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Fraud Cycles

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Abstract

Fraud is an ancient crime and one that annually causes hundreds of billions of dollars in losses. We examine the behavioral patterns over time of different types of frauds, which illustrate cyclical frequencies. We develop an evolutionary theory that suggests cyclic behavior in frauds should be common.

JEL classifications: D92, L15, P17

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1. Introduction

Fraud is as old as civilization itself.¹ Aristotle recounts a story of financial fraud in sixth century BC Greece.² Minor Greek sculptors carved the signatures of Phidias and Praxiteles into their works prepared for export to Roman collectors.³ Fraud begets anti-fraud: “forensic accounting is one of the oldest professions and dates back to the Egyptians. The ‘eyes and ears’ of the king was a person who basically served as a forensic accountant for Pharaoh, watchful over inventories of grain, gold, and other assets.”⁴ Modern frauds include Ponzi and pyramid schemes, securities frauds, corporate accounting financial scandals, medical and automobile insurance frauds, sophisticated art forgeries, the shell game, and the “Nigerian scam,” to name just a few.

Fraud is very big business. According to the FBI, fraudulent billings are estimated to be between three and ten percent of total health care expenditures⁵ or approximately \$70–\$230 billion annually.⁶ The Coalition Against Insurance Fraud estimates that insurance fraud costs Americans approximately \$80 billion a year.⁷ The

¹ The exact definition of fraud varies across different professions. In accounting, for example, fraud means misrepresentation of fact, while misappropriation of assets is termed defalcation. See, e.g., Matsumura and Tucker (1992). Here we use the word “fraud” in a broader sense to include all acts intended to swindle their victims.

² Aristotle, “Constitution of Athens,” *The Complete Works of Aristotle*, vol. 2, Princeton University Press (1984), pp. 2341-2383. (“As soon as he was at the head of affairs, Solon liberated the people once and for all, by prohibiting all loans on the security of the debtor’s person; and in addition he made laws and cancelled all debts, public and private. This measure is commonly called the Seisachtheia [removal of burdens] since thereby the people had their loads removed from them. In connexion with it some persons try to traduce the character of Solon. It so happened that, when he was about to enact the Seisachtheia, he communicated his intention to some members of the upper class, whereupon, as the partisans of the popular party say, his friends stole a march on him; while those who wish to attack his character maintain that he too had a share in the fraud himself. For these persons borrowed money and bought up a large amount of land, and so when, a short time afterwards, all debts were cancelled, they became wealthy; and this, they say, was the origin of the families which were afterwards looked on as having been wealthy from primeval times.”) Id. at pp. 2343-44.

³ <http://education.yahoo.com/reference/encyclopedia/entry/forg-art>.

⁴ Singleton, T. and Singleton, A. (2010), *Fraud Accounting and Forensic Accounting*, John Wiley & Sons, Inc., 4th ed., p. 3.

⁵ “Financial Crimes Report to the Public, Fiscal Year 2009,” FBI, http://www.fbi.gov/publications/financial/fcs_report2009/financial_crime_2009.htm.

⁶ U.S. Department of Health and Human Services, <https://www.cms.gov/NationalHealthExpendData/downloads/tables.pdf>, reporting 2008 calendar year total national health expenditures, private and public.

⁷ http://www.insurancefraud.org/fraud_backgrounder.htm. Insurance fraud has been estimated to

recent unraveling of Bernie Madoff's Ponzi scheme, reportedly the largest in history, caused approximately \$20 billion in losses among its thousand-plus investors.⁸ Corporate financial scandals based on fraudulent accounting practices have caused even more economic mayhem. The collapse of Enron in the fall of 2001 cost investors and employees over \$70 billion in lost capitalization and retirement benefits.⁹

Forensic accountants specializing in fraud detection and deterrence find that the frequency of fraud is a cyclical phenomenon.¹⁰ New frauds lead to new or modified anti-fraud measures, which lead to innovations in fraudulent activities, which lead to updated antifraud measures, and so on. A leading fraud accounting textbook explains how following the "explosion of fraud around the last half of the 1990s and the early 2000s . . . the U.S. Congress passed the Sarbanes-Oxley Act due to [the Enron, WorldCom, and other corporate accounting financial frauds.] More recently, the housing and real estate boom of the 2000s has led to increased fraud particularly in the area of mortgage fraud. . . . Various pieces of legislation have been passed in response, continuing the cycle of evolving frauds and attempts to control them. . . . The fraud environment can be and often is viewed as a pendulum, swinging from one extreme to the other with little time in between at the proper balancing point. This cycle (pendulum swing) is a natural result of human nature, business cycles, and the nature of legislation and regulation. The cycle can certainly be influenced and controlled to some extent, but it will probably never cease."¹¹

increase annual household insurance premiums by \$300, of which increased automobile insurance premiums account for \$180. See Arizona Insurance Council, "Insurance Fraud," available at <http://www.azinsurance.org/fraud.html>.

⁸ Henriques, D., "Deal Recovers \$7.2 Billion for Madoff Fraud Victims," *New York Times*, Dec. 17, 2010.

⁹ The Handbook of Texas Online, "Enron Corporation," <http://www.tshaonline.org/handbook/online/articles/EE/doi8.html>.

¹⁰ Cyclic behavior of prices and outputs has been observed in other economic markets, e.g., the canonical hog cycle. See, e.g., Shonkwiler and Spreen (1986). Agricultural cycles are distinct because they have a natural lag created by production processes; such processes are not part of our analysis.

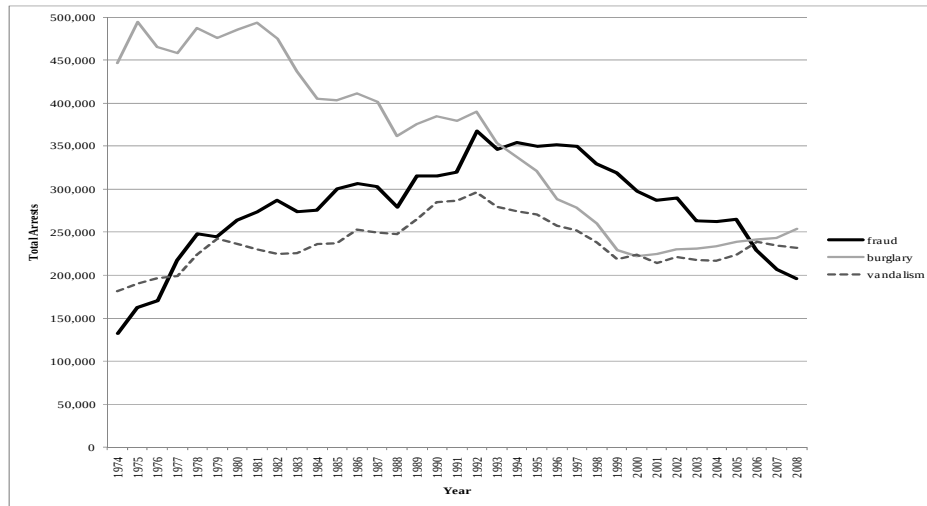
¹¹ Singleton, T. and Singleton, A. (2010), *Fraud Accounting and Forensic Accounting*, John Wiley & Sons, Inc., 4th ed., pp. 5-7. See also Simic (2005). ("There are three stages in the [credit card fraud] cycle. Stage 1 represents familiarity with weaknesses in cards and technology which drives up the value of fraud. Fraud begins to rise as new technologies and new weaknesses are found. Stage 2 represents new solutions implemented to reduce fraud. The solutions are not implemented immediately,

Textbook accounts have some empirical support. FBI data on the annual number of individuals arrested for fraud are shown in Figure 1. The data show a substantial increase in the number of individuals arrested for fraud in the 1990s. The total number of arrests declines in the 2000s, with the notable exception of two increases in 2002 and 2005. For purposes of comparison, we select two other types of crime (burglary and vandalism) that have roughly the same annual number of arrests, but that do not appear to have feedback mechanisms like that for fraud. (That is, new types of burglary or vandalism do not appear to be developed, leading to new anti-burglary or anti-vandalism measures, and so on.) The data show that in 1974, the number of arrests for fraud was lower than the number of arrests for burglary and vandalism. By 1977, the number of arrests for fraud surpassed the number of arrests for vandalism, and by 1994, fraud had the most number of arrests, which lasted until 2006, at which time fraud had the fewest number of arrests. We can measure changes over time in the rankings for total arrests for these three types of crime using the Spearman rank correlation. The rank correlation between burglary and vandalism equals 0.16, but the rank correlations between fraud and burglary or vandalism equal -0.89 and -0.58, respectively. Thus, the number of arrests for fraud changes over time in ways that are uncorrelated with the number of arrests for burglary and vandalism.

