A THEORY OF STABILITY IN MANY-TO-MANY MATCHING MARKETS

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Abstract. We develop a theory of stability in many-to-many matching markets. We give conditions under wich the setwise-stable set, a core-like concept, is nonempty and can be approached through an algorithm. The setwise-stable set coincides with the pairwise-stable set, and with the predictions of a non-cooperative bargaining model. The set-wise stable set possesses the canonical conflict/coincidence of interest properties from many-to-one, and one-to-one models. The theory parallels the standard theory of stability for many-to-one, and one-to-one, models. We provide results for a number of core-like solutions, besides the setwise-stable set.

1. Introduction

Consider a collection of firms and consultants. Each firm wishes to hire a set of consultants, and each consultant wishes to work for a set of firms. Firms have preferences over the possible sets of consultants, and consultants have preferences over the possible sets of firms. This is an example of a "many-to-many" matching market. A matching is an assignment of sets of consultants to firms, and of sets of firms to consultants, so that firm f is assigned to consultant w if and only if w is also assigned to f. The problem is to predict which matchings can occur as a result of bargaining between firms and consultants.

Many-to-many matching markets are worse understood than many-to-*one* markets—markets where firms hire many workers, but each worker works for only one firm. The many-to-one market model seems to describe most labor markets, so why should one study many-to-many markets? There are two reasons.

First, some important real-world markets are many-to-many. One example is firms/consultants. But the best-know example is probably the market for medical interns in the U.K. (Roth and Sotomayor, 1990).

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The medical-interns example is important because it works through centralized matching mechanisms. And many-to-one theory has helped understand and shape centralized matching mechanisms for medical interns in the U.S. (Roth and Peranson, 1999). Another example is the assignment of teachers to highschools in some countries (35% of teachers in Argentina work in more than one school). The assignment of teachers to highschools is a clear candidate for a centralized solution guided by theory. Finally, one can view many-to-many matching as an abstract model of long-term contracting between firms and providers.

Second, even a few many-to-many contracts can make a crucial difference, and most labor markets have at least a few many-to-many contracts. We present an example (Section 2.2) where the number of agents can be arbitrarily large, but still one many-to-many contract changes the contracting outcome for all agents. In the U.S., 76% of total employment is in industries with 5% or more multiple jobholders (Source: U.S. Bureau of Labor Statistics). If even a few multiple jobholders (many-to-many contracts) make an important difference, we need a many-to-many model to understand the bulk of the labor markets in the U.S.

Finally, characterizing the core in many-to-many matchings is listed in Open Problem 6 in Roth and Sotomayor (1990, page 246)—the classic on matching markets. We give an answer to this problem.

We shall first give an overview of our results. Then we place our results in the related literature.

1.1. **Overview of results.** We argue that the core is not the right solution for many-to-many markets. One problem is that the core may not be "individually rational," in the sense that there are core matchings where a firm—for example—would be better off firing some worker. Another problem is that the core may not be "pairwise stable," in the sense that there may be a firm f and a worker w that are not currently matched, but where w would like to work for f, and f would like to hire w. In one-to-one, and many-to-one, matching markets, the standard solution is the set of pairwise-stable, and individually-rational, matchings. In those markets, the solution coincides with the core.

We consider alternatives to the core. One alternative is the *setwise-stable set* of Roth (1984) and Sotomayor (1999): the set of individually-rational matchings that cannot be blocked by a coalition who forms new links only among themselves—but may preserve their links to agents that are not in the coalition. A second alternative is the *individually-rational core* (defined implicitly by Sotomayor (1999)): the set of individually-rational matchings that cannot be blocked using an individually-rational matching. A third alternative is the *pairwise-stable set*, described in the previous paragraph. A fourth alternative is a set we denote by $\mathcal{E}$: matchings where each agent a is choosing her best set of partners, out of the set of potential partners who, given their current match, are willing to link to a. The definition of $\mathcal{E}$ is circular.

Matching theory proceeds normally by adding hypotheses on agents' preferences. We shall work with two hypotheses. The first is "substitutability"—first introduced by Kelso and Crawford (1982), and used extensively in the matching literature. The second is a strengthening of substitutability that we call "strong substitutability."

Let f be a firm. Substitutability of f's preferences requires: "if hiring w is optimal when certain workers are available, hiring w must still be optimal when a subset of workers are available." Strong substitutability requires: "if hiring w is optimal when certain workers are available, hiring w must still be optimal when a worse set of workers are available." Using a sports analogy, substitutability means that if w has a contract with the L.A. Lakers, and is chosen to play for the West's All Star Team, then w should play in the Lakers' first team. Strong substitutability means that, if w is good enough to play for the Lakers, w must be good enough to play for the L.A. Clippers.

Strong substitutability is stronger than substitutability. But it is weaker than separability, and not stronger than responsiveness—two other assumptions used in matching theory (separability is also used extensively in social choice).

We now enumerate and briefly discuss our main results. For economy of exposition, we present results as results on $\mathcal{E}$. The implications for the other solutions should be clear at all times.

If preferences are substitutable, $\mathcal{E}$ is nonempty, and we give an algorithm for finding a matching in $\mathcal{E}$; $\mathcal{E}$ equals the set of pairwise-stable matchings; a basic non-cooperative bargaining game—firms propose to workers, then workers propose to firms—has $\mathcal{E}$ as its set of subgame-perfect equilibrium outcomes; a matching in $\mathcal{E}$ which is blocked (in the sense of the core) must be blocked in a "non-individually-rational way;" through a coalition of agents that all have incentives to deviate from the block.

If firms' preferences are substitutable, and workers' preferences are strongly substitutable, $\mathcal{E}$ equals the set of setwise-stable matchings, and a matching in $\mathcal{E}$ must be in the individually-rational core. Thus setwise-stable matchings exist, the individually-rational core is non-empty, and our algorithm finds a matching in the individually-rational core that is setwise stable.

If preferences are substitutable, $\mathcal{E}$ has certain properties one can interpret as worker-firm conflict of interest, and worker-worker (or firm-firm) coincidence of interest: $\mathcal{E}$ is a lattice. That $\mathcal{E}$ is a lattice implies that there is a "firm-optimal" matching in $\mathcal{E}$—a matching that is simultaneously better for all firms, and worse for all workers, than any other matching in $\mathcal{E}$—and a "worker-optimal" matching in $\mathcal{E}$—one that is best for all workers, and worse for all firms. Besides the lattice-structure, there are other conflict/coincidence of interest properties from one-to-one and many-to-one markets (Roth, 1985). We extend these properties to many-to-many markets. If preferences are strongly substitutable, the lattice operations on $\mathcal{E}$ are the canonical lattice operations from one-to-one matching markets.

If firms' preferences are substitutable, and workers' preferences are strongly substitutable, the theory of many-to-many matchings parallels the theory of many-to-one matchings: the setwise-stable set equals the pairwise-stable set, and the setwise-stable set is a non-empty lattice. In the standard many-to-one model, firms' preferences are substitutable, and workers' preferences are trivially strongly substitutable. So our model encompasses and explains standard many-to-one theory.

In sum, we give conditions (substitutability, strong substitutability) under which our alternatives to the core are non-empty and can be approached through an algorithm. The setwise-stable set, $\mathcal{E}$, and the pairwise-stable set are identical, and possess a lattice structure. The setwise-stable set, $\mathcal{E}$, and the pairwise-stable set coincide with the outcomes of a simple non-cooperative bargaining model. We reproduce and extend conflict/coincidence of interest properties.

1.2. **Related Literature.** Setwise stability was first defined by Roth (1984). Sotomayor (1999) emphasizes the difference between setwise stability, pairwise stability, and the core. Sotomayor (1999) presents examples where the setwise-stable set is empty; preferences in her examples are not strongly substitutable (see our Example 22). Sotomayor (1999) refers to a definition of core that in fact coincides with our individually-rational core. We are the first to prove that the setwise-stable set and the individually-rational core are non-empty.

Recently, in independent work, Konishi and Ünver (2003) proved that a concept they call strong group-wise stability is equivalent to pairwise stability. Strong group-wise stability is a concept from the theory of network formation, and it seems to be similar to setwise stability. Konishi and Ünver require preferences to be responsive and satisfy a separability assumption; their structure is neither stronger or weaker than our. Konishi and Ünver also make the point that the core is not a good prediction in many-to-many markets.

Roth (1984) proved that, with substitutable preferences, the pairwise-stable set is nonempty, and that there are firm- and worker-optimal pairwise-stable matchings. Blair (1988) proved that the pairwise-stable set has a lattice structure. We reproduce Roth's and Blair's results using fixed-point methods—similar methods have been used in matching contexts by Adachi (2000), Echenique and Oviedo (2003), Milgrom (2003), Roth and Sotomayor (1988), and Fleiner (2001). But none of these papers address many-to-many matchings. Roth (1985) discusses conflict/coincidence of interest beyond the lattice property of pairwise-stable matchings. We extend Roth's results to many-to-many matchings.

Alcalde, Pérez-Castrillo, and Romero-Medina (1998) and Alcalde and Romero-Medina (2000) prove that the core is implemented in certain many-to-one models by simple mechanisms, similar to the one we present in Section 7.

A precedent to our results on a bargaining set (defined in Section 4.3) is Klijn and Massó (2003). Klijn and Massó study Zhou's bargaining set for the one-to-one matching model. The bargaining set we propose is different from Zhou's bargaining set.

Finally, Martínez, Massó, Neme, and Oviedo (2003) present an algorithm that finds all the pairwise-stable matchings in a many-to-many matching market.

2. Motivating examples

We give two motivating examples. The first example shows that the core may be a problematic solution. The second example shows that many-to-many matchings can be important, even when they are rare.

2.1. **The problem with the core.**

Example 1. Suppose the set of workers is $W = \{w_1, w_2, w_3\}$, and the set of firms is $F = \{f_1, f_2, f_3\}$. Workers' preferences are

$$\begin{array}{ll} P(w_1): & f_3, f_2f_3, f_1f_3, f_1, f_2 \\ P(w_2): & f_1, f_1f_3, f_1f_2, f_2, f_3 \\ P(w_3): & f_2, f_1f_2, f_2f_3, f_3, f_1. \end{array}$$

The notation means that w_1 prefers $\{f_3\}$ to $\{f_3, f_2\}$, $\{f_3, f_2\}$ to $\{f_3, f_1\}$, $\{f_3, f_1\}$ to $\{f_1\}$, and so on. If $A \subseteq F$ is not listed it means that $\emptyset$ is preferred to A. Firms' preferences are

$$\begin{array}{ll} P(f_1): & w_3, w_2w_3, w_1w_3, w_1, w_2 \\ P(f_2): & w_1, w_1w_3, w_1w_2, w_2, w_3 \\ P(f_3): & w_2, w_1w_2, w_2w_3, w_3, w_1. \end{array}$$

Consider the matching $\hat{\mu}$ defined by $\hat{\mu}(w_1) = \{f_2, f_3\}$, $\hat{\mu}(w_2) = \{f_1, f_3\}$, and $\hat{\mu}(w_3) = \{f_1, f_2\}$.

Note that $\hat{\mu}$ is a core matching: To make f_1—for example—better off than in $\hat{\mu}$, f_1 should hire only w_3, which would make w_2 hired only by f_3, so w_2 would be worse off. Now, if f_1 is in a blocking coalition $\mathcal{C}$, w_3 must be in $\mathcal{C}$. Then f_2 must be in $\mathcal{C}$, or w_3 would only be hired by f_1 and thus worse off. But f_2 in $\mathcal{C}$ implies that w_1 must be in $\mathcal{C}$. Then f_3 must be in $\mathcal{C}$, so w_2 must also be in $\mathcal{C}$—a contradiction, as w_2 is worse off.

But is $\hat{\mu}$ a reasonable prediction? Under $\hat{\mu}$, f_1 is matched to w_2 and w_3, but would in fact prefer to fire w_2. The problem is that f_1 is not "allowed" to fire w_2 because—as argued above—f_1 would have to form a block that includes w_2, and w_2 is worse off if she is fired.

Example 1 shows that core matchings need not be "individually rational." Because there are actions, like firing a worker, that an agent should be able to implement on its own, but that the definition of core ends up tying into a larger coalition.

An additional problem, pointed out by Blair (1988) and Roth and Sotomayor (1990, page 177) is that core matchings may not be pairwise stable (see also Sotomayor (1999)).

2.2. **Many-to-many vs. many-to-one.** There is a large literature on one-to-one and many-to-one matchings. Many-to-many matchings are a more general model. We have argued that, in many cases, the generality matters. Here we present an an example supporting our argument; we observe that the presence of a few many-to-many contracts can change the matching outcome for all agents.

In Example 2, if *one* worker is allowed to match with more than one firm, the resulting stable/core matching changes for a large number of agents. Thus, even in markets where one-to-one, or many-to-one, is the rule, a few many-to-many contracts can make a big difference.

Example 2. Let $W = \{\overline{w}, w_1, \ldots w_{2K}\}$ and $F = \left\{f_1, \ldots f_K, \overline{f}\right\}$. The preferences of workers w_k, for $k = 1, \ldots 2K$, are the same:

$$P(w_k) : f_1, f_2, \ldots f_K, \overline{f}.$$

The preferences of $\overline{w}$ are

$$P(\overline{w}) : f_1\overline{f}, \overline{f}, f_1.$$

The preferences of firms f_k for $k = 2, \ldots K$ are

$$P(f_k) : w_{2k-2}w_{2k-1}, w_{2k-1}w_{2k}, \overline{w}w_{2k}.$$

Firms f_1 and f_K have preferences

$$\begin{array}{rl} P(f_1): & \overline{w}w_1, w_1w_2 \\ P(\overline{f}): & \overline{w}, w_1, w_2, \ldots, w_K. \end{array}$$

Consider matchings μ and μ' defined by

$$\begin{array}{rccccccc} & f_1 & f_2 & \ldots & f_k & \ldots & f_K & \overline{f} \\ \mu = & w_1w_2 & w_3w_4 & \ldots & w_{2k-1}w_{2k} & \ldots & w_{2K-1}w_{2K} & \overline{w} \\ \mu' = & \overline{w}w_1 & w_2w_3 & \ldots & w_{2k-2}w_{2k-1} & \ldots & w_{2K-2}w_{2K-1} & \overline{w} \end{array}$$

First, if $\overline{w}$ is not allowed to match with more than one firm, then μ is the unique core (and stable) matching. If $\overline{w}$ is allowed to match with more than one firm, then $\langle \{\overline{w}\}, \{f_1, \overline{f}\}, \mu' \rangle$ blocks μ. Further, μ' is the unique core matching.

The story behind the example should be familiar to academics. Suppose that firms are universities. All workers w_k agree about the ranking of firms: f_1 is the best, followed by f_2, etc. Firm $\overline{f}$ is the worst. However, worker $\overline{w}$, an established and coveted researcher in her field, has a strong desire to work at $\overline{f}$ for geographic reasons ($\overline{f}$ is in the town where $\overline{w}$ grew up, and that is where her family lives). If part-time (many-to-many) appointments are not allowed, $\overline{w}$ will only work for $\overline{f}$ and the resulting matching is, in all likelihood, μ. On the other hand, if $\overline{w}$ is allowed to have part-time appointments at $\overline{f}$ and f_1, μ' will result.

3. Preliminary definitions

3.1. **Lattices and preference relations.** A (strict) *preference relation* P on a set X is a complete, anti-symmetric, and transitive binary relation on X. We denote by R the weak preference relation associated to P; so xRy if and only if $x = y$ or xPy. If A is a set, we refer to a list of preference relations $(P(a))_{a \in A}$ as a *preference profile*.

Let X be a set, and B a partial order on X—a transitive, reflexive, and antisymmetric binary relation. Let $A \subseteq X$. Denote by $\inf_B A$ the greatest lower bound, and by $\sup_B A$ the lowest upper bound, on A in the order B. Say that the pair $\langle X, B \rangle$ is a *lattice* if, whenever $x, y \in X$, both $x \wedge_B y = \inf_B \{x, y\}$ and $x \vee_B y = \sup_B \{x, y\}$ exist in X. A subset $A \subseteq X$ is a *sublattice* of $\langle X, B \rangle$ if, whenever $x, y \in A$, both $x \wedge_B y \in A$ and $x \vee_B y \in A$. A lattice $\langle X, B \rangle$ is a *chain* if, for any $x, y \in X$, xBy or yBx, or both, are true.

