Contingent Preference for Flexibility: Eliciting Beliefs from Behavior

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October 10, 2010
ERID Working Paper Number 83

Economic Research Initiatives at Duke
WORKING PAPERS SERIES

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Abstract

Following Kreps (1979), I consider a decision maker who is uncertain about her future taste. This uncertainty leaves the decision maker with a preference for flexibility: When choosing among menus containing alternatives for future choice, she weakly prefers menus with additional alternatives. Standard representations accommodating this choice pattern cannot distinguish tastes (indexed by a subjective state space) and beliefs (a probability measure over the subjective states) as different concepts. I allow choice between menus to depend on objective states. My axioms provide a representation that uniquely identifies beliefs, provided objective states are sufficiently relevant for choice. I suggest this result as a choice theoretic foundation for the assumption, commonly made in the (incomplete) contracting literature, that contracting parties who know each others' ranking of contracts, also share beliefs about each others' future tastes in the face of unforeseen contingencies.

Keywords: Preference for Flexibility, Unique Beliefs, Unforeseen Contingencies, Incomplete Contracts

1. Introduction

The expected utility model of von Neumann and Morgenstern (1944, henceforth vNM) explains choice under risk by considering probabilities and tastes separately. In the context of choice under subjective uncertainty, the corresponding separation of beliefs and tastes is a central concern. For the extreme case where all subjective uncertainty can be captured by objective states of the world, the works of Savage (1954) and Anscombe and Aumann (1963, henceforth AA) achieve this separation. In the opposing extreme, where none of the subjective uncertainty can be captured by objective states, uncertainty can be modeled with

*I thank Roland Benabou and Wolfgang Pesendorfer for their invaluable advice. I am also grateful to David Dillenberger, Faruk Gul, R. Vijay Krishna, Mark Machina, Eric Maskin, Tymon Tatur, and Justin Valasek for helpful lessons and discussions.

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a subjective state space. Kreps (1979, henceforth Kreps) and Dekel, Lipman and Rustichini (2001, henceforth DLR; a relevant corrigendum is Dekel et al. [2007, henceforth DLRS])[1] find that the separation is not possible in this case.

In the general case, some, but potentially not all, subjective uncertainty can be captured by objective states. This paper analyzes a model of choice under such general subjective uncertainty, which features the AA and DLR models as special cases.[2] The model separately identifies tastes and beliefs over those tastes, provided that objective states are 'relevant enough.' A tight behavioral characterization of relevant enough is given.

The timing of choice is as follows: In period 1, the decision maker (DM) chooses an opportunity act. An opportunity act specifies a menu of alternatives for future choice contingent on the objective state. Between periods 1 and 2 an objective state realizes. In period 2 the act is evaluated and DM gets to choose from the resulting menu. Only period 1 choice is observed. If objective states do not account for all subjective uncertainty that resolves between periods 1 and 2, then DM has contingent uncertainty about her future taste. In that case, commitment to a contingent plan of period 2 choice is costly and one should observe contingent preference for retaining flexibility: All else being equal, DM prefers an act that assigns a menu with additional alternatives to any particular state.

This paper provides a representation of such preferences, labeled a representation of Contingent Preference for Flexibility (CPF). As in DLR, subjective uncertainty is modeled via a subjective state space, which collects all possible tastes that might govern DM's choice in period 2. I call it the taste space. DM conditions her beliefs about her future tastes on objective states. For any particular state, choice over menus has a subjective expected utility representation, as in DLR. I show that the central new axiom, *Relevant Objective States*, is equivalent to the unique identification of utilities and conditional beliefs in this representation.

To be more specific, let I be the objective state space. An opportunity act, g, assigns a contingent menu of lotteries over prizes, $g(i)$, to every objective state, i, in I. Accordingly, the taste space, S, collects all possible vNM rankings of lotteries over prizes. In the case of finite I, choice over acts has a CPF representation, if it can be represented by

$$V(g) = \sum_{i \in I} \phi(i) \left[\int_S \left(\max_{\alpha \in g(i)} U_s(\alpha) \right) d\mu(s \,|i) \right],$$

where ϕ is a probability measure on I, the realized vNM utility function U_s represents taste

[1] Throughout the paper I refer to the version of their model that represents preference for flexibility.

[2] In the Savage and Kreps models there is no objective uncertainty (or risk), while AA, DLR, and the present paper consider a combination of subjective and objective uncertainty.

s, and $\mu(s|i)$ is a probability measure on S. The representation suggests that, while the menu of alternatives DM expects to choose from in stage 2 depends on i and not s, she anticipates a utility function that depends on s and not i. She also expects to learn s and i prior to choosing an alternative. The measure ϕ is interpreted as DM's prior over I and $\mu(s|i)$ is interpreted as the belief that taste s occurs, contingent on i.

Theorem 1 takes the CPF representation and the distribution ϕ as given.[3] It establishes that conditional beliefs $\mu(s|i)$ are unique and utilities U_s are unique in an appropriate sense, if and only if choice between opportunity acts satisfies the Relevant Objective States axiom. The axiom is formulated in terms of DM's induced state contingent ranking of menus, which is derived from her choice over acts. Say that two menus are *the same for DM*, if for every contingent ranking the union of those menus is as good as either of the menus individually. Objective states are relevant, if for any two menus that are not the same for DM, there is a contingent ranking under which one is preferred over the other.

Theorem 2 states that choice over opportunity acts has a CPF representation if and only if it satisfies the immediate extensions of the AA and DLR axioms. These axioms are necessary for a more general representation, where both beliefs and utilities depend on objective states. For the separation of beliefs and tastes, however, it is important that only beliefs condition on objective states. Theorem 2 implies that this interpretation is always possible, as it does not constrain period 1 choice.

The CPF representation is a description of period 1 choice, where DM behaves *as if* she held beliefs about possible tastes that might govern period 2 choice. Theorem 1 relates beliefs, which are parameters of the representation, to period 1 choice behavior. The natural inductive step is to also employ her beliefs about future tastes to forecast period 2 choice behavior. On the one hand this requires evaluating the appropriateness of the representation for a particular application, on the other hand the model can be refuted if its forecasts do not agree with observation.

Being able to forecast behavior can be important in strategic situations, for example when one party's valuation of a contract depends on future actions taken by the other party. Contracting models usually have to assume that, first, parties know each others' ranking of contracts and that, second, they share common beliefs about future utility-payoffs when writing the contract. The first assumption raises the complex game theoretic question of how parties learn each others' ranking of contracts; this question is usually not formally addressed in applied models and is not my focus here. Instead, I am concerned with the second assumption. If two parties write a contract in the face of indescribable or unforeseen

[3] ϕ could be objective. If ϕ is subjective as suggested above, it must also be elicited from choice. I address this case in Theorem 3.

contingencies,[4] it seems natural that there might be asymmetric information about those contingencies. In a survey on incomplete contracts, Tirole (1999) speculates that *"... there may be interesting interaction between "unforeseen contingencies" and asymmetric information. There is a serious issue as to how parties [...] end up having common beliefs ex ante."* Beliefs that are elicited from a party's ranking of contracts give choice theoretic substance to the assumption of common beliefs.[5]

As an illustrative example of a CPF representation, consider a retailer, who writes a contract with her supplier today about tomorrow's order. The demand, s, facing the retailer tomorrow will be either high (h) or low (l). Today s is unknown to both parties, tomorrow it will become the private knowledge of the retailer. The only relevant public information that becomes available tomorrow is consumer confidence, i, a general market indicator, which will also be either high (H) or low (L). Thus, a contract, g, can only condition on consumer confidence, not on demand. The most efficient contract might give the retailer some choice of supply quantities, q, contingent on consumer confidence; consider this type of contract. From the retailer's perspective, the contract is an act in the terminology of this paper. Routinely one might write down the following objective function for the retailer's choice between contracts:

$$V(g) = \sum_{i \in \{H,L\}} \phi(i) \left[\sum_{s \in \{h,l\}} \mu(s \,|i) \max_{q \in g(i)} (U_s(q)) \right].$$

First, take consumer confidence, $i \in \{H, L\}$, as given. The retailer can then order any quantity in $g(i)$. If tomorrow she faces demand $s \in \{h, l\}$, she will choose the quantity q that maximizes her profits, $U_s(q)$.[6] Today she does not know tomorrow's demand, but she can assign probabilities conditional on consumer confidence, $\mu(s\,|i)$. She values the menu $g(i)$ at its expected value, $\sum_{s \in \{h,l\}} \mu(s\,|i) \max_{q \in g(i)} (U_s(q))$. Second, she takes an expectation over different levels of consumer confidence according to a probability distribution ϕ. This is an example of a CPF representation.[7]

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[4] Kreps (1992) points out that a subjective taste space naturally accounts for contingencies that are not just unobservable or indescribable, but unforeseen, at least by the observer.

[5] Dekel, Lipman and Rustichini (1998-a) note that *"... there are very significant problems to be solved before we can generate interesting conclusions for contracting [...] while the Kreps model (and its modifications) seems appropriate for unforeseen contingencies, [...] there are no meaningful subjective probabilities. A refinement of the model that pins down probabilities would be useful."*

[6] While 'demand' is actually observable in many situations, unobservable demand levels here simply serve as convenient labels for the different unobservable profit functions the retailer can conceive of.

[7] The CPF representation also evaluates more general contracts, where, contingent on consumer confidence, the retailer is given some choice between non-degenerate lotteries, α, over different quantities. For example, the contract might specify an action which has probabilistic consequences.

The example also speaks to the possible strategic value of uniquely idetified beliefs. Suppose the retailer has private knowledge about tomorrow's demand, contingent on consumer confidence. Demand may affect the supplier's profit indirectly, through tomorrow's choice of quantity by the retailer. Therefore, when evaluating contracts, the supplier would like to forecast demand based on the retailer's beliefs. I show that if the supplier knows the retailer's ranking of contracts, then he is able to identify the retailer's beliefs, $\mu(s\,|i)$, if and only if consumer confidence is *relevant*.

Section 2 demonstrates that beliefs might be identified in the example above. Section 3 lays out the model and establishes Theorems 1 and 2, first for a finite and then for a general topological objective state space. Section 4.1 contains Theorem 3, which combines the two results and elicits ϕ from choice. For the case where ϕ corresponds to objective probabilities, section 4.2 establishes robustness of elicited beliefs to misspecifications of ϕ. Section 4.3 points out a unique representation without aggregation of contingent preferences over I. Section 5 characterizes two simple examples of behavioral comparisons in terms of the unique beliefs. Section 6 discusses related literature. Section 7 comments in more detail on possible implications for contracting. Section 8 concludes.

2. Illustration of Identification of Beliefs

In this section, I consider three cases of a CPF representation: when none of the subjective uncertainty can be captured by objective states (irrelevant objective states); when all of the subjective uncertainty can be captured by objective states (no preference for flexibility); and the general case, where some, but not all, of the subjective uncertainty can be captured by objective states (preference for flexibility and relevant objective states). To illustrate these cases, I use the setup of the above example, but where final outcomes are lotteries, α, over quantities.

• Irrelevant objective states: Suppose that the retailer's beliefs are independent of consumer confidence; that is $\mu(h\,|H) = \mu(h\,|L) = \mu(h)$. In this case, her induced ranking of menus is independent of consumer confidence and it is without loss of generality to consider only contracts with $g(H) = g(L)$. If g is such a non-contingent contract, then

$$V(g) = \sum_{s\in\{h,l\}} \mu(s) \max_{\alpha\in g(H)} (U_s(\alpha)).$$

This is an example of DLR's representation. To see that beliefs are not identified, consider a different probability distribution $\widehat{\mu}(s)$ on $S = \{h, l\}$ and rescaled utilities

$$\widehat{U}_s(x) = U_s(x) \frac{\mu(s)}{\widehat{\mu}(s)}.$$

Then

$$\sum_{s \in \{h,l\}} \mu(s) \left(\max_{\alpha \in g(H)} U_s(\alpha) \right) \equiv \sum_{s \in \{h,l\}} \widehat{\mu}(s) \left(\max_{\alpha \in g(H)} \widehat{U}_s(\alpha) \right).$$

This is the fundamental indeterminacy in the Kreps and DLR models and variations of those models.

• No preference for flexibility: Suppose that $\mu(h|H) = 1$ and $\mu(h|L) = 0$. Now subjective uncertainty is perfectly captured by the objective states, and it is without loss of generality to identify h with H and l with L. This implies that none of the contingent rankings exhibit preference for flexibility. One can confine attention to contracts with lotteries, instead of menus, as outcomes. If $g(i) = \alpha_i$ is such a fully specified contract, then

$$V(g) = \sum_{i \in \{H,L\}} \phi(i) U_i(\alpha_i).$$

This is an example of AA's state dependent representation.

• Preference for flexibility and relevant objective states: Lastly, suppose the retailer believes that the probability of high demand is increasing with consumer confidence; that is $1 > \mu(h|H) > \mu(h|L) > 0$. Further suppose that there is another representation of the same ranking of contracts based on the same prior over objective states, ϕ, but with beliefs $\widehat{\mu}(s|i)$ and tastes $\widehat{U}_s$:

$$\widehat{V}(g) = \sum_{i \in \{H,L\}} \phi(i) \left[\sum_{s \in \{h,l\}} \widehat{\mu}(s|i) \max_{\alpha \in g(i)} \left(\widehat{U}_s(\alpha) \right) \right].$$

V and $\widehat{V}$ have to generate the same ranking of contracts.

Consider two quantities (or degenerate lotteries) q_h and q_l such that the retailer prefers to receive q_h if demand is high and q_l if demand is low, that is, $U_h(q_h) - U_h(q_l) > 0$ and $U_l(q_h) - U_l(q_l) < 0$. Slightly abusing notation, I denote a lottery that gives q_h with probability α and q_l with probability $1 - \alpha$ by α. I denote the menu that contains lotteries α and β by $\{\alpha, \beta\}$.

