Numerical Simulation of the Overlapping Generations Models with Indeterminacy

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Abstract

In this paper we explore the computation and simulation of stochastic overlapping generation (OLG) models. To do so we compute all Markovian equilibria adopting a recently developed numerical algorithm. Among the models we studied, the indeterminacy in deterministic OLG model results in many different equilibrium paths corresponding to the initial condition that all asymptotically converge to the same steady state. The uncertainty introduces indeterminacy with infinite dimension due to the existence of numerous selections of transition and policy functions from the equilibrium set. Each selection correspondences a sequential competitive equilibrium that may present excessive volatile movements in asset price. It is possible to construct a continuum of recursive equilibrium. However our numerical simulations suggest that it is problematic to look at recursive equilibrium in which the volatility of asset price is solely determined by the distribution of the shock.

KEYWORDS: Stochastic OLG, Indeterminacy, Computation, Simulation.

1 Introduction

In this paper we provide a global analysis to diagnose the existence of indeterminate equilibria in overlapping generations models (OLG) and discuss the computation and numerical simulation for these economies.

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It is well known that an overlapping generations economy may have a continuum of equilibria. There are well-established results that provide sufficient conditions for the uniqueness of the equilibrium in these economies. Gale (1973) has demonstrated that gross substitution in consumption precludes indeterminacy in OLG models with one good in each period and a single two-period-loved consumer in each generation. This result has been extended to a multi-commodity economy where a single two-period-lived consumer is characterized by log-linear preference [Balasko and Shell (1981)] and intertemporally separable preferences [Geanakoplos and Polemarchakis (1984), Kehoe and Levine (1984)]. Kehoe, Levine, Mas-Colell, and Woodford (1991) further extend these results to a multi-commodity, multi-agent, non-monetary, pure-exchange economy and show that gross substitutability of excess demand ensures the determinacy of perfect-foresight equilibria.

Unlike the deterministic OLG model, less is known about indeterminacy in the stochastic OLG model. The conditions for precluding the existence of indeterminacy in this economy are not available. There are several studies that provide examples of the existence of a continuum of stationary recursive equilibria in stochastic OLG economy [c.f. Farmer and Woodford (1984), Spear, Srivastava, and Woodford (1990)]. However our study suggests that it will be problemtic if we only look at stationary recursive equilibria. The volatility of aggregate variables will be solely determined by the distribution of the exogenous shock in this equilibrium. While we find competitive equilibrium sequences that present excessive movements in aggregate variables, which depends on the selection of transition and policy functions from the equilibrium set.

Indeterminacy is not a unique phenomenon in OLG model. It has been introduced to study the propagation mechanism of business cycle. In monetary models, indeterminacy has been used to explain the monetary transmission. In growth theory, there are models that use indeterminacy to understand growth paths difference among economies with identical fundamentals [c.f. Benhanbib and Farmer (1998) for a comprehensive survey]. The existence of indeterminate equilibria poses challenge for the economists who are interested in conducting comparative statics analysis. Researchers try to carefully choose their models to avoid the indeterminacy. However, the condition for uniqueness in the model of the type discussed in this paper is not guaranteed based on the empirical evidents provided in Mankiw, Rotemberg and Summers (1985).

Previous studies have been largely confined to the local analysis that linearizes the model around the steady state. However, what is true of the linearized system only applies for an open neighborhood of the steady state. In practice, we do not know the size of this open neighborhood as pointed out by Kehoe and Levine (1990). Furthermore, we can not say anything outside of this neighborhood based on this linearized system. Gomis-Porqueras and Haro (2003) present some techniques to characterize all manifolds of a given dynamical system in OLG setting. Their method is only applicable to the deterministic model.

In this paper, we provide a global analysis by computing all Markovian equilibria using the numerical procedure set forth in Feng et al. (2009). Therefore, our approach provides a straightforward way to diagnose the indeterminacy by simply checking the numerically obtained set of equilibrium. Our analysis departures from the economy considered by Kehoe and Levine (1990), in which we know that the economy has indeterminate equilibria around one steady state. We construct a stochastic variation of the above model in which agents have uncertain endowments and the stochastic process determining the endowments is Markovian. In both cases, we verify the existence of indeterminacy by computing all equilibria set.

It has been hypothesized that a continuum of equilibria will all converge to the same steady state asymptotically in deterministic OLG model with indeterminate equilibrium [see Spear, Srivastava, and Woodford (1990), Wang (1993)]. To our best knowledge, this has not been verified due to the lack of a robust algorithm to compute the equilibrium set of these models. We propose a new method to simulate these economies by computing all Markov equilibrium. We then study whether or not and how the existence of indeterminacy may pose impacts on the long run behavior of the economy.

Numerical simulations indicate that in deterministic OLG model we considered, all equilibrium paths asymptotically converge to the same steady state. However their choices of initial conditions will translate into different equilibrium paths before they converge to the long-run equilibrium. The introduction of uncertainty produces at least two type of indeterminacies. First, there exists numerous selections of transition and policy functions from the equilibrium set. Consequently, the simulated economies that correspond to different selection may present different simulated moments. This property holds even though they start with identical initial conditions. Similar to the deterministic counterpart, the initial conditions won't pose any effect on the long-run behavior of the economy if we fix the selection of the transition functions. The selection of the transition functions may have important welfare implication. As in one example we can improve the aggregate social welfare by 1.3% through altering the transition function. Second, if we allow the agent to switch between these transition and policy functions along the simulation, it will support a continuum of equilibrium paths starting with the same initial conditions.

We proceed as follows. In section 2, we extend the economy considered by Kehoe and Levine (1990) by introducing uncertainty in the flow of endowments. Section 3 present a numerical method to identify indeterminacy in these economies. We also consider two examples to illustrate the application. In section 4, we explain how to compute stochastic OLG models with indeterminate equilibria. Sections 5 is devoted to the discussion of indeterminacy and long run behavior of the economy. Some further comments and extensions follow in our final section.

2 Model

The economy is conformed by a sequence of overlapping generations that live for three periods. The primitive characteristics of the economy are defined by a stationary Markov chain. Time and uncertainty are represented by a countably infinite tree $\mathbf{\Sigma}$. Each node of the tree, $\sigma \in \mathbf{\Sigma}$, is a finite history of shock $\sigma = s^t = (s_0, s_1, ..., s_t)$ for a given initial shock s_0. The process of shocks (s_t) is assumed to be a Markov chain with finite support $\mathbf{S}$.

At every time period $t = 0, 1, 2, \cdots$, a new generation is born. Each generation is made up of a representative agent, who is present in the economy for three periods. The individuals is identified by the date event of her birth, $\sigma = (s^t)$, and the age of an individual is $a = 0, 1, 2$. For an agent born at node s^t, she consumes and has endowments at all nodes $s^{t+a} \geq s^t$, $a = 0, 1, 2$. An agent's individual endowments are a function of the shock and her age alone, i.e. for all $a = 0, 1, 2$, $e^{s^t}(s^{t+a}) = \mathbf{e}^a(s_{t+a})$ for some function $\mathbf{e}^a : \mathbf{S} \to \mathbf{R}_+$.

At each node s^t, there exist spot markets for the consumption good. The financial market is incomplete and there is only one bond in zero net supply[1]. The bond pays one unit consumption good at next period regardless of the state of the world. Prices of the bond is $q(s^t) \in \mathbf{R}$ and agent a's bond holding is $\theta^a(s^t) \in \mathbf{R}$. At $t = 0$, the initial conditions of the economy are determined by the bond holding of the initially alive agents of ages $a = 1, 2$.

For simplicity, we assume that every utility function U is separable over consumption of different dates. For an agent born at s^{t-a}, let $c^a(s^t)$ and $\theta^a(s^t)$ denote her consumption and portfolios. Then the intertemporal objective U is defined as

$$U(c) = \sum\nolimits_{a=0}^{2} \beta^a \sum\nolimits_{s^{t+a} \geq \sigma} \pi(s^{t+a}|\sigma) u(c^a(s^{t+a})) \tag{1}$$

The one-period utility u satisfies the following conditions:

Assumption 2.1 *For each $s \in \mathbf{S}$ the one-period utility functions $u(\cdot, s) : \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$ are increasing, strictly concave, and continuous. These functions are also continuously differentiable at every interior point $c > 0$.*

Definition 1 *A sequential competitive equilibrium is given by a collection of prices and choices of individuals $\{q(s^t), (c^a(s^t), \theta^a(s^t))_{a=0,1,2}\}_{s^t}$ such that:*

[1] We certainly can consider the complete financial market with $\mathbf{S}$ Arrow-securities. However we only include one asset in the numerical examples for the sake of computational cost.

(i) Financial market clearing:

$$\sum_{a=0}^{2} \theta^a(s^t) = 0 \tag{2}$$

(ii) For each s^t, individual $\sigma = s^t$ maximizes utility:

$$\begin{aligned} (c^a(s^t), \theta^a(s^t)) &\in \arg\max U(c), \quad s.t. && (3)\\ c^0(s^t) + q(s^t)\cdot\theta^0(s^t) &\leq e^0(s^t) && (4)\\ c^1(s^{t+1}) + q(s^{t+1})\cdot\theta^1(s^{t+1}) &\leq e^1(s^{t+1}) + \theta^0(s^t) && (5)\\ c^2(s^{t+2}) &\leq e^2(s^{t+2}) + \theta^1(s^{t+1}) && (6) \end{aligned}$$

The existence of a SCE can be established by standard methods [e.g., Balasko and Shell (1980), and Schmachtenberg (1988)]. Moreover, by similar arguments used by these authors it is easy to show that every sequence of equilibrium asset prices $(q(s^t))_{t\geq 0}$ is bounded.

