among the firms that weakly prefer worker w_i to their no signaling match $\Delta(w_i) = (f_j \in \mathcal{F} : w_i \succeq_{f_j} w_j)$:

$$s_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}}(f \in \Delta(w_i)).$$

Firms use the following strategies.

1. *Firm f_1 makes an offer to worker w_1, if it receives a signal from her; otherwise, it makes an offer to worker w_2.*

2. *Firm f_2 makes an offer to worker w_1, if it receives a signal from her. Firm f_2 makes an offer to worker w_3, if either worker w_1 sends a signal to firm f_1 and worker w_2 sends a signal to firm f_3 or worker w_1 sends a signal to firm f_3 and worker w_2 sends a signal to firm f_2. In all other cases, firm f_2 makes an offer to worker w_2.*

3. *Firm f_3 makes an offer to the best worker from whom it receives a signal. If it receives no signals, it makes an offer to worker w_3.*

Each firm's beliefs on the equilibrium path are consistent with agents' strategies and each firm off-equilibrium beliefs coincide with priors.

Let us consider the strategies outlined in Example 4. Mismatches happen when both worker w_1 and worker w_2 are atypical. If at least two atypical workers send their signals to the same firm, only one worker receives an offer from it. Since, signals are public, all other firms infer that the other worker is atypical. This creates a considerable amount of uncertainty about the worker preferences. Since, this worker could be a good one, firms have incentives to compete for her.

Another reason for excessive competition among firms is that signals may not transmit information about workers' top firms. Some workers send their signals to firms that differ from their top ones in an equilibrium, because they want to attract feasible offers. Therefore, several firms could have incentives to make an offer to a given worker. This creates competition among firms, which again lead to mismatches.

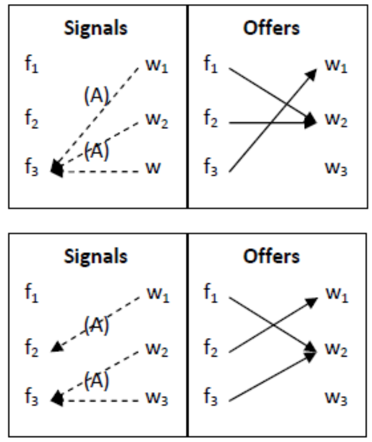

Figure II. Public signaling.

Proposition 4 formally proves that the set of strategies in Example 4 constitutes a sequential equilibrium.

Proposition 4 *The set of strategies in Example 4 constitutes a sequential equilibrium.*

The implications of the above example can be summarized by way of two observations. First, public signals do not transmit enough information about workers' preferences. This could introduce uncertainty about workers preferences and induce excessive firm competition for the same workers. This results in mismatches.

In addition, mismatches in the offer game with public signals occur only if there at least two atypical workers, which happens only with probability of the order ε^2. In contrast, mismatches in the offer game with private signals occur with the probability of the order ε. Therefore, mismatches happen less often when signals are public.

7 Conclusion

There is a general belief that preference signaling should facilitate match formation (see Crawford and Sobel, 1982; Roth, 2008b; AEA, 2005). This belief is also supported by Coles et al. (2009) who show that the introduction of signals increases the expected number of matches and welfare of workers. We show in this paper that this belief can be erroneous for some matching markets. We exemplify an environment in which the introduction of signals harms matching markets: it decreases the expected number of matches and ambiguously affects the welfare of firms and the welfare of workers. Based on this example, we argue that signals play controversial roles in match formation. Though they help to transmit information about participants' preferences, they also facilitate information asymmetry among them. While the former effect reduces coordination failures and facilitates better match formation, the latter effect acts in the opposite direction. Finally, we show that making signals observable to all agents does not change the welfare applications. We leave here as an open empirical question which effect dominates in real matching markets.

In conclusion, we want to discuss some assumptions of our model. We analyze the introduction of signals to congested decentralized matching markets, as we believe the job market for new Ph.D. economists to be. The fact that we do not analyze any centralized clearinghouse mechanism and stable matches captures the idea of decentralized markets. The fact that we analyze a one-period model captures the idea of congestion. Moreover, several (but finite) periods of interactions between firms and workers would give an opportunity for firms to secure better matches; but signals would still introduce information asymmetry. If each worker sent several signals, these would transmit information to a greater number of firms, but each signal would be less informative. Several vacant positions would make only the preferences of firms more complicated and would not influence the results. Overall, the roles of signals in match formation are robust to these modifications.

