On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games*

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Abstract

We extend and strengthen both Athey's (2001) and McAdams' (2003) results on the existence of monotone pure strategy equilibria in Bayesian games. We allow action spaces to be compact locally-complete metrizable semilatttices and can handle both a weaker form of quasisupermodularity than is employed by McAdams and a weaker single-crossing property than is required by both Athey and McAdams. Our proof — which is based upon contractibility rather than convexity of best reply sets — demonstrates that the only role of single-crossing is to help ensure the existence of monotone best replies. Finally, we do not require the Milgrom-Weber (1985) absolute continuity condition on the joint distribution of types.

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1. Introduction

In an important paper, Athey (2001) demonstrates that a monotone pure strategy equilibrium exists whenever a Bayesian game satisfies a Spence-Mirlees single-crossing property. Athey's result is now a central tool for establishing the existence of monotone pure strategy equilibria in auction theory (see e.g., Athey (2001), Reny and Zamir (2004)). Recently, McAdams (2003) has shown that Athey's results, which exploit the assumed total ordering of the players' one-dimensional type and action spaces, can be extended to settings in which type and action spaces are multi-dimensional and only partially ordered. This permits new existence results in auctions with multi-dimensional signals and multi-unit demands (see McAdams (2004)).

At the heart of the results of both Athey (2001) and McAdams (2003) is a single-crossing assumption. Roughly, the assumption says that if a player prefers a high action to a low one given his type, then the high action remains better than the low one when his type increases. That is, as a function of his type, the difference in a player's payoff from a high action versus a low one crosses zero at most once and from below — it undergoes a "single crossing" of zero.

It is not difficult to see that a single-crossing condition of the sort described above is virtually necessary if one's goal is to establish the existence of a monotone pure strategy equilibrium. After all, if the condition fails, then a higher type sometimes prefers a lower action, and ruling this out in equilibrium would require very special additional assumptions.

One of the roles of single-crossing, therefore, is to ensure that players possess monotone best replies. However, previous research suggests that this is not its only role, and perhaps not even its most central role. Indeed, McAdams remarks (2003, p. 1202), "... existence of a monotone best response is far from guaranteeing monotone equilibrium." This comment reflects the fact that the proof techniques of both Athey and McAdams rely on a sufficiently strong version of single-crossing, one that not only helps ensure that monotone best replies exist, but also helps ensure that each player's entire set of optimal actions is, as a function of his type, increasing in the strong set order.[1] The import of the latter requirement is that it renders a player's set of monotone pure-strategy best-replies convex, in a sense pioneered by Athey and extended by McAdams. Specifically, Athey observed that

[1] When actions are real numbers, this means that if some action is optimal given one's type and another action is optimal when one's type changes, then the higher of the two actions is optimal for the higher type and the lower of the two actions is optimal for the lower type.

monotone functions from the unit interval into a finite totally ordered action set are characterized by their jump points and that even though sets of monotone best reply functions are not convex, their associated sets of jump points *are* convex when the strong set-order property holds. This impressive convexity result permits Athey and McAdams to establish the existence of a monotone pure strategy equilibrium through an application of Kakutani's theorem in the case of Athey, and Glicksberg's theorem in the case of McAdams.

In the present paper, we provide a generalization of the results of both Athey and McAdams and we do so through a new route which, by avoiding the convexity issue altogether, furnishes additional insights into the role of the single-crossing assumption and eliminates the need to view monotone strategies as a collection of jump points, a view that is helpful only when action spaces are restricted to finite sets. In particular, our main result (Theorem 4.1) can be specialized to show that when action spaces are one-dimensional (as in Athey) or are such that distinct dimensions of a player's own action vector are complementary (as in McAdams), then the existence of monotone best replies alone *does* guarantee the existence of a monotone pure strategy equilibrium.[2] Hence, we find that the role of the single-crossing assumption in establishing the existence of monotone pure strategy equilibria is simply to ensure the existence of monotone best replies, nothing more. In particular, there is no need to impose a more restrictive single-crossing assumption so as to ensure that players' sets of optimal actions are increasing in the strong set order as their types vary. Thus, while the strong set-order property remains important for comparative statics exercises (see Milgrom and Shannon (1994)), we find that it is unrelated to the existence of monotone pure strategy equilibria per se. Moreover, because we work directly with the monotone strategies themselves, not their jump points, we are able to handle infinite action spaces with no difficulty.

The key to our result is to abandon the use of both Kakutani's and Glicksberg's theorems. In their place, we instead employ a corollary (Theorem 5.1) of a fixed point theorem due to Eilenberg and Montgomery (1946). Whereas Athey and McAdams impose additional assumptions to obtain convexity of the players' sets of best replies, we instead take advantage of the fact that the Eilenberg-Montgomery corollary only requires best reply sets to be *contractible*, a property that is remarkably straightforward to establish in the class of Bayesian games we study. In particular, so long as action spaces are compact locally-complete metriz-

[2] Actions of distinct *players* need not be complementary. That is, we do not require the game to be one of strategic complements.

able semilattices, our approach applies whether action spaces are finite (as in both Athey and McAdams) or infinite; whether they are one dimensional and totally ordered (as in Athey), finite dimensional and partially ordered (as in McAdams), or infinite dimensional; and whether they are sublattices of Euclidean space (as in both Athey and McAdams) or not. Further, the transparency of our proof of contractibility is in rather stark contrast to the technique cleverly employed by McAdams to extend Athey's convexity results from one-dimensional totally-ordered to finite-dimensional partially-ordered types and actions. By focusing on contractibility rather than convexity of best reply sets, and by relying upon a more powerful fixed point theorem, we obtain a more direct proof under strictly weaker hypotheses.

If in addition to our assumptions on payoffs, the actions of distinct players are strategic complements, Van Zandt and Vives (2003) have shown that even stronger results can be obtained. They prove that monotone pure strategy equilibria exist under more general distributional, type-space and action-space assumptions than we impose here, and demonstrate that such an equilibrium can obtained through iterative application of the best reply map.[3] In our view, Van Zandt and Vives (2003) have obtained perhaps the strongest possible results for the existence of monotone pure strategy equilibria in Bayesian games when strategic complementarities are present. Of course, while many interesting economic games exhibit strategic complements, many do not. Indeed, most auction games satisfy the single-crossing property required to apply our result here (see e.g., Athey (2001), McAdams (2004), Reny and Zamir (2003)), but fail to satisfy the strategic complements condition.[4] The two approaches are therefore complementary.

The remainder of the paper is organized as follows. Section 2 provides a simple first-price auction example satisfying the hypotheses of our main result but not those of Athey (2001) or McAdams (2003). The essential ideas behind the present technique are also provided here. Section 3 describes the formal environment, including semilattices and related issues. Section 4 contains our main result and a corollary which itself strictly generalizes the results of both Athey and McAdams. Section 5 provides the corollary of Eilenberg and Montgomery's (1946) fixed point

[3]Related results can be found in Milgrom and Roberts (1990) and Vives (1990).

[4]In a first-price IPV auction, for example, you might increase your bid if your opponent increases his bid slightly when his private value is high. However, for sufficiently high increases in his bid at high private values, you might be better off reducing your bid (and chance of winning) to obtain a higher surplus when you do win. Such non monotonic responses to changes in the opponent's strategy are not possible under strategic complements.

theorem that is central to our approach, and Section 6 contains the proof of our main result.

2. An Example

We begin with a simple example highlighting the essential difference between the approach taken by Athey and McAdams and that which we adopt here.

Consider a first-price auction between two bidders for a single item. Bidder 1's value is $v_1 = 7/2$ and is public information. Bidders 1 and 2 each receive a private signal, x and y, respectively. Bidder 2's value, $v_2(x, y)$, depends upon both x and y, and is nondecreasing in each argument. The signals x and y are either each drawn independently and uniformly from $[0, 1/2)$, or each drawn independently and uniformly from $[1/2, 1]$, with each of these two possibilities being equally likely. Consequently, the signals are affiliated.

For the purposes of this example, bidder 1 must submit a bid from the set $\{1, 2, 3\}$, while bidder 2 must submit a bid from the set $\{0, 1, 2, 3, 4\}$. Ties are broken randomly and uniformly.