2.1 Indeterminacy in OLG economy

It is well known that an overlapping generations economy may have a continuum of equilibria. Unfortunately, the existing studies have been largely confined to the local analysis that linearizes the model around the steady state. More specifically, economists linearize the equilibrium conditions around a steady state and then solve the linearized version of the model.

To better illustrate the point, we abstract from the uncertainty and work on the deterministic case of the above economy. It is convenient to build market-clearing into the first order conditions and the steady state can be characterized by the solutions of the following equations.

$$u_c(e^0 - q_t\theta_t) - \beta u_c(e^1 + \theta_t - q_{t+1}\theta_{t+1}) = 0 \tag{7}$$

$$u_c(e^1 + \theta_{t-1} - q_t\theta_t) - \beta u_c(e^2 + \theta_t) = 0 \tag{8}$$

$$q_t = q_{t+1} = q^* \tag{9}$$

$$\theta_{t-1} = \theta_t = \theta_{t+1} = \theta^* \tag{10}$$

Evidently, the system always has a solution with $q^* = 1$, corresponding to the golden rule monetary steady state. However, it is more interesting to check how many other solutions, refereed as real steady states in Kehoe and Levine (1990), exist in the economy. Following

Kubler and Schmedders (2009), we isolate the variable q and characterize all real steady states as the positive real solution ($q^* \neq 1$) to the following equation,

$$f(q) = 0. \tag{11}$$

It is convenient to rewrite the equilibrium conditions (7) and (8) as

$$F(\theta_{t-1}, \theta_t, \theta_{t+1}) = 0 \tag{12}$$

Then we linearize the system as

$$D_1 F \theta_{t-1} + D_2 F \theta_t + D_3 F \theta_{t+1} = 0 \tag{13}$$

Here $D_i F$ is evaluated at the steady state (q^*, θ^*). Rewriting this linearized equilibrium condition as a first-order difference equation, we obtain:

$$\begin{bmatrix} \theta_{t+1} \\ \theta_t \end{bmatrix} = \begin{bmatrix} -D_1 F_1 \cdot D_2 F_1^{-1} & 0 \\ 0 & -D_1 F_2 \cdot D_2 F_2^{-1} \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta_{t-1} \end{bmatrix} \tag{14}$$

Indeterminacy of the linearized system manifests itself as too many stable eigenvalues of the matrix in (14). The advantage of this approach is that indeterminacy of the linearized system is easy to diagnose. However, what is true of the linearized system only applies for an open neighborhood of the steady state. In practice, we do not know the size of this open neighborhood as pointed out by Kehoe and Levine (1990). Furthermore, we can not say anything outside of the open neighborhood around the steady state based on this linearized system for both cases. When we consider the uncertainty, then the problem gets more profound as shown by the examples we studied below. One task of this paper is to provide a general approach to identify the existence of indeterminacy for OLG models. As we describe our approach in the following section, we will also explain a global analysis for the impact of indeterminacy on the long-run behavior of the economy.

3 Diagnose the indeterminacy

Feng et al. (2009) develop a method to approximate the set of all Markovian equilibria for dynamic equilibrium models. Their study suggests that one can identify the existence of continuous Markov equilibrium by computing the boundary of the set of all Markov equilibria. As the numerical example in section (6.3) of their paper shows that the distance between the upper boundary and the lower one will go to zero whenever the equilibrium is unique.

This result has important implication for diagnosing the indeterminacy in OLG models.

In this section, we first briefly explain the main results in their paper. We then detail the procedure of computing the smallest possible convex-valued correspondence that contains the set of all Markov equilibria. Baed on this approximation, we derive two propositions to identify the indeterminacy.

3.1 Markov equilibrium correspondence

In our model economy, the equilibrium equations consist of first order conditions, budget constraints, and market-clearing conditions.

$$c^0(s) = e^0(s) - q(s)\theta^0(s) \tag{15}$$
$$c^1(s) = e^1(s) + \theta^0(s_-) - q(s)\theta^1(s) \tag{16}$$
$$c^2(s) = e^2(s) + \theta^1(s_-) \tag{17}$$
$$q(s)u_c(c^0(s)) = \beta\Sigma\pi(s_+|s)u_c(c^1(s_+)|s) \tag{18}$$
$$q(s)u_c(c^1(s)) = \beta\Sigma\pi(s_+|s)u_c(c^2(s_+)|s) \tag{19}$$
$$\theta^0(s_-) + \theta^1(s_-) = 0 \tag{20}$$
$$\theta^0(s) + \theta^1(s) = 0 \tag{21}$$

It is useful to build market-clearing into the endogenous choice of security θ^0, θ^1. Then the natural state space $\boldsymbol{\Theta}$ consists of beginning-of-period bond-holding of the middle aged θ. Let m denote a vector of consumption of the middle-aged[2]

$$m = c^1. \tag{22}$$

We define the Markov equilibrium correspondence $\mathbf{V}^* : \boldsymbol{\Theta} \times \mathbf{S} \to \mathbf{R}^S$ as

$$\mathbf{V}^*(\theta_0, s_0) = \{m(s_0) : (q, (c^a, \theta^a)_{a=0,1,2}) \text{ is a SCE}\}. \tag{23}$$

From the above results on the existence of SCE for OLG economies, we obtain

Proposition 1 *Correspondence* $\mathbf{V}^*$ *is nonempty, compact-valued, and upper semi-continuous.*

From this correspondence $\mathbf{V}^*$, we can generate recursively the set of equilibria since $\mathbf{V}^*$ is the fixed point of an operator $\mathbf{B} : \mathbf{V} \to \mathbf{B}(\mathbf{V})$ that links state variables to future equilibrium

[2]In their original paper, m is defined as the vector of shadow values of the marginal return to investment for all assets: $m = qu'(c^1)$. Here we define m as c for the convenience purpose of computation only. It turns out to be equivalent in all examples we considered in this paper.

states. This operator embodies all equilibrium conditions. More precisely, let $\mathbf{B}(\mathbf{V})(\theta, s)$ be the set of all values m with the following property: for any given (θ, s) there exists $\theta_+ = g^m(\theta, s, m)$ and $m_+(\theta_+, s_+) \in \mathbf{V}(\theta_+, s_+)$ with $s_+ \in \mathbf{R}^S$ such that

$$q(s) \cdot u_c(c^0(s)) = \beta \Sigma \pi(s_+|s) \cdot u_c\left(m_+(\theta_+, s_+)\right) \tag{24}$$

and market clearing conditions.[3] The following result is proved in Feng et. al. (2009).

Theorem 1 *(convergence) Let $\mathbf{V}_0$ be a compact-valued correspondence such that $\mathbf{V}_0 \supset \mathbf{V}^*$. Let $\mathbf{V}_n = \mathbf{B}\left(\mathbf{V}_{n-1}\right), n \geq 1$. Then, $\mathbf{V}_n \to \mathbf{V}^*$ as $n \to \infty$. Moreover, $\mathbf{V}^*$ is the largest fixed point of the operator $\mathbf{B}$, i.e., if $\mathbf{V} = \mathbf{B}(\mathbf{V})$, then $\mathbf{V} \subset \mathbf{V}^*$.*

3.2 Outer approximation of $\mathbf{V}^*$

To simplify exposition, we abstract from uncertainty in this section. There is only one asset θ available in the economy and the price is q. The flow of endowment is pre-determined as $\{\mathbf{e}^a(s_{t+a})\}_{a=0}^2 = \{e^0, e^1, e^2\}$.

The numerical implementation of the operator $\mathbf{B}$ is consisted of two parts. In the first step, we construct an operator $\mathbf{B}^{\mathbf{h},\mu}$ and obtain the smallest possible convex-valued correspondence $\tilde{\mathbf{V}}^*$ containing the equilibrium set $\mathbf{V}^*$. Then we fully discretize $\tilde{\mathbf{V}}^*$ and obtain an outer-approximation of $\mathbf{V}^*$ as detailed in the next section.

We start with an initial correspondence $\tilde{\mathbf{V}}_0 \supseteq \mathbf{V}^*$. We pick arbitrary small positive number h, μ, and we partition the state space $\mathbf{\Theta}$ into N closed intervals $\mathbf{\Theta}^i$ of uniform length h such that $\cup_i \mathbf{\Theta}^i = \mathbf{\Theta}$ and $int(\mathbf{\Theta}^{i_1}) \cap int(\mathbf{\Theta}^{i_2}) = \emptyset$ for every pair $\mathbf{\Theta}^{i_1}$, $\mathbf{\Theta}^{i_2}$, where $i_1, i_2 \in \{1, 2, ..., N\}$. We use N right rectangular parallelepiped $\tilde{\mathbf{V}}_0^i := \mathbf{\Theta}^i \times \left[\inf_{\theta \in \mathbf{\Theta}^\mathbf{i}} \tilde{\mathbf{V}}_\mathbf{0}(\theta), \sup_{\theta \in \mathbf{\Theta}^\mathbf{i}} \tilde{\mathbf{V}}_\mathbf{0}(\theta)\right]$ to approximate $\tilde{\mathbf{V}}_0 = \cup_i \tilde{\mathbf{V}}_0^i$. It turns out to be convenient to characterize $\tilde{\mathbf{V}}_0^i$ by two functions $\mathbf{m}_0^{\sup}(\theta) = \sup_{\theta \in \mathbf{\Theta}^i} \tilde{\mathbf{V}}_\mathbf{0}(\theta)$, $\mathbf{m}_0^{\inf}(\theta) = \inf_{\theta \in \mathbf{\Theta}^\mathbf{i}} \tilde{\mathbf{V}}_\mathbf{0}(\theta)$. Then $\tilde{\mathbf{V}}_0^i$ is defined as $\tilde{\mathbf{V}}_0^i := \left\{m(\theta) | \theta \in \mathbf{\Theta}^i, m \in \left[\mathbf{m}_0^{\inf}(\theta), \mathbf{m}_0^{\sup}(\theta)\right]\right\}$.