To be clear, we do not contend that the frequency of fraud is completely independent of general trends in the frequency of crime. Indeed, the trends for fraud, burglary, and vandalism in Figure 1 illustrate the general decline in crime in the U.S. since the mid-1990s. However, the large swings in the number of arrests of fraud over periods as short as a year or two, relative to changes in the number of arrests for burglary and vandalism, are consistent with the conclusion of Singleton and Singleton (2010) that fraud is a cyclical phenomenon.

and therefore Stage 3 represents time lag for solutions to take effect.”) Id., p. 4. See also Reinstein and Bayou (1998). (“Fraudsters use many clever schemes to misappropriate company assets and misstate financial statements. Analyzing fraud as a mere historical event can provide an inadequate basis to detect (or prevent) fraud, given its multidimensional, cyclical, and dynamic nature.”) Id., p. 20.

Figure 1: Total Arrests: 1974-2008
Fraud, Burglary, and Vandalism



Source: FBI, *Uniform Crime Reporting Program Data, Master Files, Arrests Monthly, 1974-2008*.

One potential drawback to the FBI data in Figure 1 is that the data are aggregated for all types of fraud. However, a new type of bankruptcy fraud, for example, that leads to a new anti-fraud measure designed to deter bankruptcy fraud may have little effect on the frequency of insurance fraud. Data on the frequency of specific types of fraud are not publicly available. Therefore, we estimate the frequency of a specific type of fraud by the number of articles in major news sources discussing that type of fraud.

We use the LexisNexis news database for this purpose, which allows us to conduct full-text keyword searches of all archived articles from major news sources. We assume that the frequency of articles on a given type of fraud reflects the underlying level of that type of fraudulent activity. Thus, we assume crimes and news about crimes are correlated and the latter may be used as a proxy to measure the former. This methodology is used by Krueger and Pischke (1997), who note that it has a long history of use in sociology, e.g., Danziger (1975), Inverarity (1976), and Olzak (1989).

We test the methodology by regressing the number of articles published in a local newspaper that contain the relevant keyword for a specific type of crime, e.g.,

“burglary,” on the actual number of incidents of that type of crime. In particular, we use monthly crime data from 1991 through 2008 for aggravated assault, simple assault, burglary, and larceny as reported in the U.S. Department of Justice’s Uniform Crime Reporting Program for the city of Houston, Texas.¹² We then determine the number of articles per month containing the relevant keywords using LexisNexis to search through the archives of the *Houston Chronicle*. The estimated regressions show statistically significant results for each of the four types of crime.¹³ These results provide some empirical support for our assumption that the number of articles discussing a given type of fraud serves as a proxy for the actual number of incidents of that type of fraud.

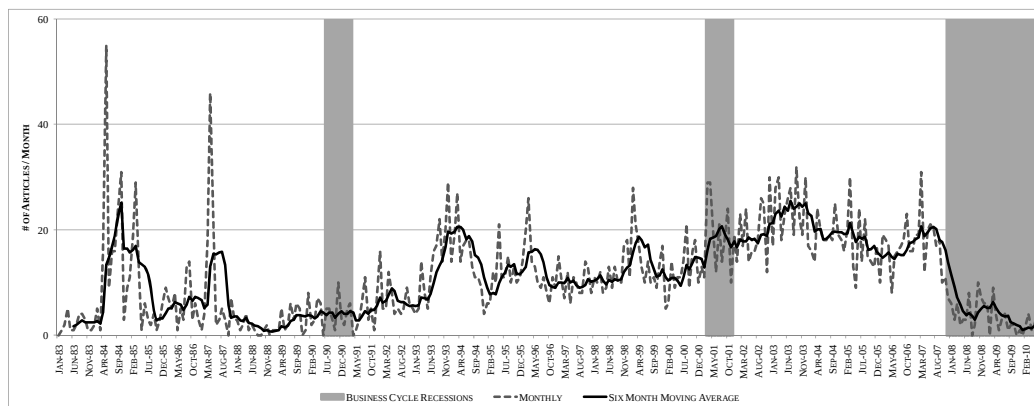
Figure 2 shows the number of articles per month containing the keywords “bankruptcy fraud,” as well as the simple six-month moving average of that series.¹⁴ The results indicate that the frequency of occurrences is not monotonical. Instead significant peaks and troughs are visible. The frequency of occurrences is not significantly correlated with recessions in the U.S. economy. The evidence presented here and in section II indicates that some types of fraud display cyclical patterns.

¹² U.S. Department of Justice, Uniform Crime Reporting Program, Houston, TX, 1991-2008.

¹³ The correlations between the monthly number of newspaper articles and crimes are as follows: aggravated assault, 0.14; simple assault, 0.46; burglary, 0.17; and larceny, 0.52. The regression and correlation results are available at www.mcafee.cc/Papers/PDF/Fraudcycles/.

¹⁴ For each of the different types of fraud analyzed here, we searched LexisNexis using the criterion that the word “fraud” had to be within one word of the word indicating the type of fraud. For example, in the search for “bankruptcy fraud,” the word “fraud” had to be either next to or within one word of the word “bankruptcy.” The news sources used in Figure 2 are *American Banker*, *Associated Press*, *Business Wire*, *Facts on File World News Digest*, *Journal of Commerce*, *New York Times*, *PR Newswire*, *United Press International*, *Wall Street Journal*, and *Washington Post*. We also used an expanded list of news sources, consisting of all sources in the LexisNexis category “U.S. Newspapers and Wires.” In 2010, that database contained 691 news sources. The results, which are available at www.mcafee.cc/Papers/PDF/Fraudcycles/Graphs.pdf, are similar in terms of indicating the presence of cycles for different types of frauds. The expanded list of news sources results in upward trends in the frequency of articles on different types of frauds, caused by duplicate articles discussing the same frauds as reported in our more restricted list of news sources.

Figure 2: Frequency of “Bankruptcy Fraud” in Selected U.S. Newspapers and Wires



Source: LexisNexis.

Why do frauds come and go? One possible answer is the public’s short memory. The earliest theoretical study of fraud-related phenomenon is Lui (1986), who developed a model of corruption deterrence in an overlapping generation structure. Lui assumes that when corruption becomes more prevalent in the economy, effectively auditing a corrupt official becomes more difficult. This reinforcement leads to variations in government’s effectiveness to deter corruption, resulting in cyclical patterns of fraud over time. A related topic, the intertemporal variation in business ethics, was studied by Noe and Rebello (1994). They modeled the dynamic interaction between business ethics and economic activities, generating cycling of ethics behavior. Another approach correlates certain types of frauds with the business cycle. For example, corporate financial misrepresentation can be concealed by a boom that an ensuing bust reveals. Povel et al. (2005) developed a theoretical model with financial misrepresentation cycles based on investors’ vigilance level fluctuating with the boom and bust cycle of the economy. All of this work is consistent with the general model we develop. Our approach emphasizes cyclicity created endogenously, rather than driven off an external cycle, but our approach is consistent with such external influences.

Our model is most closely related to that of Berentsen and Lengwiler (2004). They used replicator dynamics and developed a model that predicts fraud cycles; their model is approximately a linear version of a special case of our theory (see Section 3).

Separately, Sutter (2003) provided a theoretical model to study election fraud, where he showed the relevance of both demand and cost factors in the elimination of corrupt election practices. Hyman (2001) analyzed the complexity of relevant parties' differing interests in health care fraud. As a follow-up commentary on Hyman (2001), Feldman (2001) argued that the root cause of fraud in medical programs is distorted higher prices coupled with agents' efficiency-seeking activities under price distortion. There is also a sizable literature in accounting regarding the detection of fraud through audits, including Matsumura and Tucker (1992), Morton (1993), Hansen et al. (1996), and Bonner et al. (1998).