There is no reason not to expect a monotone pure strategy equilibrium to exist here, and in fact, at least one does exist. Nevertheless, the proof techniques of Athey and McAdams, which rely upon the convexity (up to a homeomorphism) of the players' monotone best reply sets, cannot be directly applied.[5] Indeed, we will show that, in this example, there is a monotone bid function for bidder 2 against which bidder 1's set of monotone best replies is not mapped onto a convex set by the ingenious mapping introduced by Athey (2001). Indeed, we will show that there is no homeomorphism mapping 1's set of monotone best replies onto a convex set.

[5] It is possible to perturb this simple example so that the Athey-McAdams approach can be applied to the perturbed game, and then take limits to obtain existence in the unperturbed game. In general, however, our main result cannot be obtained through a limiting argument based upon Athey's or Mcadams' results (e.g., see Remark 5).

2.1. A Nonconvex Set of Monotone Best Replies

Consider the following monotone bid function for bidder 2, as a function of his signal, y.

$$\beta(y) = \begin{cases} 0, & \text{if } y \in [0, 1/18), \\ 3, & \text{if } y \in [1/18, 1/2), \\ 4, & \text{if } y \in [1/2, 1]. \end{cases}$$

Our interest lies in the set of monotone best replies for bidder 1, as a function of his signal, x. Note that when $x < 1/2$, bidder 1 knows that y is uniform on $[0, 1/2)$ and so knows that 2's bid is 0 with probability 1/9 and 3 with probability 8/9. Consequently, bidder 1 is indifferent between bidding 1 and 3, each of which is strictly better than bidding 2. On the other hand, when $x > 1/2$, bidder 1 knows that y is uniform on $[1/2, 1]$ and so knows that bidder 2 bids 4. Hence any bid, 1, 2, or 3, is optimal since each bid loses with probability one. All in all, bidder 1's best reply correspondence is as follows:

$$B(x) = \begin{cases} \{1, 3\}, & \text{if } x \in [0, 1/2), \\ \{1, 2, 3\}, & \text{if } x \in [1/2, 1]. \end{cases}$$

Consequently, a monotone best reply for bidder 1 is any monotone step function of x taking the values 1, 2, or 3, such that a bid of 2 occurs only when $x \geq 1/2$.

As observed by Athey, totally-ordered actions and signals permit monotone step functions to be represented by the signals at which they jump from one action to the next. With three actions, namely the bids 1, 2, and 3, just two jump points are required. Therefore, every monotone bid function for bidder 1 can be mapped into a vector in $[0, 1]^2$ lying weakly above the diagonal. Conversely, each vector in $[0, 1]^2$ lying weakly above the diagonal, let is call this set D, determines a monotone bid function for bidder 1.[6] Consequently, each monotone bid function for bidder 1 can be mapped to a vector in the compact convex set D and vice versa. Moreover, for an appropriate topology on monotone bid functions, the mapping is continuous in both directions and so is a homeomorphism. Following McAdams (2003), let us call this very useful map the "Athey-map."

The Athey-map shows that 1's set of monotone bid functions (and similarly 2's) is homeomorphic to a convex set. Athey's (2001) main insight is noting that, under a sufficiently strong form of single-crossing, each player's set of best replies

[6] For the purposes of determining ex-ante payoffs, it is not necessary to specify bids at the finitely many jump points themselves because each particular jump point has prior probability zero.

is increasing in the strong set order as a function of his type, and that this implies that each player's set of monotone best replies is also homeomorphic (again via the Athey-map) to a convex set. This permits Athey to apply Kakutani's fixed point theorem to obtain the existence of a monotone pure strategy equilibrium.

In the present example however, Athey's technique (and therefore McAdams' also) does not work. The underlying reason is that 1's best reply correspondence is not increasing in the strong set order. Indeed, a bid of 3 is best for any particular signal less than 1/2 and a bid of 2 is best for any particular signal greater than 1/2. However, the smaller of these two bids, namely 2, is not best for the smaller of the two particular signals, which is less than 1/2. A consequence of this is that the image under the Athey-map of bidder 1's set of monotone best replies against bidder 2's strategy is not convex, and this precludes the all important application of Kakutani's theorem. Let us now demonstrate the nonconvexity.

According to the Athey-map, bidder 1's set of monotone best replies against the above monotone bidding function of bidder 2 is mapped into those vectors, $(x_1, x_2) \in [0,1]^2$, in the set

$$\{0 \leq x_1 = x_2 \leq 1/2\} \cup \{1/2 \leq x_1 \leq x_2 \leq 1\}.$$ [7]

This flag-shaped set, depicted in Figure 2.1, is clearly not convex.

But just because one particular homeomorphism, the Athey-map, fails to map 1's set of monotone best replies into a convex set, does not mean that some other homeomorphism might not do so. That is, it may still be the case that bidder 1's set of monotone best replies is homeomorphic to a convex set and so one might still ultimately be able to apply Kakutani's theorem, which, in some sense, is the heart of the Athey-McAdams approach. But this too fails, as we show next.

To see that bidder 1's set of monotone best replies is not homeomorphic to a convex set, it suffices to show that the set in Figure 2.1, to which it is homeomorphic, is itself not homeomorphic to a convex set. To see this, let us suppose that it were. Then, the convex set would have to be two dimensional, because dimension is preserved under a homeomorphism. Hence, the Figure 2.1 set, let us call it C, would have to be homeomorphic to a disc. But then $C \backslash \{(1/2, 1/2)\}$ would be homeomorphic to the disc minus the image of $(1/2, 1/2)$. But this is impossible

[7] The first set corresponds to monotone best replies that jump from a bid of 1 to a bid of 3 at some signal weakly less than 1/2. The second set corresponds to monotone best replies that either jump from 1 to 3 at a signal weakly above 1/2, or that jump from 1 to 2 at a signal $x_1 \geq 1/2$ and then from 2 to 3 at a signal $x_2 > x_1$.

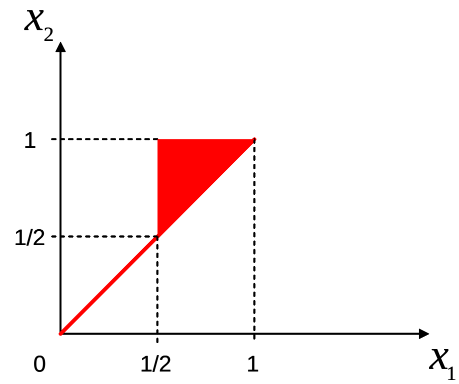

Figure 2.1: Non-Convex Bext Reply Set

since the latter set is connected, while $C\backslash\{(1/2,1/2)\}$ is not, and connectedness is preserved under a homeomorphism.

2.2. An Alternative Approach

Perhaps the main contribution of the approach taken here lies in moving away from imposing conditions that ensure that sets of best replies are homeomorphic to convex sets as in Athey (2001) and McAdams (2003)). Indeed, the only reason for insisting upon convexity of best reply sets is to prepare for an application of Kakutani's (or Glicksberg's) fixed point theorem. But there are more powerful fixed point theorems one can instead rely upon, theorems which do not require convexity. Rather, these theorems rely upon the more permissive condition of contractibility.

Loosely, a set is contractible if it can be continuously shrunk, within itself, to one of its points. Formally, a subset X of a topological space is *contractible* if for some $x_0 \in X$ there is a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X$, $h(0,x) = x$ and $h(1,x) = x_0$. We then say that h is a *contraction* for X.

Note that every convex set is contractible since, choosing any point x_0 in the set, the function $h(\tau,x) = (1-\tau)x+\tau x_0$ is a contraction. On the other hand, there are contractible sets that are not convex (e.g., any curved line in $\mathbb{R}^2$ that does not intersect itself). Hence, contractibility is a strictly more permissive condition than convexity.

Returning to the auction example, it is not difficult to show that, against the given bidding function for bidder 2, bidder 1's set of best replies, while not homeomorphic to a convex set, is contractible. One way to see this is to first apply the Athey-map to 1's set of best replies, leading to the homeomorphic set in Figure 2.1. It then suffices to show that this latter set is contractible since contractibility is preserved under homeomorphism. But the set in Figure 2.1 is clearly contractible. Consider, for example, the contraction that shrinks the set radially into the point $(1/2, 1/2)$.

