Willpower and the Optimal Control of Visceral Urges*

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Abstract

Psychologists and behavioral economists have documented that individuals often make intertemporal choices that are inconsistent with the conventional economic model. To explain intertemporal choices better, psychologists have offered, and provided experimental support for, a "willpower depletion" model which predicts that a person who exerts self-discipline in one activity will behave as if he has less self-discipline available to exert in other activities. We formalize a willpower depletion model and investigate how willpower constraints affect the canonical problem of how to divide a cake (or paycheck or workload) over time to maximize utility. We find that a consumer behaving optimally subject to willpower constraints acts in ways that others have described as anomalous. This consumer reveals a preference for increasing paths of consumption, a preference for commitment, and time-inconsistency in preferences. We also study the optimal allocation of willpower between the intertemporal saving activity and other activities that require self-discipline. Finally, we show how the ability to build willpower by its exercise further influences the optimal path of consumption.

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1 Introduction

Although consumers making intertemporal choices are conventionally modeled as maximizing an exponentially-weighted sum of additively-separable, stationary utilities, psychologists and behavioral economists have recently pointed out that, in both the field and the laboratory, people behave in ways that model cannot explain. For example, people often prefer increasing sequences of consumption and, when commencing a new project, often prefer to get some hard work out of the way at the outset. In addition, in many settings consumers seem to exhibit time-inconsistent behavior and to procrastinate severely. *Time and Decision* (2003) edited by George Loewenstein, Roy F. Baumeister and Daniel Read, constitutes an up-to-date catalogue of such "anomalies." Most of the contributions suggest, as the editors note, that "the road to understanding intertemporal choice is not through developing better discount functions but through understanding the variety of psychological processes that enter into future-based decision making ...." (Loewenstein *et al.,* 2003).

One chapter in this collection surveys experiments conducted by Baumeister and colleagues over a decade to test the so-called "willpower depletion" or "strength depletion" model. According to this model, if a person exerts self-discipline in one activity, he behaves as if he has less self-discipline available to exert in other activities. Even if these activities are completely unrelated, therefore, how a person behaves in each of them is linked.

Anecdotes consistent with the existence of such linkages come readily to mind. It is commonplace that people quitting smoking violate their diets. People often temporarily abandon diet or smoking regimes when preparing for an exam or an important presentation. Profligate spending or drinking to excess is often the "reward" for a hard week at work, even when that work generated no extra wealth.

Baumeister and his colleagues have gone beyond such anecdotes, however, and demonstrated such linkages experimentally. Their experiments have two phases. Every subject in the experiment participates in the second phase but only a subset of the subjects, randomly selected, participates in the first phase; the remainder is used as the control group. In the first phase, subjects are asked to perform some task that depletes their willpower; in the second phase, their endurance in an entirely unrelated activity is measured. For example, in the first phase subjects have been asked, while reporting every thought that occurs to them, to suppress thoughts of white bears. This nearly impossible task of thought suppression—which reportedly preoccupied Tolstoy, Dostoyevsky and other Russian writers (Wegner, 1994)—erodes a subject's willpower; when asked in the second phase to solve insoluble anagrams or geometric puzzles or to squeeze the handgrip of a muscle exerciser, these subjects give up long before their counterparts in the control group. The experiments have used a variety of other first-phase activities to erode willpower. Besides thought suppression, subjects have been asked to regulate their emotions, to divert their attention from movie subtitles, to restrain their appetites in the presence of tempting food, and so forth. As Baumeister and Vohs summarize the results: "... our findings very consistently supported the willpower theory—that is, performing a first act of self-regulation weakened people's ability to regulate themselves subsequently. This pattern was found repeatedly, with many different manipulations and measures of self-regulation." (Baumeister and Vohs, 2003.)

Recently, willpower depletion has also been shown to have effects on impulsive buying. In one experiment (Vohs and Faber, 2004) members of the treatment group were given the standard task of suppressing thoughts of white bears, paid \$10, dismissed and then offered the immediate opportunity to purchase items from the campus bookstore displayed on a table in the lab. A control group which was also given \$10 and the same opportunity, did not have to tax themselves with thought suppression. Subjects in the treatment group chose to buy more items and to spend more total dollars than their counterparts in the control group. Reviewing this and other studies, Baumeister wrote "Self-control research seemingly has much to offer researchers interested in consumer behavior... [T]he processes that undermine self-control should lead to more buying and more impulsive buying." (Baumeister, 2002.)

Given the experimental and anecdotal evidence that a willpower constraint may have important economic implications, and given the fact that, in the absence of such constraints, maximization of the conventional objective function cannot explain a set of observed behaviors, it is natural to consider how behavior in the conventional formulation would change if willpower constraints were introduced. In this paper, we formalize a willpower depletion model and investigate how willpower constraints would affect the canonical problem of how to divide a cake (or paycheck or workload) over a fixed time horizon to maximize utility. In our formulation of the cake-eating problem, the greater restraint the consumer exercises the faster his willpower erodes. In addition, to capture Baumeister's view that "further persistence demands ever greater exertions of willpower," (Baumeister *et al.,* 1994) we assume that a given level of consumption depletes willpower at a faster rate when a person's reserves of willpower are lower.

A consumer behaving optimally subject to binding willpower constraints will act in ways that others have characterized as "anomalous." For example, even with positive discounting, the agent may prefer a path of consumption which *increases* over time. Equivalently, if the consumption good is leisure, the consumer may *prefer* to work hard *early* and enjoy more leisure as time passes (although, beyond some point, the agent must work incessantly to accomplish what he has deferred).[1]

[1] Fischer (2001) finds that the discount factor required to explain procrastination in a model where geometrically-

Loewenstein and Prelec (1993) and others have observed preference for improving sequences in the laboratory, even among agents who—when asked which of two items they want to consume first—prefer the one generating the larger utility. Such behavior could be exhibited by the consumer in our model.

The modified model may explain other "anomalies" as well. The existence of savings clubs which hold a person's money and, for a fee, dole it back to him gradually over time has been regarded as inconsistent with the standard model. But our consumer would strictly prefer to have his "cake" or paycheck doled out to him by a "savings club" rather than have the entire amount available in his wallet where he would need to use up scarce willpower to resist spending it. Indeed, he would be willing to pay a fee for this service.

As is well-known, time-inconsistent behavior cannot occur in a model where a single agent maximizes exponentially discounted, additively separable utilities. Suppose, however, that to verify this theoretical prediction experimentally we interrupted our consumer in the middle of his optimal program, administered a short questionnaire, and let him re-optimize starting with the cake remaining. His behavior would change because his effort to complete the questionnaire would deplete his willpower. Since willpower is unobservable, this change in behavior would have the appearance of time-inconsistency.

We also consider the allocation of willpower between this intertemporal saving activity and *other* activities (e.g. cramming for exams, training for musical performances, or preparing an important presentation) that require self-discipline. We show that the optimal allocation almost never results in consumption smoothing over the entire horizon, even when it is feasible. Moreover, this formulation permits us to investigate how two agents, who differ only in the size of their initial cakes, would choose to consume in the first phase if they recognized that utility from the alternative activity depends on the amount of willpower held back during cake eating. We provide an example where the poorer agent consumes at the same rate as his richer counterpart, until his smaller cake runs out, and then transfers a larger stock of willpower for use in the alternative activity. We thus show how the poor may, in some domains, appear to exert less self-discipline, not because they have different preferences, willpower endowments or self-discipline technology but merely because they are poor.

Finally, we consider the intuitive idea that the exercise of self control in the present, while depleting willpower, may also build willpower reserves in the future. We find that the opportunity to build future willpower by constraining current consumption distorts the intertemporal profile

discounted, additively-separable utility is maximized is unrealistic; our model generates procrastination (zero consumption of leisure for the final phase of the planning horizon) for any discount rate.

of consumption; renewable willpower creates a time preference even when the consumption would have been constant (when positive), were willpower non-renewable.