A lattice $\langle X, B \rangle$ is *distributive* if, for all $x, y, z \in X$, $x \vee_B (y \wedge_B z) = (x \vee_B y) \wedge_B (x \vee_B z)$. Let $\langle X, B \rangle$ and $\langle Y, R \rangle$ be lattices. A map

$\psi : X \to Y$ is a *lattice homomorphism* if, for all $x, y \in X$, $\psi(x \wedge_B y) = \psi(x) \wedge_R \psi(y)$ and $\psi(x \vee_B y) = \psi(x) \vee_R \psi(y)$. ψ is *lattice isomorphism* if it is a bijection and a lattice homomorphism.

Remark 1. The product of lattices, when endowed with the product order, is a lattice (Topkis, 1998, page 13). The lattice operations are the product of the component lattice operations.

3.2. **The Model.** The model has three primitive components:

- a finite set W of workers,
- a finite set F, disjoint from W, of firms,
- a preference profile $P = (P(a))_{a \in W \cup F}$, where $P(a)$ is a preference relation over 2^F if $a \in W$, and over 2^W if $a \in F$.

If $a \in W \cup F$ is an agent, we shall refer to any subset $A \subseteq W \cup F$ as a *set of partners of* a. If $a \in F$, a's partners will be subsets of W, and if $a \in W$, a's partners will be subsets of F.

Denote the preference profile $(P(w))_{w \in W}$ by $P(W)$, and $(P(f))_{f \in F}$ by $P(F)$.

The assignment problem consists of matching workers with firms, allowing that some firms or workers remain unmatched. Formally,

A *matching* μ is a mapping from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that for all $w \in W$ and $f \in F$:

(1) $\mu(w) \in 2^F$.
(2) $\mu(f) \in 2^W$.
(3) $f \in \mu(w)$ if and only if $w \in \mu(f)$.

We denote by $\mathcal{M}$ the set of all matchings.

Given a preference relation $P(a)$, the sets of partners preferred by a to the empty set are called *acceptable*; so we allow that a firm prefers not hiring any worker rather than hiring unacceptable sets of workers, and that a worker prefers to remain unemployed over working for an unacceptable set of firms.

Given a set of partners S, let $Ch(S, P(a))$ denote agent a's most-preferred subset of S, according to a's preference relation $P(a)$. So $Ch(S, P(a))$ is the unique subset S' of S such that $S'P(a)S''$ for all $S'' \subseteq S$, $S'' \neq S'$.

To express preference relations in a concise manner, and since only acceptable sets of partners will matter, we will represent preference relations as lists of acceptable partners. For instance,

$$\begin{aligned} P(f_i) &= w_1 w_3, w_2, w_1, w_3 \\ P(w_j) &= f_1 f_3, f_1, f_3 \end{aligned}$$

indicates that

$$\{w_1, w_3\}\, P(f_i)\, \{w_2\}\, P(f_i)\, \{w_1\}\, P(f_i)\, \{w_3\}\, P(f_i)\, \emptyset$$

and

$$\{f_1, f_3\}\, P(w_j)\, \{f_1\}\, P(w_j)\, \{f_3\}\, P(w_j)\, \emptyset.$$

We often omit brackets ($\{\ldots\}$) when denoting sets.

3.3. **Individual rationality, stability, and Core.** Let P be a preference relation A matching μ is *individually rational* if $\mu(a)R(a)A$, for all $A \subseteq \mu(a)$, for all $a \in W \cup F$. Hence a matching is individually rational if and only if

$$\mu(a) = Ch(\mu(a), P(a)),$$

for all $a \in W \cup F$.

Individual rationality builds on the idea that links are voluntary: if agent a prefers a proper subset $A \subsetneq \mu(a)$ of partners over $\mu(a)$, then she will upset μ by severing her links to the agents in $\mu(a)\backslash A$.

Let $w \in W$, $f \in F$, and let μ be a matching. The pair (w, f) is a *pairwise block* of μ if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup \{w\}, P(f))$, and $f \in Ch(\mu(w) \cup \{f\}, P(w))$.

Definition 3. A matching μ is *pairwise stable* if it is individually rational, and there is no pairwise block of μ. Denote the set of stable matchings by $S(P)$.

Definition 4. A *block* of a matching μ is a triple $\langle W', F', \mu' \rangle$, where $F' \subseteq F$, $W' \subseteq W$, and $\mu' \in \mathcal{M}$ are such that

(1) $F' \cup W' \neq \emptyset$,
(2) $\mu'(W' \cup F') \subseteq W' \cup F'$,
(3) $\mu'(s)R(s)\mu(s)$, for all $s \in F' \cup W'$,
(4) and $\mu'(s)P(s)\mu(s)$ for at least one $s \in W' \cup F'$.

In words, a block of a matching μ is a "recontracting" between a subset of workers and firms, so that the agents who recontract are all weakly better off, and at least one of them is strictly better off. Say that $\langle W', F', \mu' \rangle$ blocks μ if $\langle W', F', \mu' \rangle$ is a block of μ.

Definition 5. A matching μ is a *core matching* if there are no blocks of μ. Denote the set of core matchings by $C(P)$.

3.4. **Substitutability.**

Definition 6. An agent a's preference relation $P(a)$ satisfies *substitutability* if, for any sets S and S' with $S \subseteq S'$,

$$b \in Ch(S' \cup b, P(a)) \text{ implies } b \in Ch(S \cup b, P(a)).$$

A preference profile $P = (P(a))_{a \in A}$ is *substitutable* if, for each agent $a \in A$, $P(a)$ satisfies substitutability. Say that $P(W)$ ($P(F)$) is substitutable if for each $a \in W$ ($a \in F$), $P(a)$ satisfies substitutability.

4. Non-core setwise stability

4.1. The setwise-stable set.

Definition 7. A *set-wise block* to a matching μ is a triple $\langle W', F', \mu' \rangle$, where $F' \subseteq F$, $W' \subseteq W$, and $\mu' \in \mathcal{M}$ are such that

(1) $F' \cup W' \neq \emptyset$,
(2) $\mu'(s) \backslash \mu(s) \subseteq F' \cup W'$, for all $s \in F' \cup W'$,
(3) $\mu'(s) P(s) \mu(s)$, for all $s \in F' \cup W'$,
(4) $\mu'(s) = Ch(\mu'(s), P(s))$, for all $s \in F' \cup W'$

Definition 8. A matching μ is in the set-wise stable set if μ is individually rational, and there are no set-wise blocks to μ. Denote the set of set-wise stable matchings by $SW(P)$.

The definition of $SW(P)$ is from Sotomayor (1999, Definition 2, pages 59–60).

Recall Example 1. We argued that the core matching $\hat{\mu}$ (in fact the unique core matching) is not a good prediction. Consider, instead, the matching defined by $\mu(w_i) = \{f_i\}$, for $i = 1, 2, 3$. It is easy, if somewhat cumbersome, to check that μ is set-wise stable.

It also has some interest to see why μ in Example 1 is not a core matching; $\langle W, F, \hat{\mu} \rangle$ blocks μ, as $\hat{\mu}(w_1) = \{f_2, f_3\} \, P(w_1) \mu(w_1)$, $\hat{\mu}(w_2) = \{f_1, f_3\} \, P(w_2) \mu(w_2)$, $\hat{\mu}(w_3) = \{f_1, f_2\} \, P(w_3) \mu(w_3)$. Similarly for firms. But this is a block from which all agents wish to unilaterally deviate. We characterize the blocks of setwise-stable matchings in Section 8.

4.2. The individually-rational core.

Definition 9. A block $\langle W', F', \mu' \rangle$ is an *individually-rational block* if $\mu'(s) = Ch(\mu'(s), P(s))$, for all $s \in W' \cup F'$.

Definition 10. A matching μ is in the *individually-rational core* if it is individually rational, and it has no individually-rational blocks. Denote the set of individually-rational core matchings by $IRC(P)$.

Sotomayor (1999) restricts attention to individually-rational matchings. So she implicitly refers to the individually-rational core.

4.3. **A bargaining set.** Let μ be a matching.

Definition 11. An *objection* to μ is a triple $\langle W', F', \mu' \rangle$, where $F' \subseteq F$, $W' \subseteq W$, and $\mu' \in \mathcal{M}$ are such that

(1) $F' \cup W' \neq \emptyset$,
(2) $\mu'(s) \backslash \mu(s) \subseteq F' \cup W'$, for all $s \in F' \cup W'$,
(3) $\mu'(s) P(s) \mu(s)$, for all $s \in F' \cup W'$.

Let $\langle W', F', \mu' \rangle$ be an objection to μ. A *counter-objection* to μ is an objection $\langle W'', F'', \mu'' \rangle$ to μ' such that $F'' \subseteq F'$ and $W'' \subseteq W'$.

Note that objections only require a partner's "consent" if they involve new links.

Definition 12. A matching μ is in the *bargaining set* if μ is individually rational, and there are no objections without counterobjections to μ. Denote the bargaining set by $B(P)$.

For the one-to-one model, Klijn and Massó (2003) prove that Zhou's bargaining set (Zhou, 1994) coincides with a weak pairwise-stability solution. The bargaining set we propose is different from Zhou's because counterobjections are only allowed from "within" the objecting coalition.

4.4. **The Blair Core.** The definition of a block (Definition 4) makes formally sense for any profile of binary relations $(B(s))_{s \in W \cup F}$. Accordingly, one can define the core matchings $C(B)$ for any profile $B = (B(s))_{s \in W \cup F}$ of binary relations.

In particular, given a preference profile $P = (P(s))$, we can construct a binary relation $R^B = (R^B(s))$ by saying that $AR^B(s)B$ if and only if $A = B$ or $A = Ch(A \cup B, P(s))$. The strict relation P^B is $AR^B(s)B$ if and only if $A \neq B$ and $A = Ch(A \cup B, P(s))$. We call the resulting core, $C(P^B)$, the *Blair-Core*, as Blair (1988) introduced the relation P^B.

Note that a matching μ is in the Blair-Core if it is immune to deviations μ' such that $\mu'(a) = Ch(\mu'(a) \cup \mu(a), P(a))$. But $\mu'(a) = Ch(\mu'(a) \cup \mu(a), P(a))$ is only sufficient, and not necessary, for $\mu'(a) P(a) \mu(a)$. So the Blair-Core contains more matchings than the core.

4.5. **More pairwise stability.** A pair $(B, f) \in 2^W \times F$, with $B \neq \emptyset$, *blocks** μ if $B \cap \mu(f) = \emptyset$, $B \subseteq Ch(\mu(f) \cup B, P(f))$, and $f \in Ch(\mu(w) \cup f, P(w))$, for all $w \in B$. Similarly, A pair $(w, A) \in W \times 2^F$, with $A \neq \emptyset$, *blocks** μ if $A \cap \mu(w) = \emptyset$, $A \subseteq Ch(\mu(w) \cup A, P(w))$, and $w \in Ch(\mu(f) \cup w, P(f))$, for all $f \in A$.

Definition 13. A matching μ is *stable** if it is individually rational and there is no pair $(B, f) \in 2^W \times F$ that blocks* μ. A matching μ is *stable*** if it is stable*, and there is no pair $(w, A) \in W \times 2^F$ that blocks* μ. Denote the set of stable* matchings by $S^*(P)$, and the set of stable** matchings by $S^{**}(P)$.

5. A fixed-point approach.

We construct a map T on the set of "pre-matchings," a superset of $\mathcal{M}$. We shall use the fixed points of T to prove results on the different notions of stability.

5.1. **Pre-matchings.** Say that a pair $\nu = (\nu_F, \nu_W)$, with $\nu_F : F \to 2^W$ and $\nu_W : W \to 2^F$, is a *pre-matching*. Let $\mathcal{V}_W$ ($\mathcal{V}_F$) denote the set of all ν_W (ν_F) functions. Thus, $\mathcal{V}_F = \left(2^W\right)^F$, $\mathcal{V}_W = \left(2^F\right)^W$. Denote the set of pre–matchings $\nu = (\nu_F, \nu_W)$ by $\mathcal{V} = \mathcal{V}_F \times \mathcal{V}_W$. We shall often refer to $\nu_W(w)$ by $\nu(w)$ and to $\nu_F(f)$ by $\nu(f)$.

A pre-matching ν is a matching if ν is such that $\nu_W(w) = f$ if and only if $w \in \nu_F(f)$.

5.2. **The map T.** Let ν be a pre-matching, and let

$$U(f, \nu) = \{w \in W : f \in Ch(\nu(w) \cup \{f\}, P(w))\},$$

and

$$V(w, \nu) = \{f \in F : w \in Ch(\nu(f) \cup \{w\}, P(f))\}.$$

The set $V(w, \mu)$ is the set of firms f that are willing to hire w, possibly after firing some of the workers it was assigned by ν. The set $U(f, \nu)$ is the set of workers w that are willing to add f to its set of firms $\nu(w)$, possibly after firing some firms in $\nu(w)$.

Now, define $T : \mathcal{V} \to \mathcal{V}$ by

$$(T\nu)(s) = \begin{cases} Ch(U(s, \nu), P(s)) & \text{if } s \in F \\ Ch(V(s, \nu), P(s)) & \text{if } s \in W. \end{cases}$$

The map T has a simple interpretation: $(T\nu)(f)$ is firm f's optimal team of workers, among those willing to work for f, and $(T\nu)(w)$ is the set of firms preferred by w, among the firms that are willing to hire w.

We shall denote by $\mathcal{E}$ the set of fixed points of T, so

$$\mathcal{E} = \{\nu \in \mathcal{V} : \nu = T\nu\}.$$

Recall Example 1 and matching μ from Section 4.1. Note that μ, the unique setwise-stable matching, is a fixed-point of T: $V(w_1, \mu) = \{f_1, f_2\}$, so $\{f_1\} = Ch(V(w_1, \mu), P(w_1))$, $V(w_2, \mu) = \{f_2, f_3\}$, so $\{f_2\} = Ch(V(w_2, \mu), P(w_2))$, and $V(w_3, \mu) = \{f_1, f_3\}$, so $\{f_3\} = Ch(V(w_3, \mu), P(w_2))$. Similarly for firms.

Further, $\hat{\mu}$, the unique core matching in Example 1, is not a fixed-point of T; as $U(f_1, \hat{\mu}) = \{w_2\}$ and $\{w_2, w_3\} \neq Ch(U(f_1, \hat{\mu}), P(f_1))$.

Definition 14. The T-*algorithm* is the procedure of iterating T, starting at some prematching ν.

Note that the T-algorithm stops at $\nu' \in \mathcal{V}$ if and only if $\nu' \in \mathcal{E}$.

Let ν_0 and ν_1 be the prematchings defined by $\nu_0(f) = \nu_1(w) = \emptyset$, $\nu_0(w) = F$, and $\nu_1(f) = W$ for all w and f. We shall consider the T-algorithm starting at prematchings ν_0 and ν_1. See Echenique and Oviedo (2003) for a discussion of the T-algorithm in the many-to-one model.

6. Non-emptiness of, and relations between, solutions.

We organize the results according to the structure needed on preferences. In some results, we impose structure on one side of the market only; we impose weakly more structure on workers' preferences. The model is symmetric, so it should be clear that appropriate versions of the results are true, interchanging the structure on workers' and firms' preferences.

6.1. Results under substitutability.

Theorem 15. $\mathcal{E} \subseteq S^{**}(P) \subseteq S^{*}(P) \subseteq S(P)$. *Further:*

1. *If $P(W)$ is substitutable, then*
$$S^{**}(P) = S^{*}(P) = \mathcal{E} \subseteq C(P^B).$$
2. *If P is substitutable, then $S(P) = \mathcal{E}$, $\mathcal{E}$ is nonempty, and the T-algorithm finds a matching in $\mathcal{E}$.*

6.2. Results under strong substitutability.

Definition 16. An agent a's preference ordering $P(a)$ satisfies *strong substitutability* if, for any sets S and S', with $S'P(a)S$,

$$b \in Ch(S' \cup b, P(a)) \text{ implies } b \in Ch(S \cup b, P(a)).$$

Say that a preference profile P is *strongly substitutable* if $P(a)$ satisfies strong substitutability for every agent a.