But is this a general property? That is, is each bidder's set of monotone best replies contractible no matter what monotone strategy is employed by the other bidder? Establishing the contractibility of a set is not, in general, trivial. However, establishing the contractibility of each bidder's set of monotone best replies, for any given monotone bidding function of the other, is rather simple. Indeed, contractibility can be established without referring to the Athey map and without considering jump points at all. The simplest approach is to consider the monotone bidding functions themselves.

So, fix some monotone bidding function for bidder 2, and suppose that $b^0 : [0,1] \to \{1,2,3\}$ is a monotone best reply for bidder 1.[8] We shall provide a contraction that shrinks bidder 1's set of monotone best replies, within itself, to the function b^0. The simple, but key, observation is that a bidding function is a best reply if and only if it it is an interim best reply for almost every signal $x \in [0,1]$.

Consider the following candidate contraction map (see Figure 2.2). For $\tau \in [0,1]$ and any monotone best reply, b, for bidder 1, define $h(\tau, b) : [0,1] \to \{1,2,3\}$ as follows:

$$h(\tau, b)(x) = \begin{cases} b(x), & \text{if } x \leq |1-2\tau| \text{ and } \tau < 1/2, \\ b^0(x), & \text{if } x \leq |1-2\tau| \text{ and } \tau \geq 1/2, \\ \max(b^0(x), b(x)), & \text{if } x > |1-2\tau|. \end{cases}$$

Note that $h(0,b) = b$, that $h(1,b) = b^0$, and that $h(\tau,b)(x)$ is always either $b^0(x)$ or $b(x)$, and so is a best reply given the signal x. The function $h(\tau,b)(\cdot)$ is also clearly monotone. It can also be shown that the monotone function $h(\tau,b)(\cdot)$ varies continuously in the arguments τ and b, when the set of monotone functions is endowed with the topology of almost everywhere pointwise convergence.[9] Consequently, h is a contraction, and we have established that, given any monotone bidding function for bidder 2, bidder 1's set of monotone best replies is contractible.

[8] We assume that a monotone best reply exists, which in fact it does. The existence of monotone best replies will be explicitly considered in the general setup of the sequel (see Section 4.1).

[9] That is, $b_n(\cdot) \to b(\cdot)$ if and only if $b_n(x) \to b(x)$ for a.e. $x \in [0,1]$.

Figure 2.2 shows how the contraction works. Three step functions are shown in each panel. The thin dashed line step function (black) is b, the thick solid line step function (green) is b^0, and the very thick solid line step function (red) is the step function determined by the contraction.

In panel (a), $\tau = 0$ and so the very thick (red) step function coincides with b. The position of the vertical line (blue) appearing in each panel represents the value of τ. When $\tau = 0$ the vertical line is at the far right-hand side, as shown in panel (a). As indicated by the arrow, the vertical line moves continuously toward the origin as τ moves from 0 to 1/2. The very thick (red) step function determined by the contraction is $b(x)$ for values of x to the left of the vertical line and is $\max(b^0(x), b(x))$ for values of x to the right; see panels (a)-(c). Note that this step function therefore changes continuously with τ, in a pointwise sense, and that when $\tau = 1/2$ this function is $\max(b^0(\cdot), b(\cdot))$.

In panels (d)-(f), τ increases from 1/2 to 1 and the vertical line moves from the origin continuously to the right. For these values of τ, the very thick (red) step function determined by the contraction is now $b^0(x)$ for values of x to the left of the vertical line and is $\max(b^0(x), b(x))$ for values of x to the right. Hence, when $\tau = 1$, the contraction yields $b^0(\cdot)$; see panel (f). So altogether, as τ moves continuously from 0 to 1, the image of the contraction moves continuously from b to b^0.

It can similarly be shown that, for any monotone bidding function of bidder 1, bidder 2's set of monotone best replies is contractible. Consequently, *so long as each player possesses a monotone best reply whenever the other employs a monotone bidding function*, an appropriate generalization of Kakutani's theorem — relying on contractible-valuedness instead of convex-valuedness — can be employed to establish that the example possesses a monotone pure strategy equilibrium.[10] Note that this is so even though the strong set order property fails to hold. Our approach goes through in general, whether or not the strong set order property holds.

Note also that single-crossing plays no role in the demonstration that best reply sets are contractible. In this totally-ordered action space example, best reply sets are contractible whether or not single-crossing holds since contractibility follows from the pointwise nature of best replies. But this does not mean that the single-crossing assumption is not useful. Recall that, for simplicity, we assumed the existence of monotone best replies. When single-crossing holds, one can prove

[10] The appropriate generalization is due to Eilenberg and Montgomery (1946). See Section 4 below.

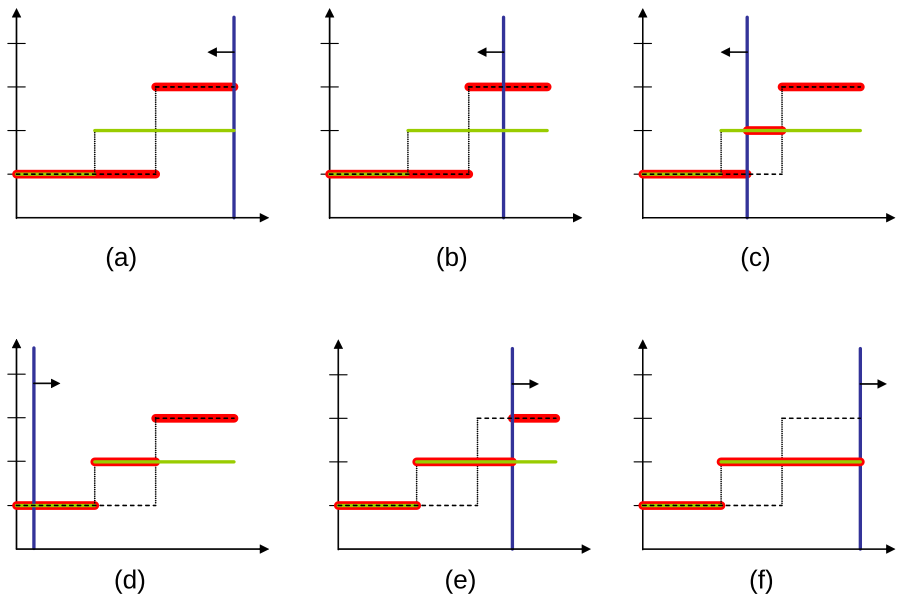

Figure 2.2: The Contraction

their existence (as we shall do in Section 4.1). But note well that when we do assume single-crossing, we shall do so only to ensure the existence of monotone best replies, nothing more. This permits us to employ a weaker single-crossing assumption than both Athey and McAdams. The remainder of the paper is based entirely upon these simple ideas.

3. The Environment

3.1. Lattices and Semilattices

Let A be a non empty set and let $\geq$ be a partial order on A.[11] For $a, b \in A$, if the set $\{a, b\}$ has a least upper bound (l.u.b.) in A, then it is unique and will denoted by $a \vee b$, the *join* of a and b. In general, such a bound need not exist. However, if every pair of points in A has an l.u.b. in A, then we shall say that $(A, \geq)$ is a *semilattice*. It is straightforward to show that, in a semilattice, every

[11]That is, $\geq$ is transitive ($a \geq b$ and $b \geq c$ imply $a \geq c$), reflexive ($a \geq a$), and antisymmetric ($a \geq b$ and $b \geq a$ imply $a = b$).

finite set, $\{a, b, ..., c\}$, has a least upper bound, which we denote by $\vee\{a, b, ..., c\}$ or $a \vee b \vee ... \vee c$.

If the set $\{a, b\}$ has a greatest lower bound (g.l.b.) in A, then it too is unique and it will be denoted by $a \wedge b$, the *meet* of a and b. Once again, in general, such a bound need not exist. If every pair of points in A has both an l.u.b. in A and a g.l.b. in A, then we shall say that $(A, \geq)$ is a *lattice.*[12]

Clearly, every lattice is a semilattice. However, the converse is not true. For example, employing the coordinatewise partial order on vectors in $\mathbb{R}^m$, the set of vectors whose sum is at least one is a semilattice, but not a lattice.