Suppose for some $\beta < \alpha$ and $\delta, \varepsilon \in (0, 1-\alpha)$ the retailer is indifferent between the two contracts

$$g = \begin{pmatrix} \{\alpha+\delta, \beta\} \text{ if } i = H \\ \{\alpha, \beta\} \text{ if } i = L \end{pmatrix}$$
$$g' = \begin{pmatrix} \{\alpha, \beta\} \text{ if } i = H \\ \{\alpha+\varepsilon, \beta\} \text{ if } i = L \end{pmatrix}.$$

$\beta < \alpha$ implies that α is relevant for the value of these contracts only under taste h. Hence, $g \sim g'$ implies that

$$\phi(H)\,\mu(h\,|H)\,\delta\,(U_h(q_h) - U_h(q_l)) = \phi(L)\,\mu(h\,|L)\,\varepsilon\,(U_h(q_h) - U_h(q_l))\,.$$

An analogous equality must hold for the parameters of $\widehat{U}$. Therefore,

$$\frac{\mu(h\,|H)}{\mu(h\,|L)} = \frac{\varepsilon\phi(L)}{\delta\phi(H)} = \frac{\widehat{\mu}(h\,|H)}{\widehat{\mu}(h\,|L)}.$$

Similarly,

$$\frac{\mu(l\,|H)}{\mu(l\,|L)} = \frac{\widehat{\mu}(l\,|H)}{\widehat{\mu}(l\,|L)}.$$

Since μ and $\widehat{\mu}$ are both probability measures, it follows immediately that $\mu \equiv \widehat{\mu}$. Standard arguments, applied to the comparison of contracts which disagree only under state i, imply that the scaling of the expected utility functions U_h and U_l is unique up to a common linear transformation. This argument illustrates how identification relies crucially on the fact that beliefs ϕ over objective states are held fixed.

The above reasoning can be generalized to any finite state space, I. If a CPF representation has the feature that there are at least as many linearly independent probability measures over the taste space, indexed by $i \in I$, as there are relevant tastes, then beliefs are uniquely identified and the scaling of utilities is uniquely identified up to a common linear transformation. For the proof of Theorem 1, however, no particular representation is given. The theorem implies that the CPF representation of any ranking that satisfies Relevance of Objective States must have this feature.

3. A Model with Unique Beliefs

Consider a two-stage choice problem, where an objective state realizes between the two stages. In period 2 DM chooses a lottery over prizes. This choice is not modelled explicitly.

In period 1 DM chooses an opportunity act. Such an act is a state contingent specification of a set of lotteries (a menu) that contains the feasible alternatives for period 2 choice.

Let Z be a finite prize space with cardinality k and typical elements x, y, z. $\Delta(Z)$ is the space of all lotteries over Z with typical elements α, β, γ. When there is no risk of confusion, x also denotes the degenerate lottery that assigns unit weight to x. Let $\mathcal{A}$ be the collection of all compact subsets of $\Delta(Z)$ with menus A, B, C as elements.[8] Endow $\mathcal{A}$ with the topology generated by the Hausdorff metric

$$d_h(A,B) = \max\left\{\max_A \min_B d_p(\alpha,\beta), \max_B \min_A d_p(\alpha,\beta)\right\}$$

where d_p is the Prohov metric, which generates the weak topology, when restricted to lotteries.

Further, let I be an objective state space with elements i, j. Let $\mathcal{F}$ be a σ-algebra on I. Two cases have to be distinguished: If I is finite, $\mathcal{F}$ is assumed to be the σ-algebra generated by the power set of I. If I is a general topological space, then $\mathcal{F}$ is the Borel σ-algebra.

Let G be the set of all opportunity acts with typical elements g, h. An opportunity act is a measurable function $g : I \to \mathcal{A}$. If state i realizes, DM gets to choose an alternative from the menu $g(i) \in \mathcal{A}$. This choice is not explicitly modeled. $\succ$ is a binary relation on $G \times G$; $\succcurlyeq$ and $\sim$ are defined the usual way. G can be viewed as a product space generated by the index set I, $G = \prod_{i \in I} \mathcal{A}$. Thus, it can be endowed with the product topology, based on the topology defined on $\mathcal{A}$.

The following concepts are important throughout the paper.

Definition 1: The convex combination of menus is defined as

$$pA + (1-p)B := \{p\alpha + (1-p)\beta \,|\alpha \in A, \beta \in B\}.$$

The convex combination of opportunity acts is defined, such that

$$(pg + (1-p)h)(i) := pg(i) + (1-p)h(i).$$

To define DM's induced ranking of menus A and B contingent on event $D \in \mathcal{F}$, consider acts g_D^A and g_D^B that give menu A or B, respectively, in event D and some arbitrary but fixed default menu, A^*, in the event not D. Comparing g_D^A and g_D^B induces a ranking $\succ_D$ over

[8] Compactness is not essential. If menus were not compact, maximum and minimum would have to be replaced by supremum and infimum, respectively, in all that follows.

menus. In the context of the model, $\succ_D$ turns out to be independent of A^*.

Definition 2: Fix an arbitrary menu $A^* \in \mathcal{A}$. For $D \in \mathcal{F}$ and $A \in \mathcal{A}$, define g_D^A by

$$g_D^A(i) := \begin{cases} A \text{ for } i \in D \\ A^* \text{ otherwise} \end{cases}.$$

Let the *contingent ranking* $\succ_D$ be the induced binary relation on $\mathcal{A} \times \mathcal{A}$, $A \succ_D B$ if and only if $g_D^A \succ g_D^B$. $\succcurlyeq_D$ and $\sim_D$ are defined the usual way. An event $D \in \mathcal{F}$ is *nonnull*, if there are $A, B \in \mathcal{A}$ with $A \succ_D B$.

In period 2, objects of choice are lotteries over the prize space. The taste space (the collection of all conceivable period 2 tastes) is the collection of all vNM rankings of lotteries. The following definition is due to DLRS.

Definition 3:

$$S = \left\{ s \in \mathbb{R}^k \,\middle|\, \sum_t s_t = 0 \text{ and } \sum_t s_t^2 = 1 \right\}$$

is the *taste space.*[9]

S collects all possible realized vNM utilities, twice normalized. Every taste in S is a vector with k components where each entry can be thought of as specifying the relative utility associated with the corresponding prize.[10]

Definition 4: Call (ϕ, μ, U) a *Contingent Preference for Flexibility (CPF)* representation of the preference relation $\succ$, if ϕ is a probability measure on I, $\mu = \{\mu(. |i)\}_{i \in I}$ is a family of probability measures on S, and $U = \{U_s(.)\}_{s \in S}$ is a family of vNM utilities where U_s represents taste s and the objective function

$$V(g) = E_\phi \left[\int_S \left(\max_{\alpha \in g(i)} U_s(\alpha) \right) d\mu(s|i) \right]$$

is well defined and represents $\succ$.

[9] DLRS refer to S as the universal state space.

[10] In the context of the representation theorem in DLRS, as in the theorems that follow, there is clearly always a larger taste space, also allowing a representation of $\succ_D$, in which multiple tastes represent the same ranking of lotteries.

E_ϕ denotes an appropriately defined expectation. The following two subsections consider I to be a finite and a general topological space, respectively.

3.1. Finite Objective State Space

Assume that I is finite and let $i, j \in \mathcal{I}$ also denote the elementary events of the $\sigma-$algebra $\mathcal{F}$. Then the CPF representation (ϕ, μ, U) corresponds to the objective function

$$V(g) = \sum_{i \in I} \phi(i) \left[\int_S \left(\max_{\alpha \in g(i)} U_s(\alpha) \right) d\mu(s|i) \right].$$

If U_s is a vNM representation of taste s, then it must have the form $U_s(\alpha) = l(s)(s \cdot \alpha) + c(s)$, where $s \cdot \alpha$ is the dot product of state s and lottery α, $l(s)$ is the 'intensity' of taste s and $c(s)$ is some constant. The relative intensity of utilities together with beliefs determines how DM trades off gains across tastes. The constants $c(s)$ have no behavioral content. In addition, any changes on measure zero subsets of S have no behavioral content. This motivates the next definition.

Definition 5: For the CPF representation (ϕ, μ, U)
i) The *space of relevant objective states,* $I^* \subseteq I$, is the minimal set with $\phi(I^*) = 1$.
$\mu = \{\mu(.|i)\}_{i \in I}$ is unique, if the measure $\mu(s|i)$ is unique for all $i \in I^*$.
ii) The *space of relevant tastes,* $S^* \subseteq S$, is the minimal set with $\mu(S^*|i) = 1$ for all $i \in I^*$.
$U = \{U_s(.)\}_{s \in S}$ is *essentially unique*, if U_s are unique up to a common linear transformation, up to the addition of constants $c(s)$ and up to changes on $S \backslash S^*$.

S^* can be thought of as the set of tastes DM considers possible.

An axiomatization of the CPF representation is given in Theorem 2. The distribution ϕ is identified from behavior in Theorem 3. The main concern, however, is to separately identify beliefs μ and tastes U, *provided* that DM's choice over acts has a CPF representation for a given distribution ϕ.

Axiom 1 *(Relevant Objective States)*: *If $A \cup B \nsim_i B$ for some $i \in I$, then there is $j \in I$ with $A \nsim_j B$.*

To paraphrase Axiom 1: whenever two menus are not the same for DM, then there is some state contingent ranking under which they are not equally good. If A and B were the same for DM, then she should be willing to choose from $A \cup B$ by simply ignoring A. This can not be the case if $A \cup B \nsim_i B$ for some $i \in I$. Implicit in the interpretation is that, ultimately, only the chosen item matters for the value of a menu. If $A \nsim_i B$, then Axiom 1 is empty. If $A \sim_i B$, then $A \cup B \nsim_i B$ implies that, contingent on i, the item chosen from $A \cup B$ must sometimes be in A and sometimes in B. Axiom 1 requires that there exists a contingent ranking for which either one or the other case becomes more relevant, namely that there is $j \in I$ with $A \nsim_j B$. Axiom 1 is not a strong assumption in the sense that it is local; it only requires breaking indifferences. For comparison, suppose instead that the state $i \in I$ was required to provide a complete description of all relevant aspects of the world, as in AA. Then the stronger assumption of state contingent strategic rationality would have to hold: If $A \cup B \succ_i B$, then $A \sim_i A \cup B$.[11] Intuitively, the experimenter might suspect that the weather (the objective state) is relevant for DM's beliefs about her ranking of, say, getting an umbrella versus getting a frisbee, but he can conceive of DM preferring a frisbee even when it rains and vice versa. This notion is weaker than the assumption of 'state contingent strategic rationality', according to which the experimenter knows that DM prefers the umbrella when it rains and the frisbee if the sun shines.

Theorem 1: *If, given a probability distribution ϕ on I, $\succ$ has the CPF representation (ϕ, μ, U), then statements i and ii below are equivalent and imply iii:*
i) $\succ$ satisfies Axiom 1,
ii) μ is unique and U is essentially unique,
iii) the cardinality of S^* is bounded above by the cardinality of I^*.

Proof: See Appendix.

If a decision maker behaves *as if* she has preference for flexibility, updates her beliefs over tastes when learning the objective state, and maximizes her expected utility according to objective probabilities over those states, then her preferences satisfy Axiom 1, if and only if her beliefs over future tastes are determined uniquely. This identification gives meaning to the description of beliefs and tastes as distinct concepts. Lack of this distinction is the central drawback of previous work on preference for flexibility, starting with Kreps.

Another difficulty in the application and interpretation of models of preference for flexibility is the generically infinite subjective state space. Theorem 1 conveniently constrains the space of relevant tastes, S^*, to be finite. Axiom 1 implies this finiteness, because I must be rich enough to distinguish between any two menus for which DM might have preference for flexibility. This implies that only finitely many lotteries can be appreciated in any menu. Section 3.2 generalizes my results to consider I to be a general topological space, lifting the

[11] Axiom 1 is immediately satisfied: $A \cup B \nsim_i B$ implies $A \nsim_i B$.

constraint on the cardinality of S^*.

Remark 1: A remark on the interpretation of tastes, or subjective states, is in order. Suppose for a moment that there is an underlying state space Ω, which provides a complete description of all relevant aspects of the world. That is, $\omega \in \Omega$ even determines DM's taste, $s \in S$. In that case, S generates a sub$-\sigma-$algebra on Ω. The question is to what extent Ω is *observable.* Let I be the collection of observable events $i \subset \Omega$, where I generates another sub$-\sigma-$algebra on Ω. Now consider a probability measure μ on Ω representing DM's beliefs. If there is no correlation between events in I and events in S, then the induced marginal distribution $\mu(s|i)$ is independent of i, and the objective state space Ω can be dropped from the description of the model, as in DLR. For example, Ω could be the product space $I \times S$ and μ a product measure. If, in the other extreme, there is perfect correlation between events in I and events in S, then I itself can play the role of the complete objective state space in (the state dependent version of) the AA model. Theorem 1 is concerned with the general case of some, but not perfect correlation. While I is naturally interpreted as the collection of all observable contingencies, I will call events that are not in I 'unobservable contingencies.'

To see how Relevant Objective States imply unique beliefs and utilities, suppose there were two CPF representations of the same preference relation, (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$. Suppose further that for the contingent ranking $\succ_i$ one could construct menus $K \sim_i \widehat{K}$, such that K generates constant utility payoff across tastes according to (ϕ, μ, U) and $\widehat{K}$ according to $\left(\phi, \widehat{\mu}, \widehat{U}\right)$. Changing the objective state from i to j only changes DM's beliefs about her future tastes. If a menu generates the same utility payoff for every taste, then the conditional value of the menu is independent of the objective state. Hence, $K \sim_j \widehat{K}$ for all $j \in I$ would have to hold. At the same time, if (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ were distinct, $\widehat{K}$ would not generate constant utility payoffs across tastes according to (ϕ, μ, U), because utility payoffs depend on the intensities of U and $\widehat{U}$, respectively. Therefore $K \cup \widehat{K} \succ_{j'} K$ for some $j' \in I$. Relevant Objective States would then imply that there is $j \in I$ with $K \nsim_j \widehat{K}$, a contradiction. This rough intuition does not quite work, because the construction of menus that generate the same utility payoff for every taste is not always possible. Because $S^* \subset S$ is finite, however, one can construct pairs of menus (A, B for (ϕ, μ, U) and $\widehat{A}, \widehat{B}$ for $\left(\phi, \widehat{\mu}, \widehat{U}\right)$) for which the difference in utility payoffs is constant across tastes. Let K be the convex combination of menus $\frac{1}{2}A + \frac{1}{2}\widehat{B}$ and let $\widehat{K} = \frac{1}{2}\widehat{A} + \frac{1}{2}B$. Then $K \sim_i \widehat{K}$ implies that $K \sim_j \widehat{K}$ for all $j \in I$. By the type of argument laid out above, $K \cup \widehat{K} \succ_{j'} K$ for some $j' \in I$. This contradicts the Relevant Objective States axiom.

If Axiom 1 fails completely, in the sense that objective states are irrelevant to the decision maker, then only the support of the probability measures $\mu(s|i)$ which allow a representation can be identified. This is the same indeterminacy encountered by DLR. Partial failures of the axiom are considered in the appendix.