Proposition 17. *If $P(a)$ satisfies strong substitutability, then it satisfies substitutability.*

Proof. Let S and S' be sets of agents, with $S \subseteq S'$. Suppose that $b \in C' = Ch(S' \cup b, P(a))$. We shall prove that $b \in Ch(S \cup b, P(a))$.

Note that $C' = Ch(C', P(a))$. Now, $S \cup b \subseteq S' \cup b$ implies that $C'R(a)S \cup b$. If $C' = S \cup b$ then $S \cup b = Ch(S \cup b, P(a))$ and we are done. Let $C'P(a)S \cup b$. Then $b \in Ch(S \cup b, P(a))$, as $P(a)$ satisfies strong substitutability. ∎

Theorem 18. *$SW(P) \subseteq \mathcal{E}$ and $B(P) \subseteq \mathcal{E}$. Further, if $P(F)$ is substitutable, and $P(W)$ is strongly substitutable, then $\mathcal{E} = SW(P) = B(P)$, and $\mathcal{E} \subseteq IRC(P)$.*

Thus, when one side of the market has strongly substitutable preferences, we can characterize the setwise-stable set. In light of Proposition 17, Theorem 18 implies that $S(P) = SW(P)$.

Theorem 19. *If $P(F)$ is substitutable, and $P(W)$ is strongly substitutable, then $S(P)$, $IRC(P)$, $SW(P)$, and $B(P)$ are non-empty. The T-algorithm finds a matching in $S(P)$, $IRC(P)$, $SW(P)$, and $B(P)$.*

Remark 2. We can weaken the definition of strongly substitutable to: For all S and S', with $S = Ch(S, P(a))$, $S' = Ch(S', P(a))$, and $S'P(a)S$,

$$b \in Ch(S' \cup b, P(a)) \text{ implies } b \in Ch(S \cup b, P(a)).$$

All our results go through under this weaker definition. We chose the stronger formulation in our exposition to make the comparison with earlier work easier. But when we check that an example violates strong substitutability, we actually check for the weaker version.

6.3. **Discussion of strong substitutability.** How strong is the assumption of strong substitutability? We lack a characterization of strong substitutability—just as a characterization of traditional (Kelso-Crawford) substitutability is unavailable. But we give a feeling for the assumption by discussing preferences that are built from preferences over individual workers.

First, strong substitutability is weaker than the assumption of separability used in matching models (Dutta and Massó, 1997; Sönmez, 1996). Separability says that, for any set of partners S, $S \cup bPS \backslash b$ if and only if $bP\emptyset$ (separability has been used quite extensively in social choice theory; e.g. Barberá, Sonnenschein, and Zhou (1991)). The proof that separability implies strong substitutability is straightforward; we omit it.

Second, it is not stronger than responsiveness, another common assumption is the matching literature (see Roth and Sotomayor (1990) for a definition of responsiveness). One can easily write examples of non-responsive preferences that satisfy strong substitutability.

Third, to give a feeling for how restrictive strong substitutability is, consider the following example with four workers and a quota of 2. [1]

Example 20. Let $W = \{w_1, w_2, w_3, w_4\}$. Suppose that a firm has preferences over individual workers $w_1 P w_2$, $w_2 P w_3$ and $w_3 P w_4$.

Suppose the firm has a quota of 2. So only sets with two or less elements are acceptable. How can we rank the sets

$$\{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_2, w_4\}, \{w_3, w_4\}$$

building from the preferences over individuals? Obviously we need $\{w_1, w_2\} P \{w_1, w_3\}$, $\{w_2, w_3\} P \{w_3, w_4\}$, and so on. There are two possibilities:

$$P_1 : w_1w_2, w_1w_3, \boxed{w_1w_4, w_2w_3,} w_2w_4, w_3w_4, w_1, w_2, w_3, w_4$$

$$P_2 : w_1w_2, w_1w_3, \boxed{w_2w_3, w_1w_4,} w_2w_4, w_3w_4, w_1, w_2, w_3, w_4$$

The ranking of $\{w_1, w_4\}$ and $\{w_2, w_3\}$ is undetermined; P_1 ranks $\{w_1, w_4\}$ first, P_2 ranks $\{w_2, w_3\}$ first. Both P_1 and P_2 are substitutable, but only P_2 is strongly substitutable: Note that $w_4 \in Ch(\{w_1 w_4\}, P_1)$, and $\{w_1, w_4\} P_1 \{w_2, w_3\}$, but $w_4 \notin Ch(\{w_2, w_3\} \cup w_4, P_1)$. So P_1 is not strongly substitutable. It is simple, if tedious, to check that P_2 is strongly substitutable.

Example 20 points to a general procedure for obtaining strongly substitutable preferences from preferences over individuals when there is a quota. Let S, $S \cup bPS \backslash b$ if and only if $\{b\} P \emptyset$ unless S and S' have the maximum number of elements allowed by the quota. If S and S' have the maximum number of elements, let SPS' if the worst agent in S is preferred to the worst agent in S'.

Finally, a trivial but important point is that, in applications, the set of acceptable partners is often quite small. And both substitutability and strong substitutability are less restrictive if fewer sets of partners are acceptable (Remark 2). For example, the set of acceptable hospitals in the National Resident Matching Program was on average 7.45, out of 3719 programs, in 2003 (source: `http://www.nrmp.org/`). In a recent proposal to match high schools and students in New York City by a Gale-Shapley algorithm, students would be required to rank 12—out of over 200—acceptable high schools. Students and parents complain that 12 is too long a list (New York Times story by David M. Herszenhorn, "Revised Admission for High Schools," on October 3rd, 2003).

[1] The right separability assumption for models with quotas is q-separability, developed by Martínez, Massó, Neme, and Oviedo (2000). But q-separability does not imply strong substitutability.

A similar point is that several well-known examples in the literature have agents with strongly substitutable preferences. Two such examples are 6.6 in Roth and Sotomayor (1990) and 5.2 in Blair (1988).

6.4. **Examples.** Example 21 shows that $C(P^B) \nsubseteq S^*(P)$.

Example 21. Let $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2\}$, with preferences

$$\begin{array}{rl} P(f_1): & w_1w_2 \\ P(f_2): & w_1w_2, w_1, w_2 \\ P(f_3): & w_1w_2, w_1, w_2 \\ P(w_1): & f_1f_2, f_2f_3, f_1, f_2, f_3 \\ P(w_2): & f_1f_2, f_2f_3, f_1, f_2, f_3. \end{array}$$

Consider matchings μ and μ' defined by

$$\mu = \begin{array}{ccc} f_1 & f_2 & f_3 \\ \emptyset & w_1w_2 & w_1w_2 \end{array}$$

Then $\mu \in C(P^B)$ but $\mu \notin S^*(P)$, as $(f_1, \{w_1, w_2\})$ blocks* μ.

Strongly substitutable preferences imply—among other things—that there are setwise stable matchings. Sotomayor (1999) presents an example where the set of setwise stable matchings is empty. We reproduce her example in 22, and show that preferences in her example are not strongly substitutable.

Preferences in Example 22 are substitutable. So the example shows that strong substitutability is strictly stronger than substitutability.

Example 22. (Example 2 of Sotomayor (1999)) Each firm (worker) may form at most q_f (q_w) partnerships. For each pair (f, w) there are two numbers a_{fw} and b_{fw}. The preferences of firms f and workers w over allowable sets of partners are determined by these numbers. Therefore, say, f prefers w to w' if and only if $a_{fw} > a_{fw'}$, and w prefers f to f' if and only if $b_{fw} > b_{f'w}$. Say that f prefers the set $S \subseteq W$ to the set $S' \subseteq W$ if and only if $\sum_{w \in S} a_{fw} > \sum_{w \in S'} a_{fw}$.

Let $F = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$. Let $q_{f_1} = 3$, $q_{f_2} = q_{f_3} = 2$, $q_{f_3} = q_{f_4} = q_{f_6} = 1$, $q_{w_1} = q_{w_2} = q_{w_4} = 2$, and $q_{w_3} = q_{w_5} = q_{w_6} = q_{w_7} = 1$. The pairs of numbers (a_{fw}, b_{fw}) are given in Table 1 below.

	w_1	w_2	w_3	w_4	w_5	w_6	w_7
f_1	(13,1)	(14,10)	(4,10)	(1,10)	(0,0)	(0,0)	(3,10)
f_2	(1,10)	(0,0)	(0,0)	(10,1)	(4,10)	(2,10)	(0,0)
f_3	(10,4)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
f_4	(10,2)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
f_5	(0,0)	(9,9)	(0,0)	(10,4)	(0,0)	(0,0)	(0,0)
f_6	(0,0)	(0,0)	(0,0)	(10,2)	(0,0)	(0,0)	(0,0)

There are pairwise-stable matchings that are not setwise stable: Consider μ, defined by

$$\mu = \begin{matrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ w_2w_3w_7 & w_5w_6 & w_1 & w_1 & w_2w_4 & w_4. \end{matrix}$$

It is easy to see that μ is pairwise-stable. But μ is not setwise stable: coalition $\{f_1, f_2, w_1, w_2\}$ causes an instability in μ.

The problem is that the resulting preference profile is not strongly substitutable: Consider the preference of firm f_1, we have that

$$\{w_1, w_2\}P(f_1)\{w_2, w_3, w_7\},$$

(because $14 + 13 > 14 + 4 + 3$), $w_4 \in Ch(\{w_1, w_2\} \cup \{w_4\}, P(f_1))$ but $w_4 \notin Ch(\{w_2, w_3, w_7\} \cup \{w_4\}, P(f_1)) = \{w_2, w_3, w_7\}$ (because $14 + 4 + 3 > max\{14 + 4 + 1, 14 + 3 + 1, 4 + 3 + 1\}$)

7. An implementation of $SW(P)$

We present a simple (non-cooperative) bargaining model. The set of subgame-perfect Nash equilibrium (SPNE) outcomes of the model coincides with the setwise-stable set; so the model fully implements $SW(P)$ in a complete-information environment.

Bargaining proceeds as follows: First, every firm f proposes a set of partners $\eta_f \subseteq W$. Firms make these proposals simultaneously. Second, after observing all the firms' proposals, each worker w proposes a set of partners $\xi_w \subseteq F$. Workers make these proposals simultaneously. Finally, a matching μ results by $w \in \mu(f)$ if and only if $w \in \eta_f$ and $f \in \xi_w$. In words, w and f are matched if and only if f proposes w as its partner, and w proposes f as its partner.

A strategy for a firm f is a proposal $\eta_f \subseteq W$. Given a profile of firms' proposals, $\eta = (\eta_f)_{f\in F}$, a strategy by a worker w is a proposal $\xi_w(\eta) \subseteq F$. A strategy profile (η, ξ) is a *subgame-perfect Nash equilibrium* (SPNE) if for all w and f,

$$\xi_w(\eta) \cap \left\{\tilde{f} : w \in \eta_{\tilde{f}}\right\} R(w)A,$$

for all $A \subseteq \left\{\tilde{f} : w \in \eta_{\tilde{f}}\right\}$; and if

$$\eta_f \cap \left\{w : f \in \xi_f(\eta)\right\} R(f) A \cap \left\{w : f \in \xi_f(A, \eta_{-f})\right\},$$

for all $A \subseteq W$. In words, (η, ξ) is a SPNE if $\xi_w(\eta)$ is an optimal proposal, given firms' proposal η, and η_f is optimal given the other firms' proposals η_{-f}, and workers' proposals.

Theorem 23. *Let $P(W)$ be substitutable. A matching μ is the outcome of a subgame-perfect Nash equilibrium if and only if $\mu \in \mathcal{E}$.*

Theorems 18 and 23 imply

Corollary 24. *Let $P(F)$ be substitutable and $P(W)$ be strongly substitutable A matching μ is the outcome of a subgame-perfect Nash equilibrium if and only if $\mu \in SW(P)$.*

The implication of Theorem 23 and Corollary 24 is that the setwise-stable matchings are exactly those consistent with a basic non-cooperative bargaining model. Thus core matchings, for example, are not guaranteed to be SPNE outcomes.

8. Blocks of setwise-stable matchings.

In Example 1, $\hat{\mu}$ blocks μ through a coordinated, and non-individually-rational, effort of all agents. The preferences in Example 1 exhibit agents a who want agents b, where b dislikes a but is willing to accept a if she gets c, who dislikes b, and so on until a cycle is closed. We shall call such a cycle an acceptance-rejection cycle.

We now show that a matching in $\mathcal{E}$ can, in fact, only be blocked through an effort of this kind.

Definition 25. Let μ be a matching. An agent a wants to *add* an agent b to her partners if

$$b \in Ch(\mu(a) \cup b, P(a)).$$

An alternating sequence of workers and firms

$$(w_0, f_0, w_1, f_1, \ldots w_K, f_K)$$

is an *acceptance-rejection cycle for μ* if, for k with $0 \leq k \leq K-1$, w_k wants to add f_k to her partners but f_k does not want to add w_k, while f_k want to add w_{k+1} to her partners, and w_{k+1} does not want to add f_k.

Theorem 26. *Let P be substitutable. If $\mu \in \mathcal{E}$, and $\langle W', F', \mu' \rangle$ is a block of μ, then there is an acceptance-rejection cycle for μ in $\mu'(W' \cup F') \backslash \mu(W' \cup F')$.*

Corollary 27. *Let $P(F)$ be substitutable, and $P(W)$ strongly substitutable. If $\mu \in SW(P)$, and $\langle W', F', \mu' \rangle$ is a block of μ, then there is an acceptance-rejection cycle for μ in $\mu'(W' \cup F') \backslash \mu(W' \cup F')$.*

9. Lattice Structure

9.1. **Partial Orders.** We shall introduce two partial orders on $\mathcal{V}$. The first (Definition 28) is the partial order introduced by Blair (1988). The second (Definition 29) is the canonical partial order on matchings from one-to-one theory.

Definition 28. Define the following partial orders on $\mathcal{V}_F$, $\mathcal{V}_W$ $\mathcal{V}$:

(1) $<_F^B$ on $\mathcal{V}_F$ by $\nu'_F <_F^B \nu_F$ if and only if $\nu'_F \neq \nu_F$ and, for all f in F, $\nu_F(f) = \nu'_F(f)$ or
$$\nu_F(f) = Ch\left(\nu_F\left(f\right) \cup \nu'_F\left(f\right), P\left(f\right)\right).$$
(2) $<_W^B$ on $\mathcal{V}_W$ by $\nu'_W <_W^B \nu_W$ if and only if $\nu'_W \neq \nu_W$ and, for all w in W, $\nu_W(w) = \nu'_W(w)$ or
$$\nu_W(f) = Ch\left(\nu_W\left(w\right) \cup \nu'_W\left(w\right), P\left(w\right)\right).$$
(3) The weak partial orders associated to $<_F^B$ and $<_W^B$ are denoted $\leq_F^B$ and $\leq_W^B$, and defined as: $\nu'_F \leq_F^B \nu_F$ if $\nu_F = \nu'_F$ or $\nu'_F <_F^B \nu_F$, and $\nu'_W \leq_W^B \nu_W$ if $\nu_W = \nu'_W$ or $\nu'_W <_W^B \nu_W$.
(4) $\leq_F^B$ on $\mathcal{V}$ by $\nu' \leq_F^B \nu$ iff $\nu_W \leq_W^B \nu'_W$ and $\nu'_F \leq_F^B \nu_F$. The strict version of $\leq_F^B$ on $\mathcal{V}$ is $\nu' <^B \nu$ if $\nu' \leq^B \nu$ and $\nu' \neq \nu$.
(5) $\leq_W^B$ on $\mathcal{V}$ by $\nu' \leq_W^B \nu$ iff $\nu \leq_F^B \nu'$.