One shortcoming of the prior literature is the focus on the supply side of frauds, without endogenizing behavior of the demand side (victims) over time. In our approach, we assume a certain percentage of businesses are fraudulent while the rest are legitimate, and a certain percentage of buyers on the demand side are vigilant while the rest are not. The interaction of these two forces over time leads to multiple steady states in equilibrium. The reason for multiple steady states is similar to that found in Freeman, Grogger, and Sonstelie (1996). Our main result concerns the convergence to a steady state. We show that cyclic behavior—specifically a spiral—is a robust feature of a large class of evolutionary adaptation models.

The intuition for cyclic behavior is that it constitutes a feedback loop, mediated by evolution. When fraud is prevalent, vigilance pays. Increased vigilance reduces the return to fraud, thereby decreasing fraud. The reduction in fraud reduces the return to vigilance, thereby increasing fraud. We make two contributions. First, we empirically examine this cyclic outcome for a variety of frauds. Second, we show that the predicted outcome (i.e., that the frequency of a given type of fraud will be cyclical) is theoretically robust, and moreover we identify cycles based on the relative response or reaction rates of the two parties—scammers and victims. Interestingly, extremely fast responses by either side tend to eliminate cycles. Moderate adjustment speeds by both sides is necessary, and with a payoff condition sufficient, for cyclic behavior.

The remainder of the paper is organized as follows. Background information

and data on fraudulent behavior are described in Section 2. We use product quality as an example to illustrate a model in Section 3 that generates cyclical behavior. We then extend the example to a general, two-variable model in Section 4 and derive the main result of the paper that characterizes the conditions for cyclical behavior of fraud occurrences. Concluding remarks are contained in Section 5. All technical proofs are presented in the Appendix.

2. Background and Data on Types of Fraud at a Nationwide Level

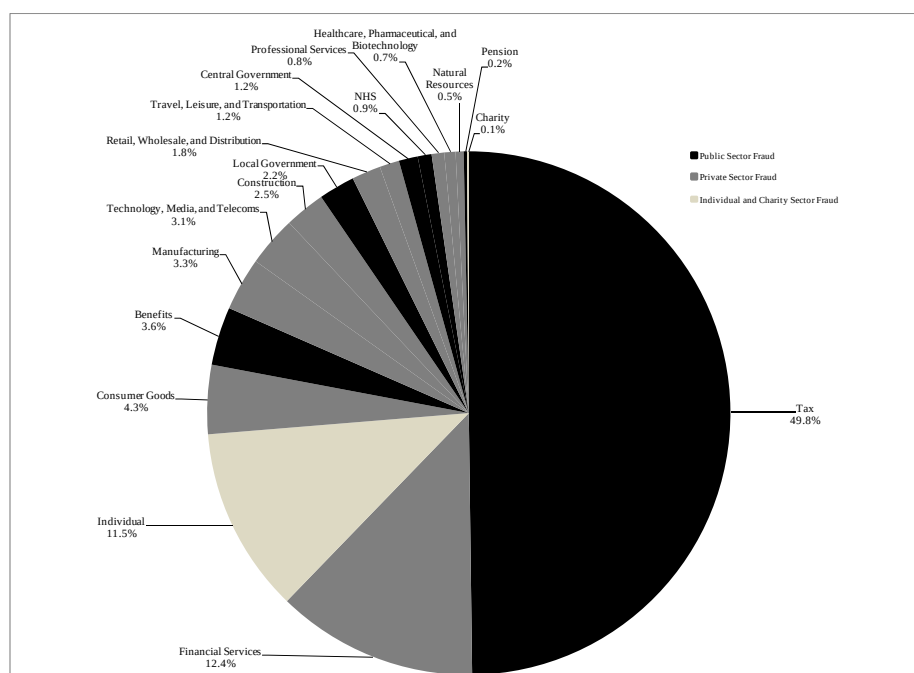
Useful statistics on fraud at a nationwide level are published by the United Kingdom's National Fraud Authority ("NFA").¹⁵ In its most recent report, the NFA estimates total fraud in the UK in 2008 at £30.5 billion¹⁶ or approximately 2.1 percent of the country's 2008 Gross Domestic Product.¹⁷ Figure 3 shows a detailed breakdown of the types of frauds occurring in the UK. The largest category is tax fraud, at £15.2 billion or approximately half of all fraud. Public sector fraud accounts for approximately 58 percent of all fraud in the UK, with the private and individual/charity sectors accounting for approximately 30 percent and 12 percent, respectively.

¹⁵ See <http://www.attorneygeneral.gov.uk/departments/pages/nfsa.aspx>.

¹⁶ See http://www.attorneygeneral.gov.uk/nfa/GuidetoInformation/Documents/NFA_fraud_indicator.pdf.

¹⁷ <http://data.worldbank.org/country/united-kingdom>. British pounds in 2008 are converted to dollars according to the U.S. Internal Revenue Service, "Yearly Average Exchange Rates," available at <http://www.irs.gov/businesses/small/international/article/0,,id=206089,00.html>.

Figure 3: Types of Frauds in the UK (2008)



Source: National Fraud Authority (January 2010), “Annual Fraud Indicator,” p. 7, available at http://www.attorneygeneral.gov.uk/nfa/GuidetoInformation/Documents/NFA_fraud_indicator.pdf.

Fraud is a worldwide crime. In 2009, the Association of Certified Fraud Examiners (“ACFE”) conducted a survey of its approximately 23,000 members and obtained detailed statistics on 1,843 occupational frauds occurring in 106 nations.¹⁸ Table 1 summarizes some of the key findings from the study. A strong inverse relationship exists between the median loss for a given type of fraud and the median duration of that type of fraud. For example, the least costly type of asset misallocation was register disbursement, with a median loss of \$23,000 and a median duration of 12 months, while the most costly type of asset misallocation was check tampering, with a median loss of \$131,000 and a median duration of 24 months. The most costly type of fraud in the survey was financial statement fraud, with a median loss of \$4.1 million and a median duration of 27 months.

¹⁸ Association of Certified Fraud Examiners (2010), “Report to the Nations on Occupational Fraud and Abuse,” available at <http://butest.acfe.com/rtn/rtn-2010.pdf>. The ACFE defines “occupational fraud,” which may occur in public or private organizations, as follows: “The use of one’s occupation for personal enrichment through the deliberate misuse or misapplication of the employing organization’s resources or assets.” Id. at p. 6.

Table 1: International Statistics on Occupational Frauds

Type of Fraud	Percentage of all Frauds ¹	Median Loss (\$)	Median Duration (Number of Months)
Asset Misallocation ²	89	135,000	12-24 ⁵
Corruption ³	33	250,000	18
Financial Statement ⁴	5	4,100,000	27

Source: Association of Certified Fraud Examiners (2010), “Report to the Nations on Occupational Fraud and Abuse,” available at <http://butest.acfe.com/rtnn/rtnn-2010.pdf>. ¹Percentages sum to more than 100 percent because several fraudulent schemes involved more than one type. ²“Asset misallocation” is the theft or misuse of an organization’s resources, e.g., skimming cash receipts, falsifying expense reports, or forging company checks. ³“Corruption” is an employee’s effort to influence business transactions in a way that violates the employee’s duty to the employer for the purpose of benefitting the employee or someone else, e.g., bribery, extortion, or conflict of interest. ⁴“Financial statement” fraud is the intentional misstatement or omission of material information in the organization’s financial records, e.g., fictitious revenues, concealing expenses or liabilities, or artificially inflating reported assets. ⁵The duration of the asset misallocation depended on the type of fraud.