Consider then any element $\mathbf{\Theta}^i$ of the state space partition and the corresponding $\tilde{\mathbf{V}}_0^j$. Given $\theta \in \mathbf{\Theta}^i$, we test whether there exists $m \in \left[\mathbf{m}_0^{\inf}(\theta), \mathbf{m}_0^{\inf}(\theta) + \mu\right]$ such that the one period temporary equilibrium conditions can be satisfied for some arbitrary small constant $\epsilon > 0$. If the answer is yes, then we set $\mathbf{m}_1^{\inf}(\theta)$= $\mathbf{m}_0^{\inf}(\theta)$, otherwise, we set $\mathbf{m}_1^{\inf}(\theta)$= $\mathbf{m}_0^{\inf}(\theta)+$

[3]For any given (θ, s) and $m \in \mathbf{V}$, θ_+ is determined as the solution to the following equations

$$m = e^1(s) + \theta - q(s)\theta_+(s) \tag{25}$$
$$q(s) \cdot u_c(m) = \beta \Sigma \pi(s_+|s) \cdot u_c(e^2(s_+) + \theta_+(s)|s). \tag{26}$$

μ. A symmetric operation is performed for the case $\theta \in \mathbf{\Theta}^i, m \in [\mathbf{m}_0^{\sup}(\theta) - \mu, \mathbf{m}_0^{\sup}(\theta)]$. The details are below.

1. At each given $\mathbf{\Theta}^i$, we set $\mathbf{m}_1^{\inf}(\theta) \equiv \mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\inf}(\theta)\right) = \mathbf{m}_0^{\inf}(\theta)$ if either

$$\min_{\substack{m \in \left[\mathbf{m}_0^{\inf}(\theta), \mathbf{m}_0^{\inf}(\theta)+\mu\right] \\ \theta \in \mathbf{\Theta}^i, \quad m_+(\theta_+) \in \tilde{\mathbf{V}}_{\mathbf{0}}(\theta_+(m))}} \left\| q \cdot u_c(e^0 - q\theta_+(m)) - \beta \cdot u_c(m_+(\theta_+)) \right\| \leq \epsilon \tag{27}$$

   or $\mathbf{m}_0^{\inf}(\theta) + \mu > \mathbf{m}_0^{\sup}(\theta)$. If any of these two conditions does not hold, then we set $\mathbf{m}_1^{\inf}(\theta) \equiv \mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\inf}(\theta)\right) = \mathbf{m}_{\mathbf{0}}^{\inf}(\theta) + \mu$. A symmetric procedure can be used to define $\mathbf{m}_1^{\sup}(\theta) \equiv \mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\sup}(\theta)\right)$.

   Notice, given θ and m, we can determine the values for q, θ_+ by solving the following functions.

$$m - (e^1 - \theta + q\theta_+) = 0 \tag{28}$$

$$q \cdot u_c(m) - \beta \cdot u_c(e^2 - \theta_+) = 0 \tag{29}$$

   In case of no solution exists, the operator $\mathbf{B}^{\mathbf{h},\mu}$ skips the above procedure and set $\mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\inf}(\theta)\right) = \mathbf{m}_{\mathbf{0}}^{\inf}(\theta) + \mu$ when $m \in \left[\mathbf{m}_0^{\inf}(\theta), \mathbf{m}_0^{\inf}(\theta) + \mu\right]$, $\mathbf{B}^{\mathbf{h},\mu}\left(\mathbf{m}_0^{\sup}(\theta)\right) = \mathbf{m}_{\mathbf{0}}^{\sup}(\theta) - \mu$ when $m \in [\mathbf{m}_0^{\sup}(\theta) - \mu, \mathbf{m}_0^{\sup}(\theta)]$.

2. Repeat step 1 until the sequence of $\mathbf{m}_n^{\inf}(\theta), \mathbf{m}_n^{\sup}(\theta)$ have converged to their limit $\mathbf{m}^{\inf^*}(\theta), \mathbf{m}^{\sup^*}(\theta)$.

Note that $\mathbf{B}^{h,\mu}$ is monotone decreasing by construction, and generates a convergent sequence of convex-valued correspondences containing the equilibrium correspondence $\mathbf{V}^*$. At the limit of the procedure, we obtain the smallest possible convex hull $\tilde{\mathbf{V}}^*_{\theta \in \mathbf{\Theta}^i}(\theta)$ that contains $\mathbf{V}^*_{\theta \in \mathbf{\Theta}^i}(\theta)$, where $\tilde{\mathbf{V}}^*_{\theta \in \mathbf{\Theta}^i}(\theta) := \left\{ m(\theta) | \theta \in \mathbf{\Theta}^i, m \in \left[\mathbf{m}^{\inf^*}(\theta), \mathbf{m}^{\sup^*}(\theta)\right] \right\}$. Finally we have $\tilde{\mathbf{V}}^* = \cup_i \tilde{\mathbf{V}}^*_{\theta \in \mathbf{\Theta}^i}(\theta)$.

3.3 Sufficient condition for determinacy in OLG economy

A straightforward application of Theorem 1 is that we cannot rule out the possibility of indeterminacy if the above procedure converges to a convex-valued set that the distance between the upper boundary and the lower one is greater than zero.

Proposition 2 *(indeterminacy) If there is indeterminacy in the model economy, then we can find $\delta \geq \mu$ such that $\max_{(\theta,s) \in \mathbf{\Theta} \times \mathbf{S}} \left\{ \mathbf{m}^{\sup^*}(\theta, s) - \mathbf{m}^{\inf^*}(\theta, s) \right\} > \delta$ as $h \to 0, \mu \to 0$.*

Proposition 3 *(determinacy) There is no indeterminacy in the model economy if for any arbitrary* $\delta > 0$ *we have* $\max_{(\theta,s)\in\mathbf{\Theta}\times\mathbf{S}} \left\{\mathbf{m}^{\sup^*}(\theta,s) - \mathbf{m}^{\inf^*}(\theta,s)\right\} \leq \delta$ *as* $h \to 0, \mu \to 0$.

Proof: In the limit, we have $\max_{(\theta,s)\in\mathbf{\Theta}\times\mathbf{S}} \left\{\mathbf{m}^{\sup^*}(\theta,s) - \mathbf{m}^{\inf^*}(\theta,s)\right\} = 0$. *This immediately implies that any perturbation will lead the economy off from the equilibrium path.*

Proposition 2 says that we can not rule out the possibility of existence of indeterminacy if there exists $\delta \geq \mu$, such that $\max_{\theta\in\mathbf{\Theta}} \left\|\mathbf{m}^{\sup^*}(\theta) - \mathbf{m}^{\inf^*}(\theta)\right\| > \eta$. Similarly, the application of proposition 3 is that there is no indeterminacy in the model if $\max_{\theta\in\mathbf{\Theta}} \left\|\mathbf{m}^{\sup^*}(\theta) - \mathbf{m}^{\inf^*}(\theta)\right\| \leq \mu$, for any $\mu > 0$.

3.4 Numerical specifications

We apply the above algorithm to two examples, and illustrate the application of the sufficient condition for determinacy.

3.4.1 Example 1

We consider the parametrization employed in Kehoe and Levine (1990). More specifically, the preference is given by $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$. We choose $\beta = 0.5$, $\gamma = 4$, and $\{\mathbf{e}^a(s_{t+a})\}_{a=0}^2 = \{e^0, e^1, e^2\} = \{3, 12, 1\}$. There is only one short-lived bond θ available and the price is given by q_t.

As Kehoe and Levine (1990) pointed out this economy has three real steady states with prices and bond-holdings given by $q_1^* = 0.176$, $\theta_1^* = 5.772$, $q_2^* = 0.793$, $\theta_2^* = 3.732$, and $q_3^* = 44.634$, $\theta_3^* = 0.183$. They show that close to the middle steady state there must be a continuum of equilibria.

In our example, we start from a big set $\tilde{\mathbf{V}}_0(\theta) = \{c^1_{\min} \leq m \leq c^1_{\max}\}$. We apply $\mathbf{B}^{h,\mu}$ to $\tilde{\mathbf{V}}_0(\theta)$ and end up with the area in left panel of Figure 1. The right panel represents the corresponding mapping from today's bond holding of middle age b_t to tomorrow's holding b_{t+1}.

[Figure 1 here.]

As in this example, the maximum distance between the boundaries of the limit of the sequence $\tilde{\mathbf{V}}_{n+1}$ is given by a constant $d \gg \mu$. Proposition 2 suggests that we can not rule out the possibility of indeterminacy in this model economy.