Our paper contributes to the recent literature on the economics of self-regulation. See, e.g., Laibson (1997), O'Donoghue and Rabin (1999), Gul and Pesendorfer, (2001), Benhabib and Bisin (2004), and Bernheim and Rangel (2004). Relative to the literature, our analysis is distinguished by an explicit model of willpower depletion and regeneration, a focus on the dynamics of willpower management, and a detailed consideration of the optimal allocation of willpower across competing activities. We offer further discussion of the self-regulation literature, and our contribution to it, in section 6.

In the next section we present our model of consumption over time with limited willpower and derive and interpret the first-order conditions characterizing the optimal program. Section 3 describes qualitative properties of the optimal program including behavior exhibiting time preference in the absence of time discounting. In section 4 we study the optimal allocation of willpower across multiple activities. In section 5 we analyze a model where willpower is renewed by the exercise of self-discipline. Section 6 discusses the related economics literature. In section 7 we show how relationships central to our predictions can be illuminated by new experiments. In section 8 we conclude.

2 Consumption with Limited Willpower

In the canonical cake-eating problem in continuous time, a consumer maximizes his discounted utility by choosing his consumption path $c(t)$ over a fixed horizon ($t \in [0, T]$). We denote the size of the cake at time t as $R(t)$ and assume that $R(0)$ is given. The rate of decline in the cake at t (denoted $-\dot{R}(t)$) is, therefore, $c(t)$.

We depart from the canonical model by assuming that the agent is endowed with a given stock of willpower $W(0)$ and depletes it when he restrains his consumption. We denote the rate of willpower depletion as $f(W(t), c(t))$. Allowing this depletion function to depend on willpower reserves captures Baumeister's observation that the same restraint depletes willpower at a faster rate when one's willpower reserves are lower. Because neither experiments nor introspection suggests the sign of the relationship, we assume that $f(\cdot)$ is not affected by the stock of cake remaining $(R(t))$.[2]

An important feature of willpower depletion is that as long as even a morsel of cake remains, one has to use willpower to consume nothing. But consuming nothing requires no willpower when there

[2] Assuming $f(\cdot)$ is not a function of $R(t)$ simplifies our analysis. If, however, f were increasing (decreasing) in $R(t)$, we conjecture that this would mitigate (reinforce) the agent's incentive to increase consumption over time.

is nothing left to eat. We refer to this as the "fundamental discontinuity of willpower depletion"and take account of it in our formulation.[3]

In anticipation of the analysis in Section 4, we assume that any willpower remaining after the conclusion of intertemporal consumption is used in an alternative activity and generates additional utility $m(\cdot)$.

Since the agent is not permitted to choose a consumption path which results in negative willpower, the willpower constraint may preclude the path which exhausts the cake while equalizing discounted marginal utility up through T. We refer to that path, which is the hallmark of the canonical model, as "perfect smoothing." Even when perfect smoothing is feasible, the agent may choose to forgo it. Our goal throughout is to investigate how the presence of the willpower constraint alters the agent's chosen consumption plan relative to the predictions of the canonical model.

2.1 Formulation of the Model

The agent chooses $c(t)$ to maximize

$$\begin{aligned}
& V(0) = \int_0^T e^{-\rho t} U[c(t)]\,dt + e^{-\rho t'} m\left(W(t')\right) \\
\text{subject to} \quad & \dot{R}(t) = -c(t) \qquad\qquad (P1) \\
& \dot{W}(t) = \begin{cases} -f(W(t), c(t)) & \text{if } R(t) > 0 \\ 0, \text{ otherwise} \end{cases} \\
& R(T) \geq 0,\ W(T) \geq 0 \\
& R(0) = \overline{R} \geq 0 \\
& W(0) = \overline{W} \geq 0
\end{aligned}$$

where ρ is the subjective rate of time discount and $t' = \sup\{t \in [0,T] : R(t) > 0\}$. Note that the law of motion for willpower is discontinuous reflecting the fundamental discontinuity of willpower depletion discussed above.

We make the following assumptions on the willpower technology f and the utility function U. We assume that, for all W, $f(W,c) > 0$ for $c \in [0,\bar{c})$ and $f(W,c) = 0$ for $c \in [\bar{c},\infty)$ for some $\bar{c} > 0$. While following a given consumption path, if the willpower stock becomes zero before time T, consumption must weakly exceed $\bar{c}$ thereafter. Therefore, this assumption guarantees that the

[3] This discontinuity would persist even if $f(\cdot)$ were a function of $R(t)$ so long as $\lim_{R(t)\to 0} f(\cdot) > 0$. While it would simplify our analysis if the discontinuity conveniently disappeared in this way, there is no reason to expect that it does. Therefore, we show how the optimal consumption path can be determined even when the discontinuity remains.

set of feasible consumption paths is nonempty. We also assume that f is strictly decreasing and weakly convex in c for $c \in [0, \bar{c})$ and weakly decreasing in W. Thus, we assume that the more the agent restrains his consumption, the faster he depletes his willpower reserves; moreover, the same level of restraint may result in faster depletion of willpower if the agent's reserves of willpower are lower. We also assume that f is twice differentiable everywhere except at $\bar{c}$ and continuous at $\bar{c}$ and $f_{cW} \geq 0$. We assume $U(0) = 0$[4], $U(c)$ is differentiable, strictly increasing and strictly concave. We are interested in modeling non-addictive behaviors like those considered in the experiments by Baumeister and colleagues; so $U(c)$ is a function only of contemporaneous consumption, and not past consumption.

We now consider a related, but more tractable problem and argue that, by solving it, we solve problem $(P1)$. In the related problem, the agent chooses both an optimal consumption path $c(t)$ and an optimal horizon $s \leq T$, where $\dot{W} = c(t) = 0$ for all $t \in (s, T]$.

$$\begin{aligned} & V(0) = \int_0^s e^{-\rho t} U[c(t)]\, dt + e^{-\rho s} m(W(s)) \\ \text{subject to} \quad & \dot{R}(t) = -c(t) \\ & \dot{W}(t) = -f(W(t), c(t)) \\ & R(s) \geq 0, \ W(s) \geq 0 \\ & R(0) = \overline{R} \geq 0 \\ & W(0) = \overline{W} \geq 0 \end{aligned} \tag{P2}$$

In problem $(P1)$, the optimal consumption path either finishes the cake at time $t' < T$, or it does not. If the cake is exhausted at time $t' < T$, then the law of motion governing the depletion of willpower jumps to zero, and for any time $t \in (t', T]$, $\dot{W}(t) = c(t) = 0$. These same paths of consumption and willpower could be achieved in problem $(P2)$ by choosing $s = t'$, and would generate the same payoff. Similarly, if in the original problem $(P1)$ the cake is not exhausted before time T $(R(t) > 0$ for $t < T)$, these paths of consumption and willpower could also be achieved in the related problem by choosing $s = T$, and would generate the same payoff. Since the two problems share objective functions and laws of motion up to time $s = t'$, any program that is feasible in the original problem is also feasible in the related problem and will generate the same payoff.

Given that any consumption path that is feasible in problem $(P1)$ is also feasible in problem $(P2)$, if the optimal consumption path in problem $(P2)$ is feasible in problem $(P1)$ then it is also optimal in problem $(P1)$.[5] A sufficient condition for any consumption path in problem $(P2)$ to be

[4] As long as $U(0) > -\infty$, we can always renormalize the utility function U so that $U(0) = 0$.

[5] To see this, suppose that the optimal consumption path in problem $(P2)$ is feasible but not optimal in problem

feasible in problem $(P1)$ is that the consumption path depletes the cake by time s $(R(s) = 0)$. After all, that consumption path generates the same willpower path up to time s in both problems and therefore $W(t) \geq 0$ up to time s. After time s in problem $(P1)$, willpower remains constant (at $W(s)$) since cake is depleted. Hence any such consumption path is feasible in problem $(P1)$ as well.