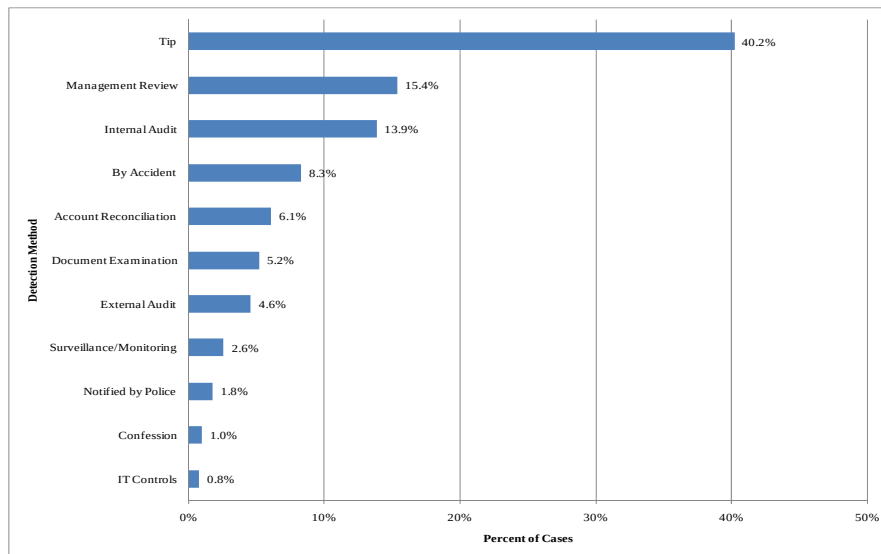
The ACFE survey asked its members to estimate the percentage of annual revenue that a typical organization loses to fraud. The median response was five percent, which implies annual global losses of hundreds of billions of dollars. This is consistent with the NFA’s estimate that fraud accounts for 2.1 percent of the UK’s GDP. Applied globally, this implies an annual cost of fraud of approximately \$1.3 trillion dollars.

In efforts to develop better methods to deter fraud, investigators have documented how frauds are detected. The AFCE survey reported that the most common form of detection was tips, which accounted for approximately 40 percent of all cases (see Figure 4). This finding was consistent with four prior studies conducted by the AFCE in 2002, 2004, 2006, and 2008. Approximately 50 percent of all tips came from employees, while 34 percent came from sources outside of the organization, e.g., customers and vendors. Management review and internal audits were the next most common forms of fraud detection.

Given the level of fraud, public agencies and private firms have taken actions to deter fraud. As summarized by ACFE: “In response to the discovery of fraud, more than 80% of the victim organizations in our study implemented or modified internal

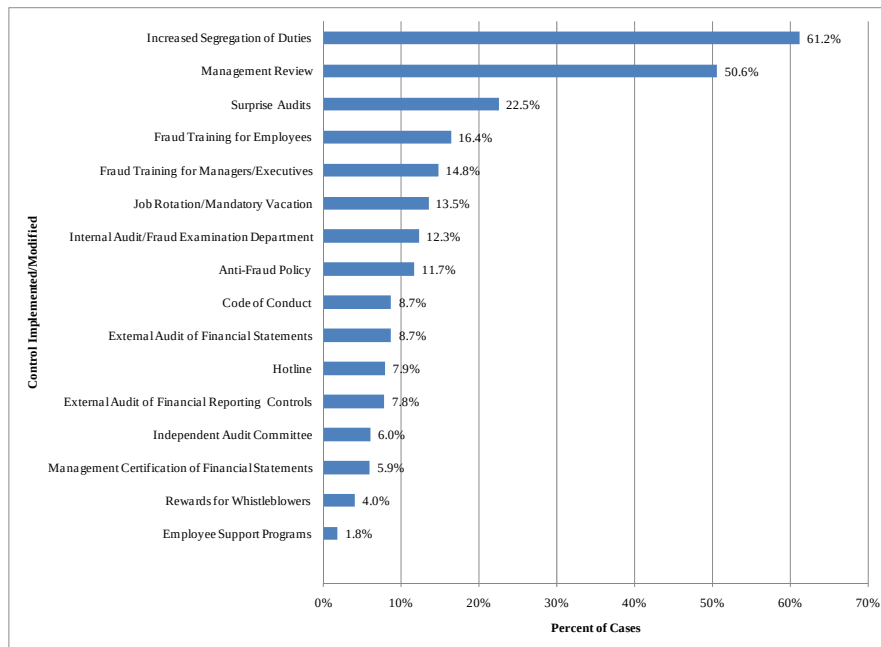
controls.” For those approximately 80 percent of organizations that did change their internal controls in response to fraud, Figure 5 shows the actions they took. The most common response was to increase employees’ segregation of duties, using the “need to know” principle. Increased management review and surprise audits were the next most commonly imposed changes in internal controls.

Figure 4: Initial Detection of Occupational Frauds



Source: Association of Certified Fraud Examiners (2010), “Report to the Nations on Occupational Fraud and Abuse,” p. 16, available at <http://butest.acfe.com/rtn/rtn-2010.pdf>.

Figure 5: Internal Controls Implemented or Modified in Response to Fraud



Source: Association of Certified Fraud Examiners (2010), “Report to the Nations on Occupational Fraud and Abuse,” p. 47, available at <http://butest.acfe.com/rtn/rtn-2010.pdf>. The percentages sum to more than 100 percent because many organizations implemented or modified more than one type of anti-fraud control.

The responses of organizations to fraud illustrates a general theme: new types of frauds are met with new or modified anti-fraud measures, which leads to innovations in types of frauds, which leads to more changes in anti-fraud measures, and so on. Forensic accountants have noted this cyclical pattern:

Fraud rolls across American history like waves move onto a beach. Fraud rises and falls with new innovations and ultimate corrections. For example, prior to the 1930s one innovative type of accounting fraud was to exaggerate the value of inventory on hand by reporting non-existent inventory. External auditors were not required to verify the physical presence of inventory on hand. The corrective measure came as a result of the famous McKesson Robbins scandal in which this company even reported inventory stocks in nonexistent warehouses. As a result of the lawsuit and intense media reporting of the bad audit, the [Certified

Public Accountant] profession instituted an auditing rule that required auditors to physically test for the existence of warehouses and inventory stocks within warehouses.

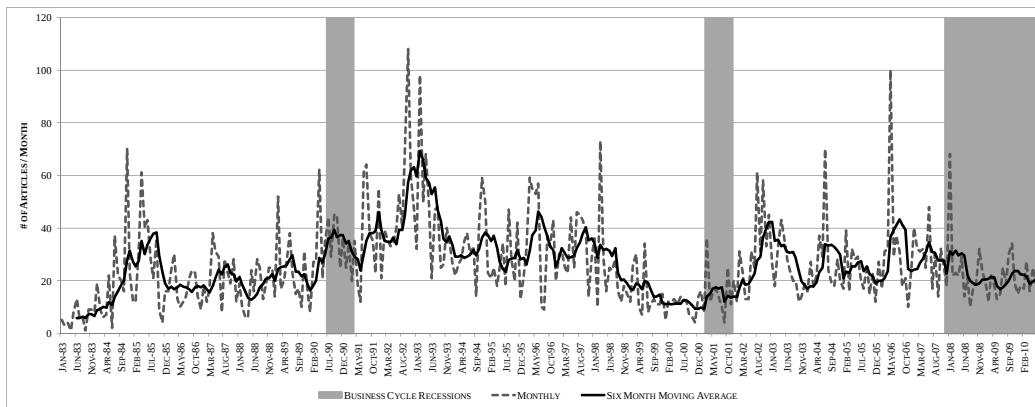
Each new corporate ploy to get around accounting and auditing rules eventually results in corrective accounting and auditing rules, which of course is why the exponentially growing set of such rules is becoming almost incomprehensible. The same thing happens with consumer and investor protection laws. When fraud finally gets so out of hand and has intense media exposure, U.S. democracy generally works. Corrective laws are eventually passed, and criminals are forced to seek newer and more innovative frauds.¹⁹

In order to obtain more insight into the frequency of fraud and the role of media exposure, we use the LexisNexis database to conduct full-text searches to determine the frequency of articles on different types of fraud. Here we report the frequency of articles containing the keywords: “bank fraud,” “wire fraud,” “mail fraud,” and “insurance fraud” (see Figures 6-9).²⁰

¹⁹ Jensen, Robert, “History of Fraud in America,” available at <http://www.trinity.edu/rjensen/FraudAmericanHistory.htm>.