Both types of exogenous uncertainty in my domain are essential for the uniqueness result: On the one hand, DLR find that preferences over menus of lotteries alone do not allow the separate identification of tastes and beliefs $\mu(s)$. There has to be some possibility of varying one, but not the other. In the CPF representation, only beliefs, $\mu(s|i)$, condition on objective states. On the other hand, Nehring (1999) finds that acts with menus of prizes as outcomes do not allow the separate identification of tastes and beliefs in the axiomatic setup developed by Savage (1954). To establish the uniqueness result, the payoff generated by a menu must be varied independently for different tastes. This is possible only because DM can be offered lotteries over prizes.

I now establish existence of a CPF representation. As mentioned above, the axioms are direct extensions of familiar assumptions. I use the general notation $D \in \mathcal{F}$ (instead of $i \in I$) to denote events, because the axioms and some results also apply to the case of a general topological objective state space and the induced $\sigma-$algebra, which I discuss in Section 3.2.

Axiom 2 *(Preference): $\succ$ is asymmetric and negatively transitive.*

Axiom 3 *(Continuity): The sets $\{g|g \succ h\}$ and $\{g|g \prec h\}$ are open in the topology defined on G for all $h \in G$.*

Axiom 4 *(Independence): If for $g, g' \in G$, $g \succ g'$ and if $p \in (0,1)$, then*

$$pg + (1-p)h \succ pg' + (1-p)h$$

for all $h \in G$.

If a convex combination of menus were defined as a lottery over menus, then the motivation of Independence in my setup would be the same as in more familiar contexts. Uncertainty would resolve before DM consumes an item from one of the menus. However, following DLR and Gul and Pesendorfer (2001), I define the convex combination of menus as the menu containing all the convex combinations of their elements. The uncertainty generated by the convex combination is only resolved after DM chooses an item from this new menu. Gul and Pesendorfer term the additional assumption needed to motivate Independence in this setup 'indifference as to when uncertainty is resolved.'

Axiom 5 *(Nontriviality): There are $g,h \in G$, such that $g \succ h$.*

The next axiom considers DM's contingent ranking of menus, $\succ_D$. As long as some subjective uncertainty is not captured by objective states, $\succ_D$ should exhibit preference for flexibility. This is captured by the central axiom in Kreps, which states that larger menus are weakly better than smaller menus:

Axiom 6 *(Monotonicity): $A \cup B \succsim_D A$ for all $A, B \in \mathcal{A}$ and all $D \in \mathcal{F}$.*

Corollary 1: *If $\succ$ satisfies Axioms 2-6, then $\succ_D$ is a preference relation and satisfies the appropriate variants of Continuity, Independence and Monotonicity for all $D \in \mathcal{F}$. Further, there is a nonnull event $D \in \mathcal{F}$.*

The proof is immediate.

Theorem DLRS *(Theorem 2 in DLRS): For $D \in \mathcal{F}$ nonnull, $\succ_D$ is a preference that satisfies Continuity, Independence and Monotonicity if and only if there is a subjective state space S_D, a positive countably*[12] *additive measure $\mu_D(s)$ on S_D, and a set of non-constant, continuous expected utility functions $U_{s,D} : \Delta(Z) \to \mathbb{R}$, such that*

$$V_D(A) = \int_{S_D} \max_{\alpha \in A} U_{s,D}(\alpha)\, d\mu_D(s)$$

represents $\succ_D$ and every vNM ranking of lotteries $\alpha \in \Delta(Z)$ corresponds to at most one state in S_D.

Because $U_{s,D}(\alpha)$ are realized vNM utility functions, the subjective state space S_D can be replaced by the taste space S for all $D \in \mathcal{F}$. Note that the taste space does not include the taste where DM is indifferent between all prizes, implicitly assuming nontriviality of the ex-post preferences over prizes.

Theorem 2: *$\succ$ satisfies Axioms 2-6 if and only if it has a CPF representation.*

Proof: See Appendix.

[12] See footnote 3 in DLRS.

The proof first establishes an additively separable representation of $\succ$, confined to acts with support in the convex subsets of $\Delta(Z)$, via the Mixture Space Theorem. Because those acts are order dense in G, this representation pins down an additively separable representation of $\succ$ on G; that is, $V(g) = \sum_{i \in I} v_i(g(i))$ represents $\succ$ for some family of utility functions, $\{v_i\}_{i \in I}$, on $\Delta(Z)$. Now suppose V_i is a linear representation of $\succ_i$. Because of the uniqueness implied by the Mixture Space Theorem, V_i must agree with v_i up to scaling. The scaling is absorbed by $\phi(i)$, which is then normalized to be a probability distribution. Thus, an act is evaluated by

$$V(g) = \sum_{i \in I} \phi(i) V_i(g(i)).$$

Note that this is AA's representation, with the exception that opportunity acts have menus as outcomes, while AA acts have lotteries as outcomes. Indeed, Axioms 2-4 imply AA's axioms. Furthermore, Axioms 2-6 imply DLRS' axioms, as shown in Corollary 1. According to Theorem DLRS, $\succ_i$ can then be represented by

$$\widehat{V}_i(A) = \int_S \max_{\alpha \in A} (U_{s,i}(\alpha))\, d\mu_i(s),$$

where $U_{s,i}$ is a vNM utility function that represents taste s, that is, $U_{s,i}$ and $U_{s,j}$ are identical up to a positive affine transformation. Pick any $j \in I$ and define $U_s := U_{s,j}$. Rescaling $\mu_i(s)$ allows $\succ_i$ to be represented by

$$V_i(A) = \int_S \max_{\alpha \in A} U_s(\alpha)\, d\mu_i(s)$$

for all $i \in I$. Since V_i is linear, there is a CPF representation (ϕ, μ, U); that is,

$$V(g) = \sum_{i \in I} \phi(i) \left[\int_S \max_{\alpha \in g(i)} U_s(\alpha)\, d\mu(s \,|i) \right]$$

represents $\succ$. The intensity of each taste is endogenous, but it is fixed across objective states.

Clearly Axioms 2-6 are also necessary for the generic combination of the AA and DLRS representations,

$$\widehat{V}(g) = \sum_{i\in I} \phi(i)\, \widehat{V}_i(g(i)) = \sum_{i\in I} \phi(i) \left[\int_S \max_{\alpha\in g(i)} (U_{s,i}(\alpha))\, d\mu_i(s) \right]$$

where objective states impact not only probabilities, $\mu_i(s)$, but also the intensities of tastes. Theorem 2 implies that there is a CPF representation of $\succ$ whenever the more general representation $\widehat{V}$ exists. Therefore, the assumption that only beliefs condition on objective states does not constrain period 1 choice.

Remark 2: So far I have argued for the intuitive appeal of the CPF representation. This remark argues that the CPF representation is also 'minimal,' generalizing the formal argument for using the state independent representation in AA. Let the subjective state space $\widetilde{S} = S \times \mathbb{R}_+$ collect all pairs of vNM rankings *and* intensities. Suppose

$$\widetilde{V}(g) = \sum_{i\in I} \phi(i) \left[\int_{\widetilde{S}} \max_{\alpha\in g(i)} (U_{\widetilde{s}}(\alpha))\, d\widetilde{\mu}(\widetilde{s}\,|i) \right]$$

represents $\succ$. This representation is even more general than the representation $\widehat{V}$ above. Theorem 2 implies that there is a family of probability measures $\widetilde{\mu} = \{\widetilde{\mu}(\widetilde{s}\,|i)\}_{i\in I}$ on $\widetilde{S}$ that allows a representation of $\succ$ and for which every taste, $s \in S$, corresponds to at most one subjective state in its support. It is straightforward to verify that this $\widetilde{\mu}$ has the smallest possible support $\widetilde{S}^* \subset \widetilde{S}$ among all measures that allow a representation of $\succ$. Thus, restricting attention to CPF representations is equivalent to considering those representations based on the subjective state space $\widetilde{S}$ which utilize only a minimal amount of subjective states in the sense of DLR. According to Theorem 1, $\widetilde{\mu}$ is unique.

3.2. Topological Objective State Space

If the objective state space I is finite, then Axiom 1 limits the cardinality of the space of relevant tastes, S^*. This is no longer the case when I is infinite. This sub-section generalizes the previous one by considering I to be a general topological space. The reader may choose to proceed directly to Section 4 without a loss in the continuity of ideas. Here and in the proofs, definitions and results that generalize those in the previous sub-section are distinguished by a prime on their label.

Recall that $\mathcal{F}$ is the Borel σ-algebra on I. The expectation under probability measures on $\mathcal{F}$ can only be calculated directly for simple functions.[13] For general functions it is defined as an appropriate limit:

Definition 6 *(Based on Definition 10.12 in Fishburn (1970)):* For a countably additive probability measure π on $\mathcal{F}$ and a bounded measurable function $\varphi : I \to \mathbb{R}$, let $\langle \varphi_n \rangle$ be a sequence of simple functions, $\varphi_n : I \to \mathbb{R}$, that converge from below to φ. Then define

$$E_\pi [\varphi] := \sup \{E_\pi [\varphi_n] \, | n = 1, 2, ...\}$$

to be the *expectation of* φ *under* π.

Fishburn establishes that this expectation is well defined.

Definition 5': For the CPF representation (ϕ, μ, U)
i) $\mu = \{\mu (. \, |i)\}_{i \in I}$ is unique if the measure $\mu (s \, |D) := E_\phi [\mu (s \, |i) \, |D]$ is unique up to ϕ-measure zero changes for all $D \in \mathcal{F}$.
ii) $U = \{U_s (.)\}_{s \in S}$ is essentially unique, if U_s are unique up to a common linear transformation, up to the addition of constants $c(s)$, and up to changes on a set $S' \subset S$ with $E_\phi \left[\int_{S'} d\mu (s \, |i) \right] = 0$.

The next definition provides a measure of how much set A is preferred over set B in terms of how much the menu corresponding to the entire prize space, Z, is preferred over the worst prize.

Definition 7: Given $D \in \mathcal{F}$, let $\underline{z}$ be the worst prize: $A \succsim_D \{\underline{z}\}$ for all $A \in \mathcal{A}$. For $A, B \in A$, define $p_{A,B} (D) \in (-1, 1)$, such that
i) for $A \succsim_D B$, $p = p_{A,B} (D)$ solves

$$\frac{1}{1+p} A + \frac{p}{1+p} \{\underline{z}\} \sim_D \frac{1}{1+p} B + \frac{p}{1+p} Z,$$

ii) for $B \succ_D A$, $p_{A,B} (D) = -p_{B,A} (D)$.
Call $p_{A,B} (D)$ the *cost of getting to choose from* B *instead of* A *under event* D.

If $\succ$ can be represented by a CPF representation, then the prize $\underline{z}$ must exist because

[13] The value of a simple function φ_n depends only on some finite and measurable partition $\{D_t \, |t \in \{1, .., T\}\}$ of I . $E_\pi [\varphi_n] := \sum_{t=1}^{T} \pi (D_t) \varphi_n (D_t)$.

Z is finite and because $\succ_D$ must obviously satisfy *Monotonicity*. Note that $p_{A,B}(D) \neq 0$ implies that D is nonnull.

If two sequences of menus, $\langle A_n \rangle$ and $\langle B_n \rangle$, approach each other, then the cost of getting to choose from B_n instead of A_n vanishes under every event. However, the ratio of such costs may have a well defined limit.

Axiom 1' *(Relevance and Tightness of Objective States): If $\langle A_n \rangle, \langle B_n \rangle, \langle C_n \rangle \subseteq \mathcal{A}$ converge in the Hausdorff topology, then*

$$\frac{p_{C_n, A_n \cup B_n}(D)}{p_{C_n, B_n}(D)} \nrightarrow 1$$

for some $D \in \mathcal{F}$ implies that there is $D' \in \mathcal{F}$, such that

$$\frac{p_{C_n, A_n}(D')}{p_{C_n, B_n}(D')} \nrightarrow 1.$$

Axiom 1' implies Axiom 1, where i is substituted by D. To see this, note that Axiom 1 holds trivially unless there is $D \in \mathcal{F}$, such that $A \cup B \nsim_D B$ and $A \sim_D B$. This implies $p_{C,B}(D) = p_{C,A}(D)$ and $p_{C,A \cup B}(D) \neq p_{C,B}(D)$. Define the constant sequences $A_n := A$ and $B_n := B$ and let $C_n := C \succ_D A$. Then $\frac{p_{C_n, A_n \cup B_n}(D)}{p_{C_n, B_n}(D)} \nrightarrow 1$. Thus, according to Axiom 1', there is $D' \in \mathcal{F}$ with $\frac{p_{C_n, A_n}(D')}{p_{C_n, B_n}(D')} \nrightarrow 1$. Hence $A \nsim_{D'} B$, and Axiom 1 is satisfied. If $p_{C_n, B_n}(D) \nrightarrow 0$, then Axiom 1 also trivially implies Axiom 1'. Thus, Axiom 1' is only stronger than Axiom 1 for $p_{C_n, B_n}(D) \to 0$.

Theorem 1': *If, given $\phi : I \to \mathbb{R}_+$, $\succ$ has the CPF representation (ϕ, μ, U), then $\succ$ satisfies Axiom 1' if and only if μ is unique and U is essentially unique.*

Proof: See Appendix.

The discussion of Theorem 1 applies here. The intuition for the proof of Theorem 1 involves identifying taste $s \in S^*$ via two menus, where one is preferred over the other under taste s, but they generate the same payoff under every other relevant taste. If S is continuous, however, then there is the complication that making a menu less preferred by a finite amount under one taste will invariably make it worse under similar tastes (where tastes are viewed as vectors in $\mathbb{R}^k_+$,) too. Therefore, individual tastes can only be identified in the limit where the less preferred and the more preferred menu approach each other. In this limit, the cost of getting to choose from the less preferred menu instead of the more preferred menu tends to zero. Axiom 1' allows statements about the limit of the ratio of these costs for two different pairs of menus. The main idea of the proof of Theorem 1' is the same as for Theorem 1.

To construct a similar argument here, menus are best described in terms of their support functions.[14]

In addition to Axioms 2-6, an axiomatization of the CPF representation requires that $\succ_D$ does not change too much for small changes in D.

Axiom 7 *(Event-Continuity): For any $B \in \mathcal{A}$, the set $\{A \,|A \succ_D B\}$ is continuous in D.*

Theorem 2': *$\succ$ satisfies Axioms 2-7 if and only if it has a CPF representation.*

Proof: See Appendix.

Straightforward changes to the proof of Theorem 2 establish the result for $\succ$ constrained to all simple acts.[15] The simple acts are shown to be dense in G under the topology defined on G. Ensuring that Definition 6 applies here completes the proof.

4. Probabilities over Objective States

Theorems 1 and 1' take the distribution ϕ on I and a CPF representation (ϕ, μ, U) as given and establish that μ and U are unique in the appropriate sense if and only if objective states are relevant. ϕ might be objective in the sense that it corresponds to observed frequencies of objective states, or it might be subjective. Alternatively, one can also consider a representation that does not aggregate preferences over I at all.