To see that the optimal consumption path in $(P2)$ exhausts the cake at time s, suppose the contrary—that the "optimal" program leaves $R(s) > 0$ cake uneaten at s. To dominate this program, consider a different program that duplicates the "optimal" consumption path up to time $s - \Delta$, for any $\Delta > 0$, and consumes $\frac{R(s)}{\Delta}$ more during the remaining interval of length Δ. This exhausts the cake, draws willpower down to the same level at $s - \Delta$ and depletes less willpower during $(s - \Delta, s)$. Indeed, one can always choose Δ small enough that $\frac{R(s)}{\Delta} \geq \bar{c}$. In this case, depletion of willpower during $(s - \Delta, s)$ ceases altogether. Not only would utility from the alternative activity be weakly larger than on the "optimal" path but utility from intertemporal consumption would be strictly larger. This follows since the alternative consumption path is uniformly higher throughout and strictly higher from time $s - \Delta$ to s. This contradicts the claim that any feasible path with $R(s) > 0$ can be optimal. Hence, without loss of generality, we can confine our attention to problem $(P2)$.

The Hamiltonian for problem $(P2)$ is given by

$$H(c(t), R(t), W(t), t, \alpha(t), \lambda(t)) = e^{-\rho t} U(c(t)) - \alpha(t) c(t) - \lambda(t) f(W(t), c(t)).$$

To reduce notation, we shall refer to this Hamiltonian as $H(t)$ when no confusion arises. The first-order conditions are:

$$c(t) \geq 0,\ e^{-\rho t} U'(c(t)) - \alpha(t) - \lambda(t) f_c \leq 0 \text{ and c.s.} \tag{1}$$

$$\dot{W}(t) = -f \tag{2}$$

$$\dot{\alpha}(t) = 0 \tag{3}$$

$$\dot{\lambda}(t) = \lambda(t) f_W \tag{4}$$

$$T - s \geq 0,\ \ H(s) - \rho e^{-\rho s} m(W(s)) \geq 0 \text{ and c.s.} \tag{5}$$

$$R(s) \geq 0,\ \ \alpha(s) \geq 0 \text{ and c.s.} \tag{6}$$

$$W(s) \geq 0,\ \ \lambda(s) - m'(W(s)) \geq 0 \text{ and c.s.} \tag{7}$$

$(P1)$. Then there is a strictly preferred consumption path that is feasible in problem $(P1)$. But by our earlier argument this path is also feasible in problem $(P2)$ and would dominate the program we claimed was optimal in problem $(P2)$. Hence we have a contradiction.

It should be noted for future use that, whenever willpower is strictly positive, consumption varies continuously with time. This result follows because (1) the Hamiltonian is strictly concave in c; (2) α and λ vary continuously with time when $W(t) > 0$; (3) U' is continuous in c; (4) f_c is continuous in both c and W; and (5) W varies continuously with time.

The first-order conditions can be interpreted intuitively. When $c(t) > 0$, we can rewrite (1) as:

$$\underbrace{e^{-\rho t}U'(c(t))}_{\text{direct marginal benefit}} + \underbrace{\lambda(t)(-f_c)}_{\text{indirect marginal benefit}} = \underbrace{\alpha(t)}_{\text{marginal cost}} .$$

Consuming at a slightly faster rate at time t generates two marginal benefits and one marginal cost. The direct marginal benefit is that the utility flow at time t increases by $e^{-\rho t}U'(c(t))$. The indirect marginal benefit is that $-f_c$ units of willpower are saved for every unit of additional consumption and the increased willpower has a value of $\lambda(t)$ (in terms of utils at $t = 0$). The marginal cost of consuming at a faster rate at t is the utility lost ($\alpha(t)$) because the additional cake consumed at t can no longer be consumed at some other time. At an interior optimum ($c(t) > 0$), the sum of the two marginal benefits equals the marginal cost.

It is also instructive to interpret (4), which indicates that the imputed value of additional willpower at t ($\lambda(t)$) weakly declines. When $f_W = 0$, willpower is equally useful no matter when it arrives and so its imputed value is constant. But when $f_W < 0$, additional willpower is more valuable the earlier it arrives because the rate of depletion of willpower slows from the moment the additional increment arrives. In that case, $\dot{\lambda}(t) < 0$.

In the presence of willpower constraints, consumption may vary over time even in the absence of time discounting. Henceforth, suppose $\rho = 0$. If $c(t) > 0$ for $t \leq s$, then by condition (1)

$$\begin{aligned} U''(c)\,\dot{c} &= \dot{\alpha} + \dot{\lambda} f_c + \lambda f_{cc}\dot{c} + \lambda f_{cW}\dot{W} \\ &= \lambda f_W f_c + \lambda f_{cc}\dot{c} - \lambda f_{cW} f \\ &= \lambda\left(f_W f_c + f_{cc}\dot{c} - f_{cW} f\right) \end{aligned}$$

$\Longleftrightarrow$

$$\dot{c}\left[U''(c) - \lambda f_{cc}\right] = \lambda\left(f_W f_c - f_{cW} f\right)$$

$\Longleftrightarrow$

$$\dot{c} = \lambda(t)\,\frac{f_W f_c - f_{cW} f}{U''(c) - \lambda(t) f_{cc}} \tag{8}$$

Since U is strictly concave in c while f is weakly convex in c, the denominator of the right hand side of equation (8) is strictly negative. So if $\lambda = 0$, equation (8) yields the classical conclusion that consumption is constant as long as it is positive. On the other hand, if $\lambda > 0$, then when $c(t) > 0$, $\dot{c} \gtreqless 0$ as $(f_{cW} f - f_W f_c) \gtreqless 0$. Intuitively, consumption increases over time if and only if the indirect marginal benefit of additional consumption increases over time: $\frac{d}{dt}\lambda(-f_c) > 0$. In that case, the direct marginal benefit must decrease over time and this entails increasing consumption.

3 Optimal Consumption When Willpower Has No Alternative Use

We consider first the case where willpower has no alternative use besides the regulation of intertemporal consumption ($m(W) \equiv 0$). When willpower can only be used to regulate intertemporal consumption, perfect smoothing is optimal whenever feasible. Intuitively, when perfect smoothing is feasible, there is no shortage of willpower and $\lambda(t) = 0$ for $t \geq 0$. In the absence of discounting, perfect smoothing implies that consumption is constant over time for $t \in [0, T]$.

More generally, Proposition 1 describes the qualitative properties of the optimal consumption path when $\rho = m(W) = 0$.

Proposition 1 *Let W_H be the minimum level of initial willpower such that setting $c(t) = \frac{\overline{R}}{T}$ for $t \in [0, T]$ is feasible. Denote the optimal consumption path as $c^*(t)$. If $\overline{W} \geq W_H$ then the cake is exhausted, and $c^*(t) = \frac{\overline{R}}{T}$ for $t \in [0, T]$. If $\overline{W} < W_H$ then both the cake and willpower are exhausted ($R(s) = W(s) = 0$), and when consumption is strictly positive it is strictly increasing (resp. constant, strictly decreasing) if and only if $f_{cW} f - f_W f_c$ is strictly positive (resp. zero, strictly negative).*

Proof. From (3) $\alpha(t)$ is a constant function, and with a slight abuse of notation we denote this constant as $\alpha \geq 0$.

First assume $\overline{W} \geq W_H$. Consider the case where $\lambda(t) = 0$ for all $t \in [0, s]$. Since $U'(\cdot) > 0$, (1) requires $\alpha > 0$ and $c(t)$ constant. With a slight abuse of notation, we denote this constant as $c \geq 0$. Since $\alpha > 0$, (6) requires $R(s) = 0$. Since $R(0) = \overline{R} > 0$, then $c > 0$. If $T - s > 0$, (5) would require $U(c) - U'(c)c = 0$. But since $U'(\cdot) > 0$ and $U''(\cdot) < 0$ this can not occur for $c > 0$. It follows that when $\lambda(t) = 0$ for all $t \in [0, s]$ then $s = T$ and $c = \frac{\overline{R}}{T}$. By definition of W_H, $W(T) \geq 0$. Thus (7) is satisfied. This proves the first statement in the proposition.