Definition 29. Define the following partial orders on $\mathcal{V}_F$, $\mathcal{V}_W$ and $\mathcal{V}$:

(1) $\leq_F$ on $\mathcal{V}_F$ by $\nu'_F \leq_F \nu_F$ if $\nu_F(f)R(f)\nu'_F(f)$, for all $f \in F$. The strict version of $\leq_F$ on $\mathcal{V}_F$ is $\nu'_F <_F \nu_F$ if $\nu'_F \leq_F \nu_F$ and $\nu'_F \neq \nu_F$.
(2) $\leq_W$ on $\mathcal{V}_W$ by $\nu'_W \leq_W \nu_W$ if $\nu_F(w)R(w)\nu'_W(w)$, for all $w \in W$. The strict version of $\leq_W$ on $\mathcal{V}_W$ is $\nu'_W <_W \nu_W$ if $\nu'_W \leq_W \nu_W$ and $\nu'_W \neq \nu_W$.
(3) $\leq_F$ on $\mathcal{V}$ by $\nu' \leq_F \nu$ iff $\nu_W \leq_W \nu'_W$ and $\nu'_F \leq_F \nu_F$. The strict version of $\leq$ on $\mathcal{V}$ is $\nu' < \nu$ if $\nu' \leq \nu$ and $\nu' \neq \nu$.
(4) $\leq_W$ on $\mathcal{V}$ by $\nu' \leq_W \nu$ iff $\nu \leq_F \nu'$.

Definitions 28 and 29 abuse notation in using each symbol ($\leq_F^B$, $\leq_F$, $\leq_W^B$, and $\leq_W$) for two different orders. The abuse of notation does not—we believe—confuse.

To simplify the notation in the sequel, let $(\leq^B, \leq) \in \left\{(\leq_F^B, \leq_F), (\leq_W^B, \leq_W)\right\}$. All statements that follow are true both with $(\leq^B, \leq) = (\leq_F^B, \leq_F)$ and $(\leq^B, \leq) = (\leq_W^B, \leq_W)$.

Remark 3. $\leq^B$ is a coarser order than $\leq$, as $\nu' \leq^B \nu$ implies that $\nu' \leq \nu$.

Remark 4. $\langle \mathcal{V}, \leq_F \rangle$ is a lattice (see Remark 1), and the lattice operations are

$$\nu \vee_F \nu' (f) = \begin{cases} \nu(f) & \text{if} \quad \nu(f) R(f) \nu'(f) \\ \nu'(f) & \text{if} \quad \nu'(f) P(f) \nu(f) \end{cases}$$

and

$$\nu \vee_F \nu' (w) = \begin{cases} \nu'(w) & \text{if} \quad \nu(w) R(w) \nu'(w) \\ \nu(w) & \text{if} \quad \nu'(w) P(w) \nu(w). \end{cases}$$

$\nu \wedge_F \nu'$ is defined symmetrically; giving f the worst of $\nu(f)$ and $\nu'(f)$, and giving w the best of $\nu(w)$ and $\nu'(w)$.

$\langle \mathcal{V}, \leq_W \rangle$ is a lattice, and the lattice operations are analogous to $\vee_F$ and $\wedge_F$.

Blair's order incorporates strong substitutability:

Proposition 30. *If $P(a)$ is substitutable, then $P^B(a)$ is strongly substitutable.*

Proof. Let $b \in Ch(S' \cup b, P(a))$ and $S' P^B(a) S$. Note that

$$\begin{aligned} b \in Ch(S' \cup b, P(a)) &= Ch(Ch(S \cup S', P(a)) \cup b, P(a)) \\ &= Ch(S \cup S\prime \cup w, P(f)) \end{aligned}$$

Where the first equality is by definition of P^B and the second equality is a property of choice. Finally, $b \in Ch(S \cup S' \cup b, P(a))$ and substitutability implies that $b \in Ch(S' \cup b, P(a))$. ∎

9.2. **Lattice Structure.** With substitutable preferences, T is a monotone increasing map under order $\leq^B$. Tarski's fixed point theorem then delivers a lattice structure on $\mathcal{E}$. With strongly substitutable preferences, T is a monotone increasing map under order $\leq$. Tarski's fixed point theorem gives a lattice structure on $\mathcal{E}$ under order $\leq$. We discuss the implications below.

Theorem 31. *Let P be substitutable. Then*

1. *$\langle \mathcal{E}, \leq^B \rangle$ is a non-empty lattice;*
2. *the T-algorithm starting at ν_0 stops at $\inf_{\leq^B_F} \mathcal{E}$, and the T-algorithm starting at ν_1 stops at $\sup_{\leq^B_F} \mathcal{E}$.*

Further, if P is strongly substitutable, $\langle \mathcal{E}, \leq \rangle$ is a non-empty lattice, $\inf_{\leq^B_F} \mathcal{E} = \inf_{\leq_F} \mathcal{E}$, and $\sup_{\leq^B_F} \mathcal{E} = \sup_{\leq_F} \mathcal{E}$.

Theorem 32. *Let $P(F)$ be substitutable and $P(W)$ be strongly substitutable. Then*

(1) *if $\nu, \nu' \in \mathcal{E}$ are such that $\nu'(w)R(w)\nu(w)$ for all $w \in W$, then $\nu(f)R(f)\nu'(f)$ for all $f \in F$.*
(2) *Further, let $P(F)$ be strongly substitutable. If $\nu, \nu' \in \mathcal{E}$ are such that $\nu'(f)R(f)\nu(f)$ for all $f \in F$, then $\nu(w)R(w)\nu'(w)$ for all $w \in W$.*

By definition of $\leq^B_F$, $\leq_F$, $\leq^B_W$, and $\leq_W$, we get $\inf_{\leq_F} \mathcal{E} = \sup_{\leq_W} \mathcal{E}$, $\inf_{\leq_W} \mathcal{E} = \sup_{\leq_F} \mathcal{E}$, $\inf_{\leq^B_F} \mathcal{E} = \sup_{\leq^B_W} \mathcal{E}$, and $\inf_{\leq^B_W} \mathcal{E} = \sup_{\leq^B_F} \mathcal{E}$.

Theorem 31 implies Theorem 19. It also implies

Corollary 33. *If $P(F)$ is substitutable, and $P(W)$ is strongly substitutable, then $\langle SW(P), \leq^B \rangle = \langle B(P), \leq^B \rangle = \langle S(P), \leq^B \rangle$ are non-empty lattices. Further, if $P(F)$ is strongly substitutable, $\langle SW(P), \leq \rangle = \langle B(P), \leq \rangle = \langle S(P), \leq \rangle$ are non-empty lattices*

Theorems 31 and 32 have an interpretation in terms of worker-firm "conflict" and worker-worker (or firm-firm) "coincidence" of interests (Roth, 1985).

First, Theorem 31 implies that there are two distinguished matchings in $\mathcal{E}$. One is simultaneously better for all firms, and worse for all workers, than any other matching in $\mathcal{E}$. The other is simultaneously worse for all firms, and better for all workers, than any other matching in $\mathcal{E}$. The lattice structure thus implies a coincidence-of-interest property.

Second, Theorem 32 reflects a worker-firm conflict of interest; for any two matchings in $\mathcal{E}$, if one is better for all firms it must also be worse for all workers, and vice versa. Roth (1985) proved that Statement 1 in Theorem 32 holds in the one-to-one model and in the many-to-one model. Roth also proved that Statement 2 in Theorem 32 holds in the one-to-one model. Here we extend Roth's results, as workers' preferences are trivially strongly substitutable in the many-to-one model, and all agents' preferences are trivially strongly substitutable in the one-to-one model. [2]

In light of Theorem 15, Theorem 33 implies that $\langle S(P), \leq^B_F \rangle$ is a lattice when preferences are substitutable—a result first proved by Blair (1988). Blair shows with an example that $\langle S(P), \leq_F \rangle$ may not be a lattice. Preferences in Blair's example are not strongly substitutable; we discuss Blair's example in Section 9.4.

In the one-to-one model, the lattice-structure of $\langle S(P), \leq_F \rangle$ is known since at least Knuth (1976) (Knuth attributes the result to J. Conway).

[2]By Proposition 30, we also extend Blair's (1988) version of Roth's result (Blair's Lemmas 4.3 and 4.4).

Theorem 31 extends the result to the many-to-many model, as preferences are trivially strongly substitutable in the one-to-one model.

9.3. **Further conflict/coincidence properties.** There are two additional features of many-to-one and one-to-one matchings that merit attention.

9.3.1. *Stronger coincidence-of-interest property.* Roth (1985) presents a stronger version of the coincidence-of-interest property implicit in the result that $\langle \mathcal{E}, \leq_F^B \rangle$ is a lattice. He proves that, if μ and μ' are pairwise-stable matchings in the many-to-one model, the matching that gives each firm f its best subset out of $\mu(f) \cup \mu'(f)$ is stable, and worse than both μ and μ' for all workers.

Roth's stronger coincidence of interest property does not extend to the many-to-many model with strongly substitutable preferences. Example 5.2 in Blair (1988) is a counterexample—we discuss this example in Section 9.4.

But note

Proposition 34. *Let $P(F)$ be substitutable. Let $\mu, \mu' \in S(P)$. Define the matching $\hat{\mu}$ by $\hat{\mu}(f) = Ch(\mu(f) \cup \mu'(f), P(f))$, for all $f \in F$. If $\hat{\mu}(w) \in \{\mu(w), \mu'(w)\}$, for all $w \in W$, then $\hat{\mu} \in S(P)$. Further, if $P(W)$ is substitutable, $\mu(w) R(w) \hat{\mu}(w)$ and $\mu'(w) R(w) \hat{\mu}(w)$, for all $w \in W$.*

Proof. The proof that $\hat{\mu} \in S(P)$ is a minor variation of Roth's (1985) proof of the coincidence-of-interest property in the many-to-one model.

First, $\hat{\mu}$ is individually rational: μ and μ' are individually rational so $\hat{\mu}(w) = Ch(\hat{\mu}(w), P(w))$ for all w; by definition of $\hat{\mu}$, $\hat{\mu}(f) = Ch(\hat{\mu}(f), P(f))$ for all f.

Second, there are not pairwise blocks of $\hat{\mu}$. If (w, f) is a pairwise block of $\hat{\mu}$. Then $w \notin \hat{\mu}(f)$, $f \in Ch(\hat{\mu}(w) \cup f, P(w))$, and $w \in Ch(\hat{\mu}(f) \cup w, P(f))$. Without loss of generality, let $\hat{\mu}(w) = \mu(w)$. So $f \notin \mu(w)$ and $f \in Ch(\mu(w) \cup f, P(w))$. But $w \in Ch(\hat{\mu}(f) \cup w, P(f))$ implies

$$w \in Ch\left(Ch\left(\mu(f) \cup \mu'(f), P(f)\right) \cup w, P(f)\right) = Ch\left(\mu(f) \cup \mu'(f) \cup w, P(f)\right).$$

By substitutability of $P(f)$, $w \in Ch(\mu(f) \cup w, P(f))$. Then $f \notin \mu(w)$ and $f \in Ch(\mu(w) \cup f, P(w))$ implies that (w, f) is also a pairwise block of μ.

So $\mu, \mu' \in S(P)$ implies that there are no pairwise blocks of $\hat{\mu}$.

When $P(W)$ is substitutable, $S(P) = \mathcal{E}$ and it is routine to verify that

$$\hat{\mu}(w) \subseteq V(w, \hat{\mu}) \subseteq V(w, \mu) \cap V(w, \mu').$$

Then $\mu, \mu' \in \mathcal{E}$ implies that $\mu(w)R(w)\hat{\mu}(w)$ and $\mu'(w)R(w)\hat{\mu}(w)$. ∎

In light of Proposition 34, what seems to be behind Roth's result is the many-to-one-ness of the many-to-one model; it does not seem that we can capture the stronger coincidence-of-interest property in a many-to-many model.

9.3.2. *Distributive property of lattice operations.* The set of stable matchings in the one-to-one model is a distributive lattice (Knuth, 1976). The distributive property of the one-to-one model does not extend to our many-to-many model: In Blair's (1988) Example 5.2, the set of stable many-to-many matchings is not a distributive lattice, and all agents' preferences in Blair's example satisfy strong substitutability (see Section 9.4).

We identify why the distributive property fails in the many-to-many model. The problem is that the lattice operations (see Remark 4) in $\langle \mathcal{V}, \leq \rangle$ may not preserve the property that matchings in $\mathcal{E}$ are matchings—not only prematchings. That is, if $\mu \vee \mu' \in \mathcal{M}$ and $\mu \wedge \mu' \in \mathcal{M}$ for all μ and μ' in $\mathcal{E}$, then $\langle \mathcal{E}, \leq_F \rangle$ is a distributive lattice. This result does extend the one-to-one result.

Let us order $\mathcal{V}$ by set-inclusion; let $\nu' \sqsubseteq \nu$ if $\nu'(f) \subseteq \nu(f)$ and $\nu(w) \subseteq \nu'(w)$, for all f and w. Then $\langle \mathcal{V}, \sqsubseteq \rangle$ is a lattice (see Remark 1). The lattice operations are $\sqcup$ and $\sqcap$, defined by $(\nu \sqcup \nu')(f) = \nu(f) \cup \nu'(f)$, and $(\nu \sqcap \nu')(f) = \nu(f) \cap \nu'(f)$, for all f, and $(\nu \sqcup \nu')(w) = \nu(w) \cap \nu'(w)$, and $(\nu \sqcap \nu')(w) = \nu(w) \cup \nu'(w)$, for all w.

Let $\psi : \mathcal{V} \to \mathcal{V}$ be

$$(\psi\nu)(a) = \begin{cases} U(f,\nu) & \text{if} \quad a = f \in F \\ V(w,\nu) & \text{if} \quad a = w \in W \end{cases}$$

Theorem 35. *Let P be strongly substitutable. The map ψ is a lattice homomorphism of $\langle \mathcal{V}, \leq \rangle$ into $\langle \mathcal{V}, \sqsubseteq \rangle$. Further, if $\mu \vee \mu' \in \mathcal{M}$ and $\mu \wedge \mu' \in \mathcal{M}$ for all $\mu, \mu' \in \mathcal{E}$, then*

(1) *$\langle \mathcal{E}, \leq \rangle$ is a distributive sublattice of $\langle \mathcal{V}, \leq \rangle$,*
(2) *$\psi|_{\mathcal{E}}$ is a lattice isomorphism of $\langle \mathcal{E}, \leq \rangle$ onto $\langle \psi\mathcal{E}, \sqsubseteq \rangle$.*

The partial order $\leq$ on $\mathcal{V}$ depends on the profile P of preferences. In Theorem 35, we translate $\leq$ into an order that does not depend on P: set-inclusion. We interpret the result as showing how the lattice structure on $\mathcal{V}$ and, under additional assumptions, $\mathcal{E}$ is inherited from the lattice structure of set inclusion.

The interest of Theorem 35 is, first, that it shows why distributivity fails in the many-to-many model. Second, it shows how the distributive property in the one-to-one model is inherited from the distributive property of set inclusion on $\mathcal{V}$; it is easy to verify that the one-to-one model satisfies that $\mu \vee \mu' \in \mathcal{M}$ and $\mu \wedge \mu' \in \mathcal{M}$ for all $\mu, \mu' \in \mathcal{E}$. In fact the verification is carried out in Knuth (1976, page 56), as a first step in the proof that $\langle S(P), \leq_F \rangle$ is a distributive lattice.

Note that $\langle \mathcal{E}, \leq_F \rangle$ being a sub-lattice of $\langle \mathcal{V}, \leq_F \rangle$ means that the lattice operations $\vee_F$ and $\wedge_F$ on $\langle \mathcal{V}, \leq_F \rangle$ (see Remark 4) are also the lattice operations of $\langle \mathcal{E}, \leq_F \rangle$. Martínez, Massó, Neme, and Oviedo (2001) assuming substitutable (and q-separable) preferences, show that the stable matchings are not a lattice under $\vee_F$ and $\wedge_F$.

9.4. **Examples 5.1 and 5.2 in Blair (1988).** We do not reproduce the examples here.