The results of this paper are in terms of a sufficiently small ε. However, what we really need is the uniqueness of equilibrium in the benchmark model without signals. The multiplicity of equilibria does not allow a clean comparison between models with and without signals. One could check that there is a unique equilibrium in no signaling model with uniform distribution of atypical preferences if $\varepsilon < \min(\min_i(\frac{u(i)-u(i+1)}{u(i)}), \frac{u(F)}{u(F)+0.5u(1)})$.

We should also highlight that we rely on cardinal utility in our analysis. Our results could not be extended to an ordinal framework, because an Ordinal Bayesian Nash Equilibrium (see d'Aspremont and Peleg, 1988) may not exist in our environment.

A Appendix

Proof of Proposition 1.

Let us show this statement for each firm sequentially. Each worker w_i has preferences $\theta_0 = (f_1, ..., f_F)$ with probability $1-\varepsilon$ and with complementary probability some preferences distributed according to $A(\Theta_{\mathcal{W}})$. Let us consider firm f_1 which has some preferences θ_{f_1}. If it makes an offer to worker $w_1 = max_{\theta_{f_1}}(w|w \in \mathcal{W})$, its offer will be the best worker w_1's offer with probability at least $1-\varepsilon$. Hence, its expected utility from making an offer to worker w_1 equals at least $(1-\varepsilon)u(1)$ which is greater than $u(2)$ for sufficiently small ε. Hence, independently on other firms' strategies, firm f_1's optimal strategy is to make an offer to its best worker.

Let us assume that each firm f_k, $k < j$, makes its offer to worker w_k. Now we consider the decision of firm f_j. The expected payoff from making an offer to some worker among $\{w_1, ...w_{j-1}\}$ is less than $\varepsilon u(1)$. In the same time the expected payoff from making offer to some worker among $\mathcal{W} \backslash \{w_1, ...w_{j-1}\}$ is at least $(1-\varepsilon)u(j)$. Hence, given the strategies of other firms and sufficiently small ε, the optimal strategy of firm f_j is to make an offer to its best worker among $\mathcal{W} \backslash \{w_1, ...w_{j-1}\}$.$\square$

Proof of Proposition 2.

The only undominated strategy of a worker at the last stage is to choose the best offer among available ones. Then, under the condition that firm f does not respond to any signal, for any $h \subset \mathcal{W}$ reached in an equilibrium $s_f(h) = const$. Let us assume that there exists a realization of agents' preferences such that firm f_1 is matched to some worker w_i, $i > 1$, in the equilibrium. Hence, for any $h \subset \mathcal{W}$, reached in the equilibrium, $s_{f_1}(h) = w_i$. Hence, the expected firm 1's payoff equals at most $u(2)$. However, the strategy $s_{f_1}(h) = w_1$ for any $h \subset \mathcal{W}$ is compatible with assumption that firm f_1 does not respond to any signals and gives payoff $(1-\varepsilon)\, u(1)$ independently of strategies of other firms. Hence, $s_{f_1}(h) = w_i$ cannot be an equilibrium strategy. Similar argument could be applied to any other firm f_j, $j = 2, ..., F$.$\square$

Proof of Theorem 1.

We prove the theorem by way of several lemmata. In the proof of the lemmata we presume that ε is sufficiently small. First, we establish that a firm believes about a particular worker is typical with probability more than $1-\varepsilon$ either when it receives her signal or when it does not receive her signal. Second, we show that firms do not make their offers to a worker better than no signaling match if they do not receive her signal. The third lemma proves that if a firm does receive a signal from a worker better than its no signaling match, it makes its offer to the best such worker. Finally, using the statements of lemmata we show that the set of strategies stated in the theorem constitutes a unique very informative equilibrium.

First two lemmata do not require the assumption that each firm f_j, $j = 1, ..., F$, responds to all signals from workers better or equal to worker w_j according to its preferences.

Lemma A1 *For any worker* $w \in \mathcal{W}$, *any firm* $f \in \mathcal{F}$, *and any* $h \subset \mathcal{W}$ *either* $\mu_f(\theta_w = \theta_0 | h \cup w) \geq 1 - \varepsilon$ *or* $\mu_f(\theta_w = \theta_0 | h \backslash w) \geq 1 - \varepsilon$. *Similarly, either* $\mu_f(\theta_w \neq \theta_0 | h \cup w) \leq \varepsilon$ *or* $\mu_f(\theta_w \neq \theta_0 | h \backslash w) \leq \varepsilon$.