Now assume $\overline{W} < W_H$. Then $\lambda(t) > 0$ for all $t \in [0, s]$,for suppose to the contrary that $\lambda(t) = 0$ for some $t \in [0, s]$. Equation (4) implies that $\lambda(t)$ is weakly decreasing and can be written as:

$$\lambda(t) = \lambda(0)\, e^{\int_{n=0}^{t} f_W(W(n), c(n)) dn}.$$

Since $e^{\int_{n=0}^{t} f_W(W(n), c(n)) dn} > 0$ for all t, $\lambda(t) = 0$ for some $t \in [0, s]$, implies that $\lambda(t) = 0$ for all $t \in [0, s]$. But as seen above the conditions above then imply that $c = \frac{\overline{R}}{T}$ which is infeasible when

$\overline{W} < W_H$. So if $\overline{W} < W_H$ then $\lambda(t) > 0$ for all $t \in [0, s]$, and by (7) $W(s) = 0$. To satisfy (1) with $U'(\cdot) > 0$ and $-\lambda f_c > 0$ requires $\alpha > 0$; and, again, since $\alpha > 0$, (6) requires that the cake is entirely consumed $(R(s) = 0)$. In this case, by our earlier observations, $(f_{cW} f - f_W f_c) \gtreqless 0$ for all $t \in [0, s]$ if and only if $\dot{c} \gtreqless 0$ for all t where $c(t) > 0$. This concludes the proof of the proposition. ■

To illustrate, consider any act of will function of the following form:

$$f = A + K(W)\, g(c) \text{ with } K' \leq 0 \text{ and } g' < 0.$$

Note that,

$$\begin{aligned} f_W &= K'g \\ f_c &= Kg' \\ f_{cW} &= K'g' \end{aligned}$$

Thus,

$$\begin{aligned} \text{sign}(\dot{c}) &= \text{sign}\left(-K'gKg' + K'g'(A + Kg)\right) \\ \text{sign}(\dot{c}) &= \text{sign}\left(K'g'A\right) \end{aligned}$$

Since g is strictly decreasing and K is weakly decreasing, there are two possibilities. At any t where $c(t) > 0$, $\dot{c} = 0$ if $K' = 0$ and, if $K' < 0$ then $\dot{c} \gtreqless 0$ as $A \gtreqless 0$.

4 Optimal Consumption When Willpower Has Alternative Uses

We next consider the case were willpower may have an alternative use besides the regulation of intertemporal consumption $(m(W) > 0)$. In particular, we assume that $m(\cdot)$ is strictly increasing and weakly concave.

Our investigation of this case is motivated by results in experimental psychology which indicate that when subjects appear to have nearly exhausted their willpower, they in fact are holding willpower in reserve for future activities. In one experiment (Muraven, 1998), some subjects were given two tasks to be performed consecutively and some were told in advance that a third task would follow the first two. When performing the second task, those who anticipated the third task gave up sooner and thus appeared as though they had less willpower.

As shown in the previous section, when willpower has no alternative use, the agent smoothes his consumption perfectly whenever that is feasible. That is, whenever $\bar{W} \geq W_H$, $c(t) = \frac{\bar{R}}{T}$ for $t \in [0, T]$. Denote perfectly smooth consumption as c_H. When willpower has alternative uses, however, perfect smoothing may be avoided even when feasible. In order to identify conditions sufficient for perfect smoothing to be avoided, we assume throughout this section that $\overline{W} > W_H$. To isolate the influence of willpower concerns we will continue to assume $\rho = 0$.

To find conditions sufficient for perfect smoothing to be suboptimal, we determine when $c(t) = c_H$ for $t \in [0, T]$ violates one of the necessary conditions for an optimum. Denote by $\hat{W}$ the willpower available at T if perfect smoothing (c_H) has been implemented; clearly $\hat{W}$ depends on the initial levels of willpower and cake $(\bar{W}, \bar{R})$ but we suppress this dependence for simplicity.

If perfect smoothing has been implemented, (1) implies that $U'(c_H) - \lambda(T) f_c\left(\hat{W}, c_H\right) = \alpha$. Multiplying both sides by c_H we get,

$$\left[U'(c_H) - \lambda(T) f_c\left(\hat{W}, c_H\right)\right] c_H = \alpha c_H. \tag{9}$$

Solving (9) and substituting into (5), we obtain:

$$U(c_H) - \left[U'(c_H) - \lambda(T) f_c\left(\hat{W}, c_H\right)\right] c_H - \lambda(T) f\left(\hat{W}, c_H\right) \geq 0.$$

Rearranging we obtain,

$$\lambda(T) \leq \frac{U(c_H) - U'(c_H) c_H}{f\left(\hat{W}, c_H\right) - f_c\left(\hat{W}, c_H\right) c_H}, \tag{10}$$

with equality if $s < T$. Define $m_H(W, c) = \frac{U(c) - U'(c)c}{f(W,c) - f_c(W,c)c}$. Then, we can rewrite (10) as

$$\lambda(T) \leq m_H(\hat{W}, c_H). \tag{11}$$

with equality if $s < T$.

4.1 A Necessary Condition

As the following proposition makes clear, the agent almost never chooses to smooth perfectly when willpower has an alternative use:

Proposition 2 *Perfect smoothing is almost never optimal. Specifically, the agent chooses to smooth perfectly only if*

(1) $f_W f_c - f_{cW} f = 0$, *for* $c = \frac{\bar{R}}{T}$, $W \in \left[\hat{W}, \bar{W}\right]$, *and*

(2) $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$.

Proof. If smoothing is optimal then $c(t) = \frac{\bar{R}}{T}$, implying $\dot{c} = 0$ and $c(t) > 0$ for $t \in (0, T)$. Since $\bar{W} > W_H$ and therefore $\hat{W} > 0$, (7) implies that $\lambda(T) = m'\left(\hat{W}\right) > 0$, which in turn implies that $\lambda(t) > 0$ for all $t \leq T$. It then follows from (8) that perfect smoothing requires $f_W f_c - f_{cW} f = 0$ where $f(\cdot, \cdot)$ and its partial derivatives are evaluated at $c = \frac{\bar{R}}{T}$ and any $W \in \left[\hat{W}, \bar{W}\right]$. This confirms condition (1) of the proposition. Even then, since $\lambda(T) = m'\left(\hat{W}\right) > 0$, (11) requires $m'(\hat{W}) \leq m_H(\hat{W}, c_H)$, confirming condition (2) of the proposition. ■

It should be emphasized that while necessary conditions (1) and (2) of Proposition 2 are extremely restrictive, they may be insufficient to insure perfect smoothing. Even if they hold, the agent may finish the cake before time T and consume nothing for the remainder of the time horizon.

We conclude this section by noting a surprising implication of the theory. Suppose the poor and the rich have the same preferences over consumption profiles, the same initial endowment of willpower, and the same willpower technology ($f = K(\bar{c} - c)$). Assume, moreover, that the rich and the poor have the same constant marginal utility (m) if additional willpower is used in the alternative activity. The poor are assumed to differ from the rich in only one respect: the poor have smaller cakes.

Then, by Proposition (1), optimal consumption paths are constant for both rich and poor as long as consumption is strictly positive. To distinguish whether a variable pertains to the poor or the rich, we append a subscript "p" or "r", respectively. Thus, for consumption we write c_i (for $i = p, r$). Since, in this example, members of each group have the same utility function and self-control technology, the *function* $m_H(\cdot, \cdot)$ is identical for the two groups and there is no need to add the subscript i. Assume that, for $i = p, r$, $m > m_H(\hat{W}_i, c_{H,i})$. Then, by Proposition (2), the optimal consumption path does not involve perfect smoothing. Since consumption is constant and exhausts the cake, $s_i < T$ and $c_i > c_{H,i}$, for $i = p, r$.