A *topological semilattice* is a semilattice endowed with a topology under which the join operator, $\vee$, is continuous as a function from $A \times A$ into A.[13,14] Clearly, every finite semilattice is a topological semilattice. Note also that because in a semilattice $b \geq a$ if and only if $a \vee b = b$, in a topological semilattice $\{(a, b) \in A \times A : b \geq a\}$ is closed. When the topology on A rendering the join operator continuous is metrizable we say that $(A, \geq)$ is a *metrizable semilattice.* When the topology on A renders A compact, we say that $(A, \geq)$ is *compact.*

A semilattice $(A, \geq)$ is *complete* if every non empty subset S of A has a least upper bound, $\vee S$, in A. A topological semilattice $(A, \geq)$ is *locally complete* if for every $a \in A$ and every neighborhood U of a, there is a neighborhood W of a contained in U such that every non empty subset S of W has a least upper bound, $\vee S$, contained in U.[15]

Many metrizable semilattices are locally complete. For example, local completeness holds trivially in any finite semilattice, and more generally in any metrizable semilattice $(A, \geq)$ where A is a compact subset of $\mathbb{R}^K$ and $\geq$ is the coordinatewise partial order (see Lemma C.3). On the other hand, infinite-dimensional compact metrizable semilattices need not be locally complete.[16] Indeed, it can be

[12] Defining a semilattice in terms of the join operation, $\vee$, rather than the meet operation, $\wedge$, is entirely a matter of convention.

[13] Product spaces are endowed with the product topology throughout the paper.

[14] For example, the set $A = \{(x, y) \in \mathbb{R}^2_+ : x + y = 1\} \cup \{(1, 1)\}$ is a semilattice with the coordinatewise partial order. But it is not a topological semilattice when supplemented with, say, the Euclidean metric because whenever $a_n \neq b_n$ and $a_n, b_n \to a$, we have $(1, 1) = \lim(a_n \vee b_n) \neq (\lim a_n) \vee (\lim b_n) = a$.

[15] We have not found a reference to the concept of local completeness in the lattice-theoretic literature.

[16] Whether or not every compact metrizable semilattice is locally complete was to us an open question until a recent visit to The Center for the Study of Rationality at The Hebrew University of Jerusalem. Shortly after we posed the question, Sergiu Hart and Benjamin Weiss settled the matter by graciously providing a subtle and beautiful example of a compact metrizable

shown (see Lemma C.2) that a compact metrizable semilattice $(A, \geq)$ is locally-complete if and only if for every $a \in A$ and every sequence $a_n \to a$, $\lim_m (\vee_{n \geq m} a_n) = a$.[17] A distinct sufficient condition for local completeness is given in Lemma C.4.

Finally, we wish to mention that because our main result requires only a notion of least upper bound, we have found it natural to consider semilattices rather than lattices in most of our formal development. On the other hand, a development within the confines of a lattice structure would entail little loss of generality since any complete semilattice becomes a complete lattice when supplemented with single point that is deemed less than all others.[18] However, our assumptions would then have to be stated with explicit reference to the join operator. For example, local completeness in a semilattice as defined above is equivalent to local completeness "with respect to the join operator" in a lattice. Such a qualification would be important since local completeness "with respect to the meet operator," is a substantive additional restriction that is not necessary for our results. Our choice to employ semilattices is therefore largely a matter of convenience as it avoids the need for such qualifications.

3.2. A Class of Bayesian Games

Consider any Bayesian game, G, described as follows. There are N players, $i = 1, 2, ..., N$. Player i's type space is $T_i = [0,1]^{k_i}$ endowed with the Euclidean metric and the coordinatewise partial order, and i's action space is a partially ordered topological space A_i. All partial orders, although possibly distinct, will be denoted by $\geq$. Player i's bounded and measurable payoff function is $u_i : A \times T \to \mathbb{R}$, where $A = \times_{i=1}^N A_i$ and $T = \times_{i=1}^N T_i$. The common prior over the players' types is a probability measure μ on the Borel subsets of T. This completes the description of G.

A subset C of $[0,1]^m$ is a *strict chain* if for any two points in C, one of them is strictly greater, coordinate by coordinate, than the other. We shall make use

semilattice that is not locally complete (see Hart and Weiss (2005)). In contrast, such examples are not difficult to find if compactness is not required. For instance, no $\mathcal{L}_p$ space is locally complete when $p < +\infty$.

[17] Hence, compactness and metrizability of a lattice under the order topology (see Birkohff (1967, p.244) is sufficient, but not necessary, for local completeness of the corresponding semilattice.

[18] Given any subset B of a complete semilattice, its set of lower bounds is either non empty or empty. In the former case, completeness implies that the join of the lower bounds exists and is then the g.l.b. of B. In the latter case, the added point is the g.l.b. of B.

of the following assumptions, where μ_i denotes the marginal of μ on T_i. For every player i, and every Borel subset B of T_i,

G.1 $\mu_i(B) = 0$ if $B \cap C$ is countable for every strict chain C in T_i.

G.2 $(A_i, \geq)$ is a compact locally-complete metrizable semilattice.

G.3 $u_i(\cdot, t) : A \to \mathbb{R}$ is continuous for every $t \in T$.

Assumptions G.1-G.3 strictly generalize the assumptions in Athey (2001) and McAdams (2003) who assume that each A_i is a finite sublattice of Euclidean space and that μ is absolutely continuous with respect to Lebesgue measure.[19]

Note that G.1 implies that each μ_i is atomless because we may set $B = \{t_i\}$ for any $t_i \in T_i$. Note also that we do not require the familiar absolute continuity condition on μ introduced in Milgrom and Weber (1985). For example, when each player's type space is $[0, 1]$ with its usual total order, G.1 holds if and only if μ_i is atomless. In particular, G.1 holds when there are two players, each with unit interval type space, and the types are drawn according to Lebesgue measure conditional on any one of finitely many positively or negatively sloped lines in the unit square. Assumption G.1 helps ensure the compactness of the players' sets of monotone pure strategies (see Lemma 6.1) in a topology in which ex-ante payoffs are continuous. This assumption therefore plays the same role for monotone pure strategies as the Milgrom-Weber (1985) absolute-continuity assumption plays for mixed strategies.

It can be shown (see Lemma A.1) that every compact metrizable semilattice is equivalent to a compact semilattice in the Hilbert cube, $[0, 1]^\infty$, with the coordinatewise partial order and the coordinatewise Euclidean metric. On the other hand, assumption G.2 as stated above is more easily verified in practice than its Hilbert cube counterpart because the natural description of the players' action spaces might not be as subsets of the Hilbert cube (e.g., when players are consumers in an exchange economy with private information and their actions are demand functions to submit to an auctioneer). As mentioned in the previous subsection, G.2 holds for example whenever $(A_i, \geq)$ is a compact metrizable semilattice in Euclidean space with the coordinatewise partial order (see Lemma C.3).

[19] Absolute continuity of μ implies G.1 because, if for some Borel subset B of i's type space, $B \cap C$ is countable for every strict chain C, then $B \cap C$ is countable for every strict chain of the form $[0, 1]t_i$ with $t_i \in T_i$. But then Fubini's theorem implies that B has Lebesgue measure zero, and so $\mu_i(B) = 0$ by absolute continuity.

A *pure strategy* for player i is a measurable function, $s_i : T_i \to A_i$. Such a pure strategy is *monotone* if $t_i' \geq t_i$ implies $s_i(t_i') \geq s_i(t_i)$.[20]

An N-tuple of pure strategies, $(\hat{s}_1, ..., \hat{s}_N)$ is an *equilibrium* if for every player i and every pure strategy s_i',

$$\int_T u_i(\hat{s}(t), t) d\mu(t) \geq \int_T u_i(s_i'(t_i), \hat{s}_{-i}(t_{-i}), t) d\mu(t),$$

where the left-hand side, henceforth denoted by $U_i(\hat{s})$, is player i's payoff given the joint strategy $\hat{s}$, and the right-hand side is his payoff when he employs s_i' and the others employ $\hat{s}_{-i}$.

It will sometimes be helpful to speak of the payoff to player i's type t_i from the action a_i given the strategies of the others, s_{-i}. This payoff, which we will refer to as i's *interim* payoff, is

$$V_i(a_i, t_i, s_{-i}) \equiv \int_T u_i(a_i, s_{-i}(t_{-i}), t) d\mu_i(t_{-i}|t_i),$$

where $\mu_i(\cdot|t_i)$ is a version of the conditional probability on T_{-i} given t_i. A single such version is fixed for each player i once and for all.