Blair presents Example 5.1 as an example where $\langle S(P), \leq \rangle$ is not a lattice. In Example 5.1 there are 13 firms and 12 workers; $F = \{1, 2, \ldots 13\}$, $W = \{a, b, \ldots q\}$. Firm 10's preference relation is not strongly substitutable:

$$P(10) : mp, bnp, m, \ldots,$$

where $\ldots$ means that there are other acceptable sets of workers not listed. Note that $\{b, n, p\} \, P(10) \, \{m\}$ and $b \in Ch(\{b, n, p\} \cup \{b\}, P(10))$, but that $b \in Ch(\{m\} \cup \{b\}, P(10)) = \{m\}$.

Thus Blair's Example 5.1 illustrates that, with non-strongly substitutable preferences, $\langle \mathcal{E}, \leq \rangle$ may not be a lattice.

Blair presents Example 5.2 as an example where $\langle S(P), \leq^B \rangle$ is not a distributive lattice. Preferences in Example 5.2 are strongly substitutable—this is easy, if tedious, to verify. Blair's example thus illustrates that $\langle \mathcal{E}, \leq \rangle$ and $\langle \mathcal{E}, \leq^B \rangle$ may not be distributive lattices (the lattice operations in $\langle \mathcal{E}, \leq \rangle$ and $\langle \mathcal{E}, \leq^B \rangle$ might not coincide, but in this example they do).

We show that Example 5.2 does not satisfy the property that $\mu \vee \mu' \in \mathcal{M}$ and $\mu \wedge \mu' \in \mathcal{M}$ for all $\mu, \mu' \in \mathcal{E}$. So the example is not in the hypotheses of Theorem 35.

In Example 5.2 there are 7 firms and 10 workers; $F = \{1, 2, \ldots 7\}$, $W = \{a, b, \ldots j\}$. Consider matchings

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mu_1 & bcd & ae & af & j & h & i & g, \\ \mu_2 & bcd & ae & i & ag & h & f & j. \end{array}$$

Then $\mu_1 \vee \mu_2$ is:

$$\begin{array}{lcccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & & & \\ \mu_1 \vee \mu_2 & bcd & ae & i & j & h & f & g & & & \\ & a & b & c & d & e & f & g & h & i & j \\ \mu_1 \vee \mu_2 & 24 & 1 & 1 & 1 & 2 & 6 & 7 & 5 & 3 & 4. \end{array}$$

But $\mu_1 \vee \mu_2$ is not a matching, as $4 \in \mu_1 \vee \mu_2(a)$ while $a \notin \mu_1 \vee \mu_2(4)$.

Finally, we show that Blair's Example 5.2 also violates Roth's stronger conflict-of-interest property. From μ_1 and μ_2, constructing matching $\hat{\mu}$ by $\hat{\mu}(f) = Ch(\mu_1(f) \cup \mu_2(f), P(f))$, for all f, gives

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hat{\mu} & bcd & ae & i & j & h & f & g. \end{array}$$

Now, $\hat{\mu}$ is blocked by the pair $(1, a)$, so $\hat{\mu}$ is not pairwise stable. Note that $\hat{\mu}(a) = \{2\} \notin \{\mu_1(a), \mu_2(a)\}$. Thus Example 5.2 is not in the hypotheses of Proposition 34.

10. Proof of Theorem 15

The following proposition is immediate, but useful in some of our proofs.

Proposition 36. *A pair* $(B, f) \in 2^W \times F$ *blocks** μ *if and only if, for all* $w \in B$*, there is* $D_w \subseteq \mu(w)$ *such that*

$$[D_w \cup f]\, P(w) \mu(w),$$

and there is $A \subseteq \mu(f)$ *such that*

$$[A \cup B]\, P(f)\, \mu(f).$$

In words, (B, f) blocks* μ if firm f is willing to hire the workers in B—possibly after firing some of its current workers in $\mu(f)$—and all workers w in B prefer f, possibly after rejecting some of the firms in $\mu(w)$.

We present the proof of Theorem 15 in a series of lemmas. The first statement in Theorem 15 follows because $S^{**}(P) \subseteq S^*(P)$ is immediate, and from Lemmas 37 and 39. Item i) of the theorem follows from Lemmas 40 and 41. Item ii) follows from Lemma 42, and from Theorem 31.

Lemma 37. $S^*(P) \subseteq S(P)$

Proof. Let $\mu \notin S(P)$. We shall prove that $\mu \notin S^*(P)$. If μ is not individually rational there is nothing to prove; assume then that μ is individually rational. $\mu \notin S(P)$ implies that there is $(w,f) \in W \times F$ such that $w \notin \mu(f)$ (or $f \notin \mu(w)$),

$$f \in Ch(\mu(w) \cup \{f\}, P(w)) \tag{1}$$

$$w \in Ch(\mu(f) \cup \{w\}, P(f)). \tag{2}$$

Statements (1), (2), and $w \notin \mu(f)$ imply that

$$Ch(\mu(w) \cup \{f\}, P(w))\, P(w)\, \mu(w) \tag{3}$$

and

$$Ch(\mu(f) \cup \{w\}, P(f))\, P(f)\, \mu(f). \tag{4}$$

Let $B = \{w\}$, $D_w = Ch(\mu(w) \cup \{f\}, P(w)) \cap \mu(w)$, and

$$A = Ch(\mu(f) \cup \{w\}, P(f)) \cap \mu(f).$$

We shall prove that (B,f) blocks* μ. Since $B = \{w\}$, statement (3) implies that

$$\begin{aligned} D_w &= Ch(\mu(w) \cup \{f\}, P(w)) \cap \mu(w) \\ &= Ch(\mu(w) \cup \{f\}, P(w)) \setminus \{f\}. \end{aligned}$$

So statement (3) implies that

$$[D_w \cup \{f\}] = Ch(\mu(w) \cup \{f\}, P(w))\, P(w)\, \mu(w),$$

which gives us the first part of the definition of block*. Also,

$$\begin{aligned} A &= Ch(\mu(f) \cup \{w\}, P(f)) \cap \mu(f) \\ &= Ch(\mu(f) \cup \{w\}, P(f)) \setminus \{w\} \\ &= Ch(\mu(f) \cup \{w\}, P(f)) \setminus B. \end{aligned}$$

So statement (4) implies that

$$[A \cup B] = Ch(\mu(f) \cup \{w\}, P(f))\, P(f)\, \mu(f),$$

and we have the second part of the definition of block*. Thus, $\mu \notin S^*(P)$. ∎

Remark 5. In general, $S^*(P) \neq S(P)$.

We use Lemma 38 in many of our results, starting with Lemma 39.

Lemma 38. *If $\nu \in \mathcal{E}$ then ν is a matching and ν is individually rational.*

Proof. Let $\nu = (\nu_F, \nu_W) \in \mathcal{E}$.

Fix $w \in \nu_F(f)$, we shall prove that $f \in \nu_W(w)$. $\nu \in \mathcal{E}$ implies that

$$w \in \nu_F(f) = (T\nu)(f) = Ch(U(f, \nu), P(f)).$$

Thus $w \in U(f, \nu)$.

The definition of $U(f, \nu)$ implies

$$f \in Ch(\nu_W(w) \cup \{f\}, P(w)) R(w) \nu_W(w). \tag{5}$$

Now, $\nu_F(f) \cup \{w\} = \nu_F(f)$ and $\nu \in \mathcal{E}$, imply that

$$\nu_F(f) = (T\nu)(f) = Ch(U(f, \nu), P(f)). \tag{6}$$

So

$$\begin{aligned} Ch(\nu_F(f), P(f)) &\overset{(1)}{=} Ch(Ch(U(f, \nu), P(f)), P(f)) \\ &\overset{(2)}{=} Ch(U(f, \nu), P(f)) \\ &\overset{(3)}{=} \nu_F(f). \end{aligned}$$

Equalities (1) and (3) follow from statement (6). Equality (2) is a simple property of choice sets: $Ch(Ch(S, P(f)), P(f)) = Ch(S, P(f))$. Hence we have that

$$\nu_F(f) = Ch(\nu_F(f), P(f)). \tag{7}$$

Now $w \in \nu_F(f)$ implies that $Ch(\nu_F(f), P(f)) = Ch(\nu_F(f) \cup \{w\}, P(f))$. So statement (7) implies that

$$f \in V(w, \nu). \tag{8}$$

But

$$\nu_W(w) = (T\nu)(w) = Ch(V(w, \nu), P(w)),$$

so

$$\nu_W(w) \subseteq V(w, \nu). \tag{9}$$

But statements (8) and (9) give

$$V(w, \nu) \supseteq \nu_W(w) \cup \{f\} \supseteq Ch(\nu_W(w) \cup \{f\}, P(w)).$$

The definition of choice set implies

$$\nu_W(w) R(w) Ch(\nu_W(w) \cup \{f\}, P(w)). \tag{10}$$

Statements (5), (10) and anti-symmetry of preference relations imply that, $f \in \nu_W(w)$.

Let $f \in \nu_W(w)$, the proof that $w \in \nu_F(f)$ and that

$$\nu_F(f) = Ch(\nu_F(f), P(f)) \tag{11}$$

is entirely symmetric to the proof for workers above.

Thus, $w \in v_F(f)$ if and only if $f = v_W(w)$. So, ν is a matching.

Statements (7) and (11) imply that ν is individually rational. ∎

Lemma 39. $\mathcal{E} \subseteq S^{**}(P)$

Proof. Let $\mu \in \mathcal{E}$. By Lemma 38 we know that μ is an individually rational matching. Fix $f \in F$, $B \subseteq W$ such that $B \neq \emptyset$. We assume that, for all $w \in B$ there exist $D_w \subseteq \mu(w)$ such that $\{f\} \cup D_w P(w) \mu(w)$. μ is individually rational, so $\mu(f) = Ch(\mu, P(f))$. Then $\{f\} \cup D_w P(w) \mu(w)$ implies that

$$f \in Ch(\mu(w) \cup \{f\}, P(w)); \tag{12}$$

for all $w \in B$. By the definition of $U(f, \mu)$, we have that

$$B \subseteq U(f, \mu). \tag{13}$$

Let $A \subseteq \mu(f)$. $\mu \in \mathcal{E}$ implies that $\mu(f) = (T\mu)(f) \subseteq U(f, \mu)$; so statement (13) gives

$$A \cup B \subseteq U(f, \mu). \tag{14}$$

Now, $\mu \in \mathcal{E}$ and statement (14) implies

$$\mu(f) R(f) Ch(A \cup B, P(f)) R(f) [A \cup B]; \tag{15}$$

as $\mu(f) = Ch(U(f, \mu), P(f))$.

Statements (12) and (15) show that there is no (B, f) that blocks* μ. The proof that there is no $(w, A) \in W \times 2^F$ that blocks* μ is symmetric. Thus $\mu \in S^{**}(P)$. ∎

Lemma 40. *If $P(W)$ is substitutable, $S^*(P) \subseteq \mathcal{E}$*

Proof. Let $\mu \in S^*(P)$ and assume that $\mu \notin \mathcal{E}$, so $\mu \neq T\mu$. We shall first prove that $(T\mu)(f) \neq \mu(f)$, for some f, yields a contradiction, and then that $(T\mu)(w) \neq \mu(w)$, for some w, yields a contradiction. Note that, by the asymmetric situation of firms and workers in the definition of $S^*(P)$, the proof of the two statements is not analogous.

First assume that there exist $f \in F$ such that

$$\mu(f) \neq (T\mu)(f) = Ch(U(f, \mu), P(f)) = C \subseteq U(f, \mu).$$

Let $A = C \cap \mu(f)$, and $B = C \backslash \mu(f)$. Because μ is an individually rational matching we have that $\mu(w) = Ch(\mu(w), P(w)) = Ch(\mu(w) \cup w, P(w))$, for all $w \in \mu(f)$. Hence, $\mu(f) \subseteq U(f, \mu)$, so $(T\mu)(f) P(f) \mu(f)$ implies that $B \neq \emptyset$.

Now,

$$A \cup B = C P(f) \mu(f). \tag{16}$$

Also, for all $w \in B$, $w \in U(f, \mu)$; so $f \in Ch(\mu(w) \cup f, P(w))$ by the definition of $U(f, \mu)$. For any $w \in B$ let $D_w = Ch(\mu(w) \cup f, P(w)) \cap \mu(w)$. Since $f \notin \mu(w)$ we have that

$$\{f\} \cup D_w P(w) \mu(w) . \tag{17}$$

Statements (16) and (17) imply that (B, f) block* μ, which contradicts that $\mu \in S^*(P)$.

Hence, for all $f \in F$,

$$\mu(f) = (T\mu)(f) . \tag{18}$$

Now assume that there exists $w \in W$ such that

$$\mu(w) \neq (T\mu)(w) = Ch(V(w, \mu), P(w)) = G \subseteq V(w, \mu) .$$

If $f \in G$, then

$$w \in Ch(\mu(f) \cup \{w\}, P(f)) , \tag{19}$$

by the definition on $V(w, \mu)$. Because μ is an individually rational matching we have—by the same argument as above—that $\mu(w) \subseteq V(w, \mu)$. We can assume that $G \nsubseteq \mu(w)$; for, if $G \subseteq \mu(w)$, then $\mu(w) \subseteq V(w, \mu)$ and the Choice Property,[3] imply that

$$G = Ch(V(w, \mu), P(w)) = Ch(\mu(w), P(w)) = \mu(w) ,$$

where the last equality follows because μ is an individually rational matching—but this would contradict that $G \neq \mu(w)$, hence we can assume $G \nsubseteq \mu(w)$.

Let $\overline{f} \in G \backslash \mu(w)$. μ is a matching, so $w \notin \mu(\overline{f})$. Now, statement (19) implies that

$$w \in Ch\left(\mu\left(\overline{f}\right) \cup \{w\}, P\left(\overline{f}\right)\right) = C.$$

Let $A = C \cap \mu(f) = C \backslash \{w\}$, and $B = \{w\}$. Then

$$C = [A \cup B] P(f) \mu(f) . \tag{20}$$

Now, $\overline{f} \in G \backslash \mu(w)$; so substitutability of $P(w)$ implies that there exists $D_w = Ch(V(w, \mu), P(w)) \cap \mu(w)$ such that

$$\left[\overline{f} \cup D_w\right] P(w) \mu(w) . \tag{21}$$

Statements (20) and (21) imply that $\left(\overline{f}, \{w\}\right)$ blocks* μ, which contradicts $\mu \in S^*(P)$. Hence, for all $w \in W$,

$$\mu(w) = (T\mu)(w) . \tag{22}$$

Statements (18) and (22) imply that $\mu = T\mu$. Hence $\mu \in \mathcal{E}$. ∎

Lemma 41. *If $P(W)$ is substitutable then $S^*(P) \subseteq C(P^B)$.*

[3] $Ch(A, P(s)) \subseteq B \subseteq A$, then $Ch(A, P(s)) = Ch(B, P(s))$

Proof. Let $\mu \in S^*(P)$, and suppose that $\mu \notin C(P^B)$. Let $F' \subseteq F$, $W' \subseteq W$ with $F' \cup W' \neq \emptyset$, and let $\hat{\mu} \in \mathcal{M}$ such that, for all $w \in W'$, and for all $f \in F'$

$$\hat{\mu}(w) \subseteq F', \text{ and } \hat{\mu}(f) \subseteq W', \tag{23}$$

$$\hat{\mu}(w) R^B(w) \mu(w), \tag{24}$$

$$\hat{\mu}(f) R^B(f) \mu(f), \tag{25}$$

and

$$\hat{\mu}(s) P^B(s) \mu(s) \text{ for at least one } s \in W' \cup F'.$$

We shall need the following

Claim. There exists $f \in F'$, such that $\hat{\mu}(f) P^B(f) \mu(f)$ if and only if there exists $w \in W'$ such that $\hat{\mu}(w) P^B(w) \mu(w)$.

Proof of the claim. Let $\hat{\mu}(f) P^B(f) \mu(f)$. Because μ is individually rational, we have that $\hat{\mu}(f) \nsubseteq \mu(f)$, so let $\overline{w} \in \hat{\mu}(f) \backslash \mu(f)$. By condition (23), we have that $\overline{w} \in \hat{\mu}(f) \subseteq W'$; but then $\overline{w} \notin \mu(f)$ and condition (24) implies that

$$\hat{\mu}(\overline{w}) P^B(\overline{w}) \mu(\overline{w}).$$

Similarly we show that if $\hat{\mu}(w) P^B(w) \mu(w)$ then there exist $\overline{f}$ such that $\hat{\mu}(\overline{f}) P^B(\overline{f}) \mu(\overline{f})$. This proves the claim.