Since, by assumption, both the rich and the poor carry some willpower into the second activity, additional willpower must have the same marginal utility in the two activities and, from (7), $\lambda_i(s_i) = m$. However, since $s_i < T$, (11) requires that $\lambda_i(s_i) = m_H(W_i, c_i)$. Hence,

$$m_H(W_i, c_i) = m \text{ for } i = p, r. \tag{12}$$

Since the left-hand side of (12) is independent of its first argument and strictly increasing in its second argument, this equation defines the same consumption for the two groups: $c_p = c_r$ as long as both consumptions are positive. Since the poor have strictly smaller cakes ($\bar{R}_r > \bar{R}_p$), they must run out of cake sooner ($s_p < s_r$), after which their consumption drops to zero. Having started with the same willpower and having exercised the same restraint over a shorter time interval, the poor will have more willpower remaining to invest in the alternative activity (e.g. athletics): $W_p(s_p) > W_r(s_r)$.[6]

[6] If $\bar{W}$ is reduced sufficiently, only the poor will carry willpower into the alternative activity, contrary to the assumption in the text. In this case, the rich will consume at a faster rate than the poor ($c_r > c_p$) as long as the consumption of each group is strictly positive, and the imputed value of using additional willpower in consumption will exceed the value of using it in the alternative activity ($m_H(W_r, c_r) > m$).

The poor in this example may appear to a casual observer to have less willpower (smaller $\bar{W}$) than the rich or at least to have less ability to control visceral impulses (larger f functions). Consuming as rapidly as the rich despite smaller paychecks while conserving scarce willpower for athletic or other activities may even seem self-destructive and requiring better skill in personal management. But in fact, the behavior of the poor is optimal and the only policy intervention in this example which can improve their welfare is an increase in initial cake ($\bar{R}$).

5 Optimal Consumption When the Exercise of Willpower Creates More Willpower in the Future: the Muscle Model

In the previous sections we have assumed that willpower is non-renewable. Common intuition and experimental psychology indicate, however, that willpower can be built up if it is exercised. Indeed, it can be built up in one arena and then used to advantage in other arenas. A striking example of this is the self-discipline that body builder and governor Arnold Schwarzenegger brings to public speaking. According to *Vanity Fair,* before Arnold Schwarzenegger delivered his speech at the Republican Convention "staffers had been startled to see him give a flawless performance of the speech, then repeat it exactly minutes later." When asked about that, Schwarzenegger replied "I rehearse like that with everything I do. I had already rehearsed it dozens of times...For me, it is like reps." p.160 (Jan. 2005).

Governor Schwarzenegger may be on to something. Experimental psychologists have also found that, while exerting self-control depletes willpower over the short term, regular exercise of self-restraint may eventually build willpower. In one experiment (Muraven *et al.,* 1999), subjects who participated in two-week self-control drills (regulating moods, improving posture, etc.) later showed significant increases in the length of time the would squeeze a handgrip relative to those who did not participate in the drills.

In this section, we introduce the notion that, like muscle, the prior exertion of willpower may strengthen the ability to self-regulate i.e., slow down future depletion of willpower. We will see that muscle management introduces variation in consumption over time even when willpower itself does not. To this end we introduce a third state variable, muscle, the level of which is denoted by $M(t)$. We augment our earlier model in two ways. First the rate of change of willpower is given by $\dot{W}(t) = \gamma M(t) - f(W(t), c(t))$. As before, willpower is depleted by restraining consumption, $f(W(t), c(t))$, but this depletion is moderated by the service flow γ from the stock of muscle, $M(t)$. Since the stocks of willpower and muscle cannot jump, the only way to alter the rate of willpower depletion immediately is by altering contemporaneous consumption. In the future, however, the muscle may develop and provide additional willpower at rate γ.The rate at which muscle develops (or deteriorates) is given by $\dot{M} = f(W(t), c(t)) - \sigma M(t)$. To interpret this equation note that

$$M(t) = M(0) + \int_{u=0}^{u=t} e^{-\sigma(t-u)} f(W(u), c(u)) du.$$

The idea is that exercising willpower today contributes to one's future muscle but the contributions decay.

The agent chooses $c(t)$ to maximize

$$V(0) = \int_0^T e^{-\rho t} U[c(t)]\, dt \tag{P3}$$

subject to

$$\dot{R}(t) = -c(t)$$

$$\dot{W}(t) = \begin{cases} \gamma M(t) - f(W(t), c(t)) & \text{if } R(t) > 0 \\ \gamma M(t), \text{ otherwise} \end{cases} \tag{13}$$

$$\dot{M}(t) = \begin{cases} f(W(t), c(t)) - \sigma M(t) & \text{if } R(t) > 0 \\ -\sigma M(t), \text{ otherwise} \end{cases} \tag{14}$$

$$R(t) \geq 0,\ W(t) \geq 0,\ M(t) \geq 0 \text{ for } t \in [0, T] \tag{15}$$

$$R(0) = \overline{R} > 0,\ W(0) = \overline{W} > 0,\ M(0) = \overline{M} > 0$$

Once again we consider a related, but more tractable problem and argue that, by solving it, we solve problem $(P3)$. In the related problem, the agent chooses both an optimal consumption path $c(t)$ and an optimal horizon $s \leq T$, where $\dot{W}(t) = \dot{M}(t) = c(t) = 0$ for all $t \in (s, T]$,to maximize:

$$V(0) = \int_0^s e^{-\rho t} U[c(t)]\, dt \tag{P4}$$

subject to the constraints of problem $(P3)$ except constraints (13)-(15) which are replaced by:

$$\dot{W}(t) = \gamma M(t) - f(W(t), c(t))$$

$$\dot{M}(t) = f(W(t), c(t)) - \sigma M(t)$$

$$R(t) \geq 0,\ W(t) \geq 0,\ M(t) \geq 0 \text{ for } t \in [0, s]$$

As before, we must argue that the more tractable problem we have formulated has the same consumption path for $t \in [0, s]$ as the solution to the actual problem $(P3)$. Once again this will be the case if we can show that in problem $(P4)$ the entire cake is consumed by $t = s$ in the optimal solution. Our previous argument suffices. If $R(s) > 0$ in the optimal program then $M(s) > 0$ and $W(s) \geq 0$. But then we could choose Δ small enough that $\frac{R(s)}{\Delta} \geq \bar{c}$. We could then duplicate the proposed optimal path until $s - \Delta$ and augment it by $\frac{R(s)}{\Delta}$ in this final interval. The payoff would be strictly higher and the program would be feasible since the willpower left at $s - \Delta$ is the same in the two programs $(W(s - \Delta) \geq 0)$ and no willpower is depleted in the final interval.

Having established that the solution of our related problem ($P4$) solves the actual problem ($P3$), we make two observations which simplify the analysis. Since the muscle is initially positive and decays exponentially even if it is never augmented, muscle will be strictly positive and will at no time violate the nonnegativity constraint. Moreover, since the stock of cake can only decline, requiring that it is nonnegative at s insures that it will be nonnegative previously.