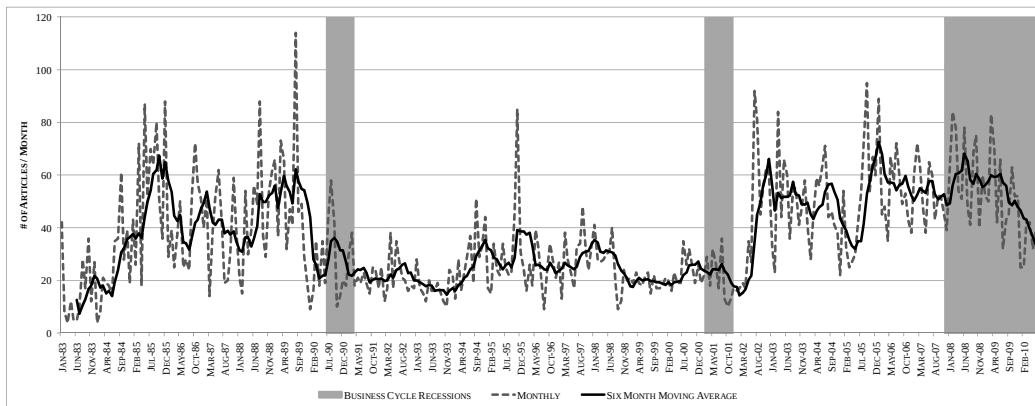
²⁰ See footnote 14 for sources and comments. Graphs illustrating the frequency of articles on other types of frauds are available at www.mcafee.cc/Papers/PDF/Fraudcycles/Graphs.pdf.

Figure 6: Frequency of “Bank Fraud” in U.S. Newspapers and Wires



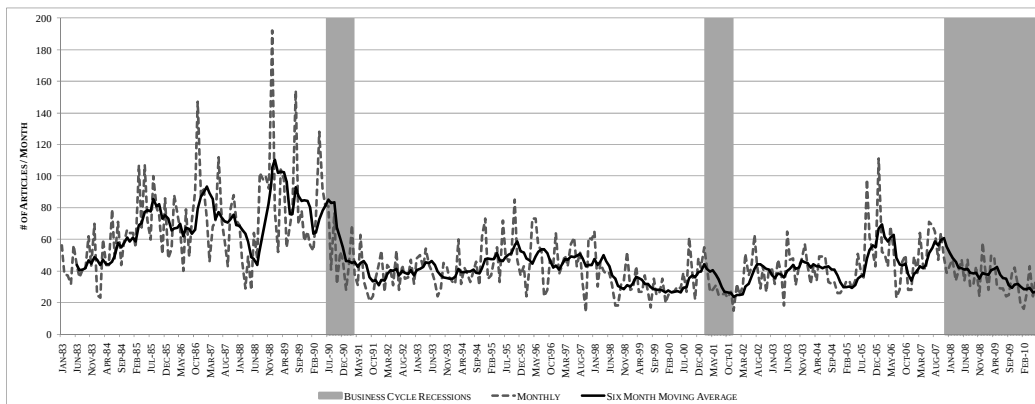
Source: LexisNexis.

Figure 7: Frequency of “Wire Fraud” in U.S. Newspapers and Wires



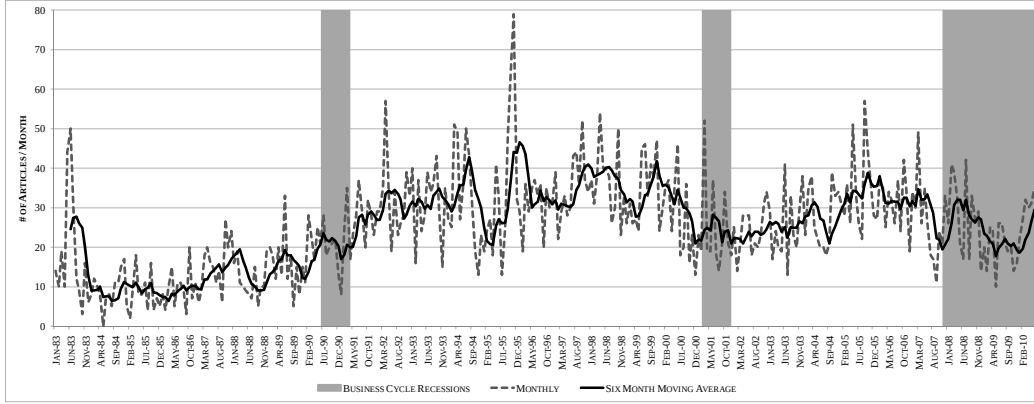
Source: LexisNexis.

Figure 8: Frequency of “Mail Fraud” in U.S. Newspapers and Wires



Source: LexisNexis.

Figure 9: Frequency of “Insurance Fraud” in U.S. Newspapers and Wires



Source: LexisNexis.

The graphs illustrate that the frequency of different types of fraud have substantial local peaks and troughs. Moreover the monthly frequencies of the different types of frauds are uncorrelated with each other²¹ and with the business cycle. In sum, the data indicate that the frequencies of different types of frauds are cyclical.

3. A Product Quality Model

In this section, we provide an example where fraudulent behavior displays cyclical patterns. Suppose sellers in a market offer products of either high or low quality, and y is the fraction of sellers with high quality. All of these products are sold as high quality. Buyers can verify the quality by incurring a verification cost c , and x is the fraction of verifiers. We assume that verifiers never buy the low-quality product. We let v be the net utility of the high-quality product; the net utility of the low-quality product equals zero.

A verifier who encounters a high-quality product, which occurs with probability y , buys, while a verifier who encounters a low-quality product does not buy but instead searches again, discounting utility due to delay by a rate δ . This means the verifier obtains a utility in satisfying $u = yv + (1 - y)\delta u - c$. The non-verifier obtains utility yv .

²¹ For the ten correlations on the five types of fraud, the correlations in the monthly number of articles range from -0.27 to 0.41.

We focus on verification and so rule out signaling, introductory prices, reputation, and other solutions given in the literature. These solutions are to some extent consistent with the model. For example, verification could entail checking a firm's reputation. The buyer's utility is:

$$u = \begin{cases} yv + (1 - y)\delta u - c & \text{if verify} \\ yv & \text{if not} \end{cases}$$

which simplifies to:

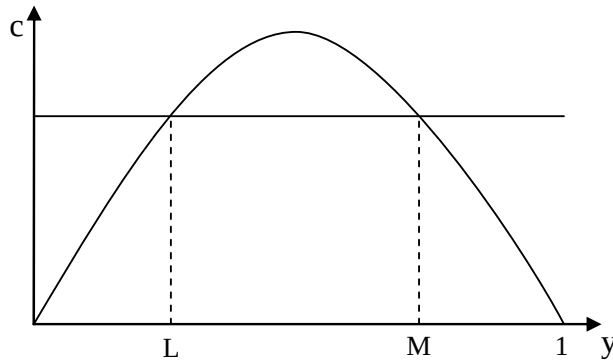
$$u = \begin{cases} \frac{yv - c}{1 - \delta(1 - y)} & \text{if verify} \\ yv & \text{if not} \end{cases} \quad (3.1)$$

The net utility gain to verification for a consumer equals:

$$\frac{yv - c}{1 - \delta(1 - y)} - yv = \frac{\delta y(1 - y)v - c}{1 - \delta(1 - y)} \quad (3.2)$$

The existence of at least two equilibria is apparent from (3.2), because there are typically two levels of y in which the consumer is indifferent between verifying and not. In Figure 10, we graph the net utility of verifying as a function of the proportion of high-quality firms. This utility is zero at both $y = 0$ and $y = 1$, because there is nothing to learn. Thus, if there is a level of y in which the value of verifying exceeds the cost, there will usually be two such levels. Call them L and M . Below L and above M verification does not pay, so the fraction of verifiers will tend to fall.

Figure 10: Net Benefits to Vigilance



The values of L and M can be derived by letting the numerator of (3.2), $\delta y(1-y)v - c$, equal zero, which yields two solutions, $\frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4c}{\delta v}} \right)$. This means L and M are on the opposite sides of $\frac{1}{2}$, assuming $\frac{c}{v} < \frac{1}{4} \delta$ to guarantee non-overlapping real values for L and M . We maintain this assumption throughout the section.