3.4.2 Example 2

When the distribution of endowment is given by $\{\mathbf{e}^a(s_{t+a})\}_{a=0}^2 = \{e^0, e^1, e^2\} = \{3.5, 6, 1.5\}$, one can verify that there is only one real steady state. We apply the above algorithm at different value of μ. The algorithm always converges to the case that $\max_{\theta\in\mathbf{\Theta}}\left\|\mathbf{m}^{\sup^*}(\theta)-\mathbf{m}^{\inf^*}(\theta)\right\|\leq\mu$. The proposition 3 implies that there is no indeterminacy in the model economy.

[Figure 2 here.]

4 Computing all Markov equilibria $\mathbf{V}^*$

Numerical simulations have been used to study the long-run behavior of the economy. It will be useful if we can simulate the economy with indeterminate equilibria so that we can derive some quantitative implications of indeterminacy. To our knowledge, this has never been done in the literature, largely due to the lack of a robust algorithm to compute the set of all equilibria.

The convex-valued correspondence obtained in the previous section delivers important message from which we can diagnose the existence of indeterminacy. However, this correspondence has limited application for us to conduct simulation. This is because it may contain non-equilibrium points. It will be misguiding if we start the simulation from those non-equilibrium points. Therefore it is necessary to compute the exact equilibrium set, rather than the convex hull that contains the set. In this section, we describe $\mathbf{B}^{h,\mu,N}$, a fully discritized version of operator $\mathbf{B}^{h,\mu}$ from which we obtain an approximation of the set of all Markov equilibria. Then we construct a stochastic OLG model with indeterminacy and apply the operator $\mathbf{B}^{h,\mu,N}$.

4.1 Discretization of operator $\mathbf{B}^{h,\mu}$

We use $\tilde{\mathbf{V}}^*$, the fixed point of the operator $\mathbf{B}^{h,\mu}$, as initial condition for an operator on discrete correspondences defined as follows:

The vector of possible values for bond-holding and shocks are given by $\hat{\mathbf{\Theta}}=\left\{\theta_0^{i_1}\right\}_{i_1=1}^{N_\theta}$, $\hat{\mathbf{S}}=\left\{s_0^{i_2}\right\}_{i_2=1}^{N_s}$, and for each pair of the bond-holding and shock grids, $\left(\theta_0^{i_1},s_0^{i_2}\right)$, we also define a finite vector of possible values for $\hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^{i_1},s_0^{i_2}\right)=\left\{m_0^{i_1,i_2,j}\right\}_{j=1}^{Nv}$.[4] Notice, $\lim_{N_\theta\to\infty}\hat{\mathbf{\Theta}}=\mathbf{\Theta}$, $\lim_{Nv\to\infty}\hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^{i_1},s_0^{i_2}\right)=\tilde{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^{i_1},s_0^{i_2}\right)$. Finally, we construct the discrete version of operator $\mathbf{B}^{h,\mu,N}$ by eliminating points that cannot be continued (in the Euler equation, for a predetermined tolerance $\epsilon>0$) as follows:

1. Given $\left(\theta_0^{i_1},s_0^{i_2}\right)$, pick a point $m_0^{i_1,i_2,j}$ in the vector $\hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^{i_1},s_0^{i_2}\right)$. From $m_0^{i_1,i_2,j}$ we can

[4]Notice the portfolio of the household has $\mathbf{S}$ components in stochastic case. In case of two shocks, $\theta_0=(\theta_{0,1},\theta_{0,2})$.

determine the values of $\left(\theta_+^{i_1,i_2,j}, q^{i_1,i_2,j}\right)$ by solving for

$$\begin{aligned} m_0^{i_1,i_2,j} - \left(e^1(s_0^{i_2}) + \theta_0^{i_1} - q^{i_1,i_2,j}\theta_+^{i_1,i_2,j}\right) &= 0. & (30) \\ q^{i_1,i_2,j} \cdot u_c\left(m_0^{i_1,i_2,j}\right) - \beta \sum_{s_+} \pi(s_+|s_0) u_c\left(e^2(s_+) + \theta_+^{i_1,i_2,j}\right) &= 0 & (31) \end{aligned}$$

Thus, if for all $m_+ \in \hat{\mathbf{V}}_0^{\mu,\varepsilon}(\theta_+^{i_1,i_2,j}, s_+) = \left\{m_+^l(\theta_+^{i_1,i_2,j}, s_+)\right\}_{l=1}^{N_V}$ we have

$$\min_{m_+ \in \left\{m_+^l\right\}_{l=1}^{N_V}} \left\| q^{i_1,i_2,j} \cdot u_c\left(e^0(s_0^{i_2}) - q^{i_1,i_2,j}\theta_+^{i_1,i_2,j}\right) - \beta \sum \pi(s_+|s_0^{i_2}) u_c\left(m_+\right)\right\| > \epsilon \quad (32)$$

then $\hat{\mathbf{V}}_1^{\mu,\varepsilon}\left(\theta_0^{i_1}, s_0^{i_2}\right) = \hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^{i_1}, s_0^{i_2}\right) - m_0^{i_1,i_2,j}$.

2. Iterate over all possible values $m_0^{i_1,i_2,j} \in \hat{\mathbf{V}}_0^{\mu,\varepsilon}\left(\theta_0^{i_1}, s_0^{i_2}\right)$, and all possible $\left(\theta_0^{i_1}, s_0^{i_2}\right) \in \hat{\boldsymbol{\Theta}} \times \hat{\mathbf{S}}$.

3. Iterate until convergence is achieved $\sup \left\|\hat{\mathbf{V}}_n^{\mu,\varepsilon} - \hat{\mathbf{V}}_{n-1}^{\mu,\varepsilon}\right\| = 0$.

At the limit of the above algorithm, we have $\lim_{n\to\infty} \hat{\mathbf{V}}_n^{\mu,\varepsilon} = \hat{\mathbf{V}}^{\mu,\varepsilon^*}$.

4.2 ϵ-Equilibrium

It is important to notice that the sequence of $\mathbf{V}_{n+1}^{h,\mu,N} := \mathbf{B}^{h,\mu,N}(\mathbf{V}_n^{h,\mu,N})$, $\mathbf{V}_0^{h,\mu,N} = \tilde{\mathbf{V}}^*$, asymptotically converges to $\mathbf{V}^*$ as stated in the following theorem provided by Feng et. al (2009).

Theorem 2 *For given h, μ, N, and initial condition $\mathbf{V}_0 \supseteq \mathbf{V}^*$, consider the recursive sequence $\{\mathbf{V}_{n+1}^{h,\mu,N}\}$ defined as $\mathbf{V}_{n+1}^{h,\mu,N} = \mathbf{B}^{h,\mu,N}\mathbf{V}_n^{h,\mu,N}$. Then, (i) $\mathbf{V}_n^{h,\mu,N} \supseteq \mathbf{V}^*$ for all n; (ii) $\mathbf{V}_n^{h,\mu,N} \to \mathbf{V}^{*,h,\mu,N}$ uniformly as $n \to \infty$; and (iii) $\mathbf{V}^{*,h,\mu,N} \to \mathbf{V}^*$ as $h \to 0$, $\mu \to 0$ and $N \to \infty$.*

In numerical work, because of rounding and truncation errors, it is not feasible to compute the exact equilibria in finite time. Kubler and Schmedders (2005) overcome this hurdle by constructing a Markov ϵ-equilibrium as a collections of policy function and transition function such that the maximum error in agents' equilibrium conditions are below some predetermined ϵ when evaluated at all possible points in the finite state space. They also provide error bound for this ϵ-equilibrium. However, the non-existence of stationary Markov equilibria in OLG models has been demonstrated in Kubler and Polemarchakis (2004). Under fairly mild assumptions, the Generalized Markov equilibria exists as in our example. While Markov

ϵ-equilibrium does not necessarily approximate generalized Markov equilibria. Therefore we define Markov ϵ-equilibrium correspondence as an approximation of the Generalized Markov equilibria.

Definition 2 *A Makov ϵ-equilibrium correspondence consists of a finite state space Θ, a correspondence $\mathbf{V} : \boldsymbol{\Theta} \times \mathbf{S} \rightarrow \mathbf{R}^S$, such that any measurable selection of policy function $\theta_+ = g^\theta(\theta, s, m)$, and transition function $m_+(s_+) = g^m(\theta, s, m; s_+)$ from $\mathbf{V}$ constitute a recursive ϵ-equilibrium.*

In all numerical examples we considered, we are able to find the Markov ϵ-equilibrium correspondence at any given $\epsilon > 0$.

4.3 Numerical specifications

4.3.1 Example 1, revisited

We apply the operator $\mathbf{B}^{h,\mu,N}$ to the model economy in example 1. We use $\tilde{\mathbf{V}}_{n+1}$ as the initial condition for our iteration procedure. Our numerical result indicates that $\tilde{\mathbf{V}}_{n+1}$ is the fixed point of $\mathbf{B}^{h,\mu,N}$. As shown in the left panel of Figure 3, the area between two solid lines represent the approximate solution. We also include the solution obtained from backward shooting algorithm in the same graph, which is presented by dots. The numerical experiment also suggests that the equilibrium set $\mathbf{V}$ is a convex-valued correspondence.

[Figure 3 here.]