Given that for $t \in [0, s)$ these two state variables must be nonnegative, we can simplify our formulation by replacing $R(t) \geq 0$ and $W(t) \geq 0$ by $R(s) \geq 0$ and $W(s) \geq 0$. However, since $W(s) \geq 0$ no longer implies that $W(t) \geq 0$ for $t < s$, the conditions which must necessarily hold at an optimum whenever $W(t) > 0$ will differ from those that hold while $W(t) = 0$. Given our focus, we consider only the former situation in detail.[7] The Hamiltonian for this problem is:

$$
\begin{aligned}
&H\left(c\left(t\right), R\left(t\right), W\left(t\right), t, \alpha\left(t\right), \lambda\left(t\right), \pi\left(t\right)\right) \\
&= e^{-\rho t} U\left(c\left(t\right)\right) - \alpha\left(t\right) c\left(t\right) + \lambda\left(t\right)\left(\gamma M\left(t\right) - f\left(W\left(t\right), c\left(t\right)\right)\right) + \\
&\pi\left(t\right)\left(f\left(W\left(t\right), c\left(t\right)\right) - \sigma M\left(t\right)\right).
\end{aligned}
$$

The first order conditions are given by,

$$c\left(t\right) \geq 0,\ e^{-\rho t} U'\left(c\left(t\right)\right) - \alpha\left(t\right) - \left(\lambda\left(t\right) - \pi\left(t\right)\right) f_c \leq 0 \text{ and c.s.} \tag{16}$$

$$\dot{W}\left(t\right) = \gamma M\left(t\right) - f \tag{17}$$

$$\dot{M}\left(t\right) = f - \sigma M\left(t\right) \tag{18}$$

$$\dot{\alpha}\left(t\right) = 0 \tag{19}$$

$$\dot{\lambda}\left(t\right) = f_W\left(\lambda\left(t\right) - \pi\left(t\right)\right) \tag{20}$$

$$\dot{\pi}\left(t\right) = \pi\left(t\right)\sigma - \lambda\left(t\right)\gamma = -\gamma\left(\lambda\left(t\right) - \frac{\sigma}{\gamma}\pi\left(t\right)\right) \tag{21}$$

$$T - s \geq 0,\ H(c\left(s\right), R\left(s\right), W\left(s\right), s, \alpha\left(s\right), \lambda\left(s\right), \pi\left(s\right)) \geq 0 \text{ and c.s.} \tag{22}$$

$$W\left(t\right) > 0 \text{ and c.s.} \tag{23}$$

$$R\left(s\right) \geq 0,\ \alpha\left(s\right) \geq 0 \text{ and c.s.} \tag{24}$$

$$W\left(s\right) \geq 0,\ \lambda\left(s\right) \geq 0 \text{ and c.s.} \tag{25}$$

$$M\left(s\right) \geq 0,\ \pi\left(s\right) \geq 0 \text{ and c.s.} \tag{26}$$

[7] To derive conditions which must hold across both cases, Seierstad and Sydsœter (1987), and also in Léonard and Long (1992)) begin by forming the Lagrangean $H + \Theta(t)W(t)$, where $\Theta(t)$ is a Lagrange multiplier. In such problems the multiplier (λ) on the state variable (willpower) may jump discontinuously as the nonnegativity constraint is just reached or as it becomes slack. If that multiplier does jump, then consumption would jump as well at such dates.

To analyze the dynamics of consumption, it will be useful to sign $\lambda(t)-\pi(t)$ and $\lambda(t)\left(\frac{\gamma}{\sigma}\right)-\pi(t)$. First assume that $\gamma/\sigma \leq 1$. In this case, we show that each of the preceding terms is weakly positive while $\dot{\lambda}$ and $\dot{\pi}$ are weakly negative. These results can be most readily understood using the phase diagram depicted in figure (1). Provisionally assume that $f_W < 0$ and $\gamma > 0$. Then we can plot the locus of (λ, π) pairs such that $\dot{\lambda} = 0$. By the $\dot{\lambda}$ equation (20), these points lie on the 45^0 line $\pi = \lambda$. Horizontal motion above this locus is to the right and below it is to the left. Similarly, we can plot the locus of (λ, π) pairs such that $\dot{\pi} = 0$. By the $\dot{\pi}$ equation (21), these points lie on a flatter ray provided $\frac{\gamma}{\sigma} < 1$. In the extreme case where $\frac{\gamma}{\sigma} = 1$, the two rays coincide. Vertical motion above the $\dot{\pi} = 0$ locus is upward and below this locus it is downward. As long as muscle exists at any time in the program, some will remain at the end $(M(T) > 0)$, because it at most decays exponentially and therefore never reaches zero. It follows from condition (26) that the endpoint condition $\pi(T) = 0$ is satisfied. As long as willpower considerations matter $(\lambda(0), \pi(0) \neq 0)$,the endpoint condition and dynamics preclude initial multipliers set at or above the lower of the two rays since then $\dot{\pi} \geq 0$ implying $\pi(T) > 0$. Thus $\pi(T) = 0$ requires that the initial multipliers be set below the lower of the two rays. But this in turn implies that $\lambda - \pi > 0, \frac{\gamma}{\sigma}\lambda - \pi > 0, \dot{\lambda} < 0$, and $\dot{\pi} < 0$.

Now consider the case where $\gamma/\sigma > 1$, depicted in figure (2). The endpoint condition $\pi(T) = 0$, together with these dynamics imply that there will be a final phase in which the multipliers will lie strictly below the $\dot{\lambda} = 0$ locus, and thus, again, $\lambda - \pi > 0, \frac{\gamma}{\sigma}\lambda - \pi > 0, \dot{\lambda} < 0$, and $\dot{\pi} < 0$.

The optimal consumption path in the muscle model shares some qualitative features with that in the model without muscle. First, the cake is entirely consumed. To see this, note that having assumed $U'(\cdot) > 0$ and $f_c < 0$, and shown $(\lambda - \pi) > 0$, we can satisfy (16) only if $\alpha > 0$; and then (24) requires that the cake be exhausted by time s. Second, for every initial level of muscle there is a willpower level $W_{\tilde{H}}$ above which the optimal path entails perfect smoothing. This is true because, for every initial stock of muscle there is an initial stock of willpower sufficiently large such that the Hotelling path is feasible and therefore optimal. Indeed, if the initial muscle level is large enough, the agent will be able to achieve perfect smoothing without any initial willpower. Intuitively, in any case where perfect smoothing is feasible, there will be no shortage of willpower or muscle and $\lambda(t) = \pi(t) =$ for $t \geq 0$.

In the more interesting cases, if we start with an initial level of willpower sufficient for perfect smoothing, decreases in that stock will eventually lead to a willpower level $\left(W_{\tilde{H}}\right)$ where any further reduction in the initial stock of willpower will make the perfectly smooth path infeasible. Since $\lambda(0) \geq 0$, we know that utility is increasing in the initial stock of willpower. Because perfect smoothing is the optimal path in the absence of willpower concerns path, it follows that such a path is infeasible for any $W(0) < W_{\tilde{H}}$.

Next, we ask how recognition that the exercise of willpower builds muscle alters our conclusions about the time path of optimal consumption in the absence of discounting $(\rho = 0)$. Intuitively, since condition (16) holds for $s < T$, the *sum* of the direct and indirect marginal benefits of increased consumption must remain equal to α at all times. Hence, if consumption is strictly increasing over time, which would depress the direct marginal benefits of consumption, then the indirect marginal benefits $((\lambda - \pi)(-f_c))$ must also be strictly increasing. More formally, differentiating condition (16), we obtain:

$$\begin{aligned} U''(c)\,\dot{c} &= \dot{\alpha} + \left(\dot{\lambda} - \dot{\pi}\right) f_c + (\lambda - \pi)\left(f_{cc}\dot{c} + f_{cW}\dot{W}\right) \\ &= \left(f_W(\lambda - \pi) - (\pi\sigma - \lambda\gamma)\right) f_c + (\lambda - \pi)\left(f_{cc}\dot{c} + f_{cW}\dot{W}\right) \\ &= (\lambda - \pi)\left(f_W f_c + f_{cc}\dot{c} + f_{cW}\dot{W}\right) - (\pi\sigma - \lambda\gamma) f_c \end{aligned}$$

which implies

$$\dot{c} = \overbrace{\frac{(\lambda - \pi)(f_W f_c - f_{cW} f)}{\Delta}}^{\text{direct willpower effect}} + \overbrace{\frac{(\lambda - \pi) f_{cW}\gamma M(t)}{\Delta}}^{\text{muscle service flow}} + \overbrace{\frac{\left(\lambda\left(\frac{\gamma}{\sigma}\right) - \pi\right)\sigma f_c}{\Delta}}^{\text{muscle building}} \tag{27}$$

where $\Delta = [U''(c) - (\lambda - \pi) f_{cc}] < 0$. The inequality follows because c maximizes the Hamiltonian. If $\lambda(t) = \pi(t) = 0$ for $t \geq 0$, equation (27) yields the classical result that consumption is constant as long as it is positive and equation (22) requires that it be positive until T.