A company that makes the high-quality product earns a per unit profit denoted π_g and any buyer who shops with that seller purchases the high-quality product. Low-quality companies sell only to non-verifiers, but capture a share λ of the gain in value v of the high-quality product, in addition to the normal per unit profit π_g .

Verifiers stay in the market longer than non-verifiers, as there are on average $1/y$ searches per verifier, but just one search per non-verifier. Thus, the proportion of non-verifiers per search is $\frac{1-x}{1-x+x/y}$. The profit of low-quality companies is $\frac{1-x}{1-x+x/y} (\pi_g + \lambda v)$, and the profit of high-quality companies is π_g . The net gain to being a high-quality company, per arriving searcher, is then

$$\pi_g - \frac{1-x}{1-x+x/y} (\pi_g + \lambda v) = \frac{x\pi_g/y - (1-x)\lambda v}{1-x+x/y} \quad (3.3)$$

Before further extending our analysis, we briefly introduce a class of market evolution models pioneered by Farrell (1970).²² In general, when there are n types of interacting market forces indexed by i with market share z_i and utility u_i , the standard replicator dynamics (e.g., Hopkins 2002; Montgomery 2010) are given by:

$$z'_i = z_i(u_i - \sum_{j=1}^n z_j u_j) \quad (3.4)$$

In these models, agents following behavioral strategies that offer utility greater

²² Evolutionary models first appeared in biology and were later introduced to economics. Our model in this section is similar to the example cited in Friedman (1991), p. 641, footnote 9. For a comprehensive treatment of evolutionary games, see, e.g., Samuelson (1998).

than the average utility gain market share, while the others lose share. While such models are clearly appropriate for the study of evolution, where utility means “surviving offspring,” they are also reasonable for economic situations where people adapt slowly to changing circumstances. Slow adaptation appears empirically relevant, and indeed might be rational in a larger game where either information or attention has limited availability (e.g., Lucas 1974).

When there are only two actions, then $z_1 = 1 - z_2$, and equation (3.4) devolves to $z_1' = z_1(1 - z_1)(u_1 - u_2)$. Applied to our environment for the product quality model, we have two dynamic variables interacting with each other to constitute a system of differential equations. Based on (3.2) and (3.3), the two differential equations are:

$$x' = \alpha x(1 - x) \frac{\delta y(1-y)v-c}{1-\delta(1-y)} \quad (3.5)$$

$$y' = \beta y(1 - y) \frac{x\pi_g - (1-x)\lambda v y}{y - xy + x} \quad (3.6)$$

where α and β are parameters that permit us to vary the speed of adjustment; they correspond to a scaling of the utility of the buyers and sellers, respectively. There are two steady states in the interior of the unit square as shown in (3.7) and (3.8), in addition to (0, 0) and (1,1), which are also steady states. Their derivation is relegated to the appendix.

$$x^* = 1 - \frac{\pi_g}{\frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4c}{\delta v}} \right) \lambda v + \pi_g} \quad (3.7)$$

$$y^* = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4c}{\delta v}} \right) \quad (3.8)$$

Provided $1 > \frac{4c}{\delta v}$, there are three values of y consistent with a constant value of x , and moreover x is increasing only in the interval $\left(\frac{1 - \sqrt{1 - \frac{4c}{\delta v}}}{2}, \frac{1 + \sqrt{1 - \frac{4c}{\delta v}}}{2} \right)$. The

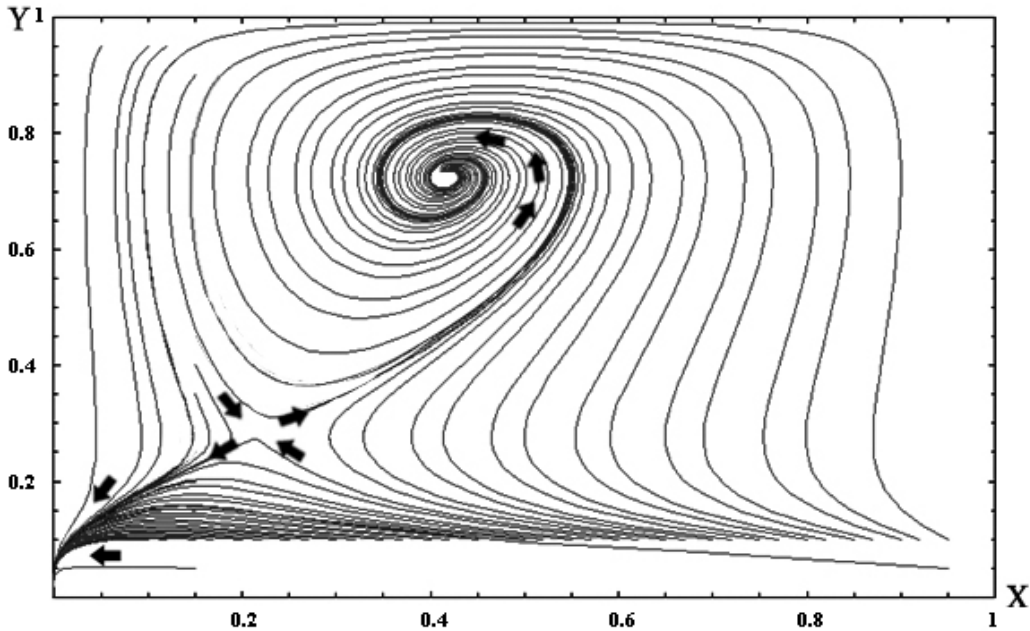
intuition is that if y is large enough, there is no point in verifying, since few companies sell low quality products. On the other hand, if y is small, it is too costly to verify since the expected cost of verifying is c/y . The steady state equilibrium defined above has two solutions, one with $y^* > 1/2$, and $y^* < 1/2$. The lower value of y^* is unstable, while the high value is a stable spiral.

Proposition 1: The higher value steady state in (3.7) and (3.8) is a stable spiral, when the following condition is satisfied:

$$\frac{c}{v} < \frac{1}{4} \delta - \frac{1}{4} \delta \left(\frac{2-\delta}{2+8\delta\frac{\alpha}{\beta}-2\delta} \right)^2 \quad (3.9)$$

All proofs are in the Appendix. Of the three steady states shown in Figure 11, the steady state producing the highest utility is locally stable and is a spiral. Thus our product quality model demonstrates the trait of periodic convergence or cyclicity to a stable steady state under external shocks.

Figure 11: Product Quality Model Phase Diagram



The (0,0) steady state is also locally stable and is always a node. It cannot be cyclic for the simple reason that market shares cannot be negative. The lower interior steady state is a saddle. It is unstable in the sense that unless the shares line up on one of the

arms pointing inward, it cannot be reached, and these arms are a set of measure zero in the space of shares. Unlike common dynamic models where one of the variables is a price or shadow price that can make a discrete jump, both variables in the present model evolve via their equations of motion (3.5) and (3.6). Thus, in the present model, saddle stable solutions are very unlikely to be observed.

Condition (3.9) can be compared to our original assumption that guarantees interior steady states, $\frac{c}{v} < \frac{1}{4} \delta$. First consider the case of no speedup where $\alpha = \beta = 1$. In that case, (3.9) becomes $\frac{c}{v} < \frac{1}{4} \delta - \frac{1}{4} \delta \left(\frac{2-\delta}{2+6\delta} \right)^2$. Note that the second term on the right hand side of this condition is always positive, so that the condition is satisfied for small values of c/v . Moreover, in a sense (3.9) is not “much” stronger in realistic settings, as it reduces the original upper bound of $1/4\delta$ by $(2 - \delta)^2/(2 + 6\delta)2$. This translates into a c/v upper bound reduction of less than 5% for δ above $2/3$. Condition (3.9) is easier to satisfy as customers react more rapidly, and harder to satisfy as the scammers react more rapidly.