4.3.2 Example 3

We consider a stochastic version of the model in example 1. There is an exogenous shock that affects the endowments of the household. We assume that the shock takes two values: boom or bust. Accordingly, the endowments of the old oscillate. More specifically, we assume $\{\mathbf{e}^a(s_t)\}_{a=0}^2 = \{3, 12, 1 \pm \epsilon\}$, where $\epsilon = 0.05$. The transition matrix that governs the Markov chain is given by

$$\pi = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}.$$

The financial market is incomplete and that there is a single bond available for trade. At each s^t, the bond is in zero net supply. Its price is denoted by $q(s^t) \in \mathbf{R}_+$, and agent a's bond-holding is $\theta^a(s^t) \in \mathbf{R}$.

At the root node, s_0, there are individuals of all ages s^{-a} with initial wealth $\theta^a(s^0)$, where $a = 0, 1, 2$. These determine the "initial condition" of the economy.

The equilibrium equations consist of first order conditions, budget constraints, and market-clearing conditions.

$$
\begin{aligned}
c^0(s) &= e^0(s) - q(s)\theta^0(s) && (33)\\
c^1(s) &= e^1(s) + \theta^0(s_-) - q(s)\theta^1(s) && (34)\\
c^2(s) &= e^2(s) + \theta^1(s_-) && (35)\\
q(s)\cdot u_c(c^0(s)) &= \beta \sum\nolimits_{i=1}^{S} \pi(s_+|s)\cdot u_c(c^1(s_+)|s) && (36)\\
q(s)\cdot u_c(c^1(s)) &= \beta \sum\nolimits_{i=1}^{S} \pi(s_+|s)\cdot u_c(c^2(s_+)|s) && (37)\\
\theta^0(s_-) + \theta^1(s_-) &= 0 && (38)\\
\theta^0(s) + \theta^1(s) &= 0 && (39)
\end{aligned}
$$

Now we apply the operator $\mathbf{B}^{h,\mu,N}$ for given $h, \mu > 0$. The equilibrium set is given by figure 4. The two figures on the top represent the equilibrium set when the current shock is given by $e^2 = 1.05$. While the bottom two are for the case when $e^2 = 0.95$. We also plot the equilibrium set of $\left\{m, \{m_+(s_+)\}_{s_+=1}^{\mathbf{S}}\right\}$ at given $\{\theta, s\}$ in left panel of figure 5. The application of the proposition 2 suggests that we can not rule out the possibility of existence of indeterminacy in this example.

[Figure 4 here.]

[Figure 5 here.]

The right panel of figure 5 represent the set of $\{m_+(s_+)\}_{s_+=1}^{\mathbf{S}}$ at given $\{\theta, s, m\}$. Here we set $\theta = 3.0$, $m = 4.427$. If there is no uncertainty, then m_+ is unique. We find that $m_+ = 5.155$ if $\{\mathbf{e}^a\}_{a=0}^2 = \{3, 12, 1.05\}$ and $m_+ = 5.257$ when $\{\mathbf{e}^a\}_{a=0}^2 = \{3, 12, 0.95\}$. The introduction of uncertainty brings some room to choose $\{m_+(s_+)\}_{s_+=1}^{\mathbf{S}}$ that satisfies the Euler equation (24). Given the persistence of the shock, there is less freedom to vary $m_+(s_+ = s)$ when the current shock is s.

5 Indeterminacy and long run behavior of the economy

In this section, we propose a simulation method for the model with indeterminate equilibria based on Peralta and Santos (2009). We then simulate the above model economies. From the simulation we derive some implication of indeterminacy on the long run behavior of the economy.

5.1 Simulation for OLG without uncertainty

In the case of deterministic OLG economies, previous studies conjectured that a continuum of equilibria will all converge to the same steady state asymptotically [see Spear, Srivastava, and Woodford (1990), Wang (1993)]. However this hypothesis has never been tested because there is no robust algorithm to compute the equilibria set for this type of economies. One contribution of this paper is that we test the validity of this statement in one example by computing all equilibrium set and conducting simulations. In what follows, we detail the simulation algorithm based on the equilibrium correspondence $\hat{\mathbf{V}}^{\mu,\varepsilon^*}$ obtained through the procedure in the previous section.

Assume that the economy begins with an initial bond-holding by the middle age $\theta_{-1} \in \hat{\mathbf{\Theta}}$ at time 0.

- At period $t = 0$, we pick an arbitrary $m_0 \in \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_{-1})$. Given $\{\theta_{-1}, m_0\}$, we solve for $\{q_0, \theta_0\}$ from the following equations

$$m_0 = e^1 + \theta_{-1} - q_0\theta_0 \tag{40}$$

$$q_0 \cdot u_c(m_0) = \beta u_c(e^2 + \theta_0), \tag{41}$$

and we can infer the value of m_1 using the Euler equation

$$q_0 \cdot u_c(e^0 - q_0\theta_0) = \beta u_c(m_1). \tag{42}$$

- At period $t > 0$, we can solve for $\{q_t, \theta_t\}$ at given $\{\theta_{t-1}, m_t\}$ from equations

$$m_t = e^1 + \theta_{t-1} - q_t\theta_t \tag{43}$$

$$q_t \cdot u_c(m_t) = \beta u_c(e^2 + \theta_t), \tag{44}$$

and infer m_{t+1} from

$$q_t \cdot u_c(e^0 - q_t\theta_t) = \beta u_c(m_{t+1}). \tag{45}$$

For the economy described in example 1, we start with $\theta_{-1} = 2.0$. As we can see from the left panel of figure 6, all simulated paths lead to the steady state $\theta^* = 3.73238$ as long as we pick $m_0 \in [4.0182, 6.0364]$, which implies that $\theta_0 \in [1.3126, 4.1277]$.

[Figure 6 here.]

The indeterminacy in this simulation exercise illustrates that existence of numerous equilibrium paths in deterministic OLG can be indexed by specifying initial condition for the shadow value of investment in bond $m_0 \in \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_{-1})$, which gives rise to the prices. After that, the Euler equation will uniquely pin down the equilibrium path. More specifically, the temporary equilibrium conditions yield a unique value of m_{t+1} for any given θ_t, m_t as in equation (45), hence the sequence of $\{\theta_t, q_t, m_t\}$.

Now imagine there are two different economies with exactly the same initial condition θ_{-1}. If their parameterizations are given by the one described in example 2, then they will converge to the steady state with the same speed, and same volatility as there is only one path leads to the long-run equilibrium. However, they may behave quite different if there is indeterminacy as in the above economy. The freedom of choosing m_0 brings them distinct equilibrium paths that lead to the same steady state. As a matter of fact, the economy (A henceforth) that chooses $m_0 = 4.0182$ will converge to the steady state in 105 periods with $mean(\theta_t) = 3.7088$, and $mean(q_t) = 0.8328$. While the economy (A henceforth) with $m_0 = 6.0364$ will converge to the steady state in 109 periods with $mean(\theta_t) = 3.7799$, and $mean(q_t) = 0.7740$.[5] The social planner in economy B chooses lower price (higher interest rate) in order to achieve smoother consumption across generations than in economy A. Consequently, the aggregate welfare level is 3.51% higher, which translates into a welfare gain equivalent to increase consumption by 1.2%.

	$std(\theta)$	$mean(\theta)$	$std(p)$	$mean(p)$	$mean(u)$	$max(ee)$
Simulation 1	0.3644	3.7088	0.3959	0.8328	-0.0073	$1.52 * 1e^{-12}$
Simulation 2	0.3725	3.7799	0.1123	0.7740	-0.0071	$1.44 * 1e^{-12}$

Table 1: Simulation for OLG without uncertainty

5.2 Simulation for stochastic OLG with incomplete market

Distinct from its deterministic counterpart, the economy may experience long-run equilibrium indeterminacy when multiple equilibria exist in the case of stochastic OLG economies.

When there is uncertainty and the financial market is incomplete, it is impossible to pin down m_{t+1} from the Euler equation as we did in the deterministic case. Let's assume that the economy begins with initial condition $\{\theta_{-1}, s_0\}$. As in the previous section, there is a continuum of choice of $m_0 \in \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_{-1}, s_0)$. Let's call it indeterminacy in initial condition. Similarly, we can solve for $\{q_0(s_0), \theta_0(s_0)\}$ at given $\{\theta_{-1}, s_0, m_0\}$ from the following equation

[5] Here convergence is defined as $\|\theta_t - \theta^*\| \le 10^{-12}$.

system.

$$m_0 - \left[e^1(s_0) + \theta_{-1} - q_0\theta_0\right] = 0 \tag{46}$$

$$q_0 \cdot u_c(m_0) - \beta \sum_{s_1} \pi(s_1|s_0) u_c \left[e^2(s_1) + \theta_0\right] = 0. \tag{47}$$

If the equilibrium is unique, or after the the initial m_0 is chosen in deterministic case, then there exists unique $\{m_1(s_1)\}_{s_1=1}^{\mathbf{S}}$ that satisfies the following Euler equation at given values of $\{q_0, \theta_0\}$.

$$q_0 \cdot u_c(e^0 - q_0\theta_0) - \beta \sum_{s_1} \pi(s_1|s_0) u_c(m_1(s_1)) = 0 \tag{48}$$

However, uncertainty will introduce an extra indeterminacy into the economy. According to the construction of the operator $\mathbf{B}^{h,\mu,N}$, there exists at least one pair of $\{m_1(s)\}_{s=1}^{\mathbf{S}} \in \dot{\hat{\mathbf{V}}}^{\mu,\varepsilon^*}(\theta_{-1}, 1) \times ... \times \dot{\hat{\mathbf{V}}}^{\mu,\varepsilon^*}(\theta_{-1}, \mathbf{S})$ (corresponding to the given $\{\theta_{-1}, s_0, m_0\}$) that satisfies all temporary equilibrium conditions. Right panel fo Figure 5 presents the typical equilibrium set of $\{m_1(s)\}_{s=1}^{\mathbf{S}}$ at given $\{\theta_{-1}, s_0, m_0\}$. As we can freely choose m_0 from the equilibrium correspondence $\hat{\mathbf{V}}^{\mu,\varepsilon^*}$, we have the freedom of picking arbitrary pair of $\{m_1(s_1)\}_{s_1=1}^{\mathbf{S}}$ that constitutes an equilibrium sequence. Consequently, even though we start the simulation with the same initial condition $\{\theta_{-1}, s_0\}$ and pick the same m_0, there still exists numerous equilibrium paths which may present different long run behavior. In what follows, we first explain the simulation algorithm and then we present some numerical results.