To clarify how muscle building influences the time path of optimal consumption when perfect smoothing is infeasible, we consider two special cases where, in the absence of muscle, the optimal consumption path is particularly simple. Specifically, we consider first the case where the rate of willpower depletion is determined only by rate of consumption and not by the willpower remaining $(f_W = 0)$, and second the case where $(f_W f_c - f_{cW} f) = 0$. In each of these cases, when muscle is absent, optimal consumption is constant until some time s when consumption drops to zero.

Referring to equation (27), case 1 $(f_W = 0)$ implies that both the direct willpower effect and the muscle service flow are absent. The time path of consumption is therefore determined only by muscle building. First consider the case where muscle decays at a faster rate than it contributes to willpower $[(\gamma/\sigma) \leq 1]$. In this case, the muscle building term of equation (27) is always positive (see Figure (1) and the associated discussion). Thus when $(\gamma/\sigma) \leq 1$ and $f_W = 0$,consumption is always increasing. Relative to the optimal path in the absence of muscle, the ability to build willpower through its exercise leads the agent to bear down at the beginning of the program in order to enjoy a greater willpower later.

When muscle decays more slowly $[(\gamma/\sigma) > 1]$, more complex consumption paths may emerge. If the multipliers λ and π are initial located in regions II or III of Figure (2), optimal behavior has the same qualitative features as when $(\gamma/\sigma) \leq 1$.Consumption is always increasing. If, however, the multipliers are initially located in region I of Figure (2), consumption will decrease with time until the multipliers pass into region II. That is, the consumption profile is $\cup$ - shaped.

Referring again to equation (27), case 2 $[(f_W f_c - f_{cW} f) = 0]$ implies that only the direct willpower effect is inactive. Relative to the optimal path in the absence of muscle, the ability to build of willpower with exercise again induces time preference. Consider, for example, a situation where the initial muscle stock is zero $(M(0) = 0)$,and thus, at the beginning of the program, the muscle service flow term is inactive. In the beginning of the program, optimal behavior in this case is like that in the case where $f_W = 0$. For example, when $(\gamma/\sigma) \leq 1$,consumption will increase in these early stages of the optimal path.

6 Contribution to the Economics Literature

We have introduced into the standard economic model of intertemporal consumption the intuitive idea that the ability to self-regulate is a depletable resource. The existence of this resource is supported by more than a decade of psychology experiments. Given that this resource—commonly dubbed "willpower"—is scarce, self-regulation today affects choices tomorrow even if the choice set tomorrow is fixed. Having completed our analysis, we wish to explain how our work contributes to the literatures on self-regulation. We discuss the behavioral economics literature in this section and the experimental psychology literature in the next section.

One strand of the literature on the economics of self-regulation (Laibson, 1997; O'Donoghue and Rabin, 1999) focuses on the present-biased choices which result from hyperbolic discounting while another strand (Gul and Pesendorfer, 2001, 2004a,b; Benhabib and Bisin, 2004; Dekel, Lipman and Rustichini, 2005) models self-control problems which arise from temptation costs. In the hyperbolic discounting models, when an individual's preferred level of consumption in the present exceeds the level to which he would have liked to commit earlier, his prior selves *never have any direct control* over the amount his present self consumes. On the other hand, the decision-maker in temptation-cost models *always has perfect control* over consumption although he must pay a temptation cost if he wants to consume less than what his "other self" would prefer. In our model, the decision-maker has perfect control as long as any willpower remains but has virtually no control when willpower is exhausted. On the issue of control, therefore, our willpower model steers an intermediate course between the extremes envisioned in the hyperbolic discounting and temptation-cost approaches. However, both models fundamentally differ from ours: neither predicts, for the consumption of non-addictive goods,[8] that current acts of self-regulation influence future choices from a given

[8] There is a related literature (Bernheim and Rangel, 2004; Laibson 2001) concerning self-regulation when the good consumed is addictive. The agent in these models is assumed to have perfect self-control in a cool state but none

set of options. Hence, neither formulation can account for the behaviors induced repeatedly by Baumeister and his colleagues in their laboratories.

Nor can the model of Benabou and Tirole (2004) explain these behaviors. Benabou and Tirole consider the role of state-dependent self-regulatory ability on optimal decision making. Their paper identifies the ability to self-regulate with the degree of present-bias in time discounting, and focuses on the effects of imperfect recall and self-reputation on optimal choices. Despite its potential importance in other contexts, imperfect recall cannot plausibly explain the findings of Baumeister and his colleagues in their laboratories since no experiment lasts more than a few hours.

In a recent working paper, Fudenberg and Levine (2005) model an agent whose long-run self may, at a direct utility cost, exert control over the choices of a sequence of short-term selves. They explicitly rule out by assumption (Assumption 5) the existence of a depletable cognitive resource like willpower, but discuss how their model might be altered to accommodate such a resource.

Several contributors to this literature treat self-regulation as having a direct contemporaneous utility cost. In discussing an appropriate formulation to capture Baumeister's experiments, Loewenstein and O'Donoghue (2004) regard the exertion of willpower as generating disutility although Loewenstein (2000) regards the matter as an open question. Whether or not willpower depletion has a utility cost seems difficult to resolve empirically. What we have resolved analytically, however, is that including the stock of willpower (or its rate of change) in the utility function is not necessary to capture either the behaviors psychologists have documented in their laboratories or to explain a number of prominent "anomalies" behavioral economists have reported in the field. For these purposes, it is sufficient to append to the conventional formulation of intertemporal utility maximization the additional constraint that the consumption path chosen must not overexhaust the agent's willpower.

7 Implications for Psychology Experiments

Because our theory of intertemporal consumption relies on the series of experiments performed by Baumeister and his colleagues over more than a decade, we are well-positioned to suggest how relationships central to our predictions can be illuminated by further experimentation.

Both introspection and evidence from the laboratory indicate that lowering consumption results

in a hot state. By changing current consumption, the agent may alter the probability that he enters the hot state, thereby indirectly exerting future self control. Such models are not intended to explain behavior in the experiments of Baumeister and his colleagues, which do not involve addictive goods. Moreover, since the good in our model is assumed to be nonaddictive, its consumption is not well-captured by state-dependent utility with a persistent state. Self-control in our model is direct and current choices affect future choices only through their effect on willpower.

in faster depletion of willpower, so we assumed $f_c < 0$ throughout our analysis. Since experimental evidence leaves ambiguous whether a given restraint on consumption results in faster or unchanged depletion of willpower when willpower reserves are lower, we took *both* possibilities into account: $f_W \leq 0$. However, since neither experiment nor introspection suggests how the amount of cake left uneaten at a given point affects the rate of willpower depletion (holding constant the level of restraint and the willpower stock) we assumed as a simplification that willpower depletion at time t is independent of cake remaining then. We now describe two experiments which would illuminate the issues which remain ambiguous.

To determine whether a given level of restraint depletes willpower more rapidly when willpower reserves are low, we propose the following variation on Baumeister's two-phase experiments. In the first phase, subjects would be asked to perform a quantifiable willpower-depleting activity (A), such as attempting to solve an insolvable puzzle. In the second phase, the same subjects would be asked to perform another, quantifiable willpower-depleting activity (B). The treatment group would perform activity A before B, and the control group would perform B before A. All subjects would be informed ahead of time about the nature of the two activities and the order in which they would be performed. If the level of willpower reserves does not affect its rate of depletion $(f_W = 0)$ the two groups should, on average, last as long on a given activity regardless of whether it precedes or follows the other activity. On the other hand, if depletion is anticipated to be more rapid when reserves are low, the subjects should restrain themselves more with a given activity when it occurs first: that is, they should squeeze the handgrip longer when it is first and quit sooner when it is last; similarly, they should quit the puzzle later when it is first and sooner when it is last.