4. The Main Result

Call a set of player i's pure strategies *join-closed* if for any pair of strategies, s_i, s_i', in the set, the strategy taking the action $s_i(t_i) \vee s_i'(t_i)$ for each $t_i \in T_i$ is also in the set.[21] We can now state our main result, whose proof is provided in Section 6.

Theorem 4.1. *If G.1-G.3 hold and for each player i, whenever the others employ monotone pure strategies, player i's set of monotone pure best replies is non empty and join-closed, then G possesses a monotone pure strategy equilibrium.*

[20] Note that both definitions involve the entire set T_i, not merely a set of full μ_i-measure. This is simply a matter of convention. In particular, if a strategy, s_i, is monotone on a subset, C, having full μ_i-measure, the strategy, $\hat{s}_i$, defined by $\hat{s}_i(t_i) = \vee\{s_i(t_i') : t_i' \leq t_i, t_i' \in C\}$, coincides with s_i on C and is monotone on all of T_i. When $(A_i, \geq)$ is a compact metrizable semilattice, as we shall assume, Lemma C.1 ensures that $\hat{s}_i$ is well-defined, and Lemma A.4 takes care of measurability.

[21] Note that when the join operator is continuous, as it is in a metrizable semilattice, the resulting function is measurable, being the composition of measurable and continuous functions.

Remark 1. *In any setting in which the action sets are totally ordered (as in Athey (2001)), each player's set of monotone best replies is automatically join-closed.*

Remark 2. *Athey (2001) assumes that the A_i are finite and totally ordered, and McAdams (2003) assumes that each $(A_i, \geq)$ is a finite sublattice of $\mathbb{R}^k$ with the coordinatewise partial order. This additional structure, which we do not require, is necessary for their Kakutani-Glicksberg-based approach.*[22]

It is well-known that within the confines of a lattice, quasisupermodularity and single-crossing conditions on interim payoffs guarantee the existence of monotone best replies and that sets of monotone best replies are lattices and hence join-closed. In the next section, we provide slightly weaker versions of these conditions and, for completeness, show that they guarantee that the players' sets of monotone best replies are non empty and join-closed.

4.1. Sufficient Conditions on Interim Payoffs

Suppose that for each player i, $(A_i, \geq)$ is a lattice. We say that player i's interim payoff function V_i is *weakly quasisupermodular* if for all monotone pure strategies s_{-i} of the others, all $a_i, a_i' \in A_i$, and every $t_i \in T_i$

$$V_i(a_i, t_i, s_{-i}) \geq V_i(a_i \wedge a_i', t_i, s_{-i}) \text{ implies } V_i(a_i \vee a_i', t_i, s_{-i}) \geq V_i(a_i', t_i, s_{-i}).$$

This weakens slightly Milgrom and Shannon's (1994) concept of quasisupermodularity by not requiring the second inequality to be strict if the first happens to be strict. McAdams (2003) requires the stronger condition of quasisupermodularity. When actions are totally ordered, as in Athey (2001), interim payoffs are automatically supermodular, and hence both quasisupermodular and weakly quasisupermodular.

[22] Indeed, consider the lattice $A = \{(0,0), (1,0), (1/2,1/2), (0,1), (1,1)\}$ in $\mathbb{R}^2$, with the coordinatewise partial order and note that A is not a sublattice of $\mathbb{R}^2$. It can be shown that the set of monotone functions from $[0,1]$ into A, endowed with the topology of almost everywhere pointwise convergence, is not homeomorphic to a convex set. Indeed, it is instead homeomorphic to three triangles joined at a common edge to form a three-bladed arrowhead in $\mathbb{R}^3$. Hence, if each player has action set A, then neither Kakutani's nor Glicksberg's theorems can be applied to the game's monotone best-reply correspondence. On the other hand, the set of monotone functions in this example is an absolute retract (see Lemma 6.2), which is sufficient for our approach.

A simple way to verify weak quasisupermodularity is to verify supermodularity. For example, it is well-known that V_i is supermodular in actions (hence weakly quasisupermodular) when $A_i = [0,1]^K$ is endowed with the coordinatewise partial order, and the second cross-partial derivatives of $V_i(a_{i1}, ..., a_{iK}, t_i, s_{-i})$ with respect distinct action dimensions are nonnegative. Thus, complementarities in the distinct dimensions of a player's *own* actions are natural economic conditions under which weak quasisupermodularity holds.[23]

We say that i's interim payoff function V_i satisfies *weak single-crossing* if for all monotone pure strategies s_{-i} of the others, for all player i action pairs $a'_i \geq a_i$, and for all player i type pairs $t'_i \geq t_i$,

$$V_i(a'_i, t_i, s_{-i}) \geq V_i(a_i, t_i, s_{-i})$$

implies

$$V_i(a'_i, t'_i, s_{-i}) \geq V_i(a_i, t'_i, s_{-i}).$$ [24]

To ensure that each player's set of monotone best replies is homeomorphic to a convex set, both Athey (2001) and McAdams (2003) assume that V_i satisfies a more stringent single-crossing condition. In particular they each require that, in addition to the above, the second single-crossing inequality is strict whenever the first one is. Returning to the example of Section 2, bidder 1's interim payoff function there satisfies weak single-crossing but it fails to satisfy Athey's and McAdams' single-crossing condition because, for example, a bid of 3 is strictly better than a bid of 2 for bidder 1 when his signal is low, but it is only weakly better when his signal is high. Because of this, bidder 1's set of monotone best replies is not homeomorphic to a convex set and the results of Athey and McAdams cannot be directly applied.

In contrast, the following corollary of Theorem 4.1 states that pure monotone equilibria exist if each V_i is weakly quasisupermodular and satisfies weak single-crossing.

Corollary 4.2. *If G.1-G.3 hold, each* $(A_i, \geq)$ *is a lattice, and the players' interim payoffs are weakly quasisupermodular and satisfy weak single-crossing, then G possesses a monotone pure strategy equilibrium.*

[23] Complementarities between the actions of distinct *players* is not required. This is useful because, for example, many auction games satisfy only own-action complementarity.

[24] For conditions on the joint distribution of types, μ, and the players' payoff functions, $u_i(a, t)$, leading to the weak single-crossing property, see Athey (2001, pp.879-81), McAdams (2003, p.1197) and Van Zandt and Vives (2005).

Proof. By Theorem 4.1, it suffices to show that weak quasisupermodularity and weak single-crossing imply that whenever the others employ monotone pure strategies, player i's set of monotone pure best replies is non empty and join-closed. To see join-closedness, note that if against some monotone pure strategy of the others, actions a_i and a_i' are interim best replies for i when his type is t_i, then weak quasisupermodularity implies that so too is $a_i \vee a_i'$. Since two pure strategies are best replies for i if and only if they specify interim best replies for almost every t_i, join-closedness follows. (Because the join operator is continuous in a metrizable semilattice, the join of two measurable functions is measurable, being the composition of measurable and continuous functions.)

Fix a monotone pure strategy, s_{-i}, for player i's opponents, and let $B_i(t_i)$ denote i's interim best reply actions against s_{-i} when his type is t_i. By action-continuity, $B_i(t_i)$ is compact and non empty, and by the argument in the previous paragraph $B_i(t_i)$ is a subsemilattice of $(A_i, \geq)$. Define $\bar{s}_i : T_i \to A_i$ by setting $\bar{s}_i(t_i) = \vee B_i(t_i)$ for each $t_i \in T_i$. Lemma C.1 together with the compactness and subsemilattice properties of $B_i(t_i)$ imply that, for every t_i, $\bar{s}_i(t_i)$ is well defined and $\bar{s}_i(t_i) \in B_i(t_i)$.

We next show that $\bar{s}_i$ is monotone. Suppose that $t_i' \geq t_i$. Then

$$V_i(\bar{s}_i(t_i), t_i, s_{-i}) \geq V_i(\bar{s}_i(t_i) \wedge \bar{s}_i(t_i'), t_i, s_{-i}), \tag{4.1}$$

since $\bar{s}_i(t_i) \in B_i(t_i)$. By weak single-crossing, (4.1) implies that

$$V_i(\bar{s}_i(t_i), t_i', s_{-i}) \geq V_i(\bar{s}_i(t_i) \wedge \bar{s}_i(t_i'), t_i', s_{-i}). \tag{4.2}$$

Hence, applying weak quasisupermodularity to (4.2) we obtain

$$V_i(\bar{s}_i(t_i') \vee \bar{s}_i(t_i), t_i', s_{-i}) \geq V_i(\bar{s}_i(t_i'), t_i', s_{-i}),$$

from which we conclude that $\bar{s}_i(t_i') \vee \bar{s}_i(t_i) \in B_i(t_i')$. But $\bar{s}_i(t_i') = \vee B_i(t_i')$ is the largest member of $B_i(t_i')$. Hence $\bar{s}_i(t_i') \vee \bar{s}_i(t_i) = \bar{s}_i(t_i')$, implying that $\bar{s}_i(t_i') \geq \bar{s}_i(t_i)$ as desired.