4.1. Subjective Probabilities over Objective States

Consider first the case where ϕ is subjective and must also be elicited from behavior. Determining ϕ uniquely is analogous to the classical problem addressed by AA. Their unique identification of probabilities of observable states is based on the assumption of *state independence* of the ranking of outcomes. The difference is that they consider acts with lotteries (instead of menus of lotteries) as outcomes, so there is no room for preference for flexibility in their setup. In my setup, the combination of *objective state independence* and Axiom 1 would rule out any preference for flexibility. Thus, the independence assumption has to be confined to a proper subset $\Psi \subset \mathcal{A}$ to be useful here. Having assumed state independent rankings, AA move on to consider only cardinally state independent rankings (or state inde-

[14] The introduction of support functions to the analysis of choice over menus is a major contributuion of DLR.

[15] The outcome of a simple act depends only on the event D in some finite partition $\{D_t \,|t \in \{1, .., T\}\}$.

pendent utilities). This cannot be assumed in terms of an axiom. Instead it is a constraint on the class of representations for which they establish their uniqueness result. For the CPF representation it would amount to requiring that $\int_S \max_{\alpha \in A} U_s(\alpha)\, d\mu(s\,|i)$ is independent of $i \in I$ for all $A \in \Psi$. But if $\Psi \subset \mathcal{A}$ is a generic collection of menus, then this might not be consistent with $\succ$, which applies to all of G.[16] Thus, the requirement must be confined to a *particular* collection of menus.

Definition 8: Let $X \subseteq Z$ denote a non-degenerate set of prizes and $\Delta(X)$ the set of all lotteries with supports in X. Let $\Psi(\Delta(X)) \subseteq \mathcal{A}$ be the set of all menus of lotteries that have supports in X.

Axiom 8 *(Partial Objective State Independence): There is $X \subseteq Z$, such that for $A, B \in \Psi(\Delta(X))$, $A \succ_D B$ for some event $D \in \mathcal{F}$ implies $A \succ_{D'} B$ for all nonnull $D' \in \mathcal{F}$. If $\succ$ satisfies the same condition for $Y \subseteq Z$, then it also satisfies the condition for $X \cup Y$.*

To illustrate Axiom 8, consider $X = \{\$1, \$0\}$ to consist of the prizes '1 Dollar' and 'nothing.' The first part of Axiom 8 then requires that the ranking of menus that consist only of lotteries that pay out either \$1 or nothing must not be state-contingent. To motivate the requirement, it is sufficient to assume that the value of \$1 (versus nothing) is not state-contingent.

Once AA restrict attention to representations with state independent utilities, there is no arbitrariness in their model. In contrast, preference for flexibility implies that X is a proper subset of Z. Hence, $\succ$ could satisfy the first part of Axiom 8 for some X and Y with $X \neq Y$, but not for $X \cup Y$. Either those menus with support in X or those with support in Y could then be assigned a cardinal ranking, which is not state-contingent. While there is no inherent argument to favor one over the other, the two assumptions clearly lead to different representations. This arbitrariness would render the uniqueness result meaningless. The second part of Axiom 8 rules out this scenario, suggesting the following definition:

Definition 9: Let $X^* \subseteq Z$ be the largest set for which $\succ$ satisfies the condition in Axiom 8.

Theorem 3: *If I is finite, then $\succ$ satisfies Axioms 1-6 and Axiom 8 if and only if it has a CPF representation, (ϕ, μ, U), where the contingent value of menus $A \in \Psi(\Delta(X^*))$ is independent of $i \in I$. For this representation ϕ is unique, μ is unique and U is essentially*

[16] For a simple example of such inconsistency consider $\Psi = \{\{\alpha\}, \{\beta\}, \{\gamma\}\}$ but, for some $p \in (0,1)$ and $D, D' \in \mathcal{F}$, $\{p\alpha + (1-p)\gamma\} \succ_D \{\beta\} \succ_{D'} \{p\alpha + (1-p)\gamma\}$. Since $\int_S \max_{\alpha \in A} U_s(\alpha)\, d\mu(s\,|i)$ is linear, it can not be independent of $i \in I$.

unique with $U_s(x)$ constant across S for all $x \in X^*$.

If I is a general topological space, $\succ$ must also satisfy Axiom 1' and Axiom 7.

Proof: For CPF representations, where $\int_S \max_{\alpha\in A} U_s(\alpha)\, d\mu(s\,|i)$ does not depend on $i \in I$ for all $A \in \Psi(\Delta(X^*))$, the uniqueness of ϕ follows in complete analogy to the corresponding result in AA. Given this unique ϕ, Theorems 1 and 1' imply uniqueness of μ and essential uniqueness of U. Because a representation where $U_s(x)$ is constant across S for all $x \in X^*$ clearly exists, the unique representation must have this feature. ■

4.2. Objective Probabilities over Objective States

Now consider the alternative case, where frequencies of objective states are observable, and suppose I is finite for simplicity. An observer who observes frequencies ϕ might be willing to assume that DM bases her evaluation of acts on ϕ, as long as a CPF representation based on ϕ *exists*.[17,18]

Proposition 1: *Suppose $\succ$ satisfies Axiom 1 and can be represented by (π, μ, U), where π has minimal support in the sense that I^* has the same cardinality, $T < \infty$, as S^*.*[19] *Then there is a neighborhood of π in $[0,1]^T$, such that for any probability measure ϕ on I^* in this neighborhood there is a representation $\left(\phi, \widehat{\mu}, \widehat{U}\right)$, where $\widehat{U}$ and $\widehat{\mu}$ are locally Lipschitz continuous in ϕ around π.*

Proof: See Appendix.

This result, which shows robustness to small misspecifications of π, can be relevant in applications where beliefs are used to forecast period 2 choice: if the observer and the decision maker disagree slightly in their perception of the 'objective' probabilities, then the observer

[17] This is not always the case. For example, if (π, μ, U) represents $\succ$ and there is an event $D \in \mathcal{F}$ that is trivial according to π but not according to ϕ, then there is no CPF representation based on ϕ.

[18] It is possible to strengthen Axiom 8, such that the *unique* CPF representation in Theorem 3 is based on those frequencies:

Axiom *(Objective Probabilities): There is $\Omega \subseteq Z$, such that for $A, B \in \Psi(\Delta(\Omega))$ and nontrivial D and $D' \in \mathcal{F}$:*

$$\frac{\phi(D')}{\phi(D)+\phi(D')} h_D^A + \frac{\phi(D)}{\phi(D)+\phi(D')} h_{D'}^B \sim \frac{\phi(D)}{\phi(D)+\phi(D')} h_{D'}^A + \frac{\phi(D')}{\phi(D)+\phi(D')} h_D^B.$$

If $\succ$ satisfies the same condition for $\Omega' \subseteq Z$, then it alo satisfies the condition for $\Omega \cup \Omega'$.

This implies Axiom 8. It also implies that $V\left(g_D^A\right) - V\left(g_D^B\right) = \left(V\left(g_{D'}^A\right) - V\left(g_{D'}^B\right)\right)\frac{\phi(D)}{\phi(D')}$ for $A, B \in \Psi(\Delta(\Omega))$.

[19] If axiom 1 holds, such a representation can always be found by considering a $\sigma-$algebra $\mathcal{F}$ on I that is just fine enough to identify beliefs.

can apply Theorem 1 (there is a representation based on the observer's perception ϕ, even if DM truly bases her decisions on π.) Further, the unique subjective probabilities of future tastes provided by Theorem 1 are at least a good approximation of DM's true beliefs.

4.3. No Aggregation over Objective States

As a third alternative, one could consider a representation that does not aggregate contingent rankings as in the CPF representation. This lack of aggregation would simplify the uniqueness statement and Theorem 3 would become irrelevant.

Let $\{\succ_i\}_{i\in I}$ be a subset of $\mathcal{A} \times \mathcal{A}\times I$. Each binary relation $\succ_i$ is a subset of $\mathcal{A} \times \mathcal{A}$ and captures choice between menus in $\mathcal{A}$ under objective state $i \in I$. The adaptation of my axioms to this new domain is straightforward. A representation of *Preference for Flexibility without Aggregation* is a pair (μ, U) where μ and U are specified as in Definition 4 and

$$V_i(A) = \int_S \max_{\alpha\in A} U_s(\alpha)\, d\mu(s\,|i)$$

represents $\succ_i$. If preferences satisfy Axiom 1, then the representation (μ, U) is unique.[20]

5. Behavioral Comparisons in Terms of Beliefs - Examples

I consider two comparisons of first stage behavior.

Definition 10: $\succ_i$ values $x \in Z$ unambiguously higher than $\succ_j$, if and only if $\{x\} \succ_i \{\alpha\}$ implies $\{x\} \succ_j \{\alpha\}$ for all $\alpha \in \Delta(Z)$.

The next comparison also appears in the context of dynamic models of preference for flexibility in Higashi, Hyogo and Takeoka (2009), and in Krishna and Sadowski (2010). Intuitively, one preference values flexibility more than another if it ranks menus over singletons more often.

Definition 11: $\succ_i$ has greater preference for flexibility than $\succ_j$ if and only if

$$A \succ_i \{\beta\} \implies A \succ_j \{\beta\}$$

for all $\beta \in \Delta(Z)$ and $A \in \mathcal{A}$.

[20]This suggests that one could also consider alternatives to the CPF representation that are based on a non-additive aggregator of contingent rankings: for example, an ambiguity averse aggregator as in Ozdenoren (2002). I thank an anonymous referee for pointing out this possiblity.

Note that the comparison in Definition 11 requires that $\succ_i$ and $\succ_j$ rank singleton menus the same:

$$\{\alpha\} \sim_i \{\beta\} \Leftrightarrow \{\alpha\} \sim_j \{\beta\}$$

for all $\alpha, \beta \in \Delta(Z)$.

It would be nice to characterize the two comparisons in terms of beliefs. The comparison of probability distributions is most intuitive when they have one-dimensional support. To give a simple example in the context of Theorem 3, consider $Z = \{\$1, \$0, x\}$ and preferences with $X^* = \{\$1, \$0\}$. Subjective uncertainty then concerns the value of prize x relative to \$1 and \$0, and the taste space is isomorphic to $\mathbb{R}$. Let $\mu_{\mathbb{R}}(.\,|i)$ denote the measure on $\mathbb{R}$ that corresponds to the contingent ranking $\succ_i$, and let $\overline{\mu_{\mathbb{R}}}(i)$ denote its mean.

Corollary 2: *$\succ_i$ values x unambiguously higher than $\succ_j$ if and only if $\overline{\mu_{\mathbb{R}}}(i) \geq \overline{\mu_{\mathbb{R}}}(j)$.*

The proof is immediate.

Proposition 2: (Proposition 6.5 in Krishna and Sadowski (2010)): *$\succ_i$ has greater preference for flexibility than $\succ_j$ if and only if $\mu_{\mathbb{R}}(.\,|j)$ second order stochastically dominates $\mu_{\mathbb{R}}(.\,|i)$.*[21]

This characterization of 'greater preference for flexibility' depends on the spread of DM's beliefs. DLR suggest an alternative notion which, limited by the lack of identification in their model, must be characterizable in terms of only the support of those beliefs.

For ease of notation, both comparisons above concern contingent rankings $\succ_i$ and $\succ_j$ for the same DM. In the context of Theorem 3, the same comparison is possible across individuals.

6. Related Literature

Ozdenoren (2002) also considers Preference for Flexibility in the presence of objective states of the world. Instead of Relevant Objective States (Axiom 1), which ensures that contingent

[21] Krishna and Sadowski (2010) provide a more general characterization result for higher dimensional taste spaces in terms of dominance in the increasing convex order (their Proposition 6.4). This more general result also applies to the present model.

Subjective uncertainty in the dynamic model of Higashi et al (2009) only concerns the intertemporal discount factor. In that sense their model corresponds to a static model with a one dimensional taste space and Proposition 2 is analogous to their Theorem 4.2.

rankings are sufficiently different, he assumes that all contingent rankings are the same. Consequently, beliefs are not identified in his model.

I know of two other identification results that deliver unique beliefs over future tastes for consumption in models of preference for flexibility. First, note that AA's identification of unique beliefs over objective states does not require *full* state independence of preferences.[22] In analogy to AA's argument, beliefs over tastes in the DLR model can be identified uniquely, as long as DM has no preference for flexibility with respect to *part* of the prize space. As an example, DLR suggest the consideration of DM without preference for flexibility on one dimension of a product prize space (Shenone (2010) provides details.) Second, in a dynamic model of preference for flexibility, Krishna and Sadowski (2010) show that intertemporal tradeoff can also uniquely identify beliefs.

The domain of opportunity acts was first analyzed by Nehring (1999), and the notion of contingent menus appears in Epstein (2006). Following Nehring (1996), a companion paper to Nehring (1999), Epstein and Seo (2009) consider a domain of random menus, which are lotteries with menus as outcomes. On this domain they establish unique induced probability distributions over ex post upper contour sets as the strongest possible uniqueness statement.

Theorem 1 does not only provide unique beliefs, but also establishes the finiteness of the collection of relevant tastes, S^*. Dekel, Lipman and Rustichini (2009) and Kopylov (2009) generate finiteness of S^* in the absence of objective states by basically assuming that the number of lotteries DM can appreciate in any given menu is limited.

Finally, note that the state independent version of AA's representation can be viewed as a special case of a unique CPF representation, where there is only one taste and the intensity of this one taste is independent of the objective state. Karni and coauthors, for example Grant and Karni (2005), Karni (2008), and Karni (2009a and 2009b), elaborate the point that interpreting AA's or Savage's (1954) unique subjective probabilities of observable states as DM's true beliefs may be misleading, in case the true intensity of her only taste is actually not state independent. The CPF model is not immune to this concern: Even if choice has a CPF representation, DM's true intensities of tastes might not actually be state independent. Similarly, DM might not actually use the expected utility criterion to evaluate uncertain prospects and alternatives other than the one that is ultimately chosen might also generate utility. None of those modeling assumptions remain innocuous, once the natural inductive step of forecasting period 2 choice is taken.[23] The usefulness of the CPF model for forecasts can only be tested by comparing its predictions to observed period 2 choice frequencies.

[22] This insight also underlies the elicitation of beliefs, ϕ, over objective states in Section 4.1 of this paper.

[23] The assumption that beliefs are meaningful beyond their role in the representation of individual choice also underlies the notion of 'objective probabilities' on which all agents can agree, even if they behave differently.

The ability to forecast period 2 choice frequencies is relevant in strategic situations, for example in the context of contracts.