The simulation starts with an initial bond-holding by the middle age $\theta_{-1} \in \hat{\mathbf{\Theta}}$ and the initial value of shock is $s_0 \in \hat{\mathbf{S}}$. We pick an arbitrary $m_0 \in \dot{\hat{\mathbf{V}}}^{\mu,\varepsilon^*}(\theta_{-1}, s_0)$.

- We first select a measurable transition function $m_+ = g^m(\theta, m, s_+; s)$ from $\hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta, s)$ as detailed in the following section.

- At period $t = 0$, given $\{\theta_{-1}, s_0, m_0\}$, we solve for $\{q_0(s_0), \theta_0(s_0)\}$ from equations (46) and (47). We use a random number generator to determine the value of shock s_1 at $t = 1$. Then m_1 can be determined as $m_1 = g^m(\theta_{-1}, m_0, s_1; s_0)$.

- At period $t > 0$, we can solve for $\{q_t(s_t), \theta_t(s_t)\}$ at given $\{\theta_{t-1}, s_t, m_t\}$ using the same strategy as in the previous step. Similarly, we get the value of s_{t+1} using a random number generator and infer m_{t+1} from the transition function g^m.

5.2.1 Selecting the transition functions

Given $(\theta, s, m) \in \hat{\mathbf{\Theta}} \times \hat{\mathbf{S}} \times \hat{\mathbf{V}}^{\mu,\varepsilon^*}$, θ_+ is determined as the solution to the following equations system

$$\begin{aligned} m - \left[e^1(s) + \theta_i - q\theta_+\right] &= 0 \qquad (49) \\ q \cdot u_c(m) - \beta \sum_{i=1}^{S} \pi(s_+|s) \cdot u_c(e^2(s_+) + \theta_+|s) &= 0. \qquad (50) \end{aligned}$$

We choose $m_+(s_+)$ from the equilibrium set $\hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_+, s_+)$ such that all temporal equilibrium conditions are satisfied and the corresponding θ_{++} is the maximum possible value of bond issued by the middle-aged. Formally, we pick m_+ such that

$$\begin{aligned} m_+(s_+) = \arg\max \theta_{++}(\theta_+, s_+, m_+) \\ s.t.\ \ m_+(s_+) \in \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_+, s_+) \end{aligned} \qquad (51)$$

where θ_{++} is a function of (θ_+, s_+, m_+) since we can solve for θ_{++} as a solution to equations similar to (49, 50).

We can also pick $m_+(s_+)$ such that θ_{++} is the minimum possible value of bond issued by the middle-aged. Similarly we can select the transition function $g^m(\theta, m, s_+; s)$ which are the maximand of the young agent's utility, middle age's utility or the aggregate welfare in the next period.

In order to highlight the effect of the selection on the long run economy, we pick the maximal $m_+(s_+)$ from $\hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_+, s_+)$ in simulation 1, while we choose minimal $m_+(s_+)$ in simulation 2. The simulated paths are quite different as we can see from Figure 7.

[Figure 7 here.]

It is important to notice that the dimension of the above indeterminacy is infinite. However, we can degenerate this infinity by choosing the initial value of m_0 in all other possible contingencies. Let's assume that the value of $s_0 = s \in \hat{\mathbf{S}}$. We can solve for $q_0(s_0 = s), \theta_0(s_0 = s)$ from equations (46) and (47) once we pick the value of $m_0 \in \hat{\mathbf{V}}^{\mu,\varepsilon^*}(\theta_{-1}, s_0 = s)$. If we also select the value of m_0 at all other possible states $s_0 \in \hat{\mathbf{S}} - \mathbf{s}$, then we can solve for $\{q_0(s_0), \theta_0(s_0)\}_{s_0=1}^{\mathbf{S}}$. Therefore we have additional $(\mathbf{S} - 1)$ Euler equations corresponding to all other possible contingencies similar to equation (48). From there we can jointly solve $\{m_1(s_1)\}_{s_1=1}^{\mathbf{S}}$. By repeating this procedure, the equilibrium sequence will be exclusively determined by the flow of shocks $\{s_t\}_{t=0}^{\infty}$. The dimension of the indeterminacy is then $(\mathbf{N}_\theta + \mathbf{N}_m) \times \mathbf{S}$, where $\mathbf{N}_\theta$ and $\mathbf{N}_m$ are the number of available financial assets, and their shadow values.

5.2.2 Numerical specifications

Numerical simulation 1: the effect of selections of g^m We choose $\theta_{-1} = 3.0$, and the initial value of shock is chosen such that $e^2 = 0.95$. The initial value of m_0 is set to be 5.50. In order to examine the effect of the selection of g^m on the long run behavior of the economy, we simulate the economy twice. Each simulation chooses different selection of the transition function g^m. The simulations last for $100,000$ periods and we drop the first $50,000$ periods. As we can see from the table below, these two artificial economies behave quite different in terms of the simulated moments. It is worth to mention that the average aggregate welfare (with equal weight for all generations) in the first economy is about 3.96% higher than the second one, which translates into a welfare gain equivalent to increase consumption by 1.3%. The welfare gain comes from the choice of lower price of bond (higher interest rate) by the social planner, which encourages middle age to save and results in smooth consumption across generations.

	$std(\theta)$	$mean(\theta)$	$std(p)$	$mean(p)$	$mean(U)$	$max(ee)$
Simulation 1	0.6738	4.1319	0.2189	0.6496	-0.0071	$4.5 * 1e^{-5}$
Simulation 2	0.3624	3.6114	0.2405	0.8855	-0.0073	$4.5 * 1e^{-5}$

Table 2: Simulation for stochastic OLG

Numerical simulation 2: the effect of initial conditions We also look at the effect of the initial conditions on the behavior of the economy. We fix the selection of g^m for all of the following experiments. As a benchmark, we choose $\{\theta_{-1}, s_0, m_0\} = \{3.0, e^2 = 0.95, 5.50\}$. In what follows, we only mention the differences from the benchmark. The first experiment started with $m_0 = 5.10$, the second chooses s_0 such that $e^2 = 1.05$, while the third one picks $\theta_{-1} = 4.3128$. After we drop the first $50,000$ periods, the simulated sequences yield identical moments as in the benchmark case.

Switching between transition functions If we allow the agent to switch the selection of transition functions along the simulation. It is obvious that we can construct any equilibrium sequence which has $mean(m)$, and $mean(q)$ lie between the sequence 1 and 2 listed in table 2.

5.3 Market incompleteness, uncertainty and indeterminacy

In order to understand the implications of market incompleteness on the indeterminacy, we revisit the stochastic model with complete financial market. To illustrate the point, we define the shadow value of investment in asset i given the value of shock s^t as follows.

$$m_i(s^t) = q_i(s^t) \cdot u_c\left(e^1(s^t) + \sum_{i=1}^{\mathbf{S}} 1_{s_t=i}\theta_i(s^{t-1}) - \sum_{i=1}^{\mathbf{S}} q_i(s^t)\theta_i(s^t)\right) \tag{52}$$

In the case of two shocks, the equilibrium correspondence is given by

$$\mathbf{V}(\theta_1(s_{-1}), \theta_2(s_{-1}), s) = \{(m_1, m_2) \mid (q_1(s_t), q_2(s_t), (c^a(s_t), \theta_1^a(s_t), \theta_2^a(s_t))_{a=0,1,2}) \text{ is a SCE}\}$$

Given the initial asset holdings $\theta_1(s_{-1}), \theta_2(s_{-1})$ and the value of shock s_0 at period 0. We can select $(m_1(s_0), m_2(s_0))$ from the equilibrium set $\mathbf{V}$. We then solve for $\{q_i(s_0), \theta_i(s_0)\}_{i=1}^2$ from the following equation system.

$$m_1(s_0) = q_1(s_0) \cdot u_c\left(e^1(s_0) + \sum_{i=1}^{2} 1_{s_0=i}\theta_i(s_{-1}) - \sum_{i=1}^{2} q_i(s_0)\theta_i(s_0)\right) \tag{53}$$

$$m_2(s_0) = q_2(s_0) \cdot u_c\left(e^1(s_0) + \sum_{i=1}^{2} 1_{s_0=i}\theta_i(s_{-1}) - \sum_{i=1}^{2} q_i(s_0)\theta_i(s_0)\right) \tag{54}$$

$$q_1(s_0) \cdot u_c\left(e^1(s_0) + \sum_{i=1}^{2} 1_{s_0=i}\theta_i(s_{-1}) - \sum_{i=1}^{2} q_i(s_0)\theta_i(s_0)\right) = \beta\pi(1|1) \cdot u_c\left(e^2(s_1) + \theta_1(s_0)\right) \tag{55}$$

$$q_2(s_0) \cdot u_c\left(e^1(s_0) + \sum_{i=1}^{2} 1_{s_0=i}\theta_i(s_{-1}) - \sum_{i=1}^{2} q_i(s_0)\theta_i(s_0)\right) = \beta\pi(2|1) \cdot u_c\left(e^2(s_1) + \theta_2(s_0)\right) \tag{56}$$