Proof.

Let us denote as α_T and α_A the probabilities that typical and atypical type of worker w correspondingly send a signal to firm f. Then, if worker w sends her signal to firm f, $(1-\varepsilon)\alpha_T + \varepsilon\alpha_A > 0$, we derive its beliefs using Bayes' rule

$$\begin{cases} \mu_f(\theta_w = \theta_0 | h \cup w) = \frac{(1-\varepsilon)\alpha_T}{(1-\varepsilon)\alpha_T + \varepsilon\alpha_A} \\ \mu_f(\theta_w = \theta_0 | h \backslash w) = \frac{(1-\varepsilon)(1-\alpha_T)}{(1-\varepsilon)(1-\alpha_T) + \varepsilon(1-\alpha_A)} \end{cases}$$

One can verify that

$$\begin{cases} \mu_f(\theta_w = \theta_0 | h \cup w) \geq 1 - \varepsilon \quad \Leftrightarrow \quad \alpha_T \geq \alpha_A \\ \mu_f(\theta_w = \theta_0 | h \backslash w) \geq 1 - \varepsilon \quad \Leftrightarrow \quad \alpha_T \leq \alpha_A \end{cases}$$

Hence, either $\mu_f(\theta_w = \theta_0 | h \cup w) \geq 1 - \varepsilon$ or $\mu_f(\theta_w = \theta_0 | h \backslash w) \geq 1 - \varepsilon$. If worker w never sends her signal to firm f, $(1-\varepsilon)\alpha_T + \varepsilon\alpha_A = 0$, firm f's beliefs are $\mu_f(\theta_w = \theta_0 | h \backslash w) = 1 - \varepsilon$ and $\mu_f(\theta_w = \theta_0 | h \cup w)$ is arbitrary. The second statement directly follows from the first one. □

Lemma A2 (Offer to better workers) *If firm* f_j *does not receive a signal from worker* w *strictly better than worker* w_j, $w \succ_{f_j} w_j$ *it does not make an offer to her in an equilibrium.*

Proof.

We prove this statement for firms sequentially. Let us first show its validity for $j = 2$. The only worker that could be better than worker w_2 for firm f_2 is worker w_1 by construction. If $w_2 \succ_{f_2} w_1$ we are done. Assume that $w_1 \succ_{f_2} w_2$.

There are two possibilities: either worker $w_1(T)$ sends her signal to firm f_1, i.e. $s_{w_1}(\theta_0) = f_1$, or she does not send her signal to firm f_1, i.e. $s_{w_1}(\theta_0) \neq f_1$, in an equilibrium.

Assume worker w_1 employs strategy $s_{w_1}(\theta_0) = f_1$. If firm f_2 does not receive worker w_1 signal, firm f_2 believes she is atypical with probability less than ε, $\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \backslash w_1) \leq \varepsilon$ (Lemma A1). According to assumption $F \leq W$, firm f_2 can secure a match with some worker w_i, $i \geq 2$, with probability at least $1 - \varepsilon$. Hence, firm f_2 does not make an offer to worker w_1 in an equilibrium.

Worker w_1 employs strategy $s_{w_1}(\theta_0) \neq f_1$ in an equilibrium only if firm f_1 makes its offer to worker w_1 with probability equals to one, and firm f_2 has a chance to be matched with worker w_1 only if she is atypical. Assume firm f_2 makes an offer to worker w_1 when it does not receive her signal. If $w_1(T)$ sends her signal to firm f_2 in an equilibrium, according to Assumption PRS firm f_2 should also make an offer if it receives a signal from w_1. However, if it receives a signal from w_1, the probability that worker w_1 is atypical less than ε (Lemma A1), which contradicts equilibrium behavior.

Now, we assume that worker $w_1(T)$ does not send her signal to firm f_2 in an equilibrium. If firm f_2 does not receive worker w_1's signal, firm f_2 believes that she is atypical with probability less or equal ε, $\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \backslash w_1) \leq \varepsilon$ (Lemma A1). Therefore, it is again suboptimal for firm f_2 to make an offer to worker w_1 if it does not receive a signal from her.

We have shown above that it is suboptimal for firm f_2 to make an offer to worker w_1 if it does not receive a signal from her. Let us assume that it is suboptimal for any firm f_j, $j < k$ to make its offer to a worker w_t, $t < j$, if firm f_j does not receive a signal from it and show that the claim for firm f_k.