7. Asymmetric Information and Contracts

As illustrated by the example in the introduction, my domain has a natural interpretation in terms of contracts. At the time two parties write a contract, the space of observable contingencies, I, is describable. In addition there are unobservable or indescribable contingencies that are more relevant for one party than for the other. It seems natural that information about those contingencies is asymmetric. In order to focus on this asymmetry, I assume that each party foresees those and only those contingencies that are directly relevant to its own payoffs, and that contingencies that are foreseen by both parties are observable, and are therefore in I.

Consider a principal and an agent who want to write a contract. Actions are observable, so there is no risk of moral hazard. An action pair specifies actions to be taken by the principal and the agent, respectively. Each action pair induces a probability distribution over outcomes.[24] Only the principal's valuations depend on unobservable contingencies, which are unforeseen only by the agent. Let S denote the principal's taste space. The contract can fully address uncertainty about the agent's payoff, but not about the principal's payoff. Therefore, an efficient contract generically assigns some control rights to the principal: it specifies a collection of action pairs for every observable contingency $i \in I$, from which the principal can choose at a later time. Whether such a contract is considered incomplete is a definitional question.[25] The reduced form of the contract, $g : I \to \mathcal{A}$, specifies a menu of lotteries over outcomes for every contingency $i \in I$. The principal chooses from $g(i)$, after i arises and after uncertainty about any unobservable contingencies that are relevant for her taste over outcomes, $s \in S$, is resolved. To agree on an efficient contract, both parties must be able to rank all contracts.

From the principal's point of view, the contract is an act in the terminology of the previous sections. The principal's ranking of contracts satisfies Axiom 1 and has a CPF representation based on the objective probabilities of events, ϕ. Her choice of an alternative, α, depends only on her taste, s, not on the intensity of the utility, U_s, that represents it: $\alpha_s^*(A) := \arg\max_{\alpha \in A} (\alpha \cdot s)$ is the choice under taste s.[26] The CPF representation can be written

[24] Contingencies that impact the effect of actions on the probabilities of outcomes are considered directly relevant for both parties and are, therefore, in I.

[25] See Section 5 in Hart and Moore (1999) for a discussion.

[26] As before, $\alpha \cdot s$ denotes the dot product between lottery α and taste s. The $\arg\max$ exists, because menus are compact. If it is not unique, ties can be broken in favor of the agent.

as

$$V(g) = E_\phi \left[\int_S U_s \left(\alpha_s^* \left(g(i) \right) \right) d\mu \left(s \,|i \right) \right],$$

where μ is unique.

The agent assigns a contingent cost, $c(x,i)$, to every prize $x \in Z$. Let $c(i) \in \mathbb{R}^k$ be the vector of these costs. Further, he also assesses probabilities of observable contingencies according to the probability distribution $\phi : I \to [0,1]$. While the agent can not foresee all the contingencies underlying the formation of the principal's taste, he does know the principal's ranking of contracts, and therefore μ. Hence, the agent can rank contracts according to

$$W(g) = E_\phi \left[\int_S \left(\alpha_s^* \left(g(i) \right) \cdot c(i) \right) d\mu \left(s \,|i \right) \right].$$

Note that $W(g)$ depends on the conditional subjective probabilities, μ, as perceived by the principal but not on the intensities of her tastes, U. In my axiomatic setup these two are distinct concepts.

The assumption that each party's ranking of contracts is known by the other party is usually required in contract theory and justified by some informal story of learning from past observations. This assumption is not my focus, and I make it without doing the game theoretic complexity of the contracting problem justice. Instead I address the additional assumption required in the (incomplete) contracting literature: In order to allow both parties to rank all contracts, it has to be assumed that they believe in the same probability distribution over utility-payoffs, ex ante.[27] This ad hoc assumption is made for lack of a useful choice theoretic model of the bounded rationality involved. It is troubling in the context of unforeseen contingencies, where it seems natural that information is asymmetric. My domain is not only well suited to describe the type of (incomplete) contracts laid out above, but, for those contracts, my axioms also give choice theoretic substance to the assumption of common beliefs.

8. Conclusion

In the context of preference for flexibility, the notion of a taste space is attractive, because in principle it allows the distinction of tastes and beliefs. Identifying the two conceptually distinct components through preferences, however, has proved difficult. This paper proposes considering objective states, which are relevant enough to allow the unique identification.

[27] Section 3 in Maskin and Tirole (1999) elaborates this point.

The interpretation that I have offered so far is that objective states are chosen by Nature. I conclude by suggesting another interpretation.

The choice between logically equivalent *frames* is made by the experimenter (or the 'sender') instead of Nature, and often influences DM's contingent ranking. One possible interpretation of frames is suggested by Sher and McKenzie (2006). They propose that logically equivalent frames may not be informationally equivalent, rather they convey information about the sender's knowledge about relevant, but not explicitly specified, aspects of the choice situation. Consider the representation suggested in Section 4.3, where there is no aggregation over objective states, and call objective states $f \in I$ frames. To paraphrase the identifying assumption, 'Relevant Frames,' in this context: if there is preference for flexibility with respect to two menus that are indifferent under one frame, then the choice can be reframed so as to break the indifference. Frames are relevant if and only if the parameters of the representation are unique in the sense of Theorems 1 and 1'. The representation suggests interpreting DM's susceptibility to frames as Bayesian decision making. The underlying model is not specified, but the uniqueness result allows classifying the information content of changing frame f to frame f' by comparing the probability distributions $\mu(s\,|f)$ and $\mu(s\,|f')$ they induce. If DM truly was a Bayesian decision maker (in the sense specified by the model), then $\mu(s\,|f)$ should predict how often taste s governs her future choice under frame f. Whether and when it does is an empirical question.

9. Appendix

Subsection 1 of the Appendix collects some relevant facts about support functions. Subsection 2 specifies the indeterminacy implied by partial failures of Axiom 1. The following Subsections prove the results in the text in the order they appear.

9.1. Support Functions

Definition 12: Call $\sigma_A : S \to \mathbb{R}$ with $\sigma_A(s) := \max_{\alpha \in A} (\alpha \cdot s)$ the *support function* of A.

Support functions have the following properties:[28]

(i) $A \subseteq B$ if and only if $\sigma_A \leqslant \sigma_B$

(ii) $\sigma_{\lambda A + (1-\lambda)B} = \lambda \sigma_A + (1-\lambda)\sigma_B$ whenever $0 \leq \lambda \leq 1$

(iii) $\sigma_{A \cap B} = \sigma_A \wedge \sigma_B$ and $\sigma_{(A \cup B)} = \sigma_A \vee \sigma_B$.

Denote by A_σ the maximal menu supported by σ, $A_\sigma = \bigcap_{s \in S} \{\alpha \in \Delta(Z)\,|\alpha \cdot s \leq \sigma(s)\}$. Let

[28] For a comprehensive treatment of support functions in the context of choice over menus, see Chatterjee and Krishna (2009).

$\overline{\mathcal{A}}$ be the collection of all convex subsets of $\Delta(Z)$. Note that $A \in \overline{\mathcal{A}}$ if and only if A is maximal with respect to some support function. Let $\succ_i$ simultaneously denote the induced ranking of support functions, $\sigma \succ_i \xi$ if and only if $A_\sigma \succ_i A_\xi$.

Lemma 1: For ε small enough, $\sigma_\varepsilon := \varepsilon$ is a support function.

Proof: The $k-1$ dimensional hyperplane in $\mathbb{R}^k$ that contains S is $H_S = \left\{x \in \mathbb{R}^k \,|x \cdot \mathbf{1} = 0\right\}$. The hyperplane that contains the $k-1$ dimensional simplex of lotteries, $\Delta(Z)$, is $H_{\Delta(Z)} = \left\{x \in \mathbb{R}^k \,|x \cdot \mathbf{1} = 1\right\}$. These two hyperplanes are parallel. Choose ε small enough such that the $k-1$ dimensional ball $B_\varepsilon \subset H_{\Delta(Z)}$ with radius ε around the center of the simplex is itself inside the simplex, $B_\varepsilon \subset \Delta(Z)$. Then $\sigma_{B_\varepsilon} \equiv \varepsilon$. □

9.2. Partial Failures of Axiom 1

Suppose there is a CPF representation of $\succ$. Further suppose there is a pair of menus, $A, B \in \mathcal{A}$, such that $A \cup B \nsim_i B$ for some $i \in I$, but $A \sim_j B$ for all $j \in I$. This means there is some preference for flexibility in having both A and B available, but their comparison is non-contingent. To say this more precisely:

Definition 13:

$$c_{A,B}(s) := \max_{\alpha \in A} U_s(\alpha) - \max_{\beta \in B} U_s(\beta)$$

is the *cost of getting to choose from* $B \in \mathcal{A}$ *instead of* $A \in \mathcal{A}$ *under taste* $s \in S$.

$A \cup B \nsim_i B$ implies that $c_{A,B}(s)$ cannot be zero for all s and $A \sim_i B$ implies that it cannot be any other constant. Still, $A \sim_j B$ for all $j \in I$ means

$$\sum_{S^*} c_{A,B}(s)\, \mu(s\,|j) = 0$$

for all $j \in I$. This suggests the following Proposition.

Proposition 3: *Suppose (ϕ, μ, U) is a CPF representation of $\succ$. Then the following two conditions are equivalent:*
i) *there is a pair of menus $A, B \in \mathcal{A}$, such that $A \cup B \nsim_i B$ for some $i \in I$, but $A \sim_j B$ for all $j \in I$,*
ii) *there is a family of representations $\left\{\left(\phi, \widehat{\mu}, \widehat{U}\right)\right\}_\eta$ based on $\widehat{\mu}(s\,|i) = \frac{\left(1+\eta c_{A,B}(s)\right)\mu(s|i)}{\sum_{S^*}\left(1+\eta c_{A,B}(s)\right)\mu(s|i)}$*

and $\widehat{U}_s = \frac{U_s}{1+\eta c_{A,B}(s)}$*, indexed by* $\eta > -\frac{1}{c_{A,B}(s)}$*.*

Another pair of menus $A', B' \in A$ *satisfying i) adds additional indeterminacy, if and only if*

$$\frac{c_{A',B'}(s)}{c_{A',B'}(s')} \neq \frac{c_{A,B}(s)}{c_{A,B}(s')}$$

for some $s, s' \in S$*.*

Proof: That i) implies ii) is demonstrated in the proof of Theorem 1. The reverse follows from Theorem 1.

It remains to be shown that if there is another pair of menus, $A', B' \in A$, such that $A' \sim_j B'$ for all $j \in I$ and $A' \cup B' \succ_i B'$ for some $i \in I^*$, then another parameter is required to index the set of possible representations if and only if $\frac{c_{A',B'}(s)}{c_{A',B'}(s')} \neq \frac{c_{A,B}(s)}{c_{A,B}(s')}$ for some $s, s' \in S$. That this condition is sufficient for the existence of additional representations is obvious. To see that it is necessary, suppose there was a representation $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ with $\widehat{\mu}(s|i) \neq \frac{\left(1+\eta c_{A,B}(s)\right)\mu(s|i)}{\sum_{S^*}\left(1+\eta c_{A,B}(s)\right)\mu(s|i)}$ for all η. There must be some non-constant function $c : S \to \mathbb{R}$, such that $\widehat{\mu}(s|i) \equiv \frac{(1+\eta c(s))\mu(s|i)}{\sum_{S^*}(1+\eta c(s))\mu(s|i)}$ for some $\eta > 0$ and $c(s) \neq c_{A,B}(s)$. $\succ_i$ mandates that $\widehat{l}(s) \propto \frac{l(s)}{1+\eta c(s)}$. Because $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ represents the same preference as (ϕ, μ, U), $\sum_{S^*} c(s)\mu(s|i)$ must be constant across I. Hence, there is some non-constant function $\widetilde{c} : S \to \mathbb{R}$, with $\sum_{S^*} \widetilde{c}(s)\mu(s|i) = 0$ for all $i \in I$. Let $\widetilde{c}^+(s)$ and $\widetilde{c}^-(s)$ be the positive and negative part of $\widetilde{c}(s)$, respectively. Following the proof of Claim 1 above, choose ξ^+ such that $\xi^+ - \sigma\big|_{S^*} = \alpha\widetilde{c}^+$ and ξ^- such that $\xi^- - \sigma\big|_{S^*} = \alpha\widetilde{c}^-$. Then $A_{\xi^+} \sim_i A_{\xi^-}$ for all $i \in I$, but $A_{\xi^+} \cup A_{\xi^-} \succ_j A_{\xi^-}$ for some $j \in I$, because $c_{A',B'}(s)$ is not constant. Thus $A' := A_{\xi^+}$ and $B' := A_{\xi^-}$ violate Axiom 1. They satisfy $\frac{c_{A',B'}(s)}{c_{A',B'}(s')} \neq \frac{c_{A,B}(s)}{c_{A,B}(s')}$ by construction. ■

9.3. Proof of Theorem 1

Proof of Theorem 1, i)⇒iii): Suppose to the contrary that S^* is infinite or finite with cardinality $\#S^* > \#I$. The definition of S^* implies that one can find $\#I + 1$ Borel Sets with non-empty interior, $\{S_t\}_{t=1}^{\#I+1}$, such that for all $t \leq \#I + 1$ there exists $i \in I$ with $\mu(\text{int}(S_t)|i) > 0$. Since μ can have at most countably many atoms, one can further guarantee $\mu(Cl(S_t) \cap Cl(S_{t'})|i) = 0$ for all $t, t' \leq \#I + 1$ and all $i \in I$.

Claim 1: Given S_t, there is ε small enough and a support function ξ_t, such that $\xi_t = \varepsilon$ on $S \backslash S_t$, $\xi_t \geq \varepsilon$ on S_t and $x_t(i) := \int_S [\xi_t(s) - \varepsilon]\, d\mu(s|i) > 0$ for some $i \in I^*$.