The Euler equations (57, 58) will only provide the next period's shadow value of investment for the same asset $\{m_1(s_1 = 1), m_1(s_1 = 2)\}$ in different contingencies, which are not enough to solve for future values of bond-holdings and prices.

$$q_1(s_0) \cdot u_c\left(e^0(s_0) + \sum_{i=1}^{2} q_i(s_0)\theta_i(s_0)\right) - \beta\pi(1|s_0) \cdot \frac{m_1(s_1 = 1)}{q_1(s_1)} = 0 \tag{57}$$

$$q_2(s_0) \cdot u_c\left(e^0(s_0) + \sum_{i=1}^{2} q_i(s_0)\theta_i(s_0)\right) - \beta\pi(2|s_0) \cdot \frac{m_1(s_1 = 2)}{q_2(s_1)} = 0 \tag{58}$$

However, we can solve for $\{m_2(s_1 = 1), m_2(s_1 = 2)\}$ if we also select the value of $(m_1(s_0), m_2(s_0))$ at all other possible contingencies. From there we can uniquely determine the value of bond-holdings and prices, as well as the simulated equilibrium path. This example suggests that the market incompleteness does not have any impact on the indeterminacy introduced by uncertainty in the OLG model we considered.

5.4 Recursive equilibrium

The studies by Farmer and Woodford (1984), Spear, Srivastava, and Woodford (1990) focus on the construction of stationary recursive equilibria. We can select recursive equilibrium for the economy in example 3, which is explained by the following proposition.

Proposition 4 *There exists a continuum of recursive equilibrium in the economy example 3..*

Proof: Since the correspondence $\mathbf{V}^{*,h,\mu,N}$ is upper semi-continuous, we can choose an approximate equilibrium selection $\theta_t = g^{\theta}(\theta_{t-1}, s_t, m_t)$ and a transition function $m_{t+1} = g^m(\theta_{t-1}, s_t, m_t)$.

At period 0, for any given $(\theta_{-1}, s_0) \in \mathbf{\Theta} \times \mathbf{S}$, we pick $m_0(s_0) \in \mathbf{V}^{*,h,\mu,N}(\theta_{-1}, s_0)$. The value of $\theta_0(s_0)$ will be determined by solving equations (49, 50). If we pick the value of m_0 at all other possible contingencies, then we can solve for $\{m_1(s_1)\}_{s_1=1}^{\mathbf{S}}$ following the same strategy as in the previous section. Now we can compute the value of θ_1

$$\theta_1 \equiv g^{\theta}(\theta_0, s_1, m_1) = g^{\theta}(\theta_0, s_1, \{\theta_{-1}, s_0, m_0\}_{s_0=1}^{\mathbf{S}}) \tag{59}$$

We pick (θ_{-1}, s_0, m_0) such that the space of $\theta_0(s_0)$ is exactly $\mathbf{\Theta}$. We approximate the mapping $(\theta_0, s_1) \to \theta_1$ using a continuous function f. One can verify that the sequence $\{\theta_t, s_t; f(\theta_t, s_{t+1})\}$ satisfies all equilibrium conditions. It is important to notice that the approximated function f is indexed by the choice of $\{\theta_{-1}, s_0, m_0\}_{s_0=1}^{\mathbf{S}}$.

It is interesting to compare the long run behavior of recursive equilibrium with the competitive equilibrium as we defined before. The simulated moments of the recursive equilibrium lie between the simulation 1 and 2 as in the following table. As we can see from table 2 and table 3, the volatility of asset price in recursive equilibrium is substantially smaller than the one in sequential competitive equilibrium. This suggests that it will be misleading if we only look at the recursive equilibrium when there is indeterminacy in the model.

	std(θ)	*mean*(θ)	*std*(p)	*mean*(p)
Simulation 1	0.0789	2.9794	0.0330	1.3215
Simulation 2	0.0188	5.5682	0.0030	0.2069

Table 3: Simulation for stochastic OLG

6 Concluding remarks

In this paper, we study the indeterminacy in OLG model with and without uncertainty. We adopt the numerical method developed in Feng et al. (2009) and provide a general approach to identify the existence of indeterminacy by computing the boundary of the equilibrium set of the above economy. We implements our approach to an OLG economy with two set of different parameterizations. In one case, the upper boundary of the computed equilibrium set is always bigger than the lower boundary in the limit of the numerical procedure, which implies that we can not rule out the existence of indeterminacy. The indeterminacy was ruled out in another example as the upper boundary is identical to the lower one.

In order to derive the implication of the indeterminacy on the long-run behavior of the economy, we solve the model numerically by finding all Markov equilibria and propose a way to simulate the OLG economy with a continuum of equilibria. Numerical results suggest that the economy endowed with the same initial condition may converge to the long-run equilibrium with different path, different volatility in the case of deterministic OLG with indeterminate equilibrium. Further analysis shows that uncertainty will bring extra indeterminacy into the model as we have the freedom to select the transition and policy functions from the equilibrium correspondence. Consequently, economies with identical initial conditions may present different simulated moments. However, as long as we follow the same transition functions, the initial conditions won't pose any effect on the long-run behavior of the economy. Numerical simulations indicate that the selections of the transition and policy functions may have important welfare effect. By varying the selection of the transition function, the economy can improve the social welfare, measured by the aggregate utility, in the order of 1.3% in consumption equivalence. This leaves room for government intervention to improve welfare. It is important to understand how to design policies that will select and implement the best possible equilibrium. We leave it as future research.

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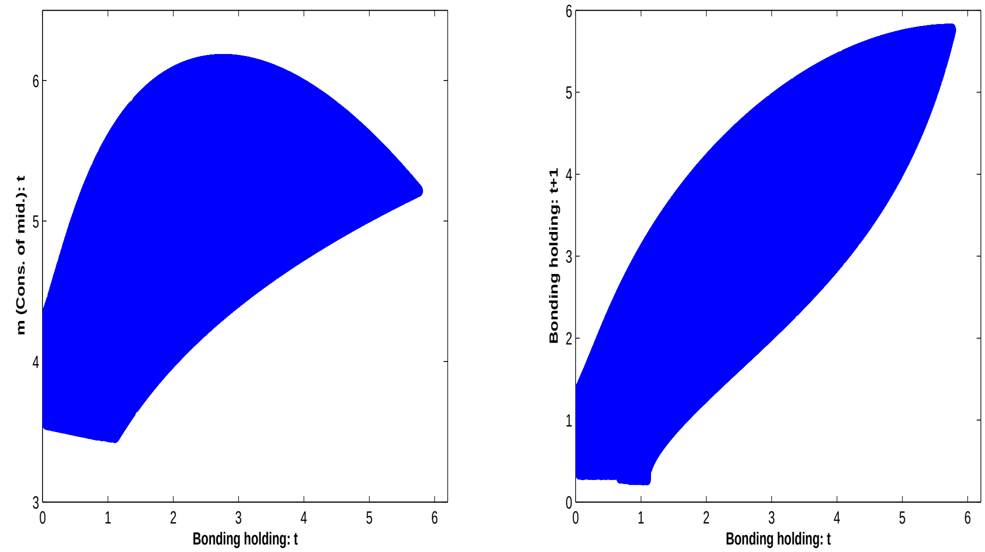