We consider some worker w_i, $i < k$. Firm f_i makes its offer to workers $\{w_1, ..., w_{i-1}\}$ with probability less than $\varepsilon\,(i-1)$. In addition, worker w_i is atypical with probability ε. Hence, firm f_k can secure a match with worker w_i with probability equals at most $i\varepsilon$ if it does not receive a signal from her. For small enough ε firm f_k's offer to worker w_i is suboptimal. $\square$

Now, we assume that each firm f_j, $j = 1, ..., F$, responds to all signals from workers better or equal to worker w_j according to its preferences. The following lemma shows that firm f_j makes its offer to some worker w better or equal to worker w_j if worker w's signal is the best signal firm f_j receives.

Lemma A3 (Response to signals) *Assume that $F > W$. Then, for any $h \subset \mathcal{W}$ $s_{f_j}(h) = max_{\theta_{f_j}}(w : w \in h)$ if $h \cap \Delta(f_j) \neq \varnothing$ in very informative equilibrium*[20].

Proof.

We prove this statement for firms sequentially. Let us consider firm f_1 and worker w_1. Assume that worker w_1 employs strategy $s_{w_1}(\theta_0) \neq f_1$. Then, firm f_1 believes that for any $h \subset \mathcal{W}$ $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \backslash w_1) \geq 1 - \varepsilon$. Therefore, for sufficiently small ε, firm f_1 always makes its offer to worker w_1, which contradicts to our assumption that it responds to worker w_1's signal. Therefore, under the assumption that firm f_1 responds to a signal from worker w_1, the only possible worker w_1's equilibrium strategy is $s_{w_1}(\theta_0) = f_1$. In this case, for any $h \subset \mathcal{W}$ firm f_1's belief is $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \cup w_1) \geq 1 - \varepsilon$. Hence, firm f_1's highest expected payoff when it receives worker w_1's signal is from making an offer to worker w_1. Hence, for any $h \subset \mathcal{W}$, firm f_1's strategy $s_{f_1}(h \cup w_1) = w_1$ is optimal.

Assume now that for any $t \leq j < k$, and for any $h \subset \mathcal{W}$, firm f_j employs strategy for $s_{f_j}(h) = max_{\theta_{f_j}}(w : w \in h)$ if $h \cap \Delta(f_j) \neq \varnothing$. We prove below that firm f_k's optimal strategy for any $h \subset \mathcal{W}$ and $s_{f_k}(h) = max(w : w \in h)$ if $h \cap \Delta(f_k) \neq \varnothing$.

There are two possibilities: either $s_{w_k}(\theta_0) \neq f_k$ or $s_{w_k}(\theta_0) = f_k$. For the former case, for any $h \subset \mathcal{W}$ $\mu_{f_k}(\theta_{w_k} = \theta_0 | h \backslash w_k) \geq 1 - \varepsilon$. Hence, it is optimal for firm f_k to make an offer to worker w_k when it receives no signals from workers better or equal to worker w_k, i.e. for any $h' \subset \mathcal{W}$ such that $h' \cap \Delta(f_k) = \varnothing$, $s_{f_k}(h') = w_k$. Hence, it is also optimal for firm f_k to make an offer to worker w_k when worker w_k's signal is the best signal it receives, i.e. for any $h'' \subset \mathcal{W}$ such that such that $h'' \cap \Delta(f_k) = w_k$, $s_{f_k}(h'') = w_k$. Therefore, firm f_k does not respond to worker w_k's signal. Contradiction.

For the latter case, $s_{w_k}(\theta_0) = f_k$, if firm f_k does not receive a signal from worker w_k, it anticipates that she is atypical. Therefore, firm f_k does not make its offer to her. If firm f_j receives signals from any worker $w_i \succeq w_k$ no other firm f_p, $p \neq j$ and $p > i$, makes its offer to worker w_i according to Lemma A2. The only offers that compete with firm f_j's offer could be the ones from the set $\{f_p, p < i\}$. However, any firm f_p, $p < i$, could make an offer to worker w_i only if worker w_p is atypical, which happens with probability ε. Hence, the interim expected payoff for firm f_j from making its offer to worker w_i equals at least $(1 - (i-1)\varepsilon)u'$, where $u' = u_{f_j}(w_i, \theta_{f_j})$. Firm f_j expected payoff from making an offer to any other worker from set $\Delta(f_j)$ is smaller than $(1 - (i-1)\varepsilon)u'$ as this worker either has not sent a signal to firm f_j or has a smaller rank in firm f_j' preferences. The expected payoff from making an offer to some worker $\mathcal{W} \backslash \Delta(f_j)$ is smaller either. Therefore, firm f_j optimal strategy is, $s_{f_j}(h) = max_{\theta_{f_j}}(w : w \in h)$ if $h \cap \Delta(f_j) \neq \varnothing$. □

[20] If $F = W$ the claim is still valid with the same assumption for all firms except firm f_F. Firm f_F should respond to a signal from any worker strictly better than the corresponding one.