Proof of Claim 1: Remember that σ_ε supports a ball, B_ε, with radius ε around the center of the simplex. The maximal menu B with $\sigma_B \leq \sigma_\varepsilon$ on $S \backslash S_t$ includes all lotteries with $p \cdot s \leq \varepsilon$ for all $s \in S \backslash S_t$. This implies $\max_{p \in B}(p \cdot s) > \varepsilon$ for all s in the non-empty interior of S_t. Hence, $\sigma_B > \sigma_\varepsilon$ must hold on $\mathrm{int}(S_t)$. Let $\xi_t := \sigma_B$. $\|$

We can solve the following system of $\#I+1$ independent linear equations with variables $\{\alpha_t\}_{t \in \{1,\#I+1\}}$ for any $n>0$ and some given t' :

$$\sum_{t=1}^{\#I+1} x_t(i)\,\alpha_t = 0 \text{ for all } i \in I \text{ and } \alpha_{t'} = \vartheta,$$

where x_t is as defined in Claim 1. Choose ϑ such that $\sum|\alpha_t| = 1$. The convex combination of finitely many menus is well defined, and by property (ii) above, the convex combination of finitely many support functions is, too. Thus one can define two support functions as

$$\begin{aligned} \xi &:= \sum_{t=1}^{\#I+1} |\alpha_t| \left(\mathbf{1}_{\alpha_t>0}\xi_t + \mathbf{1}_{\alpha_t<0}\varepsilon\right) \\ \sigma &:= \sum_{t=1}^{\#I+1} |\alpha_t| \left(\mathbf{1}_{\alpha_t>0}\varepsilon + \mathbf{1}_{\alpha_t<0}\xi_t\right) \end{aligned}$$

Then $\sum_{t=1}^{\#I+1} x_t(i)\,\alpha_t = 0$ for all $i \in I$ immediately implies that $A_\xi \sim_i A_\sigma$. At the same time, $\alpha_{t'} \neq 0$ implies that $A_\xi \cup A_\sigma \succ_i A_\xi$ for some $i \in I^*$, which contradicts Axiom 1.

Proof of Theorem 1, i)⇒ii):

Claim 2: For any positive function f on S^* there is $\overline{\alpha} > 0$ small enough, such that for any $0 < \alpha < \overline{\alpha}$ there are support functions ξ and σ with $\xi - \sigma|_{S^*} = \alpha f$.

Proof of Claim 2: List the elements of $S^* = \{s_1, s_2, ...\}$. Consider $\{S_t\}_{t=1}^{\#S^*}$ with $S_t \subseteq S$, $s_t \in S_t$ and $s_{t'} \notin S_t$ for $t' \neq t$. Construct $\xi_t(s)$ as in Claim 1. Let $x_t := \xi_t(s_t) - \varepsilon$. Choose $\{\alpha_t\}_{t=1}^{\#S^*}$ such that $\alpha_t x_t \propto f(s_t)$ and $\sum \alpha_t = 1$. Define $\overline{\xi} := \sum \alpha_t \xi_t$ and $\sigma := \varepsilon$. Then $\overline{\xi} - \sigma\big|_{S^*} \equiv \overline{\alpha} f$ for some $\overline{\alpha} > 0$. For $\alpha < \overline{\alpha}$ let $\xi := \alpha\overline{\xi} + (1-\alpha)\,\varepsilon$. Then $\xi - \sigma|_{S^*} \equiv \alpha f$. $\|$

Suppose (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ are two distinct CPF representations of $\succ$ with S^* and $\widehat{S^*}$ as the corresponding relevant taste spaces. Up to a constant, the vNM expected utility $U_s(p)$ can be written as $l(s)(s \cdot p)$. Then $\max_{\alpha \in A} U_s(\alpha) = l(s)\,\sigma_A(s)$. As in the text, $l(s)$ captures the 'intensity' of taste s. Let $f(s) \propto \frac{1}{l(s)}$ on S^*. Analogously let $\widehat{f}(s) \propto \frac{1}{\widehat{l}(s)}$ on $\widehat{S^*}$. Find α and $\widehat{\alpha}$ small enough, such that there are ξ and $\widehat{\xi}$ with $\xi - \sigma|_{S^*} = \alpha f$ and $\widehat{\xi} - \sigma\Big|_{\widehat{S^*}} = \widehat{\alpha}\widehat{f}$ and $\widehat{\xi} \sim_i \xi$. Because $f(s) \propto \frac{1}{l(s)}$ it must be true that $(\xi(s) - \sigma(s))\, l(s)$ is constant across S^*. Consequently, $\sum_{S^*} (\xi(s) - \sigma(s))\, l(s)\, \mu(s\,|j)$ must be independent of j. This independence is meaningful in terms of $\succ$. It is easy to verify that it holds if and only if

$$\frac{\phi(i)}{\phi(i)+\phi(j)} g_i^{A_\xi} + \frac{\phi(j)}{\phi(i)+\phi(j)} g_j^{A_\sigma} \sim \frac{\phi(i)}{\phi(i)+\phi(j)} g_i^{A_\sigma} + \frac{\phi(j)}{\phi(i)+\phi(j)} g_j^{A_\xi}$$

for all $i, j \in I^*$. The same argument, based on the representation $\left(\phi, \widehat{\mu}, \widehat{U}\right)$, implies that $\sum_{\widehat{S^*}} \left(\widehat{\xi}(s) - \sigma(s)\right) \widehat{l}(s)\, \widehat{\mu}(s\,|j)$ is independent of j. (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ both represent $\succ$, and therefore $\sum_{S^*} \left(\widehat{\xi}(s) - \sigma(s)\right) l(s)\, \mu(s\,|j)$ must also be independent of j. Hence, $\widehat{\xi} \sim_j \xi$ for all $j \in I$. At the same time, $\left(\widehat{\xi}(s) - \sigma(s)\right) l(s)$ is not constant across S^*, because (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ are distinct, which implies $\widehat{\alpha}\widehat{f}(s)$ is not identical to $\alpha f(s)$ on S^* or on $\widehat{S^*}$. Without loss of generality suppose they disagree on S^*. Because $\widehat{\xi} \sim_j \xi$, there must be $s', s'' \in S^*$ with $\widehat{\alpha}\widehat{f}(s') > \alpha f(s')$ and $\widehat{\alpha}\widehat{f}(s'') < \alpha f(s'')$. Hence, $A_{\widehat{\xi}} \cup A_\xi \succ_j A_\xi$ for all $j \in I$ with $\mu(s'\,|j) > 0$. This contradicts Axiom 1. Therefore, $S^* = \widehat{S^*}$ and $l(s) \propto \widehat{l}(s)$ on S^*. This establishes the essential uniqueness of U.

That the measure $\mu(.\,|i)$ is unique for all $i \in I$ with $\phi(i) > 0$ then follows immediately from the result in DLR (their Theorem 1), that $\widehat{\mu}(s\,|i)\, \widehat{l}(s) \propto \mu(s\,|i)\, l(s)$ for the case of a finite taste space.

Proof of Theorem 1, ii)⇒i): It remains to be established that Axiom 1 is also necessary. Suppose to the contrary that the representation exists with the stated uniqueness, but Axiom 1 is violated. Then, there are two menus $A, B \in \mathcal{A}$, such that $A \sim_j B$ for all $j \in I$ and $A \cup B \succ_i B$ for some $i \in I$. $A \sim_j B$ for all $j \in I$ implies $\sum_{S^*} c_{A,B}(s)\, \mu(s\,|j) = 0$ for all $j \in I$ and for $c_{A,B}(s)$ as defined in Definition 13. $A \cup B \succ_i B$ implies that $c_{A,B}(s)$ cannot be zero under all tastes, so it must be positive under some tastes and negative under others. For the proof it is important that it is not constant across tastes. Define $\widehat{\mu}(s\,|i) := \frac{\left(1+\eta c_{A,B}(s)\right)\mu(s|i)}{\sum_{S^*}\left(1+\eta c_{A,B}(s)\right)\mu(s|i)}$, where $\eta \neq 0$ is small enough, such that $1 + \eta c_{A,B}(s) > 0$ for all $s \in S^*$. Accordingly define $\widehat{l}(s) := \frac{l(s)}{1+\eta c_{A,B}(s)}$. Clearly $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ is a representation of $\succ_i$, when evaluated in acts g_i^A. As such, it is unique up to positive affine transformations. To verify that it represents $\succ$ it is, therefore, sufficient to find two menus, A and B, such that (i) $A \succ_i B$ for all $i \in I$ and (ii) the relative cost of getting g_i^B instead of g_i^A versus the cost of getting g_j^B instead of g_j^A

is the same according to $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ as according to (ϕ, μ, U) for all $i, j \in I$. Let $\widehat{V}$ and V denote the respective objective functions. Consider again A_ξ and A_σ from the proof of claim 2. Their construction immediately implies that $V\left(g_i^{A_\xi}\right) - V\left(g_i^{A_\sigma}\right) \propto \phi(i)$ and

$$\widehat{V}\left(g_i^{A_\xi}\right) - \widehat{V}\left(g_i^{A_\sigma}\right) \propto \frac{\phi(i)}{1 + \eta \sum_{S^*} c_{A,B}(s)\, \mu(s\,|i)} = \phi(i).$$

This contradicts the uniqueness statement in Theorem 1 i). Thus, Axiom 1 is necessary for this uniqueness statement. ■

9.4. Proof of Theorem 2

Definition 14: As in the proof of Theorem 1, let $\overline{\mathcal{A}}$ be the collection of all convex subsets of $\Delta(Z)$. Let $\overline{G}$ be the collection of all acts: $g: I \to \overline{\mathcal{A}}$. Call $g \in \overline{G}$ a convex act.

Lemma 2: $\succ$ *constrained to* $\overline{G}$ *satisfies Axioms 2-4 if and only if there is a family of continuous linear functions* $\{v_i\}_{i\in I}$, $v_i : \overline{\mathcal{A}} \to \mathbb{R}$, *such that* $v : \overline{G} \to \mathbb{R}$ *with* $v(g) = \sum_{i\in I} v_i(g(i))$, *represents* $\succ$ *on* $\overline{G}$.

Moreover, if there is a family of continuous linear functions $\{v_i'\}_{i\in I}$, $v_i' : \overline{\mathcal{A}} \to \mathbb{R}$, *such that* $v'(g) = \sum_{i\in I} v_i'(g(i))$ *represents* $\succ$ *on* $\overline{G}$, *then there are constants* $a > 0$ *and* $\{b_i\,|i \in I\}$, *such that* $v_i' = b_i + a v_i$ *for each* $i \in I$.

Proof: The collection of convex acts $\overline{G}$ together with the convex combination of acts as a mixture operation is a mixture space. Lemma 2 is an application of the Mixture Space Theorem (Theorem 5.11 in Kreps (1988)), where additive separability across I follows from the usual induction argument and the continuity of v_i is a consequence of Axiom 2.[29] □

Corollary 3: If $i \in I$ is nonnull, then $V_i(A)$ and $v_i(A)$ agree on $\overline{\mathcal{A}}$ up to positive affine transformations.

Proof: Evaluating $v\left(g_i^A\right)$ implies that v_i represents $\succ_i$ on $\overline{\mathcal{A}}$. v_i is linear. The Mixture Space Theorem states that any other linear representation of $\succ_i$ agrees with v_i, up to a positive affine transformation. According to Theorem DLRS, $V_i(A)$ is linear and represents $\succ_i$ on $\mathcal{A}$. □

[29] Axiom 2 (Continuity) is stronger than von Neumann-Morgenstern Continuity on $\overline{G}$, which requires that for all $g \succ g' \succ g''$ there are $p, q \in (0,1)$, such that $pg + (1-p)\, g'' \succ g' \succ qg + (1-q)\, g''$.

Let $V_i(A)$ be a representation of $\succ_i$ as provided by Theorem DLRS. For any nonnull event $i \in I$ (which exists according to Corollary 1), $V_i(A)$ and $v_i(A)$ agree on $\overline{\mathcal{A}}$ up to a positive affine transformation, as established by Corollary 3. There is an event dependent, positive scaling factor $\pi(i)$, such that, up to a constant, $v_i(A) = \pi(i) V_i(A)$ for all $A \in \overline{\mathcal{A}}$, where $\pi(i) = 0$ if and only if i is trivial. The act that always yields the entire prize space as a menu is convex and is one of the best acts. Because of Monotonicity, there is a worst act that assigns a singleton for every contingency. This act is also convex. By Continuity there is a convex act $\overline{g} \in \overline{G}$ for every $g \in G$, such that $\overline{g}(i) \sim_i g(i)$, which implies $V_i(\overline{g}(i)) = V_i(g(i))$. Let V' represent $\succ$ on G and $V' \equiv v$ on $\overline{G}$. Then, $V'(g) = V'(\overline{g}) = \sum_{i \in I} v_i(\overline{g}(i)) = \sum_{i \in I} \pi(i) V_i(\overline{g}(i)) = \sum_{i \in I} \pi(i) V_i(g(i))$. Hence, $g \succ h$ if and only if $\sum_{i \in I} \pi(i) V_i(g(i)) > \sum_{i \in I} \pi(i) V_i(h(i))$. Therefore

$$V'(g) = \sum_{i \in I} \pi(i) \left[\int_S \sigma_{g(i)}(s)\, l(s)\, d\mu_i(s) \right]$$

represents $\succ$. Since v is unique only up to positive affine transformations, $\pi(i)$ can be normalized to be a probability measure, $\phi(i)$. Interpreting $\mu(s|i) := \mu_i(s)$ as a conditional probability measure over the taste space S, define

$$V(g) := \sum_{i \in I} \phi(i) \left[\int_S \sigma_{g(i)}(s)\, l(s)\, d\mu(s|i) \right]$$

to establish the sufficiency statement in Theorem 2. That Axioms 2-6 are necessary for the existence of the representation is straightforward to verify. ■

9.5. Proof of Theorem 1'

The proof idea is the same as for Theorem 1. To show that Axiom 1 is sufficient for the uniqueness statement, I first establish the analogous claim to Claim 2. The definition of support functions (Definition 12) and all related notations remain relevant here.

Intensity of tastes $l : S \to \mathbb{R}^+$ is a strictly positive function.[30] Consider the uninformative event $I \in \mathcal{F}$. Note that $\int_{S'} l d\mu(s|I)$ exists for any measurable $S' \subset S$, because the value of the menu supported by σ_ε in Lemma 1 is $\int_S \sigma_\varepsilon l d\mu(s|I) = \varepsilon \int_S l d\mu(s|I)$.

[30] $l(s) = 0$ corresonds to the trivial state, which is not part of the CPF representation.

Lemma 3: There are support functions ξ and σ and a number $\alpha > 0$, such that $\mu(S'|I) - \int_{S'} \alpha(\xi - \sigma) l d\mu(s|I) < \varepsilon$. For $\alpha' > \alpha$ there are also support functions ξ' and σ', such that $\mu(S'|I) - \int_{S'} \alpha'(\xi' - \sigma') l d\mu(s) < \varepsilon$.

Proof of Lemma 3:

Claim 3: If f is positive and integrable, then for any $\varepsilon > 0$, there is a continuous, positive function $g : S \to \mathbb{R}$ with bounded support, such that $\int_{S'} |f - g| d\mu(s) < \varepsilon$ for every measurable set $S' \subseteq S$.