Lastly, we must ensure measurability. But for this we may appeal to Lemma A.4, which states that, because $\bar{s}_i$ is monotone, there exists a measurable and monotone $\hat{s}_i$ that coincides with $\bar{s}_i$ μ_i almost everywhere on T_i. Hence, $\hat{s}_i$ is a monotone pure strategy and is a best reply. Player i's set of monotone pure best replies is therefore non empty. ■

Remark 3. *Weak quasisupermodularity is used to ensure both join-closedness and that monotone best replies exist. On the other hand, weak single-crossing is employed only in the proof of the latter.*

Remark 4. *Finite lattices are automatically compact, locally complete, metrizable semilattices. Hence, Corollary 4.2 generalizes the main results of Athey (2001) and McAdams (2003). In fact, the corollary is a strict generalization because its hypotheses are satisfied in the example of Section 2, whereas the stronger hypotheses of Athey (2001) and McAdams (2003) are not.*

Corollary 4.2 will often suffice in applications. However, the additional generality provided by Theorem 4.1 is sometimes important. For example, Reny and Zamir (2004) have shown in the context of asymmetric first-price auctions with finite bid sets that monotone best replies exist even though weak single-crossing fails. Since action sets (i.e., bids) are totally ordered, best reply sets are necessarily join-closed and so the hypotheses of Theorem 4.1 are satisfied while those of Corollary 4.2 are not.

5. A Useful Fixed Point Theorem

The proof of Theorem 4.1 relies on a corollary of Eilenberg and Montgomery's (1946) fixed point theorem. This corollary is interesting in its own right because it is a substantial generalization of Kakutani's theorem, yet, like Kakutani's theorem, its hypotheses require only elementary topological concepts, which we now review.

Recall from Section 2 that a subset X of a metric space is *contractible* if for some $x_0 \in X$ there is a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X$, $h(0,x) = x$ and $h(1,x) = x_0$. We then say that h is a *contraction* for X.

A subset X of a metric space Y is said to be a *retract* of Y if there is a continuous function mapping Y onto X leaving every point of X fixed. A metric space (X,d) is an *absolute retract* if for every metric space (Y,δ) containing X as a closed subset and preserving its topology, X is a retract of Y. Examples of absolute retracts include closed convex subsets of Euclidean space or of any metric space, and many non convex sets as well (e.g., any contractible polyhedron).[25]

Theorem 5.1. *(Eilenberg and Montgomery (1946)) Suppose that a compact metric space (X,d) is an absolute retract and that $F : X \twoheadrightarrow X$ is an upper hemicontinuous, non empty-valued, contractible-valued correspondence. Then F has a fixed point.*

[25] Indeed, a compact subset, X, of Euclidean space is an absolute retract if and only if it is contractible and locally contractible. The latter means that for every $x_0 \in X$ and every neighborhood U of x_0, there is a neighborhood V of x_0 and a continuous $h : [0,1] \times V \to U$ such that $h(0,x) = x$ and $h(1,x) = x_0$ for all $x \in V$.

Proof. The result follows directly from Eilenberg and Montgomery (1946) Theorem 1, because every absolute retract is a contractible absolute neighborhood retract (Borsuk (1966), V (2.3)) and every non empty contractible set is acyclic (Borsuk (1966), II (4.11)). ■

6. Proof of Theorem 4.1

Let M_i denote the set of monotone pure strategies for player i, and let $M = \times_{i=1}^{N} M_i$. Let $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ denote player i's best-reply correspondence when players are restricted to monotone pure strategies. Because, by hypothesis, each player possesses a monotone best reply (among all measurable strategies) when the others employ monotone pure strategies, any fixed point of $\times_{i=1}^{n} \mathbf{B}_i : M \twoheadrightarrow M$ is a monotone pure strategy equilibrium. The following steps demonstrate that such a fixed point exists.

6.1. The M_i are Compact Absolute Retracts

We first demonstrate that each player's space of monotone pure strategies can be metrized so that it is a compact absolute retract. Without loss, we may assume that the metric d_{A_i} on A_i is bounded.[26] Given d_{A_i}, define a metric δ_{M_i} on M_i as follows:[27]

$$\delta_{M_i}(s_i, s_i') = \int_{T_i} d_{A_i}(s_i(t_i), s_i'(t_i)) d\mu_i(t_i).$$

This metric does not distinguish between strategies that are equal μ_i almost everywhere. This is natural since, from each player's ex-ante viewpoint, such strategies are payoff equivalent.

Now, suppose that s_i^n is a sequence in M_i. Then, by the semilattice-extension of Helley's theorem given in Lemma A.5, s_i^n has a μ_i almost everywhere pointwise convergent subsequence. That is, there exists a subsequence, $s_i^{n_k}$, and $s_i \in M_i$ such that

$$s_i^{n_k}(t_i) \to_k s_i(t_i) \text{ for } \mu_i \text{ almost every } t_i \in T_i.$$

[26]For any metric, $d(\cdot,\cdot)$, an equivalent bounded metric is $\min(1, d(\cdot,\cdot))$.

[27]Formally, the resulting metric space (M_i, δ_{M_i}) is the space of equivalence classes of strategies in M_i that are equal μ_i almost everywhere. Nevertheless, analogous to the standard treatment of $\mathcal{L}_p$ spaces, in the interest of notational simplicity we focus on the elements of the orginal space M_i rather than on the equivalence classes themselves.

Consequently, $d_{A_i}(s_i^{n_k}(t_i), s_i(t_i))$, a bounded function of t_i, converges to zero μ_i almost everywhere as $k \to \infty$, so that, by the dominated convergence theorem, $\delta_{M_i}(s_i^{n_k}, s_i) \to_k 0$. We have therefore established the following result.

Lemma 6.1. *The metric space (M_i, δ_{M_i}) is compact.*

The metric δ_{M_i} also renders (M_i, δ_{M_i}) an absolute retract, as stated in the next lemma, whose proof follows directly from Lemma B.3 in Appendix B.

Lemma 6.2. *The metric space (M_i, δ_{M_i}) is an absolute retract.*

Remark 5. *One cannot improve upon Lemma 6.2 by proving, for example, that M_i, metrized by δ_{M_i}, is homeomorphic to a convex set. It need not be (e.g., see footnote 21). Evidently, our approach can handle action spaces that the Athey-McAdams approach cannot easily accommodate, if at all. An economic example of this type would certainly be of some interest.*

6.2. Upper-Hemicontinuity

We next demonstrate that, given the metric δ_j on each M_j, each player i's payoff function, $U_i : M \to \mathbb{R}$, is continuous under the product topology. This immediately yields upper-hemicontinuity of best reply correspondences. To see payoff continuity, suppose that s^n is a sequence of joint strategies in M, and that $s^n \to s \in M$. By Lemma B.1, this implies that for each player i, $s_i^n(t_i) \to s_i(t_i)$ for μ_i a.e. t_i in T_i. Consequently, $s^n(t) \to s(t)$ for μ a.e. $t \in T$. Hence, since u_i is bounded, Lebesgue's dominated convergence theorem yields

$$U_i(s^n) = \int_T u_i(s^n(t), t) d\mu(t) \to \int_T u_i(s(t), t) d\mu(t) = U_i(s),$$

establishing the continuity of U_i.

Now, because each player i's payoff function, U_i, is continuous and each M_i is compact, an application of Berge's theorem of the maximum immediately yields the following result.

Lemma 6.3. *Each player i's best-reply correspondence, $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$, is non empty-valued and upper-hemicontinuous.*

6.3. Contractible-Valuedness

The simple observation at the heart of the present paper is that each player i's set of monotone best replies is contractible. A straightforward contraction map follows, where the vector of 1's is denoted by $\mathbf{1}$.