By the claim, we can assume that there exists $f \in F'$ such that $\hat{\mu}(f) \neq \mu(f)$. Let $B = \hat{\mu}(f) \backslash \mu(f)$ and $A = \hat{\mu}(f) \cap \mu(f)$ then

$$A \cup B = \hat{\mu}(f) P^B(f) \mu(f), \tag{26}$$

and $B \cap \mu(f) = \emptyset$. Let $w \in B$. Then, $f \in \hat{\mu}(w)$ and $f \notin \mu(w)$, which implies that $\hat{\mu}(w) \neq \mu(w)$. Condition (24) implies that

$$f \in \hat{\mu}(w) = Ch(\mu(w) \cup \hat{\mu}(w), P(w)).$$

By substitutability of $P(w)$, $f \in Ch(\mu(w) \cup f), P(w)$.

Now, $w \in B$ was arbitrary, so together with statement (26), this implies that (B, f) blocks* μ. Thus $\mu \notin S^*(P)$. ∎

Lemma 42. *If P is substitutable then $S(P) \subseteq \mathcal{E}$*

Proof. Let $\mu \notin \mathcal{E}$. We shall prove that $\mu \notin S(P)$. If μ is not individually rational there is nothing to prove. Suppose then that μ is individually rational. Lemma 40 and $\mu \notin \mathcal{E}$ imply $\mu \notin S^*(P)$. So, there is (f, B), with $B \neq \emptyset$ that blocks* μ. This means that, for all $w \in B$,

$$f \in Ch(\mu(w) \cup f, P(w))$$

and

$$B \subseteq Ch(\mu(f) \cup B, P(f)).$$

But $P(f)$ is substitutable, so there is $w' \in B$ with

$$w' \in Ch(\mu(f) \cup w', P(f)).$$

. Thus $\mu \notin S(P)$. ∎

11. Proof of Theorems 18 and 19

The proof of Theorem 18 follows from Lemmas 43, 44, 45, 46, and 47. Theorem 19 then follows from Theorem 31.

Lemma 43. *$SW(P) \subseteq \mathcal{E}$.*

Proof. Let μ be a matching such that $\mu \notin \mathcal{E}$. We shall prove that $\mu \notin SW(P)$. If μ is not individually rational there is nothing to prove. Suppose then that μ is individually rational matching.

Suppose, without loss of generality, that there is a $\overline{f} \in F$ such that $\mu(\overline{f}) \neq Ch(U(\overline{f}, \mu), P(\overline{f}))$. That μ is individually rational implies that $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ since, for all $w \in \mu(\overline{f})$, $\overline{f} \in \mu(w)$ so

$$Ch(\mu(w) \cup \overline{f}, P(w)) = Ch(\mu(w), P(w)) = \mu(w) \ni \overline{f}.$$

Let $F' = \{\overline{f}\}$, let $W' = Ch(U(\overline{f}, \mu), P(\overline{f})) \backslash \mu(\overline{f})$. We shall construct a $\mu' \in \mathcal{M}$ such that $\langle W', F', \mu' \rangle$ is a set-wise block of μ. Let $\mu'(\overline{f}) = Ch(U(\overline{f}, \mu), P(\overline{f}))$. For all $w \in W$, let

$$\mu'(w) = \begin{cases} Ch(\mu(w) \cup \overline{f}, P(w)) \text{ if } w \in W' \\ \mu(w) \text{ if } w \in \left[\mu'(\overline{f}) \cap \mu(\overline{f})\right] \cup \left[\mu'(\overline{f}) \cup \mu(\overline{f})\right]^c \\ \mu(w) \backslash \overline{f} \text{ if } w \in \mu(\overline{f}) \backslash \mu'(\overline{f}) \end{cases}$$

$\mu'(f)$, for $f \notin F'$, is determined from the $\mu'(w)$'s. Then μ' is a matching and $W' = \mu'(F) \backslash \mu(F)$. Note that $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ implies that $\overline{f} \in Ch(\mu(w) \cup \overline{f}, P(w))$, so $\overline{f} \in \mu'(w)$ for all $w \in W'$. So $F' = \mu'(W) \backslash \mu(W)$.

First we verify that μ' is individually rational: $\mu'(\overline{f}) = Ch(\mu'(\overline{f}), P(\overline{f}))$, as $\mu'(\overline{f}) = Ch(U(\overline{f}, \mu), P(\overline{f}))$; and $\mu'(w) = Ch(\mu'(w), P(\overline{f}))$, as $\mu'(w) = Ch(\mu(w) \cup \overline{f}, P(\overline{f}))$ for all $w \in W'$.

Finally, $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ implies that $\mu'(\overline{f}) P(\overline{f}) \mu(\overline{f})$, and $\mu'(w) = Ch(\mu(w) \cup \overline{f}, P(\overline{f}))$ implies that $\mu'(w) P(w) \mu(w)$, for all $w \in W'$. Thus the constructed $\langle W', F', \mu' \rangle$ is a set-wise block of μ, and thus $\mu \notin SW(P)$. ∎

Lemma 44. *$B(P) \subseteq \mathcal{E}$.*

Proof. Let μ be a matching such that $\mu \notin \mathcal{E}$. We shall prove that $\mu \notin B(P)$. If μ is not individually rational there is nothing to prove. Suppose then that μ is individually rational.

Suppose, without loss of generality, that there is a $\overline{f} \in F$ such that $\mu(\overline{f}) \neq Ch(U(\overline{f}, \mu), P(\overline{f}))$. Let $F' = \{\overline{f}\}$, let

$$W' = Ch(U(\overline{f}, \mu), P(\overline{f})) \backslash \mu(\overline{f}).$$

Construct μ' as in the proof of Lemma 43. We shall prove that $\langle W', F', \mu' \rangle$ is a counterobjection-free objection. Recall that

$$\mu'(\overline{f}) = Ch(U(\overline{f}, \mu), P(\overline{f})). \tag{27}$$

Note that $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ (see the proof of Lemma 43). So Statement 27 implies that $\mu'(\overline{f}) P(\overline{f}) \mu(\overline{f})$. Similarly, $W' \subseteq U(\overline{f}, \mu)$ implies that $\mu'(w) = Ch(\mu(w) \cup \overline{f}) P(w) \mu(w)$ for all $w \in W'$. Hence $\langle W', F', \mu' \rangle$ is an objection.

We now prove that there are no counterobjections to $\langle W', F', \mu' \rangle$. Let $\langle W'', F'', \mu'' \rangle$ be such that μ'' is a matching, $F'' \subseteq F'$, $W'' \subseteq W'$, and $\mu''(W'' \cup F'') \backslash \mu'(W'' \cup F'') \subseteq W'' \cup F''$.

First, let $F'' \neq \emptyset$. Then $F'' = \{\overline{f}\}$. Statement 27 implies that $\mu'(\overline{f}) R(\overline{f}) A$, for all $A \subseteq \mu'(\overline{f})$. But

$$\mu''(\overline{f}) \backslash \mu'(\overline{f}) \subseteq W' \subseteq \mu'(\overline{f}).$$

So $\mu'(\overline{f}) R(\overline{f}) \mu''(\overline{f})$. Thus $\langle W'', F'', \mu'' \rangle$ is not a counterobjection.

Second, let $F'' = \emptyset$. For all $w \in W''$, $w \in U(\overline{f}, \mu)$. So

$$\overline{f} \in Ch(\mu(w) \cup \overline{f}, P(w)) = \mu'(w) \tag{28}$$

(see the proof of Lemma 43). Also, $\mu''(w) \subseteq \mu'(w)$, as $\mu''(W'') \backslash \mu'(W'') \subseteq F'' \emptyset$. Then $\overline{f} \notin \mu''(w)$ and Statement 28 implies that $\mu'(w) P(w) \mu''(w)$. Thus $\langle W'', F'', \mu'' \rangle$ is not a counterobjection. ∎

Lemma 45. *If $P(F)$ is substitutable, and $P(W)$ is strongly substitutable, then $\mathcal{E} \subseteq SW(P)$.*

Proof. The proof is similar to the proof of Lemma 47. Let $\mu \in \mathcal{E}$. By Lemma 38, μ is an individually rational matching. Suppose, by way of contradiction, that $\mu \notin SW(P)$. Let $\langle W', F', \mu' \rangle$ be a set-wise block of μ.

Fix $\overline{f} \in F'$, so $\mu'(\overline{f}) P(\overline{f}) \mu(\overline{f})$. The matching μ is individually rational, so $\mu'(\overline{f}) P(\overline{f}) \mu(\overline{f})$ implies that

$$Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f})) \nsubseteq \mu(\overline{f})$$

Fix $\overline{w} \in Ch(\mu(\overline{f}) \cup \mu'(\overline{f})$ such that $\overline{w} \in \mu'(\overline{f}) \backslash \mu(\overline{f})$. By substitutability of $P(\overline{f})$, $\overline{w} \in Ch(\mu(\overline{f}) \cup \overline{w}, P(\overline{f}))$. So

$$\overline{f} \in V(\overline{w}, \mu). \tag{29}$$

On the other hand, $\overline{w} \in \mu'(\overline{f}) \backslash \mu(\overline{f})$ implies that $\overline{f} \in W'$, so

$$\mu'(\overline{f})P(\overline{f})\mu(\overline{f}). \tag{30}$$

$\langle W', F', \mu' \rangle$ is a set-wise block, so $\mu'(\overline{w}) = Ch(\mu'(\overline{w}), P(\overline{w}))$. Further, μ' is a matching so $\overline{f} \in \mu'(\overline{w})$. Then $Ch(\mu'(\overline{w}) \cup \overline{f}, P(\overline{w}))$. Strong substitutability of $P(\overline{w})$ and Statement 30 implies that

$$\overline{f} \in Ch(\mu(\overline{w}) \cup \overline{f}). \tag{31}$$

But Statement 29 and $\mu \in \mathcal{E}$ imply that $\mu(\overline{w}) \cup \overline{f} \subseteq V(\overline{w}, \mu)$. But then Statement 31 contradicts that $\mu(\overline{w}) = Ch(V(\overline{w}, \mu), P(\overline{w}))$. ∎

Lemma 46. *If $P(F)$ is substitutable, and $P(W)$ is strongly substitutable, then $\mathcal{E} \subseteq B(P)$.*

Proof. The proof is similar to the proof of Lemma 47. Let $\mu \in \mathcal{E}$. By Lemma 38, μ is an individually rational matching. Let $\langle W', F', \mu' \rangle$ be an objection to μ.

First, if $\mu'(s) \neq Ch(\mu'(s), P(s))$ for some $s \in F' \cup W'$, then $\langle W', F', \mu' \rangle$ has a counterobjection: let $\overline{f} \in F'$ be such that $\mu'(\overline{f}) \neq Ch(\mu'(\overline{f}), P(\overline{f}))$. Let $W'' = \emptyset$, $F'' = \{\overline{f}\}$, and let μ'' be defined by $\mu''(f) = \mu'(f)$ for all $f \neq \overline{f}$, and $\mu''(\overline{f}) = Ch(\mu'(\overline{f}), P(\overline{f}))$. The definition of $\mu''(w)$, for all $w \in W$, is implicit. Then, $\mu'(\overline{f}) \neq Ch(\mu'(\overline{f}), P(\overline{f}))$ implies that $\mu''(\overline{f})P(\overline{f})\mu'(\overline{f})$, and $\langle W'', F'', \mu'' \rangle$ is a counterobjection to $\langle W', F', \mu' \rangle$. So $\mu \in B(P)$.

Second, let $\mu'(s) = Ch(\mu'(s), P(s))$ for all $s \in F' \cup W'$. We shall prove that $\langle W', F', \mu' \rangle$ is not an objection in the first place. Suppose, by the way of contradiction, that $\langle W', F', \mu' \rangle$ is an objection to μ, so we can suppose—without loss of generality—that there is $\overline{f} \in F'$ such that $\mu'(\overline{f})P(\overline{f})\mu(\overline{f})$. The matching μ is individually rational, so $\mu'(\overline{f})P(\overline{f})\mu(\overline{f})$ implies that

$$\mu(\overline{f}) \nsubseteq Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f}))$$

Let $\overline{w} \in Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f}))$ be such that $\overline{w} \in \mu'(\overline{f}) \backslash \mu(\overline{f})$. Now, substitutability of $P(\overline{f})$ implies that

$$\overline{w} \in Ch(\mu(\overline{f}) \cup \overline{w}, P(\overline{f})).$$

Thus, $\overline{f} \in V(\overline{w}, \mu)$.

On the other hand, $\overline{w} \in \mu'(\overline{f}) \backslash \mu(\overline{f})$ implies that $\overline{w} \in W'$. So $\mu'(\overline{w})P(\overline{w})\mu(\overline{w})$, as $\langle W', F', \mu' \rangle$ is an objection. Then

$$\overline{f} \in \mu'(\overline{w}) = Ch(\mu'(\overline{w}), P(\overline{w})) = Ch(\mu'(\overline{w}) \cup \overline{f}, P(\overline{w}))$$

and strong substitutability of $P(\overline{w})$ gives $\overline{f} \in Ch(\mu(\overline{w}) \cup \overline{f}, P(\overline{w}))$. But we proved that $\overline{f} \in V(\overline{w}, \mu)$. So

$$\mu(w) \neq Ch(\mu(w) \cup \overline{f}, P(w)) \subseteq V(w, \mu).$$

A contradiction since $\mu \in \mathcal{E}$ implies that $\mu(\overline{w}) = Ch(V(\overline{w}, \mu), P(\overline{w}))$. Thus $\langle W', F', \mu' \rangle$ is not an objection, and $\mu \in B(P)$. ∎

Lemma 47. *If $P(F)$ is substitutable, and $P(W)$ is strongly substitutable, then $\mathcal{E} \subseteq IRC(P)$.*

Proof. Let $\mu \in \mathcal{E}$. By Lemma 38, μ is an individually rational matching. Suppose, by way of contradiction, that $\langle W', F', \mu' \rangle$ is an individually rational block of μ.

Without loss of generality, let $\mu'(\overline{f}) P(\overline{f}) \mu(\overline{f})$, for some $\overline{f} \in F'$. Since μ is individually rational,

$$Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f})) \nsubseteq \mu(\overline{f}).$$

Let $\overline{w} \in \mu'(\overline{f}) \backslash \mu(\overline{f})$ be such that

$$\overline{w} \in Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f})) = Ch(\mu(\overline{f}) \cup \mu'(\overline{f}) \cup \overline{w}, P(\overline{f})).$$

By substitutability of $P(\overline{f})$, $\overline{w} \in Ch(\mu(\overline{f}) \cup \overline{w}, P(\overline{f}))$. Thus

$$\overline{f} \in V(\overline{w}, \mu). \tag{32}$$

Now. $\overline{f} \in \mu'(\overline{w}) \backslash \mu(\overline{w})$ implies

$$\mu'(\overline{w}) P(\overline{w}) \mu(\overline{w}), \tag{33}$$

as $\overline{w} \in W'$, and $\mu'(\overline{w}) \neq \mu(\overline{w})$. But μ' is individually rational, so

$$\overline{f} \in Ch(\mu'(\overline{w}), P(\overline{w})) = Ch(\mu'(\overline{w}) \cup \overline{f}, P(\overline{w})).$$

Then statement 33, and strong substitutability of $P(\overline{w})$, implies that $\overline{f} \in Ch(\mu(\overline{w}) \cup \overline{f}, P(\overline{w}))$. So

$$Ch(\mu(\overline{f}) \cup \overline{f}, P(\overline{w})) P(\overline{w}) \mu(\overline{w}) \tag{34}$$

But $\mu \in \mathcal{E}$ implies that $\mu(\overline{w}) = Ch(V(\overline{w}, \mu), P(\overline{f}))$. By Statement 32, $\mu(\overline{w}) \cup \overline{f} \subseteq V(\overline{w}, \mu)$, which contradicts statement 34. Hence there are no individually rational blocks of μ, and $\mu \in IRC(P)$. ∎

12. Proof of Theorems 23 and 26

12.1. **Proof of Theorems 23.**

The proof of Theorem 23 follows from Lemmas 48, 49.