4. A General Model

In this section, we extend the example of the previous section to a more general theory. We use financial securities fraud for illustrative purposes but as will be clear, the theory itself is quite flexible. Suppose there are two types of brokerage firms in an economy: law-abiding companies selling legitimate financial products and low-quality companies that purvey scam investments. Scams can range from fraudulent to legal but with unreasonably high transaction fees. Let the demand side be characterized by a percentage of the population that is vigilant, denoted by $x(t)$, where t is time. Demand for scams, i.e., the percentage of the population susceptible to scams, is $1 - x(t)$. By vigilant, we mean that those customers check the company’s background, verify its marketing literature, and take other due diligence efforts. Vigilance is costly. Let $y(t)$ denote the percentage of high-quality companies in the market at time t . Equivalently one can think of $1 - y(t)$ as the probability

that a customer encounters a low-quality company.

Under a dynamic framework, $x(t)$ and $y(t)$ certainly interact with each other. An increase in the probability of encountering low-quality companies is likely to increase the percentage of the population who are vigilant. Likewise, the more vigilant people are, the less profitable it is for companies to purvey scams. The interaction between $x(t)$ and $y(t)$ in a dynamic market can be modeled by the following system of differential equations:

$$\begin{cases} x'(t) = \alpha f(x(t), y(t)) \\ y'(t) = \beta g(x(t), y(t)) \end{cases} \quad (4.1)$$

This model is general enough to cover a variety of fraud situations. As before, α and β are parameters to vary the speed of adjustment. When $1 - y(t)$ represents the percentage of corporate executives who misrepresent financial information, $x(t)$ can represent the percentage of vigilant investors. In this case, we have a model of corporate financial misrepresentation. Such misrepresentation may be legal, where firms mislead rather than lie. When $1 - y(t)$ represents the percentage of investment firms who plot Ponzi schemes, $x(t)$ can describe the percentage of investors who are not easily swindled. This model applies to the recent boom in Ponzi schemes led by Bernie Madoff. The variables $x(t)$ and $1 - y(t)$ also can denote the percentage of government auditors and corrupt officials, respectively, in which case we have a model to study corruption. In fact, as long as $x(t)$ and $y(t)$ are interacting forces in a market whose evolution is described by a pair of ordinary differential equations, this model can be used to study its equilibrium properties.

We are interested first in stable behavior, and second when such behavior is a spiral, in which case the convergence path displays a cyclical pattern. Mathematically both stability and cyclicity are determined by the trace and determinant of the Jacobian matrix of the system of differential equations (Luenberger, 1979).

In most differential equation models with two variables found in economics,

one variable is a state variable (e.g., capital) and the other a co-state variable (e.g., price). In such models, the co-state variable generally adjusts instantly, so solutions involve saddle paths, with the co-state variable jumping instantly to be on the path, while the state variable adjusts smoothly. In contrast, both of the relevant variables in our analysis are state variables, so saddle-stability is unstable, as there is a measure-zero set of starting points leading to the steady state.

Let x^* and y^* be a steady state, meaning that they satisfy the following:²³

$$\begin{cases} 0 = f(x^*, y^*) \\ 0 = g(x^*, y^*) \end{cases} \quad (4.2)$$

Applying Taylor's expansion around (4.2) above and ignoring higher order terms, (4.1) can be linearized as:

$$\begin{cases} x' \approx \alpha \frac{\partial f(x^*, y^*)}{\partial x} (x - x^*) + \alpha \frac{\partial f(x^*, y^*)}{\partial y} (y - y^*) \\ y' \approx \beta \frac{\partial g(x^*, y^*)}{\partial x} (x - x^*) + \beta \frac{\partial g(x^*, y^*)}{\partial y} (y - y^*) \end{cases} \quad (4.3)$$

To shorten notation, let $f_x = \frac{\partial f(x^*, y^*)}{\partial x}$, $f_y = \frac{\partial f(x^*, y^*)}{\partial y}$, $g_x = \frac{\partial g(x^*, y^*)}{\partial x}$, and $g_y = \frac{\partial g(x^*, y^*)}{\partial y}$. In matrix format, the above system can be rewritten as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \approx A \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}, \text{ where } A = \begin{bmatrix} \alpha f_x & \alpha f_y \\ \beta g_x & \beta g_y \end{bmatrix} \quad (4.3)'$$

We use the following lemma, found, e.g., in Luenberger (1979), to describe the steady state nature and its convergence patterns.

Lemma 2: A steady state to (4.1) is strictly stable if $Tr(A) < 0$ and $det(A) \geq 0$. Furthermore, a stable steady state is locally a stable spiral if $Tr(A)^2 < 4det(A)$, and a sink $Tr(A)^2 > 4det(A)$.

Looking at the model in (4.1), it is easy to see that a speedup or slowdown of

²³ We suppress the notation for the time variable t for simplicity, whenever the practice does not cause confusion.

the system, changing (α, β) , will not change the steady states and their stability conditions, as stated in Proposition 3 below. This means that changes in media technology, which might make either side react faster, will only have short-term effects. Such effects are unlikely to change the stability of the steady states, which consist of equilibrium fractions of low-quality companies and vigilant consumers, although paths to the steady states would of course change.

Proposition 3: The set of steady states and the stability of those states are invariant to the speed up parameters (α, β) .

When are stable steady states spirals? If the off-diagonal elements have the same sign, a spiral is impossible. In contrast, if the off-diagonal elements have opposite signs, then spirals exist for intermediate levels of β/α . This most important result of our general model is summarized in the following proposition.

Proposition 4: Under (4.3)' where $\det(A) \neq 0$ and $g_y \neq 0$, a stable steady state is always a sink, if $f_y g_x > 0$. If $f_y g_x < 0$, there exists an interval in β/α for which a stable steady state is a spiral. The interval is bounded away from 0 and ∞ .

Proposition 4 implies that if α and β are too different when $f_y g_x < 0$, in other words if one type of agent reacts rapidly and the other slowly, then the steady state would be a sink. When they are within intermediate ranges, a spiral results. Proposition 4 also implies the following corollary, which confirms proposition 1 for the example in the previous section in a generalized fashion.

Corollary 5: When $f_x g_y = 0$, there exists an interval in β/α for which a stable steady state is a spiral.

The critical condition for a spiral, $f_y g_x < 0$, is met if $f_y < 0$ and $g_x > 0$. Going back to the definition of these terms, $f_y < 0$ if an increase in the number of high-quality companies will reduce the rate of increase of being vigilant and verifying, while $g_x > 0$ requires that an increase in being vigilant and verifying will increase the rate of growth of high-quality companies. Both of these are reasonable conditions suggested by the class of problems motivating the analysis.

5. Concluding Remarks

We reviewed historic data on frauds and identified empirical evidence of peak and trough patterns associated with some common frauds. The data suggest that frauds tend to follow a cyclical path. We proposed a new theory to show that cyclicity follows inherently from interacting market forces to (1) shun and eliminate frauds on the demand side and (2) sustain and perpetuate frauds on the supply side. The strategic interaction between the demand and supply forces of fraud causes the cyclical path to equilibrium to be robust to outside shocks, such as government intervention. We identify conditions under which such cyclical behavior occurs.

Fundamentally, cyclicity is caused by market players' non-reinforcing responses to external shocks under stable dynamic systems. As scammers become more successful, customers react by increased wariness, reducing the pay to scamming. This cyclicity is substantially different than the supply and demand hog cycle, which was predicated on delayed reactions—hog ranchers increasing the stock in reaction to today's prices, which results in an increase in supply next season. In contrast, the cyclicity discovered in this paper is a consequence of endogenous delay, driven off the evolutionary dynamics. The present theory is general enough to investigate a variety of frauds. The characterization of local behavior greatly simplifies the understanding of system equilibrium behavior without actually solving for the solutions, which can be immensely complicated even under a simple specification like the example provided in Section 3.

Future research effort points to two directions. One is to empirically study more types of frauds to further evaluate the hypothesis of cyclical behavior. Unlike business cycles, for which data are collected systematically by macroeconomics policy authorities and other economics research institutions, fraud cycles are more subtle to discern due to their fundamentally illegal nature. Observation tends to reduce fraud through increased awareness. On the theoretical side, our model primarily focuses on the spiral steady state that results in cyclical behavior. A natural extension is the case of swindlers choosing from a set of potential frauds.