Proof: As f and μ are both weakly positive, $\int_S |f\mu(s)| ds$ exists. Thus, for every $\varepsilon > 0$, there exists a continuous function $g : S \to \mathbb{R}$ such that $\int_{S'} |g - f| d\mu(s) < \varepsilon$. See, for example, Billingsley (1995), Theorem 17.10. Since f is positive, g can be chosen to be positive.$\|$

Given $\varepsilon > 0$, Claim 3 establishes that there is a continuous, positive function g , such that $\int_{S'} |l - g| d\mu(s) < \varepsilon$ for every measurable set $S' \subseteq S$. The function $\frac{1}{g} : S \to \mathbb{R}^+$ is then positive, bounded and continuous. Thus, for any $\varepsilon > 0$, g can be chosen such that

$$\int_{S'} |g - l| \frac{1}{g} d\mu(s) \leq \left\| \frac{1}{g} \right\|_\infty \int_{S'} |g - l| d\mu(s) < \frac{\varepsilon}{2}.$$

Claim 4 (Lemma 11 in DLR): The functions that are the difference of two support functions span a cone that is dense in $C(S)$, the space of continuous functions on S, the unit sphere in $\mathbb{R}^k$.

As l is positive, $\nu(S') := \int_{S'} l d\mu(s|I)$ is itself a positive measure.[31] Claim 4 then implies that for every $\varepsilon > 0$ there are two support functions ξ and σ and a number $\alpha > 0$, such that

$$\int_{S'} \left| \frac{1}{g} - \alpha(\xi - \sigma) \right| l d\mu(s) < \frac{\varepsilon}{2}$$

for every measurable set $S' \subseteq S$.

[31] If information is ignored, in the sense that DM only gets to choose between acts that do not condition on information, then preferences can be represented as in DLRS. The measure ν corresponds to the measure featured in this representation. It is dominated by the measure $\mu(s|I)$ and the Radon-Nikodym derivative of ν with respect to $\mu(.|I)$ evaluated in s is $l(s)$, the intensity of taste s.

Hence,

$$\mu(S'|I) - \int_{S'} \alpha(\xi - \sigma) l d\mu(s|I) \leq \int_{S'} |1 - \alpha(\xi - \sigma) l| d\mu(s|I)$$

$$\leq \int_{S'} |g - l| \frac{1}{g} d\mu(s|I) + \int_{S'} \left| \frac{1}{g} - \alpha(\xi - \sigma) \right| l d\mu(s|I) < \varepsilon.$$

This establishes the first part of the lemma. To show the second part, consider $\alpha' = c\alpha$ with $c > 1$, then let $\sigma' = \sigma$ and $\xi' = \frac{1}{c}\xi + \left(1 - \frac{1}{c}\right)\sigma$. ξ' is a convex combination of support functions and therefore a support function, and $\alpha'(\xi' - \sigma') \equiv \alpha(\xi - \sigma)$. This concludes the proof of Lemma 3. □

Suppose (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ are two CPF representations of $\succ$. Following Lemma 3, one can define a sequence of support functions $\langle \xi_n \rangle$ and $\langle \sigma_n \rangle$ and a sequence of numbers $\langle \alpha_n \rangle$ such that

$$\mu(S'|I) - \int_{S'} \alpha_n(\xi_n - \sigma_n) l d\mu(s|I) < \frac{1}{n}$$

for every measurable set $S' \subseteq S$ and for all $n > 0$. Analogously define $\left\langle \widehat{\xi}_n \right\rangle$ and $\langle \widehat{\sigma}_n \rangle$ and a sequence of numbers $\langle \widehat{\alpha}_n \rangle$ based on $\left(\phi, \widehat{\mu}, \widehat{U}\right)$. According to the second part of Lemma 3 it is possible to choose $\langle \alpha_n \rangle$ and $\langle \widehat{\alpha}_n \rangle$ such that

$$\int_S (\xi_n - \sigma_n) l d\mu(s|I) = \int_S \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) l d\mu(s|I)$$

and hence $\frac{1}{2}\xi_n + \frac{1}{2}\widehat{\sigma}_n \sim_I \frac{1}{2}\widehat{\xi}_n + \frac{1}{2}\sigma_n$ according to (ϕ, μ, U) for all $n > 0$.

Rewriting $p_{A,B}(D)$ as defined in Definition 7 in terms of support functions yields $p_{A,B}(D) = \int_S (\sigma_A - \sigma_B) l d\mu(s|D)$. For the remainder of the proof, let A_n, B_n and C_n be defined such that $\sigma_{A_n} = \frac{1}{2}\xi_n + \frac{1}{2}\widehat{\sigma}_n$, $\sigma_{B_n} = \frac{1}{2}\widehat{\xi}_n + \frac{1}{2}\sigma_n$ and $\sigma_{C_n} = \frac{1}{2}\sigma_n + \frac{1}{2}\widehat{\sigma}_n$.

Claim 5: $\frac{p_{C_n,A_n}(D)}{p_{C_n,B_n}(D)} \to 1$ for all $D \in \mathcal{F}$.

Proof: First note that

$$\begin{aligned} \frac{p_{C_n,A_n}(D)}{p_{C_n,B_n}(D)} &= \frac{\int_S \frac{1}{2}(\sigma_n + \widehat{\sigma}_n - \xi_n - \widehat{\sigma}_n) l d\mu(s|D)}{\int_S \frac{1}{2}\left(\sigma_n + \widehat{\sigma}_n - \widehat{\xi}_n - \sigma_n\right) l d\mu(s|D)} \\ &= \frac{\int_S (\xi_n - \sigma_n) l d\mu(s|D)}{\int_S \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) l d\mu(s|D)} \end{aligned}$$

By definition, $\mu(S'|I) - \alpha_n \int_{S'} (\xi_n - \sigma_n) \, l d\mu(s|I) < \frac{1}{n}$ for every measurable set $S' \subseteq S$ and for all $n > 0$ implies that (i) $\lim_{n\to\infty} \left[\alpha_n \int_S (\xi_n - \sigma_n) \, l d\mu(s|I)\right] = 1$, because μ is a probability measure and (ii) $\alpha_n (\xi_n - \sigma_n) \, l \to 1$ almost everywhere according to $\mu(s|I)$. The same observations can be made for $\left\langle \widehat{\xi}_n \right\rangle$, $\langle \widehat{\sigma}_n \rangle$, $\langle \widehat{\alpha}_n \rangle$ and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$.

For every $D \in \mathcal{F}$ the measure $\mu(.|D)$ is dominated by $\mu(.|I)$ and $S' \subseteq S$ is $\mu(.|D)$ measurable if and only if it is $\mu(s|I)$ measurable. Hence, $\lim_{n\to\infty} \left[\alpha_n \int_S (\xi_n - \sigma_n) \, l d\mu(s|D)\right] = 1$ for all $D \in \mathcal{F}$. Analogously $\lim_{n\to\infty} \left[\widehat{\alpha}_n \int_S \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) \widehat{l} d\widehat{\mu}(s|D)\right] = 1$ for all $D \in \mathcal{F}$. As in the case of finite I, it is easy to verify that this independence is meaningful in terms of $\succ$. Hence, there is also a sequence of numbers $\langle \beta_n \rangle$, such that $\lim_{n\to\infty} \left[\beta_n \int_S \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) l d\mu(s|D)\right] = 1$ for all $D \in \mathcal{F}$. Since $\frac{1}{2}\xi_n + \frac{1}{2}\widehat{\sigma}_n \sim_I \frac{1}{2}\widehat{\xi}_n + \frac{1}{2}\sigma_n$ for all $n > 0$, it must be that $\frac{\alpha_n}{\beta_n} \to 1$. Together with observation (ii) above this implies that $\frac{\int_S (\xi_n - \sigma_n) l d\mu(s|D)}{\int_S \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) l d\mu(s|D)} \to 1$ for all $D \in \mathcal{F}$. $\|$

Claim 6: If (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ are two CPF representations of $\succ$ that are distinct beyond the changes permitted in the uniqueness statement of Theorem 1', then $\frac{p_{C_n, A_n \cup B_n}(D)}{p_{C_n, B_n}(D)} \nrightarrow 1$.

Proof: First note that

$$\begin{aligned}
\frac{p_{C_n, A_n \cup B_n}(I)}{p_{C_n, B_n}(I)} &= \frac{\int_S \frac{1}{2} \left(\sigma_n + \widehat{\sigma}_n - \max\left\{\xi_n + \widehat{\sigma}_n, \widehat{\xi}_n + \sigma_n\right\}\right) l d\mu(s|I)}{\int_S \frac{1}{2} \left(\sigma_n + \widehat{\sigma}_n - \widehat{\xi}_n - \sigma_n\right) l d\mu(s|I)} \\
&= \frac{\int_S \max\left\{\xi_n - \sigma_n, \widehat{\xi}_n - \widehat{\sigma}_n\right\} l d\mu(s|I)}{\int_S \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) l d\mu(s|I)}
\end{aligned}$$

It follows immediately from the uniqueness statements in Theorems 3 and 4 in DLR, that $\mu(s|D)$ and $\widehat{\mu}(s|D)$ share the same support[32] and that $l(s)\,\mu(s|D)$ is unique up to rescaling for any $D \in \mathcal{F}$. Thus, if (ϕ, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ are distinct in the sense of the claim, then the corresponding functions l and $\widehat{l}$ have to be distinct. Consequently, there is $S' \subset S$, such that $\int_{S'} \frac{l}{\widehat{l}} d\mu(s|I) \neq \mu(S'|I)$. Thus, $d := \lim_{n\to\infty} \left[\alpha_n \int_{S'} \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) l d\mu(s|D)\right] \neq \mu(S'|I)$. Without loss of generality suppose that $d > 1$. Then $\lim_{n\to\infty} \left[\alpha_n \int_S \max\left\{\xi_n - \sigma_n, \widehat{\xi}_n - \widehat{\sigma}_n\right\} l d\mu(s|D)\right] >$

[32] They induce the same space of relevant tastes, S^*. In analogy to the finite case, S^* is the minimal set of tastes with $\mu(S^*|D) = 1$ for all non-trivial $D \in \mathcal{F}$.

1, which implies

$$\frac{\int_S \max\left\{\xi_n - \sigma_n, \widehat{\xi}_n - \widehat{\sigma}_n\right\} l d\mu\left(s \left| D\right.\right)}{\int_S \left(\widehat{\xi}_n - \widehat{\sigma}_n\right) l d\mu\left(s \left| D\right.\right)} \nrightarrow 1. \;\|$$

The combination of Claims 5 and 6 provides a direct violation of Axiom 1'. Hence, Axiom 1' implies that (ϕ, μ, U) is unique in the sense of Theorem 1'.

It remains to show that Axiom 1' is also necessary. The argument requires only slight changes compared to the finite case: suppose to the contrary that the representation holds with the stated uniqueness, but Axiom 1' is violated. Then, there are sequences $\langle A_n\rangle, \langle B_n\rangle, \langle C_n\rangle \subseteq A$, which converge in the Hausdorff topology, with $\frac{p_{C_n, A_n \cup B_n}(D)}{p_{C_n, B_n}(D)} \nrightarrow 1$ for some $D \in F$ and $\frac{p_{C_n, A_n}(D')}{p_{C_n, B_n}(D')} \to 1$ for all $D' \in F$. $\frac{p_{C_n, A_n}(D')}{p_{C_n, B_n}(D')} \to 1$ for all $D' \in F$ implies that

$$\frac{\int\limits_S c_{A_n, B_n}(s)\, d\mu\left(s \left| D'\right.\right)}{\int\limits_S c_{C_n, B_n}(s)\, d\mu\left(s \left| D'\right.\right)} \to 0$$

for all $D' \in F$. $\frac{p_{C_n, A_n \cup B_n}(D)}{p_{C_n, B_n}(D)} \nrightarrow 1$ implies that there is a set $S' \subseteq S$ with $\mu\left(S' \left| D\right.\right) > 0$ and

$$\frac{\int\limits_{S'} c_{A_n, B_n}(s)\, d\mu\left(s \left| D\right.\right)}{\int\limits_S c_{C_n, B_n}(s)\, d\mu\left(s \left| D\right.\right)} \nrightarrow 0.$$

In complete analogy to the finite case, define

$$\widehat{\mu}\left(s \left| D\right.\right) := \left(1 + \eta \frac{c_{A_n, B_n}(s)}{\int\limits_S c_{C_n, B_n}(s)\, \mu\left(s \left| D\right.\right)}\right) \mu\left(s \left| D\right.\right),$$

where η is small enough such that $1 + \eta \frac{c_{A_n, B_n}(s)}{\int\limits_S c_{C_n, B_n}(s) \mu(s|D)} > 0$ for all $s \in S$. As in the finite case, there is then a CPF representation $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ for an appropriate $\widehat{U}$. Thus, Axiom 1' must hold. ■

9.6. Proof of Theorem 2':

Definition 15: Let $\{D_t \left| t \in \{1, .., T\}\right.\}$ be a finite partition of I with $D_t \in \mathcal{F}$. $\{D_t\}$ denotes a generic partition of this type. Further let $G_{\{D_t\}}$ be the collection of acts where the outcome depends only on the event $D \in \{D_t\}$. Let $G^* := \bigcup\limits_{\{D_t\}} G_{\{D_t\}}$ be the set of *simple acts.* $\overline{G} \cap G^*$

is the collection of all simple convex acts.

The support of $g \in G_{\{D_t\}}$ is a finite subset of $\mathcal{A}$.

Lemma 2': $\succ$ *constrained to* $\overline{G} \cap G^*$ *satisfies Axioms 2-4 if and only if there are continuous linear functions* $v_D : \overline{\mathcal{A}} \to \mathbb{R}$, *such that* $v : \overline{G} \cap G^* \to \mathbb{R}$ *with* $v(g) = \sum_{t=1}^{T} v_{D_t}(g(D_t))$ *for* $g \in \overline{G} \cap G_{\{D_t\}}$, *represents* $\succ$.

Moreover, if there is another collection of continuous linear functions, $v'_D : \overline{\mathcal{A}} \to \mathbb{R}$, *such that* $v'(g) = \sum_{t=1}^{T} v'_{D_t}(g(D_t))$ *represents* $\succ$ *on* $\overline{G} \cap G^*$, *then there are constants* $a > 0$ *and* $\{b_D \,|D \in \mathcal{F}\}$, *such that* $v'_D = b_D + a v_D$ *for each* $D \in \mathcal{F}$.