Lemma 48. *Let $P(W)$ be substitutable. If $\mu \in \mathcal{M}$ is the outcome of a SPNE, then $\mu \in \mathcal{E}$.*

Proof. Let (η^*, ξ^*) be a SPNE, and $\mu \in \mathcal{M}$ be the outcome of (η^*, ξ^*). For all $w \in W$,

$$\xi_w^*(\eta) \cap \left\{f : w \in \eta_f\right\} = Ch\left(\left\{f : w \in \eta_f\right\}, P(w)\right).$$

For all $f \in F$, for all η_{-f}, let

$$Y(\eta_{-f}) = \left\{w : f \in Ch\left(\left\{\tilde{f} : w \in \eta_{\tilde{f}}\right\} \cup f, P(w)\right)\right\}.$$

So, by definition of SPNE, $\eta_f^* \cap Y(\eta_{-f}^*) = Ch(Y(\eta_{-f}^*), P(f))$.

Let $(\overline{\eta}, \overline{\xi})$ be the pair of strategies obtained from (η^*, ξ^*) by having each w not propose to firms that did not propose to w, and having each f not propose to workers that will reject f. Thus, $\overline{\xi}_w(\eta) = \xi_w^*(\eta) \cap \left\{f : w \in \eta_f\right\}$ and by $\overline{\eta}_f \cap Y(\eta_{-f}^*) = \eta_f^* \cap Y(\eta_{-f}^*)$.

We shall show that $(\overline{\eta}, \overline{\xi})$ is a SPNE as well, and that its outcome is also μ. First, it is immediate that its outcome is μ: $\overline{\eta}_f = \mu(f)$, for all f, and for all $w \in \mu(f)$, $f \in \overline{\xi}_w(\overline{\eta})$ Second, given a strategy profile η for firms, each w is indifferent between proposing $\xi_w^*(\eta)$ and $\overline{\xi}_w(\eta)$, as they will both result in the same set of partners. For a firm f, $Y(\eta_{-f}^*) = Y(\overline{\eta}_{-f})$, which implies that $\overline{\eta}_f = Ch\left(Y(\overline{\eta}_{-f}), P(f)\right)$, and thus $(\overline{\eta}, \overline{\xi})$ is a SPNE. To see that $Y(\eta_{-f}^*) = Y(\overline{\eta}_{-f})$, note that $w \in Y(\eta_{-f}^*)$ if and only if $f \in Ch\left(\left\{\tilde{f} : w \in \eta_{\tilde{f}}^*\right\} \cup f, P(w)\right)$. But

$$\begin{aligned} Ch\left(\left\{\tilde{f} : w \in \eta_{\tilde{f}}^*\right\} \cup f, P(w)\right) &= Ch\left(Ch\left(\left\{\tilde{f} : w \in \eta_{\tilde{f}}^*\right\}, P(w)\right) \cup f, P(w)\right) \\ &= Ch\left(\mu(w) \cup f, P(w)\right) \\ &= Ch\left(\left\{\tilde{f} : w \in \overline{\eta}_{\tilde{f}}\right\} \cup f, P(w)\right), \end{aligned}$$

where the first equality is a consequence of substitutability of $P(W)$ (Blair, 1988, Proposition 2.3). Hence $w \in Y(\eta_{-f}^*)$ if and only if

$$f \in Ch\left(\left\{\tilde{f} : w \in \overline{\eta}_{\tilde{f}}\right\} \cup f, P(w)\right),$$

so $Y(\eta_{-f}^*) = Y(\overline{\eta}_{-f})$.

Now we shall prove that $\mu \in \mathcal{E}$. Let $f \in F$. Note that $Y(\overline{\eta}_{-f}) = \{w : f \in Ch\left(\mu(w) \cup f, P(w)\right)\}$, so $Y(\overline{\eta}_{-f}) = U(f, \mu)$. Now, by the definition of $\overline{\eta}_f$, $\mu(f) = \overline{\eta}_f = Ch(U(f, \mu), P(w))$.

Let $w \in W$. We shall first prove that

$$\mu(w) \subseteq V(w, \mu). \tag{35}$$

Let $f \in \mu(w)$, so $w \in \mu(f) = \overline{\eta}_f$. But $\overline{\eta}_f = Ch(Y(\overline{\eta}_{-f}), P(f))$, so $\overline{\eta}_f = Ch(\overline{\eta}_f, P(f))$. Then $w \in Ch(\mu(f) \cup w, P(f)) = Ch(\overline{\eta}_f, P(f))$, so $f \in V(w, \mu)$; this proves Statement 35. Second, we prove that $Ch(V(w, \mu), P(w)) \subseteq \mu(w)$, which together with Statement 35 implies that $\mu(w) = Ch(V(w,\mu), P(w))$. Let $f \in Ch(V(w,\mu), P(w))$. By Statement 35, $\mu(w) \cup f \subseteq V(w,\mu)$. Substitutability of $P(w)$ implies that

$$f \in Ch(\mu(w) \cup f, P(w)). \tag{36}$$

So $w \in U(f,\mu)$. Suppose, by way of contradiction, that $f \notin \mu(w)$. Now, $f \notin \mu(w)$ implies $w \notin \mu(f)$, so Statement 36 implies $\mu(f) \cup f P(w) \mu(f)$. But $w \in U(f,\mu)$, so $\mu(f) \cup f P(w) \mu(f)$ contradicts $\mu(f) = \overline{\eta}_f = Ch(U(f,\mu), P(w))$. The assumption $f \notin \mu(w)$ is then absurd. This finishes the proof that $\mu(w) = Ch(V(w,\mu), P(w))$. We also proved $\mu(f) = Ch(U(f,\mu), P(w))$, so $\mu \in \mathcal{E}$. ∎

Lemma 49. *If $\mu \in \mathcal{E}$, then μ is the outcome of some SPNE.*

Proof. Define $(\overline{\eta}, \overline{\xi})$ by $\overline{\eta}_f = \mu(f)$ and $\overline{\xi}_w(\eta) = Ch(\{f : w \in \eta_f\}, P(w))$. Let $\bar{(}\mu) \in \mathcal{M}$ be the outcome of $(\overline{\eta}, \overline{\xi})$. We show that $(\overline{\eta}, \overline{\xi})$ is a SPNE, and that $\bar{(}\mu) = \mu$.

Note that, for any f and w, $\left\{\tilde{f} : w \in \overline{\eta}_{\tilde{f}}\right\} \cup f = \mu(w) \cup f$. Then,

$$\begin{aligned} \left\{w : f \in Ch\left(\left\{\tilde{f} : w \in \overline{\eta}_{\tilde{f}}\right\} \cup f, P(w)\right)\right\} &= \{w : f \in Ch(\mu(w) \cup f, P(w))\} \\ &= U(f,\mu) \end{aligned}.$$

But $\mu \in \mathcal{E}$, so $\overline{\eta}_f = \mu(f) = Ch(U(f,\mu), P(f))$. Hence $\overline{\eta}_f$ is optimal given $\overline{\eta}_{-f}$. By definition of $\overline{\xi}_w$, $\overline{\xi}_w(\eta)$ is optimal for w given any profile η. Hence $(\overline{\eta}, \overline{\xi})$ is a SPNE.

Now, $f \in \mu(w)$ if and only if $w \in \mu(f) = \overline{\eta}_f$. So, $\{f : w \in \overline{\eta}_f\} = \mu(w)$. Then $\overline{\xi}_w(\overline{\eta}) = Ch(\mu(w), P(w)) = \mu(w)$, as $\mu(w) \in \mathcal{E}$ implies that μ is individually rational (Lemma 38).

Hence $w \in \overline{\mu}(f)$ if and only if $w \in \overline{\eta}_f = \mu(f)$, and $f \in \overline{\mu}(w)$ if and only if $f \in \overline{\xi}_w(\overline{\eta}) = \mu(w)$. So $\mu = \overline{\mu}$. ∎

12.2. **Proof of Theorem 26.** Let $\langle W', F', \mu' \rangle$ be a block of μ. Let $w \in W'$ be such that $\mu'(w) P(w) \mu(w)$. We shall prove that there are $f, f' \in F'$, and $w' \in W'$ such that:

- $w \neq w'$, $f \neq f'$,
- $f \in \mu'(w) \backslash \mu(w)$ $w' \in \mu'(f) \backslash \mu(f)$, and $f' \in \mu'(w') \backslash \mu(w')$
- f wants to add w' and w' wants to add f', but w' does not want to add f, and f' does not want to add w'.

Now, $\mu'(w) P(w) \mu(w)$ implies that

$$Ch(\mu(w) \cup \mu'(w), P(w)) R(w) \mu'(w) P(w) \mu(w).$$

But $\mu \in \mathcal{E}$ implies that μ is individually rational (Lemma 38); so $\mu(w)R(w)A$, for all $A \subseteq \mu(w)$. Hence

$$Ch(\mu(w) \cup \mu'(w), P(w)) \backslash \mu(w) \neq \emptyset.$$

Let $f \in Ch(\mu(w) \cup \mu'(w), P(w)) \backslash \mu(w)$. By substitutability of $P(w)$, $f \in Ch(\mu(w) \cup f, P(w))$; hence w wants to add f.

On the other hand, $f \in Ch(\mu(w) \cup f, P(w))$ implies that $w \in U(f, \mu)$. But $f \in \mu'(w) \backslash \mu(w)$ means that $w \in \mu'(f) \backslash \mu(f)$. In particular, $w \notin \mu(f)$; so, by Lemmas 39 and 37, $w \notin Ch(\mu(f) \cup w, P(f)) = \mu(f)$, as $\mu \in \mathcal{E}$ implies $\mu(f) = Ch(U(f, \mu), P(f))$ and $\mu(f) \cup w \subseteq U(f, \mu)$. Hence f does not want to add w.

But $\mu'(f) \neq \mu(f)$, and $f \in F'$, implies $\mu'(f)P(f)\mu(f)$. By an argument that is symmetric to the one above, there is $w' \in \mu'(f) \backslash \mu(f)$, and $f' \in \mu'(w) \backslash \mu(w)$ such that f wants to add w' and w' wants to add f', but w' does not want to add f, and f' does not want to add w'.

Recursively, given $w_k \in W'$ with $\mu'(w_k)P(w_k)\mu(w_k)$ let f_{k+1}, w_{k+1}, f_{k+1} be f, w' and f' obtained as above. Consider the sequence of alternating workers and firms constructed: W' is a finite set, so there must exist k and l such that $w_k = w_l$. Say $l < k$; set $\hat{w}_0 = w_l$, and $(\hat{w}_{k'}, \hat{f}_{k'}) = (w_{k'+l}, f_{k'+l})$ for $k' = 0, 1, \ldots k - l$. The resulting sequence is an acceptance-rejection cycle for μ.

13. Proof of Theorems 31, 32, and 35

13.1. **Proof of Theorem 31.** We first establish some lemmas:

Lemma 50. *Let P be substitutable. Let μ and μ' be pre-matchings. If $\mu \leq^B \mu'$ then, for all $w \in W$ and $f \in F$, $U(f, \mu) \subseteq U(f, \mu')$, and $V(w, \mu) \supseteq V(w, \mu')$.*

Proof. We shall prove that $V(w, \mu) \supseteq V(w, \mu')$. The proof that $U(f, \mu) \subseteq U(f, \mu')$ is analogous.

First, if $V(w, \mu') = \{\emptyset\}$, then there is nothing to prove, as $\{\emptyset\} = V(w, \mu') \subseteq V(w, \mu)$. Suppose that $V(w, \mu') \neq \{\emptyset\}$, and let $f \in V(w, \mu')$. Then, $w \in Ch(\mu'(f) \cup w, P(f))$.

But $\mu \leq^B \mu'$, so the definition of $\leq^B$ implies that, for all $f \in F$, either $\mu'(f) = \mu(f)$ so $w \in Ch(\mu(f) \cup w, P(f))$, or $\mu'(f) = Ch(\mu'(f) \cup \mu(f), P(f))$. Then $w \in Ch(\mu'(f) \cup w, P(f))$ implies that

$$\begin{aligned} w &\in Ch(\mu'(f) \cup w, P(f)) \\ &= Ch(Ch(\mu'(f) \cup \mu(f), P(f)) \cup w, P(f)) \\ &\overset{(1)}{=} Ch(\mu'(f) \cup \mu(f) \cup \{w\}, P(f)). \end{aligned}$$

Equality (1) is from Proposition 2.3 in Blair (1988) (Blair proves that, if P is substitutable, then $Ch(A \cup B, P(f)) = Ch(Ch(A, P(f)) \cup B, P(f))$

for all A and B). Substitutability of P implies that $w \in Ch\left(\mu(f) \cup w, P(f)\right)$. Then $f \in V\left(w, \mu\right)$, and thus $V(w,\mu) \supseteq V(w,\mu')$. ∎

Lemma 51. *Let P be strongly substitutable. Let μ and μ' be pre–matchings. If $\mu \leq \mu'$ then, for all $w \in W$ and $f \in F$, $U(f,\mu) \subseteq U(f,\mu')$, and $V(w,\mu) \supseteq V(w,\mu')$.*

Proof. We shall prove that $V(w,\mu) \supseteq V(w,\mu')$. The proof that $U(f,\mu) \subseteq U(f,\mu')$ is analogous.

First, if $V(w,\mu') = \{\emptyset\}$, then there is nothing to prove. Suppose that $V(w,\mu') \neq \{\emptyset\}$, and let $f \in V(w,\mu')$. Then, $w \in Ch(\mu'(f) \cup w, P(f))$. Strong substitutability implies then $w \in Ch(\mu(f) \cup w, P(f))$, as $\mu'(f)R(f)\mu(f)$ because $\mu \leq \mu'$. ∎

Let $\mathcal{V}' = \{\nu \in \mathcal{V} : \nu(s)R(s)\emptyset, \text{ for all } s \in F \cup W\}$. We need to work on the set $\mathcal{V}'$ instead of $\mathcal{V}$ because ν_0 and ν_1 are the smallest and largest, respectively, elements of $\mathcal{V}'$. Note that $T(\mathcal{V}) \subseteq \mathcal{V}'$, so there is no loss in working with $\mathcal{V}'$.

Lemma 52. *$T|_{\mathcal{V}'}$ is monotone increasing, when $\mathcal{V}'$ is endowed with orders $\leq^B$ or $\leq$.*

Proof. We show that $T|_{\mathcal{V}'}$ is monotone increasing when $\mathcal{V}'$ is endowed with order $\leq^B$. That is, whenever $\mu \leq^B \mu'$ we have $(T\mu) \leq^B (T\mu')$. The proof for $\leq$ follows along the same lines, using Lemma 51 instead of 50.

Let $\mu \leq^B \mu'$, and fix $f \in F$ and $w \in W$. Lemma 50 says that $U(f,\mu) \subseteq U(f,\mu')$. We first show that

$$Ch\left(U\left(f,\mu'\right), P\left(f\right)\right) = Ch\left(\left[Ch\left(U\left(f,\mu'\right), P\left(f\right)\right) \cup Ch\left(U\left(f,\mu\right), P\left(f\right)\right)\right], P(f)\right). \tag{37}$$

To see this, let $S \subseteq Ch\left(U\left(f,\mu'\right), P\left(f\right)\right) \cup Ch\left(U\left(f,\mu\right), P\left(f\right)\right)$. Then $S \subseteq U(f,\mu) \cup U(f,\mu') = U(f,\mu')$, so $Ch\left(U\left(f,\mu'\right), P\left(f\right)\right) R(f) S$. But, $Ch\left(U\left(f,\mu'\right), P\left(f\right)\right) \subseteq Ch\left(U\left(f,\mu'\right), P\left(f\right)\right) \cup Ch\left(U\left(f,\mu\right), P\left(f\right)\right)$, so we have established statement 37.