Now we are ready to prove the theorem. Let us show that the set of strategies, stated in the theorem, constitutes an equilibrium. We first prove that if all agents, except firm f_l, follow the strategies, stated in the theorem, firm f_l's strategy is optimal given its belief is consistent with the other agents' strategies. If firm f_l receives a signal from worker w_t, $t < l$, firm f_l believes that itself is the best firm among $\Delta(w_t) = \{f_j \in \mathcal{F} : w_t \succeq_{f_j} w_j\}$. Let us assume that worker w_t is the best worker who sends a signal to firm f_l. Worker w_t does not accept firm f_l's offer only if she receives an offer from some firm $f_k \in \mathcal{W} \backslash \Delta(w_t)$. However, it happens only if worker w_k is atypical, i.e. with probability less than ε. Hence, firm f_l interim expected payoff from making an offer to worker w_t equals at least $(1 - (n-1)\,\varepsilon)\,u'$, where $u' = u_{f_l}(w_t, \theta_{f_l})$. Firm f_l's offer to a worker better than worker w_t is not optimal according to Lemma A1. Firm f_l's expected payoff from making an offer to some worker w, $w_t \succ_{f_l} w$, is also smaller than making an offer to worker w_t for sufficiently small ε. Overall, firm f_l's strategy is optimal.

Let us show that, if all agents, except worker w_t, follow the strategies, stated in the theorem, worker w_t's strategy is optimal. Firm f_t does not make an offer to worker w_t when it receives a signal from a better worker. Therefore, if worker w_t is typical, her payoff from sending a signal to firm f_t equals at least $[1 - (l-1)\varepsilon]\,u(t)$. If worker w_t does not send her signal to firm f_t it loses her offer and she could get payoff at most $u(t-1)$, which is less than $[1 - (l-1)\varepsilon]\,u(t)$ for sufficiently small ε. There is also no reason for worker w_t to send her signal to a firm better than firm f_t, because this firm does not respond to her signal according to its equilibrium strategies. Hence, worker $w_t(T)$'s strategy is optimal. Using similar logic one can show that worker $w_t(A)$'s strategy is also optimal.

Now we show that the above strategies constitute the unique very informative equilibrium. Lemmata A2 and A3 imply that each firm f_l, $l = 1, ..., F$, has to follow the following strategies in an equilibrium:

$$\text{for any } h \subset \mathcal{W}, \begin{cases} s_{f_l}(h) \neq w_l \text{ if } h \cap \Delta(f_l) = \varnothing \\ s_{f_l}(h) = w_l \text{ if } h \cap \Delta(f_l) = w_l \end{cases}$$

Straightforwardly, the only worker $w_l(T)$'s optimal strategy is to send her signals to firm f_l, $s_{w_l}(\theta_0) = f_l$, otherwise, firm f_l's does not make an offer to student w_l.

Let us consider firm $f^* = \max_{\theta_{w_l}}(f' \in \Delta(w_l))$. Firm f^* responds to signals from workers better or equal than no signaling match and its equilibrium beliefs are $\mu_{f^*}(\theta_{w_l} = \theta_0 | h \backslash w_l) \geq 1 - \varepsilon$ and $\mu_{f^*}(\theta_{w_l} \neq \theta_0 | h \cup w_l) = 1$. Therefore, if firm f^* does not receive a signal better than worker w_l's one, it's optimal strategy is to make an offer to worker w_l. Taking into account that firm f^* can receive a signal from a better worker with probability less than $(l-1)\varepsilon$, worker $w_l(A)$'s optimal strategy is to send her signal to firm f^* (for sufficiently small ε). Hence, the strategies, stated in the theorem, constitute the unique equilibrium. □

Proof of Theorem 3.