To assess whether a given level of restraint depletes willpower more rapidly when the cake available is larger, we propose the following variation on Baumeister *et al.* (1998). Upon entering the lab, all subjects would be seated in front of a plate of edible, but not particularly tempting, food such as radishes or bran cereal. Subjects in the treatment group would also have a plate with a specific number (X) of cookies in front of them. The controls would be asked to take a specified amount of time to taste the untempting food and would write down their impressions of it. The treatment group would be told that they had been randomly assigned to taste the untempting food and would be instructed not to eat any of the cookies. This group would also be asked to take the same specified amount of time to write down their impressions of the food consumed. In a second phase of the experiment, both treatment and control groups would be asked to perform an easily quantifiable, endurance task such as squeezing a handgrip or solving an insolvable puzzle. Baumeister *et al.* (1998) implement this design with a fixed number (X) of cookies. In their experiment, none of the subjects in the treatment consumed any of the cookies and there is a substantial and statistically significant difference on the second task between the endurance of the treatment group and the control group. Our proposed innovation is to *vary* the number of cookies (X). If, holding consumption fixed (at zero in this case), the rate of willpower depletion depends on the size of the remaining cake, then we would predict that the difference between treatment and control groups in performance in the second phase should depend on X. That is, willpower should be more or less depleted by consuming no cookies, depending on how many cookies were available. An attractive feature of this design is that the tempting X can be varied by small amounts in the neighborhood of widely different X's. We conjecture that when X is large, small changes will have no effect on willpower depletion; when X is small, such changes seem more likely to affect willpower depletion although, the sign of the influence is not obvious *a priori*.

8 Conclusion

This paper has explored the consequences of including in a conventional model a cognitive constraint well-documented by experimental psychologists: depletable willpower. Specifically, we assumed that if a consumer has wealth to spend on current consumption and yet he does not spend it all, then exercising self-restraint requires an act of will that depletes his finite stock of willpower. This willpower constraint captures the common notion, verified by laboratory experiments, that an individual has limited, though positive, capacity to regulate his own visceral or unthinking behaviors. Willpower in our model may be interpreted as a cognitive resource which must be depleted to exercise self-restraint.

A consumer who spends his entire budget on current consumption expends no willpower since what restrains him from spending more is an empty wallet. Hence, the willpower constraint has no effect on consumer behavior in a one-period "static" problem. To investigate how the willpower constraint affects intertemporal behavior, we introduced it into the conventional model of economic decision-making over time. In that model, a cake (or paycheck or stock of leisure time) must be consumed over a finite time horizon.

A willpower-constrained consumer behaves in ways that have long puzzled economists. Willpower concerns may induce paths of consumption that are increasing until wealth is depleted, at which point consumption drops to zero. This provides one possible explanation for otherwise puzzling forms of procrastination where the time path of consumption of leisure features both some front-loading of work, and some extreme backloading as well. Specifically, if the consumer must complete a project by deadline T and foresees that completing the work will take him $T - \bar{R}$ days, then he will have $\bar{R}$ days of leisure to allocate prior to the deadline. A willpower-constrained consumer may choose to work hard at the outset, getting part of the task out of the way before enjoying increasing amounts of leisure as time passes; when only $T - s$ days remain, however, his consumption of leisure stops altogether and he works nonstop, completing the project on the instant of the deadline. Even with mild discounting, increasing consumption paths followed by zero consumption would persist as the optimal program.

When the consumer has other activities besides intertemporal consumption which benefit from self-discipline, another surprising result emerged: the optimal allocation of willpower almost never results in perfect smoothing of consumption over the entire horizon, even when willpower reserves are sufficient to permit such smoothing. To illustrate, we presented an example where neither the rich nor the poor smooth over the entire horizon. In that example, the poor consume at the same rate as the rich until their smaller paychecks are exhausted and then use their larger stock of remaining willpower on alternative activities. While such behavior might suggest to some a need for training courses in self-discipline, personal management, and planning for the future, in fact that intertemporal behavior is optimal given the smaller paychecks of the poor.

Finally, we considered what would happen if the exercise of self control in the present, while immediately depleting willpower, also builds willpower reserves in the future. We found that the ability to build future willpower by constraining current consumption further alters the intertemporal profile of consumption. Specifically, renewable willpower induces a time preference even when consumption would have been constant, if willpower had been nonrenewable.

A willpower-constrained consumer will demonstrate other behaviors that economists have often viewed as anomalous. Such a consumer would respond in surveys that he prefers one consumption path but would then pursue a different consumption path. He would value not having to exercise self-restraint and would, therefore, pay to participate in savings clubs and "fat farms" that help their customers restrain consumption. He would also make impulsive purchases after willpower reserves had been depleted. A willpower-constrained consumer would also appear to have time-inconsistent preferences. For example, if asked in the morning if he would like a large salad or a cheeseburger for lunch, the consumer might order the salad; but after a morning of problem solving that proved more difficult than anticipated, lunch time arrives and the willpower depleted consumer chooses the cheeseburger.

In sum, our analysis suggests that a number of "anomalies" of intertemporal choice can be explained without abandoning the standard economic model in which a consumer maximizes an exponentially-weighted sum of additively-separable, stationary utilities. Rather, a modified model, based on experimental psychology and acknowledging that restricting consumption requires both economic *and* cognitive resources, may explain otherwise anomalous behaviors. Indeed, the fact that these anomalous behaviors are so frequently observed may be interpreted as field evidence consistent with the "willpower depletion" model that Baumeister and colleagues have found evidence for in the lab.

As our analysis has shown, the shape of the intertemporal consumption profile depends in theory on characteristics of the willpower depletion function. If our theory is correct and if further laboratory work succeeds in determining the characteristics of this function, then the information could be used to sharpen predictions of intertemporal consumption.

9 Figures

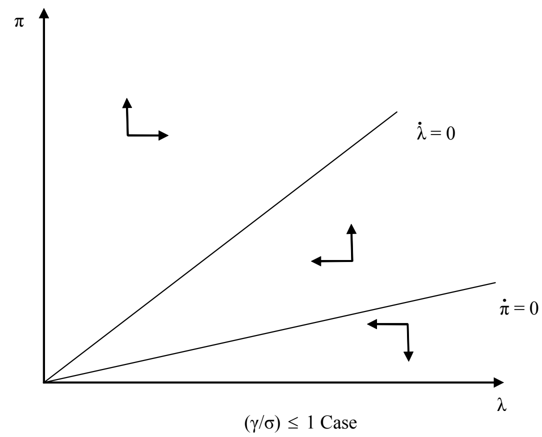

Figure 1: Phase Diagram of Costate Variables in Muscle Model Where $\frac{\gamma}{\sigma} \leq 1$.

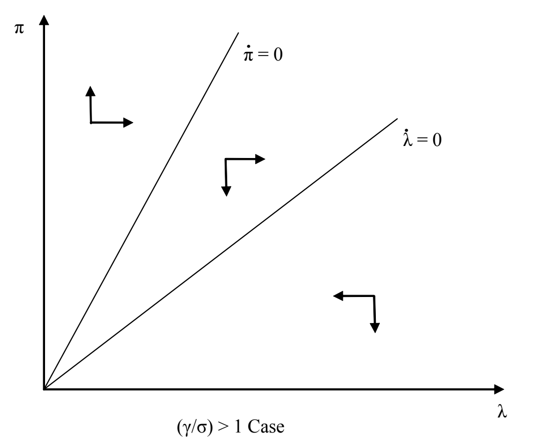

Figure 2: Phase Diagram of Costate Variables in Muscle Model Where $\frac{\gamma}{\sigma} > 1$.

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