Proof: That $v(g) = \sum_{t=1}^{T} v_{D_t}(g(D_t))$ for $g \in \overline{G} \cap G_{\{D_t\}}$ represents $\succ$ confined to $\overline{G} \cap G_{\{D_t\}}$, is implied by Lemma 2. If the simple act g is constant on each element of $\{D_t\}_{t=1}^{T}$, then it is also constant on each element of a finer partition $\{D'_t\}_{t=1}^{T'}$. For $\tau \subseteq \{1, ..., T'\}$, such that $D_t = \bigcup_{t\in\tau} D'_t$, the usual induction argument yields

$$\begin{aligned} & \frac{1}{\sharp\tau}\left(g^*(D_1), ..., g^*(D_{t-1}), A, g^*(D_{t+1}), ..., g^*(D_T)\right) + \frac{\sharp\tau - 1}{\sharp\tau} g^* \\ = & \sum_{t\in\tau} \frac{1}{\sharp\tau}\left(g^*(D'_1), ..., g^*\left(D'_{t-1}\right), A, g^*\left(D'_{t+1}\right), ..., g^*(D'_{T'})\right), \end{aligned}$$

and thus $v_{D_t}(A) = \sum_{t\in\tau} v_{D'_t}(A)$. Therefore, $v(g) = \sum_{t=1}^{T} v_{D_t}(g(D_t))$ for $g \in \overline{G} \cap G_{\{D_t\}}$ represents $\succ$ constrained to all simple acts, $g \in \overline{G} \cap G^*$.

The uniqueness statement follows immediately from the uniqueness in Lemma 2. That the representation implies continuity and linearity of v and, thus, the axioms is obvious. $\square$

As suggested in the text, I first establish the result of Theorem 2 for simple acts and then show that those are dense in the space of all acts. Once this is established, I verify that Definition 6 can be applied. Corollary 3 still holds, where i is replaced with D.

Claim 7: If $\succ$ satisfies Axioms 2-6, then there are a set of bounded positive numbers $\{l(s)\}_{s\in S}$, a collection of probability measures $\{\mu_D(s)\}_{D\in\mathcal{F}}$, and a countably additive probability measure ϕ on $\mathcal{F}$, such that, for $g \in G_{\{D_t\}}$,

$$V(g) = \sum_{t=1}^{T} \phi(D_t) \int_S \sigma_{g(D_t)}(s)\, l(s)\, d\mu_{D_t}(s)$$

represents $\succ$ on G^*. Furthermore, there is a function $v : G \to \mathbb{R}$ as in Lemma 2' that agrees with V on G^*.

Proof: In complete analogy to the proof of Theorem 2, it can be established that there is an event dependent, positive scaling factor $\pi(D)$ such that

$$v(g) = \sum_{t=1}^{T} \pi(D_t) \int_S \sigma_{g(D_t)}(s)\, l(s)\, d\mu_{D_t}(s)$$

for $g \in G_{\{D_t\}}$, where v represents $\succ$. $\pi(D) = 0$ if and only if D is trivial. This implies that $\succ_D$ can be represented by $\int_S \sigma_A(s)\, l(s)\, d\mu_D(s)$. Holding intensities, l, fixed, it is an immediate implication of Theorem 1 in DLR, that $\succ_D$ identifies $\mu_D(s)$ uniquely. Now consider a partition $\{D_t\}_{t=1}^{T}$ with $D \cup D' \in \{D_t\}_{t=1}^{T}$ and a finer partition $\{D'_t\}_{t=1}^{T'}$ with $D, D' \in \{D'_t\}_{t=1}^{T'}$. According to the proof of Lemma 2', $v_{D \cup D'}(A) = v_D(A) + v_{D'}(A)$. As $l(s)$ does not depend on D, the representation for the finer partition must then assign the same relative weight to any taste s, as the representation for the coarser partition:

$$\pi(D \cup D')\, \mu_{D \cup D'}(s) \propto \pi(D)\, \mu_D(s) + \pi(D')\, \mu_{D'}(s)$$

for all $s \in S$ and $D, D' \in F$. Since $\mu_{D \cup D'}$, μ_D and $\mu_{D'}$ are probability measures on S, integration over S yields $\pi(D \cup D') = \pi(D) + \pi(D')$. It follows inductively that

$$\pi\left(\bigcup D_t\right) = \sum \pi(D_t)$$

for $\bigcup D_t \in \mathcal{F}$. Since $\mathcal{F}$ is a σ-algebra, it includes all countable unions of its elements. Since v is unique only up to positive affine transformations, $\pi(D)$ can be normalized to be a countably additive probability measure, $\phi(D)$. For $g \in G^*$, define

$$V(g) := \sum_{t=1}^{T} \phi(D_t) \int_S \sigma_{g(D_t)}(s)\, l(s)\, d\mu_{D_t}(s)$$

which establishes Claim 7. $\|$

Claim 8: The simple acts G^* are dense in G in the topology defined on G.

Proof: I will argue that every neighborhood of an act $g \in G$ in the product topology contains a simple act. Let $p_i : G \to G_i$ be the natural projection from G to $G_i = \mathcal{A}$ and let $B_\varepsilon(A) \subseteq \mathcal{A}$ be an open ball of radius $\varepsilon > 0$ around $A \in \mathcal{A}$,

$$B_\varepsilon(A) := \{B \in \mathcal{A} \,| d_h(A, B) < \varepsilon\}.$$

It suffices to show that, for every act $g \in G$, there is a simple act in every finite intersection of sets of the form $p_i^{-1}(B_\varepsilon(g(i))) \subseteq G$.[33] Let a finite set $I' \subseteq I$ index the relevant dimensions for this intersection. I will establish that there is always a simple act h with

$$\max_{i \in I'} d_h(g(i), h(i)) < \varepsilon.$$

Given ε, let $L \subset \Delta(Z)$ be a finite set of lotteries over Z, such that for all $\alpha \in \Delta(Z)$ there is $\alpha' \in L$ with $d_p(\alpha, \alpha') < \varepsilon$. This set exists, because $\Delta(Z)$ is compact. Let $\mathcal{A}'$ be the set of all subsets of L. Then $\mathcal{A}' \subset \mathcal{A}$, and for all $A \in \mathcal{A}$ there is $A' \in \mathcal{A}'$ with $d_h(A, A') < \varepsilon$ by the definition of $d_h(A, B)$. Thus, there is an act in $\bigcap_{I'} p_i^{-1}(B_\varepsilon(g(i)))$ with support only in $\mathcal{A}'$. Because I' is finite and $\mathcal{F}$ the Borel σ-algebra, there is finite partition $\{D_t\}$ of I, such that $i, j \in I'$ and $i \in D_t$ imply $j \notin D_t$. Thus, for every $g \in G$ and for all $\varepsilon > 0$, there is a simple act in $\bigcap_{I'} p_i^{-1}(B_\varepsilon(g(i)))$. $\|$

Claim 7 establishes that on G^* it is possible to choose

$$v(g) \equiv \sum_{t=1}^{T} \phi(D_t) \int_S \sigma_{g(D_t)}(s)\, l(s)\, d\mu_{D_t}(s)$$

Claim 8 then implies that the continuous function $v(g)$ is uniquely determined on G.

To apply Definition 6, hold $l(s)$ fixed. It is bounded by construction. For a sequence of simple acts, $\langle g_n \rangle$ in G^* with $g_n \in G_{\{D_t\}}$, consider the sequence of functions $\langle \varphi_n \rangle$, where $\varphi_n : I \to \mathbb{R}$ is defined as

$$\varphi_n(i) := \int_S \sigma_{g_n(D)}(s)\, l(s)\, d\mu_D(s)$$

for $i \in D \in \{D_t\}$. Then, the task is to find a sequence $\langle g_n \rangle$ in G^*, such that φ_n converges

[33] Open sets in the product topology are the product of open sets in the topology d_h generates on $\mathcal{A}$, which coincide with $\mathcal{A}$ for cofinitely many $i \in I$.

from below to the bounded function

$$\varphi(i) := \int_S \sigma_{g(i)}(s)\, l(s)\, d\mu_i(s)$$

for a given act $g \in G$ and some measure $\mu_i(s)$. First, for $g_n \in G_{\{D_t\}}$, let $D^n(i)$ be such that $i \in D^n(i) \in \{D_t\}$. Because $g_n \in G_{\{D_t\}}$ can always be expressed by using a finer partition and because $\mathcal{F}$ is the Borel σ-algebra, it is without loss of generality to assume $\lim_{n\to\infty} D^n(i) = \{i\}$. Given $l(s)$, $\mu_D(s)$ is unique. Axiom 7 then implies that $\mu_i(s) := \lim_{g_n \to g} \mu_{D^n(i)}(s)$ is well defined. $(\alpha \cdot s)$ is continuous; thus, $\varphi_n(i) \to \varphi(i)$ for $g_n \to g$ holds by construction. Second, compactness of $\Delta(Z)$ and Continuity (Axiom 3) imply that the set of acts with only singletons in their support has a (weakly) worst element, $\underline{g}$. Axiom 6 then implies that $g \succcurlyeq \underline{g}$ for all $g \in G$. For a singleton $\{\alpha\}$,

$$\int_S \sigma_{\{\alpha\}}(s)\, l(s)\, d\mu_i(s) = \sum_{x \in Z} \left(\alpha(x) \int_S s_x l(s)\, d\mu_i(s) \right).$$

For $z \in \arg\min_{x \in Z} \left(\int_S s_x l(s)\, d\mu_i(s) \right)$, this expression is minimized in $\alpha = z$. Thus, there is a (weakly) worst element $\underline{g}$ with support in the degenerate lotteries on Z, which is a finite set. Hence $\underline{g}$ is simple.

With a simple act as a worst act, there must then be a sequence of simple acts, such that $\varphi_n(i) \to \varphi(i)$ from below. Continuity of v and Definition 6 give

$$E_\phi \left[\int_S \sigma_{g(i)}(s)\, l(s)\, d\mu_i(s) \right] = v(g).$$

Interpreting $\mu(s\,|i) := \mu_i(s)$ as a probability measure over the taste space S, conditional on the contingency $i \in I$, yields the representation in Theorem 2':

$$V(g) = E_\phi \left[\int_S \left(\max_{\alpha \in g(i)} U_s(\alpha) \right) d\mu(s\,|i) \right].$$

This completes the proof of the sufficiency statement in Theorem 2'. That the axioms are also necessary for the existence of the representation is straightforward to verify. ■

9.7. Proof of Proposition 1

Lemma 4: If I is finite and (π, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ both represent $\succ$, then

$$\frac{\phi(i)}{\phi(j)} = \frac{\pi(i)}{\pi(j)} \frac{\int_S \frac{l(s)}{\widehat{l}(s)} d\mu(s|i)}{\int_S \frac{l(s)}{\widehat{l}(s)} d\mu(s|j)}$$

has to hold for all nonnull $i, j \in I$.

Proof: For any given $i \in I$, (π, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ imply the same preference, $\succ_i$. As noted before, $\widehat{\mu}(s|i)$ must be a probability measure with $\widehat{\mu}(s|i) \propto \frac{l(s)}{\widehat{l}(s)} \mu(s|i)$ and consequently

$$\widehat{l}(s)\,\widehat{\mu}(s|i) = \frac{l(s)\,\mu(s|i)}{\int_S \frac{l(s)}{\widehat{l}(s)} d\mu(s|i)}.$$

At the same time (π, μ, U) and $\left(\phi, \widehat{\mu}, \widehat{U}\right)$ represent the same tradeoffs across I. It is easy to verify that this implies $\phi(i) \int_S \sigma_{g(i)}(s)\,\widehat{l}(s)\,d\widehat{\mu}(s|i) \propto \pi(i) \int_S \sigma_{g(i)}(s)\,l(s)\,d\mu(s|i)$ for all $g \in G$ and hence $\phi(i) = \pi(i) \int_S \frac{l(s)}{\widehat{l}(s)} d\mu(s|i)$, which establishes Lemma 4. $\square$

I^* and S^* are assumed to have finite cardinality T. According to Lemma 4, $\widehat{l}(s)$ has to solve the system of equations $\phi(i) \propto \pi(i) \sum_{S^*} \frac{l(s)}{\widehat{l}(s)} \mu(s|i)$ for all $i \in I^*$. We want to establish that there is a neighborhood of π, such that all ϕ in this neighborhood allow an alternative representation, $\left(\phi, \widehat{\mu}, \widehat{U}\right)$. Interpret π and ϕ as vectors in $\mathbb{R}_+^T$. Denote by $\mu(s) \in \mathbb{R}_+^T$ the vector with i-th component $\mu(s|i)$ and by $\pi \odot \mu(s) \in \mathbb{R}_+^T$ the component wise product of those vectors. The system of equations has a solution with $\widehat{l}(s) > 0$ if and only if ϕ is in the interior of the positive linear span of $\{\pi \odot \mu(s)\}_{s \in S^*}$.

Lemma 5: Under the conditions of Proposition 1, $\{\mu(s)\}_{s \in S^*}$ are linearly independent.

Proof: Suppose not. Let $n \in \{1, ..., T\}$ index the tastes in S^*. There must be parameters c_n for $n \in \{1, ..., T-1\}$, such that $\mu(s_T) = \sum_{n \in \{1,...,T-1\}} c_n \mu(s_n)$. Then, for some $\tau \in (0, \infty) \setminus \{1\}$, one can define $\mu'(s|i)$ to be probability measures, such that

$$\mu'(s_T) \propto \tau \mu(s_T) \text{ and } \frac{\mu'(s_n|i)}{\mu'(s_m|i)} = \frac{\mu(s_n|i)}{\mu(s_m|i)}$$

for all $n, m \in \{1, ..., T-1\}$ and all $i \in I$. Then $l'(s_n) := l(s_n) \frac{\mu(s_n|i)}{\mu'(s_n|i)}$ is well defined for all $n \in \{1, T\}$, and for $U'_s(\alpha) = l'(s) s \cdot \alpha$ the CPF representation (π, μ', U') is numerically identical to the representation (π, μ, U). At the same time, $\tau \neq 1$ implies that l' is not a linear transformation of l. This contradicts Theorem 1. $\square$

$\pi \in \mathbb{R}^T_+$. Thus, $\{\pi \odot \mu(s)\}_{s \in S^*}$ must also be linearly independent. Therefore, $\{\pi \odot \mu(s)\}_{s \in S^*}$ spans $\mathbb{R}^T$, and the positive linear span of $\{\pi \odot \mu(s)\}_{s \in S^*}$ is open in $\mathbb{R}^T_+$. π can be expressed as a linear combination, which assigns unit weight to each of those T linearly independent vectors: $\pi = \sum_{S^*} \pi \odot \mu(s)$. Hence, π is in the interior of the positive linear span of $\{\pi \odot \mu(s)\}_{s \in S^*}$. This establishes the first part of Proposition 1: under the conditions of the proposition, there is a neighborhood of π in $\mathbb{R}^T$, such that all ϕ in this neighborhood allow an alternative representation, $\left(\phi, \widehat{\mu}, \widehat{U}\right)$. The solution of a finite system of linear equations is locally Lipschitz continuous in perturbations of the parameters. This establishes the second part of Proposition 1. $\blacksquare$

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