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Appendix

DERIVATION OF THE STEADY STATES:

First note from (3.5) that $x' = 0$ arises when $y^2 - y + c/\delta v = 0$, which if it has a solution in $[0,1]$, it has two generically. These solutions are $y^* = (1 \pm \sqrt{1 - 4c/\delta v})/2$. By (3.6), the steady states in y arise when $x\pi_g/y - (1-x)\lambda v = 0$, or $x^* = \lambda v/(\lambda v + \pi_g/y^*)$. Plug in the solution for y^* to yield the two steady, nonzero states for (x^*, y^*) .

PROOF OF PROPOSITION 1:

We first linearize the system by first order approximation around the steady state:

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &\approx \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} \\ &= \begin{bmatrix} \alpha(1-2x^*)\frac{\delta y^*(1-y^*)v-c}{1-\delta(1-y^*)} & \alpha x^*(1-x^*)\frac{\delta(1-2y^*)v}{1-\delta(1-y^*)} \\ \beta y^*(1-y^*)\frac{\pi_g/y^*+\lambda v}{1-x^*+x^*/y^*} & \beta y^*(1-y^*)\frac{-(1-x^*)\lambda v/y^*}{1-x^*+x^*/y^*} \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} \end{aligned} \quad (\text{A.1})$$

Note that $a_{11} = 0$, because $\delta y^*(1-y^*)v - c = 0$. It is obvious that every product term in a_{21} is positive such that $a_{21} > 0$. The same can be said about a_{22} other than that $-(1-x^*)\lambda v/y^*$ in a_{22} is negative. Therefore, $a_{22} < 0$. Finally $a_{12} < 0$, since $1 - 2y^* < 0$ as y^* of our interest is of the higher value, and $1 - \delta(1-y^*) > 0$. Then the stability result follows Lemma 2.

To have a steady state be a spiral, one needs the condition $Tr(A)^2 - 4 \det(A) < 0$.
 $Tr(A)^2 - 4 \det(A)$

$$\begin{aligned}
&= [\beta y^*(1-y^*) \frac{-(1-x^*)\lambda v/y^*}{1-x^*+x^*/y^*}]^2 \\
&\quad + 4\alpha\beta x^*(1-x^*)y^*(1-y^*) \left[\frac{\delta(1-2y^*)v}{1-\delta(1-y^*)} \right] \left[\frac{\frac{\pi_g}{y^*} + \lambda v}{1-x^*+\frac{x^*}{y^*}} \right] \\
&= \frac{\beta(1-x^*)y^*(1-y^*)}{1-x^*+x^*/y^*} \left\{ \frac{\beta(1-x^*)(1-y^*)}{y^*(1-x^*)+x^*} (\lambda v)^2 + 4\alpha x^* \frac{\delta(1-2y^*)v}{1-\delta(1-y^*)} (\lambda v/x^*) \right\} \\
&= (\beta\lambda v^2) \frac{(1-x^*)y^*(1-y^*)}{1-x^*+x^*/y^*} \left\{ \frac{\beta(1-y^*)\lambda}{y^*(1+\lambda v/\pi_g)} + 4 \frac{\alpha\delta(1-2y^*)}{1-\delta(1-y^*)} \right\} \tag{A.1}
\end{aligned}$$

The above steps use the steady state condition $x^*\pi_g - (1-x^*)\lambda v y^* = 0$. For (A.1) to be negative, it suffices to show the third term in (A.1), what is in the large parenthesis, to be negative, since all the product terms in the front are all positive. For convenience denote $\theta = \sqrt{1 - \frac{4c}{\delta v}}$ such that the high value steady state $y^* = \frac{1}{2}(1 + \theta)$, and plug it into the last part of (A.1). We then need to show the following:

$$\frac{\beta(1-y^*)\lambda}{y^*(1+\lambda v/\pi_g)} + 4 \frac{\alpha\delta(1-2y^*)}{1-\delta(1-y^*)} = \frac{\beta(1-\theta)\lambda}{(1+\theta)(1+\lambda v/\pi_g)} - \frac{8\alpha\delta\theta}{2-\delta(1-\theta)} < 0$$

After arranging terms, we have

$$2\beta(1-\theta)\lambda - \beta\delta(1-\theta)^2\lambda - 8\alpha\delta\theta(1+\theta) \left(1 + \frac{\lambda v}{\pi_g}\right) < 0 \tag{A.2}$$

Note that (A.2) does not always hold, for example when $\theta \rightarrow 0$. But since $1 + \frac{\lambda v}{\pi_g} >$

λ , a sufficient condition for (A.2) is $2\beta(1-\theta) - \beta\delta(1-\theta)^2 - 8\alpha\delta\theta(1+\theta) < 0$.

Or,

$$\delta(8\alpha + \beta)\theta^2 + (2\beta + 8\alpha\delta - 2\beta\delta)\theta + \beta(\delta - 2) > 0 \tag{A.3}$$

Since the first term of (A.3) is always positive, we just look at a sufficient condition:

$$(2\beta + 8\alpha\delta - 2\beta\delta)\theta + \beta(\delta - 2) > 0 \tag{A.4}$$

Plugging the definition of θ and solving for c/v , it can be shown that (A.4) is satisfied under (3.9). QED.

PROOF OF LEMMA 2:

Let $Tr(A)$ denote the trace of matrix A , $\alpha f_x + \beta g_y$, and let $det(A)$ denote its determinant, $\alpha f_x \beta g_y - \alpha f_y \beta g_x$. Eigenvalues of matrix A , λ , solving $\lambda^2 -$

$(\alpha f_x + \beta g_y)\lambda + (\alpha f_x \beta g_y - \alpha f_y \beta g_x) = 0$, yields

$$\lambda = \frac{1}{2}[(\alpha f_x + \beta g_y) \pm \sqrt{(\alpha f_x - \beta g_y)^2 + 4\alpha f_y \beta g_x}], \text{ which can be rearranged as } \lambda = \frac{1}{2}[Tr(A) \pm \sqrt{Tr(A)^2 - 4det(A)}].$$

Then the behavior of the steady state depends

on the sign of $Tr(A)$ and whether the square root of the term in the parenthesis is a real or complex number. This leads to Lemma 2, which is a standard result in systems of linear differential equations (e.g., Luenberger 1979). QED.

PROOF OF PROPOSITION 3:

The steady state condition is obvious by the definition of (x^*, y^*) in (4.2).

The stability condition is also straightforward by noting that in the matrix $A = \begin{bmatrix} \alpha f_x & \alpha f_y \\ \beta g_x & \beta g_y \end{bmatrix}$ of (4.3)', $Tr(A) = \alpha f_x + \beta g_y > 0$ when $f_x + g_y > 0$ and $\alpha, \beta > 0$.

Its determinant is also positive, since $det(A) = \alpha\beta(f_x g_y - f_y g_x) \geq 0$ when $f_x g_y - f_y g_x \geq 0$ and $\alpha, \beta > 0$. QED.

PROOF OF PROPOSITION 4:

The sign of $Tr(A)^2 - 4det(A) = (\alpha f_x - \beta g_y)^2 + 4\alpha\beta f_y g_x$ is the same as the sign of $(f_x - \gamma g_y)^2 + 4\gamma f_y g_x$ where $\gamma = \beta/\alpha$. The sign of the latter can be examined by looking at its minimum, since it is convex in γ . Solving for γ minimum's first order condition yields $\gamma^* = \frac{f_x}{g_y} - \frac{2\alpha\beta f_y g_x}{g_y^2}$. Plugging this solution

back to the objective function gives:

$$\min Tr(A)^2 - 4det(A) = 4\left(\frac{\alpha f_y \beta g_x}{g_y^2}\right)(\alpha f_x \beta g_y - \alpha f_y \beta g_x) \quad (A.5)$$

Now $g_y^2 > 0$ and $\alpha f_x \beta g_y - \alpha f_y \beta g_x > 0$ by the stability condition. Thus the sign of $Tr(A)^2 - 4det(A)$ is determined by the sign of $f_y g_x$ when $\alpha, \beta > 0$. QED.