Figure 1: Final boundary of OLG with indeterminacy

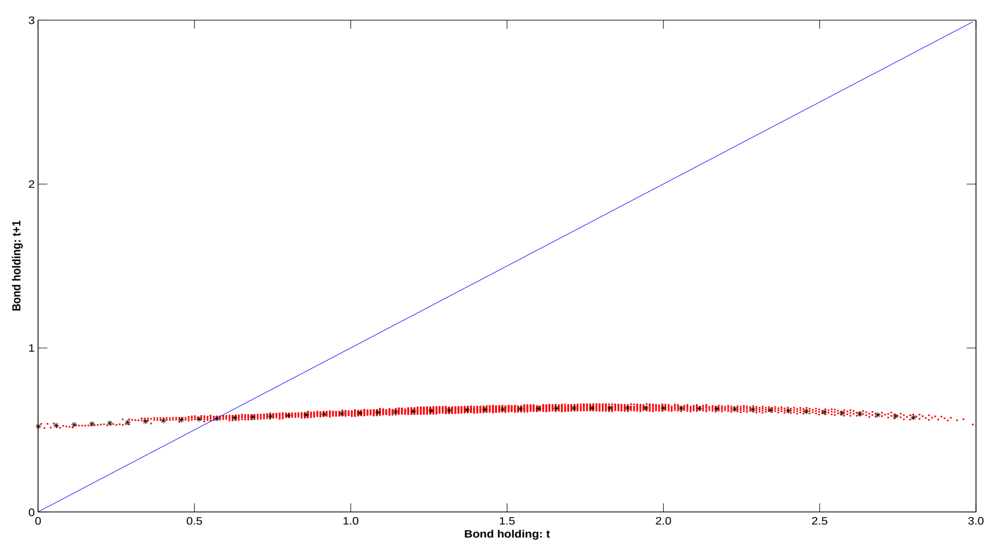

Figure 2: Final boundary of OLG with determinacy

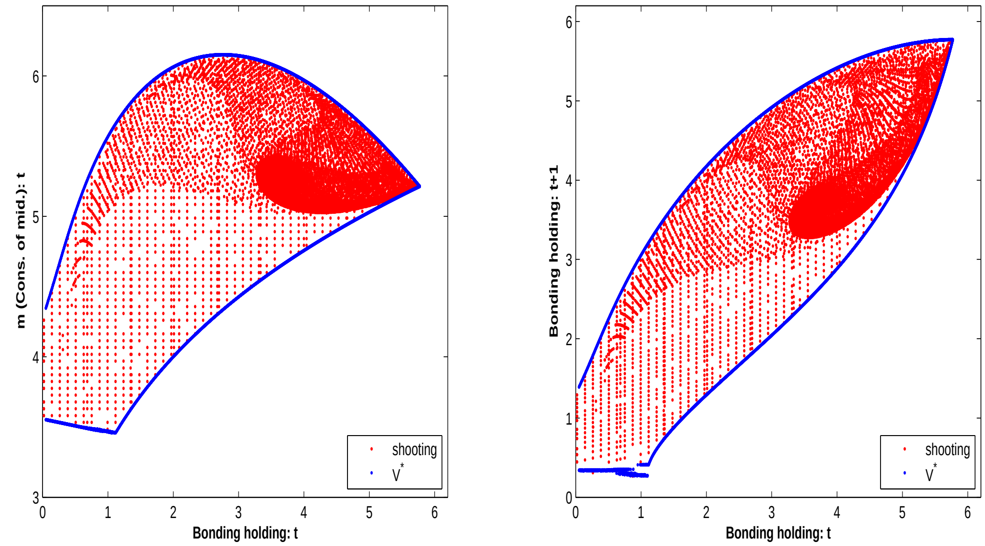

Figure 3: Final set of OLG without uncertainty

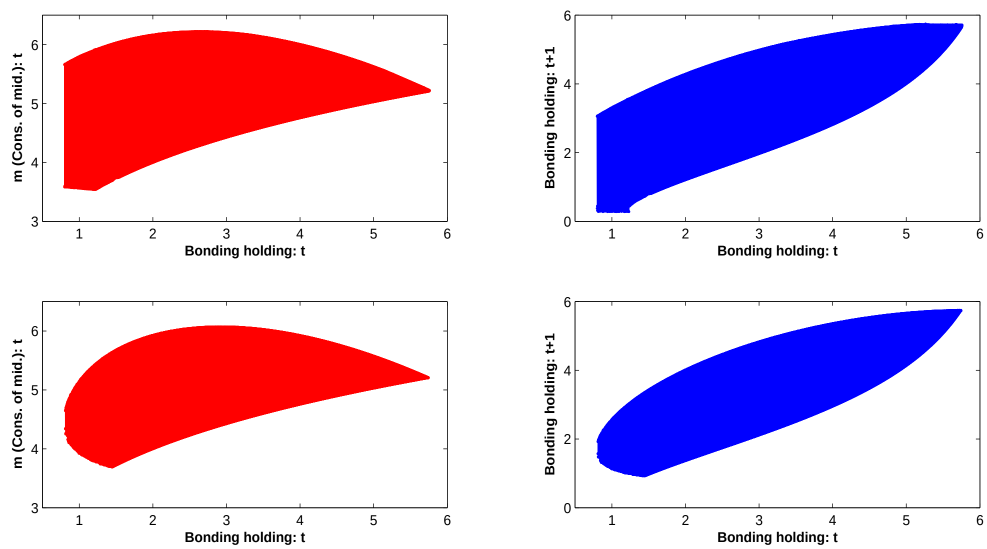

Figure 4: Final set of OLG with uncertainty

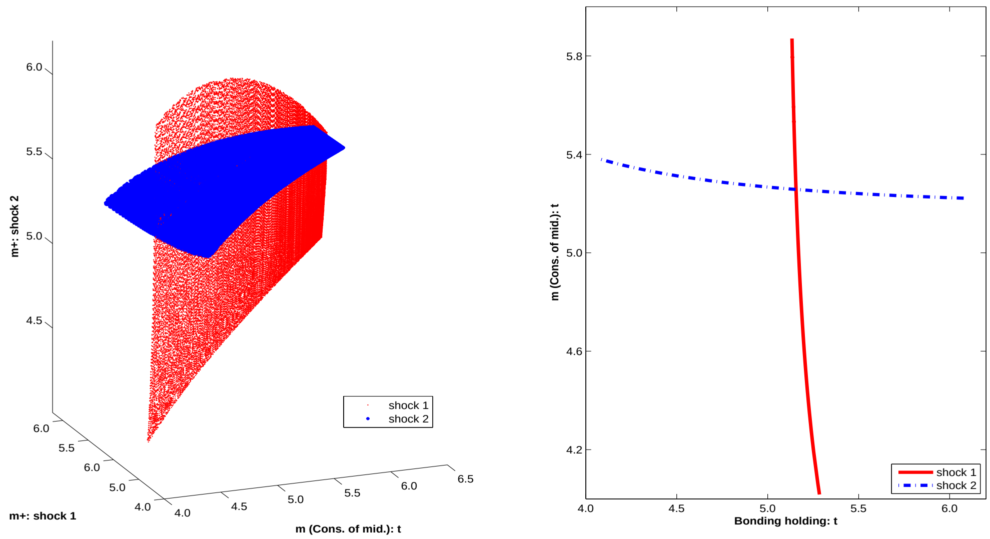

Figure 5: Equilibrium set of $\left\{\{m_{t+1}(s)\}_{s=1}^{\mathbf{S}}, m_t\right\}$ at given $\{\theta_t, s_t\}$.

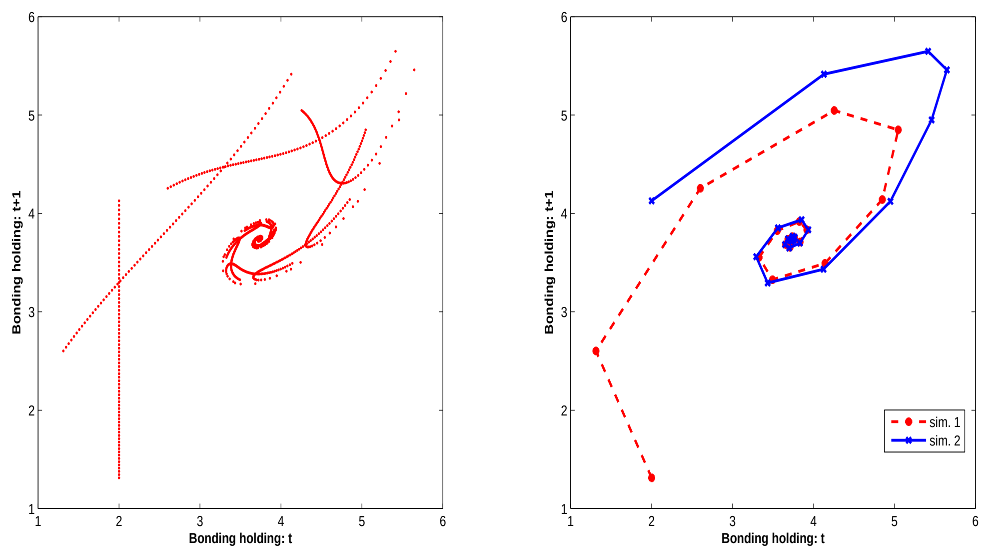

Figure 6: Simulation of OLG without uncertainty

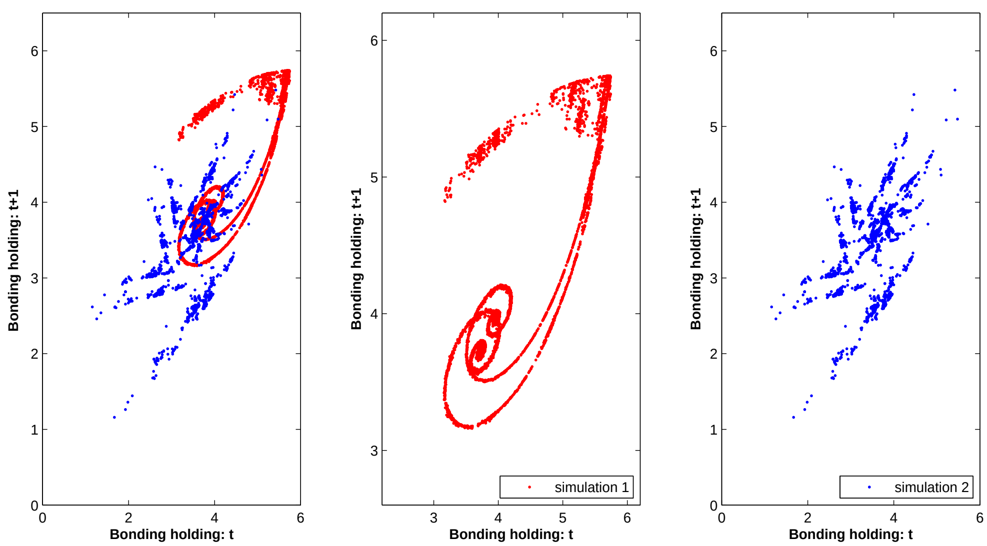

Figure 7: Simulation of stochastic OLG.