Now, $(T\mu')(f) = Ch\left(U\left(f,\mu'\right), P\left(f\right)\right)$ and $(T\mu)(f) = Ch\left(U\left(f,\mu\right), P\left(f\right)\right)$, so statement 37 implies that

$$(T\mu')(f) = Ch(\left[(T\mu')(f) \cup (T\mu)(f)\right], P(f))). \tag{38}$$

The proof for $(T\mu')(w)$ is analogous. ∎

Now $T|_{\mathcal{V}'} : \mathcal{V}' \to \mathcal{V}'$ is monotone increasing, and $\mathcal{V}'$ is a lattice (Remark 1). $T(\mathcal{V}) \subseteq \mathcal{V}'$ so $\mathcal{E} \subseteq \mathcal{V}'$, and $\mathcal{E}$ equals the set of fixed points of $T|_{\mathcal{V}'}$. So Tarski's fixed point theorem implies that $\langle \mathcal{E}, \leq^B \rangle$ and $\langle \mathcal{E}, \leq \rangle$ are non-empty lattices. Item (2) in Theorem 31 follows from standard results (Topkis, 1998, Chapter 4).

This finishes the proof of Theorem 31.

13.2. **Proof of Theorem 32.** We first prove item 1.

Let $\nu, \nu' \in \mathcal{E}$ be such that $\nu'(w)R(w)\nu(w)$ for all $w \in W$. Suppose, by way of contradiction, that there is some $\overline{f} \in F$ such that $\nu'(\overline{f})P(\overline{f})\nu(\overline{f})$. Let $C = Ch(\nu(\overline{f}) \cup \nu'(\overline{f}), P(\overline{f}))$, so $CR(\overline{f})\nu'(\overline{f})P(\overline{f})\nu(\overline{f})$. But $\nu \in \mathcal{E}$ implies that $\nu(\overline{f}) = Ch(\nu(\overline{f}), P(\overline{f}))$ (Lemma 38), so $C \nsubseteq \nu(\overline{f})$. Hence there is $\overline{w} \in C \backslash \nu(\overline{f})$; note that $w \in \nu'(f)$. Now

$$\overline{w} \in C = Ch\left(\nu(\overline{f}) \cup \nu'(\overline{f}) \cup \overline{w}, P(\overline{f})\right)$$

and substitutability of $P(\overline{f})$ implies that $\overline{w} \in Ch(\nu(\overline{f}) \cup \overline{w}, P(\overline{f}))$. So $\overline{f} \in V(\overline{w}, \nu)$.

Now, $\overline{w} \in \nu'(\overline{f}) \backslash \nu(\overline{f})$ implies $\overline{f} \in \nu'(\overline{w}) \backslash \nu(\overline{w})$. Then $\nu'(\overline{w})R(\overline{w})\nu(\overline{w})$ implies $\nu'(\overline{w})P(\overline{w})\nu(\overline{w})$, as $P(\overline{w})$ is strict. But $\nu'(\overline{w}) = Ch(\nu'(\overline{w}), P(\overline{w})) = Ch(\nu'(\overline{w}) \cup \overline{f}, P(\overline{w}))$ by Lemma 38. So strong substitutability implies that $\overline{f} \in Ch(\nu(\overline{w}) \cup \overline{f}, P(\overline{w}))$. Since $\overline{f} \notin \nu(\overline{w})$, we obtain $\nu(\overline{w}) \cup \overline{f} P(\overline{w})\nu(\overline{w})$. A contradiction with $\nu \in \mathcal{E}$, since we showed $\overline{f} \in V(\overline{w}, \nu)$ and $\nu \in \mathcal{E}$ implies $\nu(\overline{w}) = Ch(V(\overline{w}, \nu), P(\overline{w}))$.

To prove item 2 in the theorem, note that when $P(F)$ is strongly substitutable the model is symmetric, and the argument above holds with firms in place of workers, and workers in place of firms.

13.3. **Proof of Theorem 35.** We first prove that $\langle \mathcal{E}, \leq \rangle$ is a sublattice of $\langle \mathcal{V}, \leq \rangle$. That $\langle \mathcal{E}, \leq \rangle$ is distributive follows then immediately. We need to verify that the lattice operations $\vee$ and $\wedge$ in $\mathcal{V}$ are the lattice operations in $\langle \mathcal{E}, \leq \rangle$.

Let $\nu^1, \nu^2 \in \mathcal{E}$. Let $\nu = \nu^1 \vee \nu^2$ in $\mathcal{V}$. We shall prove that ν is the join of ν^1, ν^2 in $\langle \mathcal{E}, \leq \rangle$. The proof for $\nu^1 \wedge \nu^2$ is analogous.

By hypothesis ν is a matching; so

$$w \in \nu(f) \Leftrightarrow f \in \nu(w).$$

We prove that $\nu \in \mathcal{E}$. Suppose, by way of contradiction, that there is $\overline{f}$ such that $(T\nu)(\overline{f}) \neq \nu(\overline{f})$. Without loss of generality, say that $\nu(\overline{f}) = \nu^1(\overline{f})R(\overline{f})\nu^2(\overline{f})$. Since $\nu^1 \in \mathcal{E}$, ν^1 is individually rational (Lemma 38), so $\overline{f} \in Ch(\nu^1(w), P(w)) = Ch(\nu^1(w) \cup \overline{f}, P(w))$, for all $w \in \nu^1(\overline{f})$. For all w, on the other hand, $\nu^1(w)R(w)\nu(w)$. So strong substitutability gives $\overline{f} \in Ch(\nu(w) \cup \overline{f}, P(w))$ for all $w \in \nu^1(\overline{f})$. Thus $\nu^1(\overline{f}) \subseteq U(\overline{f}, \nu)$. Since, $(T\nu)(\overline{f}) = Ch(U(\overline{f}, \nu), P(\overline{f}))$, and ν^1 is individually rational, $(T\nu)(\overline{f}) \backslash \nu(\overline{f}) \neq \emptyset$.

Let $\overline{w} \in (T\nu)(\overline{f}) \backslash \nu(\overline{f})$. By substitutability, $\overline{w} \in Ch(\nu^1(\overline{f}) \cup \overline{w}, P(\overline{f}))$. Strong substitutability and $\nu^1(\overline{f})R(\overline{f})\nu^2(\overline{f})$ imply then $\overline{w} \in Ch(\nu^2(\overline{f}) \cup$

$\overline{w}, P(\overline{f}))$. So

$$\overline{f} \in V(\overline{w}, \nu^i), \tag{39}$$

$i = 1, 2$.

On the other hand $\overline{w} \in (T\nu)(\overline{f})$ implies $\overline{w} \in U(\overline{f}, \nu)$ so

$$\overline{f} \in Ch(\nu(\overline{w}) \cup \overline{f}, P(\overline{w})). \tag{40}$$

Let i be such that $\nu(\overline{w}) = \nu^i(\overline{w})$. Then Statement (39), and $\nu^i \in \mathcal{E}$, implies $\nu^i(\overline{w}) \cup \overline{f} \in V(\overline{w}, \nu^i)$.

But we assumed $\overline{w} \notin \nu(\overline{f})$, so $\overline{f} \notin \nu^i(\overline{w})$, as ν is a matching. Then $\nu^i(\overline{w}) \cup \overline{f} \neq \nu^i(\overline{w})$: a contradiction with $\nu^i \in \mathcal{E}$, given Statement (40) and that $\nu(\overline{w}) \cup \overline{f} \in V(\overline{w}, \nu^i)$.

For the rest of the theorem, we need a lemma.

Lemma 53. *Let P be strongly substitutable. For all f and w, for any ν and ν' in $\mathcal{V}$: $U(f, \nu \vee \nu') = U(f, \nu) \cup U(f, \nu')$, $U(f, \nu \wedge \nu') = U(f, \nu) \cap U(f, \nu')$, $V(w, \nu \vee \nu') = V(w, \nu) \cap V(w, \nu')$, and $V(w, \nu \wedge \nu') = V(w, \nu) \cup V(w, \nu')$.*

Proof. We shall only prove that $U(f, \nu \vee \nu') = U(f, \nu) \cup U(f, \nu')$, and that $V(w, \nu \vee \nu') = V(w, \nu) \cap V(w, \nu')$. The proof of the other statements is symmetric.

We shall first prove that $U(f, \nu \vee \nu') \subseteq U(f, \nu) \cup U(f, \nu')$. Let $w \in U(f, \nu \vee \nu')$, so $f \in Ch\left((\nu \vee \nu')(w) \cup f, P(w)\right)$. Now, $(\nu \vee \nu')(f)$ equals either $\nu(f)$ or $\nu'(f)$. If $(\nu \vee \nu')(w) = \nu(w)$, then $f \in Ch\left(\nu(w) \cup f, P(w)\right)$; so $w \in U(f, \nu)$. Similarly, if $(\nu \vee \nu')(w) = \nu'(w)$, then $w \in U(f, \nu')$. This proves that $U(f, \nu \vee \nu') \subseteq U(f, \nu) \cup U(f, \nu')$.

Second, we prove that $U(f, \nu) \cup U(f, \nu') \subseteq U(f, \nu \vee \nu')$. Let $w \in U(f, \nu)$, so $f \in Ch\left(\nu(w) \cup f, P(w)\right)$. Now $\nu(w) R(w) (\nu \vee \nu')(w)$, so strong substitutability implies $f \in Ch\left((\nu \vee \nu')(w) \cup f, P(w)\right)$. Hence $w \in U(f, \nu \vee \nu')$. This proves that $U(f, \nu) \cup U(f, \nu') \subseteq U(f, \nu \vee \nu')$. So, $U(f, \nu \vee \nu') = U(f, \nu) \cup U(f, \nu')$.

We shall now prove that $V(w, \nu \vee \nu') = V(w, \nu) \cap V(w, \nu')$. First we prove $V(w, \nu \vee \nu') \subseteq V(w, \nu) \cap V(w, \nu')$. Let $f \in V(w, \nu \vee \nu')$, so

$$w \in Ch\left((\nu \vee \nu')(f) \cup w, P(f)\right). \tag{41}$$

Without loss of generality, say $(\nu \vee \nu')(f) = \nu(f) R(f) \nu'(f)$. Then $(\nu \vee \nu')(f) = \nu(f)$ implies that $f \in V(w, \nu)$. Statement 41, and strong substitutability imply $w \in Ch\left(\nu'(f) \cup w, P(f)\right)$, as $(\nu \vee \nu')(f) R(f) \nu'(f)$. Thus $f \in V(w, \nu)$, and we obtain $V(w, \nu \vee \nu') \subseteq V(w, \nu) \cap V(w, \nu')$.

Finally, we prove that $V(w, \nu) \cap V(w, \nu') \subseteq V(w, \nu \vee \nu')$. Let Let $f \in V(w, \nu) \cap V(w, \nu')$, so $w \in Ch\left(\nu(f) \cup w, P(f)\right)$ and $w \in Ch\left(\nu'(f) \cup w, P(f)\right)$.

Now, $(\nu \vee \nu')(w)$ equals either $\nu(w)$ or $\nu'(w)$, so either way $w \in Ch\left((\nu \vee \nu')(f) \cup w, P(f)\right)$. Hence $f \in V(w, \nu \vee \nu')$. ∎

Lemma 53 implies immediately that ψ is a lattice homomorphism: Let $\nu', \nu \in \mathcal{V}$. For any f and w,

$$(\psi(\nu \vee \nu'))(f) = U(f, \nu \vee \nu') = U(f, \nu) \cup U(f, \nu') = (\psi\nu)(f) \cup (\psi\nu')(f)$$
$$(\psi(\nu \vee \nu'))(w) = V(w, \nu \vee \nu') = V(w, \nu) \cap V(w, \nu') = (\psi\nu)(f) \cap (\psi\nu')(f).$$

So $\psi(\nu \vee \nu') = \psi\nu \sqcup \psi\nu'$. That $\psi(\nu \wedge \nu') = \psi\nu \sqcap \psi\nu'$ is also trivial from Lemma 53.

We now show that $\psi|_{\mathcal{E}}$ is an isomorphism onto its range. Let $\nu, \nu' \in \mathcal{E}$. Let $\psi\nu = \psi\nu'$. Then, for all f, $U(f, \nu) = U(f, \nu')$ so $(T\nu)(f) = (T\nu')(f)$. Similarly $(T\nu)(w) = (T\nu')(w)$ for all w. So $T\nu = T\nu'$ Then $\nu, \nu' \in \mathcal{E}$ imply $\nu = \nu'$, as $v = T\nu$ and $v' = T\nu'$. Hence ψ is one-to-one, as $\psi\nu = \psi\nu'$ implies $\nu = \nu'$.

References

Adachi, H. (2000): "On a Characterization of Stable Matchings," *Economic Letters*, 68, 43–49.

Alcalde, J., D. Pérez-Castrillo, and A. Romero-Medina (1998): "Hiring Procedures to Implement Stable Allocations," *Journal of Economic Theory*, 82(2), 469–480.

Alcalde, J., and A. Romero-Medina (2000): "Simple Mechanisms to Implement the Core of College Admissions Problems," *Games and Economic Behavior*, 31(2), 294–302.

Barberá, S., H. Sonnenschein, and L. Zhou (1991): "Voting by Committees," *Econometrica*, 59(3), 595–609.

Blair, C. (1988): "The Lattice Structure of the Set of Stable Matchings with Multiple Partners," *Mathematics of Operations Research*, 13(4), 619–628.

Dutta, B., and J. Massó (1997): "Stability of Matchings When Individuals Have Preferences over Collegues," *Journal of Economic Theory*, 75(2), 464–475.

Echenique, F., and J. Oviedo (2003): "Core Many-to-one Matchings by Fixed Point Methods," Forthcoming, Journal of Economic Theory.

Fleiner, T. (2001): "A Fixed-Point Approach to Stable Matchings and Some Applications," Technical Report, Egervary Research Group on Combinatorial Optimization.

Kelso, A., and V. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504.

Klijn, F., and J. Massó (2003): "Weak Stability and a Bargaining Set for the Marriage Model," *Games and Economic Behavior*, 42(1), 91–100.

Knuth, D. E. (1976): *Marriages Stable.* Université de Montréal Press, Translated as "Stable Marriage and Its Relation to Other Combinatorial Problems," CRM Proceedings and Lecture Notes, American Mathematical Society.

Konishi, H., and U. Ünver (2003): "Credible Group-Stability in General Multi-Partner Matching Problems," Working Paper # 570 Boston College.

Martínez, R., J. Massó, A. Neme, and J. Oviedo (2000): "Single Agents and the Set of Many-to-One Stable Matchings," *Journal of Economic Theory*, 91, 91–105.

——— (2001): "On the Lattice Structure of the Set of Stable Matchings for a Many-to-one Model," *Optimization*, 50, 439–457.

——— (2003): "An Algorithm to Compute the Full Set of Many-to-Many Stable Matchings," Forthcoming in Mathematical Social Sciences.

Milgrom, P. (2003): "Matching with Contracts," Mimeo, Stanford University.

Roth, A., and M. Sotomayor (1988): "Interior Points in the Core of Two-Sided Matching Markets," *Journal of Economic Theory*, 45, 85–101.

——— (1990): *Two-sided Matching: A Study in Game-Theoretic Modelling and Analysis*, vol. 18 of *Econometric Society Monographs.* Cambridge University Press, Cambridge England.

Roth, A. E. (1984): "Stability and Polarization of Interests in Job Matching," *Econometrica*, 52(1), 47–57.

——— (1985): "Conflict and Coincidence of Interest in Job Matching: Some New Results and Open Questions," *Mathematics of Operations Research*, 10(3), 379–389.

Roth, A. E., and E. Peranson (1999): "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review*, 89(4), 748–780.

Sönmez, T. (1996): "Strategy-proofness in Many-to-one Matching Problems," *Economic Design*, 1(4), 365–380.

Sotomayor, M. (1999): "Three Remarks on the Many-to-many Stable Matching Problem," *Mathematical Social Sciences*, 38, 55–70.

Topkis, D. M. (1998): *Supermodularity and Complementarity.* Princeton University Press, Princeton, New Jersey.

Zhou, L. (1994): "A New Bargaining Set of an N-Person Game and Endogenous Coalition Formation," *Games and Economic Behavior*, 6(3), 512–526.

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