Define $h : [0,1] \times M_i \times M_i \to M_i$ as follows: For every $t_i \in [0,1]^{k_i}$,

$$h(\tau, f, g)(t_i) = \begin{cases} f(t_i), & \text{if } \mathbf{1} \cdot t_i \leq |1 - 2\tau| \, k_i \text{ and } \tau < 1/2 \\ g(t_i), & \text{if } \mathbf{1} \cdot t_i \leq |1 - 2\tau| \, k_i \text{ and } \tau \geq 1/2 \\ f(t_i) \vee g(t_i), & \text{if } \mathbf{1} \cdot t_i > |1 - 2\tau| \, k_i \end{cases} \tag{6.1}$$

Note that $h(\tau, f, g)$ is indeed monotone because, if for example $\tau < 1/2$, then $h(\tau, f, g)(t_i)$ is $f(t_i)$, a monotone function of t_i, when $\mathbf{1} \cdot t_i \leq |1 - 2\tau| \, k_i$; and is $f(t_i) \vee g(t_i)$, which is both monotone and larger than $f(t_i)$, when $\mathbf{1} \cdot t_i > |1 - 2\tau| \, k_i$. Also, note that $h(0, f, g) = f$ and $h(1, f, g) = g$. Continuity will be established below.

Figure 6.1 provides snapshots of $h(\tau, f, g)$ as τ moves from zero to unity when $k_i = 2$. The axes are the two dimensions of the type vector and the arrow within the figures depicts the direction in which the diagonal line, $\{t_i : \mathbf{1} \cdot t_i = |1 - 2\tau| \, k_i\}$, moves as τ increases locally. For example, panel (a) shows that when $\tau = 0$, $h(\tau, f, g)$ is equal to f over the entire unit square. On the other hand, panel (f) shows that when $\tau = 5/6$, $h(\tau, f, g,)$ is equal to g below the diagonal line and equal to $f \vee g$ above it.

Lemma 6.4. $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ *is contractible-valued.*

Proof. Fix $s_{-i} \in M_{-i}$. To establish the contractibility of $\mathbf{B}_i(s_{-i})$, suppose that $f, g \in \mathbf{B}_i(s_{-i})$. Because, by hypothesis, $\mathbf{B}_i(s_{-i})$ is join-closed, the monotone function, $f \vee g$, taking the action $f(t_i) \vee g(t_i)$ for each $t_i \in [0,1]^{k_i}$ is also in $\mathbf{B}_i(s_{-i})$. Consequently, $[h(\tau, f, g)](t_i)$, being equal to either $f(t_i)$, $g(t_i)$, or $f(t_i) \vee g(t_i)$, must maximize $V_i(a_i, t_i, s_{-i})$ over $a_i \in A_i$ for almost every $t_i \in [0,1]^{k_i}$, because this μ_i almost-everywhere maximization property holds, by hypothesis, for every member of $\mathbf{B}_i(s_{-i})$ and so separately for each of f, g, and $f \vee g$. But this implies that for every $\tau \in [0,1]$, $h(\tau, f, g) \in \mathbf{B}_i(s_{-i})$. So, because $h(0, f, g) = f$, $h(1, f, g) = g$ and $h(\cdot, \cdot, \cdot)$ is, by Lemma B.2, continuous, $h(\cdot, \cdot, g)$ is a contraction for $\mathbf{B}_i(s_{-i})$. ■

6.4. Completing the Proof.

The following lemma completes the proof of Theorem 4.1.

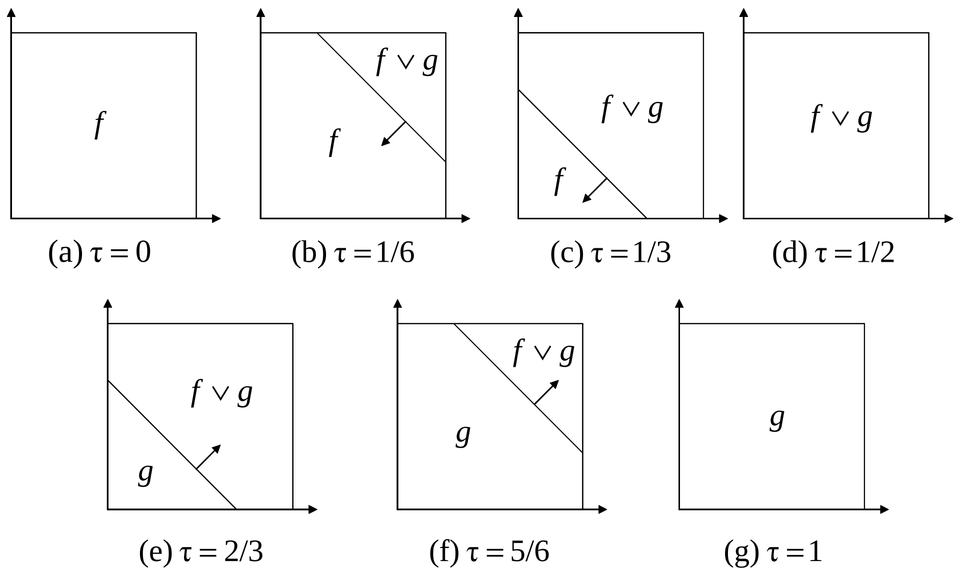

Figure 6.1: $h(\tau, f, g)$ as τ varies from 0 (panel (a)) to 1 (panel (g)) and the domain is the unit square.

Lemma 6.5. *The product of the players' best reply correspondences,* $\times_{i=1}^{n} \mathbf{B}_i : M \twoheadrightarrow M$, *possesses a fixed point.*

Proof. By Lemmas 6.1 and 6.2, each (M_i, δ_{M_i}) is a compact absolute retract. Consequently, under the product topology, M is both compact and, by Borsuk (1966) IV (7.1), an absolute retract. By Lemmas 6.3 and 6.4, $\times_{i=1}^{n} \mathbf{B}_i : M \twoheadrightarrow M$ is u.h.c., non empty-valued, and contractible-valued. Hence, applying Theorem 5.1 to $\times_{i=1}^{n} \mathbf{B}_i : M \twoheadrightarrow M$ yields the desired result. ■

Appendices

A. Canonical Semilattices

It is not difficult to see that any finite semilattice $(A, \geq)$ can be represented as a finite semilattice in Euclidean space with the coordinatewise partial order.[28] Thus, for finite semilattices, Euclidean space with its coordinatewise partial order

[28] Assign to each $a \in A$ the vector $x \in \mathbb{R}^A$ where $x_{a'} = 1$ if $a \geq a'$ and $x_{a'} = 0$ otherwise.

is canonical. We now describe a similar result for arbitrary compact metrizable semilattices.

Say that two metrizable semilattices $(A, \geq_A)$ and $(B, \geq_B)$ are *equivalent* if there is a homeomorphism ϕ mapping A onto B such that for all $a, b \in A$,

$$a \geq_A b \text{ if and only if } \phi(a) \geq_B \phi(b) \tag{A.1}$$

Recall that the Hilbert cube is the normed space $[0,1]^\infty$ with norm $\|x\| = \sum_n \frac{1}{2^n} x_n$. It is partially ordered by the coordinatewise partial order (i.e., $x \geq y$ iff $x_n \geq y_n$ all n). The following result states that given any compact metrizable semilattice $(A, \geq)$, one can assume without loss of generality that A is a compact subset of the Hilbert cube and that $\geq$ is the coordinatewise partial order.[29]

Lemma A.1. *Every compact metrizable semilattice is equivalent to a compact semilattice in the Hilbert cube.*

Proof. Let $(A, \geq_A)$ be a compact metrizable semilattice with metric d and suppose without loss that $d(a, b) \leq 1$ for all $a, b \in A$. Let $A^0 = \{a_1, a_2, ...\}$ be a countable dense subset of A. Define the function ϕ from A into the Hilbert cube $[0,1]^\infty$, by $\phi(a) = (\min_{a'} d(a' \vee a, a_1), \min_{a'} d(a' \vee a, a_2), ...)$, where each minimum is taken over all $a' \in A$. To see that ϕ is continuous, note that, for each n, Berge's theorem of the maximum and the fact that the join operator is continuous imply that the n-th coordinate function $\min_{a'} d(a' \vee a, a_n)$ is continuous in a.