Assumption that $A(\Theta_{\mathcal{W}})$ has a full support and that the strategies of the very informative equilibrium guarantee that some worker w_i sends her signals to each firm in the set $\Delta(w_i) = |\{f_j \in \mathcal{F} : w_i \succeq_{f_j} w_j\}|$ with positive probability. Then, using logic of Example 1 and Example 2 it is straightforward to show that when there are at least three firms in the set $\Delta(w_i)$ and there are at least three workers the mismatch happens with positive probability. Therefore, the expected number of matches strictly smaller than in the corresponding no signaling equilibrium. □

Proof of Proposition 3. The statement directly follows from the strategies of the very informative equilibrium. □

Example A1 (An equilibrium when assumption PRS does not hold) *Let us consider two firms and two workers. We assume that all firms have the same preferences over workers* $\theta_{f_1} = \theta_{f_2} = \{w_1, w_2\}$. *Also we assume that each typical worker has preferences* $\theta_0 = (f_1, f_2)$ *and each atypical worker has preferences* $\theta_A = (f_2, f_1)$ *with probability equal to one. Firms prefer worker* w_1 *to worker* w_2. *Agents employ the following strategies:*

- $s_{w_1}(\theta_0) = f_2,\ s_{w_1}(\theta_A) = f_1$
- $s_{w_2}(\theta_0) = f_1,\ s_{w_2}(\theta_A) = f_2$
- *for any* $h \subset \mathcal{W}$ $s_{f_1}(h) = \begin{cases} w_1 \text{ if } w_1 \notin h \\ w_2 \text{ if } w_1 \in h \end{cases}$, $s_{f_2}(h) = \begin{cases} w_1 \text{ if } w_1 \notin h \\ w_2 \text{ if } w_1 \in h \end{cases}$

Agents' believes are:

- *for any* $h \subset \mathcal{W}$ $\mu_{f_j}(\theta_{w_i} : f_j = \max_{\theta_{w_i}}(f \in \mathcal{F})|h \backslash w_i) = 1$ *and* $\mu_{f_j}(\theta_{w_i} : f_j = \min_{\theta_{w_i}}(f \in \mathcal{F})|h \cup w_i) = 1$

It is easy to show that the above strategies and the set of beliefs constitute a sequential equilibrium. One may extend this example for the environment with more firms and workers.

Proof of Proposition 4.

Let us first prove that firms' strategies are optimal. Note that if firm f receives a signal from worker w_1 it believes that it is her top firm. Therefore, it is optimal for her to make her an offer. Now, if firm f_1 that does not receive a signal from worker w_1, firm f_1 believes that worker w_1 is atypical and will not accept its offer. Then, firm f_1 strategy of making an offer to worker w_2 is optimal for any signaling pattern, because it believes that her offer will be accepted at least with probability $\frac{1}{2}$. The worst case is when worker w_2 is atypical and sends her signal to firm f_3.

Now we consider firm f_2 optimal strategy. Let us consider the case when worker w_1 sends her signal to firm f_1 and worker w_2 sends signal to firm f_3. Firm f_2 believes worker w_2 prefers firm f_3 to itself. Since, firm f_3 makes an offer to worker w_2, and firm f_1 makes an offer to worker w_1, the only worker that could accepts firm f_2 offer is worker w_3. If worker w_1 sends he signal to firm f_3 and worker w_2 sends her signal to firm f_2. In this case firm f_1 makes an offer to worker w_2, who is typical with probability $(1 - \frac{1}{3}\varepsilon)$. Since, worker w_1 most preferred firm is firm f_3, the optimal strategy of firm f_2 to make an offer to worker w_3.

Let us consider the case firm f_3 receives signals from all workers. In this case firm f_3 makes an offer to worker w_1. It is optimal for firm f_1 and firm f_2 to make an offer to worker w_2 because her preferences over these firms could be equally likely. Hence, the payoff from making an offer to worker w_2 equal $\frac{1}{2}u(2) > u(3)$. Similar, one could show that in other cases it is optimal for firm f_2 to make an offer to worker w_2.In a similar way one could show that it is always optimal for firm f_3 to make an offer to the best worker it receives a signal from.

Let us now show each worker uses optimal strategy. Worker w_1 strategy is optimal, because any firm makes her an offer upon receiving her signal. There is no incentive for worker w_2 to make an offer to firm f_1 since, all firms upon observing such behavior has believes about workers preferences that coincides with the priors. Therefore, worker w_2 optimal strategy is to send her signal to the best firms among f_1 and f_2.

Since firms do not put attention to worker w_3 signals, there is no reason for her to deviate from the equilibrium strategy. $\square$

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