We next wish to show that

$$a \geq_A b \text{ if and only if } \phi(a) \geq \phi(b), \tag{A.2}$$

where $\geq$ is the coordinatewise partial order on $[0,1]^\infty$. Before proving this, note that a corollary is that ϕ is one to one and hence, by compactness, a homeomorphism from A onto $\phi(A)$. Hence, the proof will be complete once we prove (A.2).

So, suppose first that $a \geq_A b$. Then, for each n,

$$\begin{aligned} \min_{a'} d(a' \vee a, a_n) &= \min_{a' \geq_A a} d(a', a_n) \\ &\geq \min_{a' \geq_A b} d(a', a_n) \\ &= \min_{a'} d(a' \vee b, a_n), \end{aligned}$$

[29] But note that $x \vee y = (\max(x_n, y_n))_{n=1}^\infty$ *need not hold* since the coordinatewise max need not be a member of A.

and so $\phi(a) \geq \phi(b)$. Conversely, suppose that $\phi(a) \geq \phi(b)$. Then for each n, $\min_{a'} d(a' \vee a, a_n) \geq \min_{a'} d(a' \vee b, a_n)$. In particular, this holds true along a subsequence, a_{n_k} of a_n, converging to a. Consequently, $0 = \min_{a'} d(a' \vee a, a) \geq \min_{a'} d(a' \vee b, a)$, so that $a' \vee b = a$ for some $a' \in A$. But this means that $a \geq_A b$, as desired. ■

Lemma A.1 is useful because it permits one to prove results about compact metrizable semilattices by considering compact semilattices in the Hilbert cube. The lemmas below can be proved by first proving them when $(A, \geq)$ is a compact semilattice in the Hilbert cube and then applying Lemma A.1. Since the Hilbert cube proofs amount to separate proofs for each of the countably many copies of $[0, 1]$, and the proofs for $[0, 1]$ are standard, the proofs are omitted.

In each of the lemmas below, it is assumed that $(A, \geq)$ is a compact metrizable semilattice and ν is a probability measure on $[0, 1]^m$ satisfying assumption G.1 from Section 3. Assumption G.1 is used to ensure that every measurable and monotone function $f : [0, 1]^m \rightarrow A$ is continuous v almost everywhere. This is a consequence of the fact that the restriction of f to any strict chain C is discontinuous at no more than countably many points in C, which itself implies that $D \cap C$ is countable for all strict chains C, where D denotes the set of discontinuity points of f. These latter two results are standard and so we omit their proofs.

Lemma A.2. *If a_n, c_n are sequences in A converging to a, and $a_n \leq b_n \leq c_n$ for every n, then b_n converges to a.*

Lemma A.3. *Every nondecreasing sequence and every nonincreasing sequence in $(A, \geq)$ converges.*

Lemma A.4. *If $f : [0, 1]^m \rightarrow A$ is monotone, then there is a measurable and monotone $g : [0, 1]^m \rightarrow A$ such that f and g are equal and continuous ν almost everywhere on $[0, 1]^m$.*

Lemma A.5. *(Helley's Theorem). If $f_n : [0, 1]^m \rightarrow A$ is a sequence of monotone functions, then there is a subsequence, f_{n_k}, and a measurable monotone function, $f : [0, 1]^m \rightarrow A$, such that $f_{n_k}(t) \rightarrow_k f(t)$ for ν almost every $t \in [0, 1]^m$.*

B. The Space of Monotone Functions

Throughout this appendix it is assumed that $(A, \geq)$ is a compact metrizable semi-lattice with metric d which is assumed without loss to satisfy $d(a, b) \leq 1$ for all $a, b \in A$. We also let $\mathcal{M}$ denote the set of measurable monotone functions from $[0, 1]^m$ into A, and define the metric, δ, on $\mathcal{M}$ by

$$\delta(f, g) = \int_{[0,1]^m} d(f(t), g(t))d\nu(t),$$

where ν is a probability measure on $[0, 1]^m$ satisfying assumption G.1 from Section 3.

Lemma B.1. *In $(\mathcal{M}, \delta)$, f_k converges to f if and only if in $(A, \geq)$, $f_k(t)$ converges to $f(t)$ for ν almost every $t \in [0, 1]^m$.*

Proof. (only if) Suppose that $\delta(f_k, f) \to 0$. By Lemma A.4 it suffices to show that $f_k(t) \to f(t)$ for all interior continuity points, t, of f.

Suppose that t_0 is an interior continuity point of f. Because A is compact, it suffices to show that an arbitrary convergent subsequence, $f_{k_j}(t_0)$, of $f_k(t_0)$ converges to $f(t_0)$. So, suppose that $f_{k_j}(t_0)$ converges to $a \in A$. By Lemma A.5, there exists a further subsequence, $f_{k'_j}$ and a monotone function, $g \in \mathcal{M}$, such that $f_{k'_j}(t) \to g(t)$ for a.e. t in $[0, 1]^m$. Because d is bounded, the dominated convergence theorem implies that $\delta(f_{k'_j}, g) \to 0$. But $\delta(f_{k'_j}, f) \to 0$ then implies that $\delta(f, g) = 0$ and so $f_{k'_j}(t) \to f(t)$ for a.e. t in $[0, 1]^m$.

Because t_0 is in the interior of $[0, 1]^m$, for every $\varepsilon > 0$ there exist $t_\varepsilon, t'_\varepsilon$ each within ε of t_0 such that $t_\varepsilon \leq t_0 \leq t'_\varepsilon$ and such that $f_{k'_j}(t_\varepsilon) \to_j f(t_\varepsilon)$ and $f_{k'_j}(t'_\varepsilon) \to_j f(t'_\varepsilon)$. Consequently, $f_{k'_j}(t_\varepsilon) \leq f_{k'_j}(t_0) \leq f_{k'_j}(t'_\varepsilon)$, and taking the limit as $j \to \infty$ yields $f(t_\varepsilon) \leq a \leq f(t'_\varepsilon)$, and taking next the limit as $\varepsilon \to 0$ yields $f(t_0) \leq a \leq f(t_0)$, so that $a = f(t_0)$, as desired.

(if) To complete the proof, suppose that $f_k(t)$ converges to $f(t)$ for ν almost every $t \in [0, 1]^m$. Then, because d is bounded, the dominated convergence theorem implies that $\delta(f_k, f) \to 0$. ■

Lemma B.2. *The function $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ defined by*

$$h(\tau, f, g)(t) = \begin{cases} f(t), & \text{if } \mathbf{1} \cdot t \leq |1 - 2\tau|\, m \text{ and } \tau < 1/2 \\ g(t), & \text{if } \mathbf{1} \cdot t \leq |1 - 2\tau|\, m \text{ and } \tau \geq 1/2 \\ f(t) \vee g(t), & \text{if } \mathbf{1} \cdot t > |1 - 2\tau|\, m \end{cases} \tag{B.1}$$

is continuous, where $\mathbf{1}$ denotes the vector of 1's.

Proof. Suppose that $(\tau_k, f_k, g_k) \to (\tau, f, g) \in [0,1] \times \mathcal{M} \times \mathcal{M}$. By Lemma B.1, there is a full ν measure subset, D, of $[0,1]^m$ such that $f_k(t) \to f(t)$ and $g_k(t) \to g(t)$ for every $t \in D$. There are three cases: $\tau = 1/2$, $\tau > 1/2$ and $\tau < 1/2$.

Suppose that $\tau < 1/2$. For each $t \in D$ such that $\mathbf{1} \cdot t < |1 - 2\tau|\, m$, we have $\mathbf{1} \cdot t < |1 - 2\tau_k|\, m$ for all k large enough. Hence, $h(\tau_k, f_k, g_k)(t) = f_k(t)$ for all k large enough, and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \to f(t) = h(\tau, f, g)(t)$. Similarly, for each $t \in D$ such that $\mathbf{1} \cdot t > |1 - 2\tau|\, m$, $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(\tau, f, g)(t)$, where the limit follows because $(A, \geq)$ is a metrizable semilattice. By G.1, $\nu(\{t \in [0,1]^m : \mathbf{1} \cdot t = |1 - 2\tau|\, m\}) = 0$. Consequently, if $\tau < 1/2$, $h(\tau_k, f_k, g_k)(t) \to h(\tau, f, g)(t)$ for ν a.e. $t \in [0,1]^m$ and so, by Lemma B.1, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$.

Because the case $\tau > 1/2$ is similar to $\tau < 1/2$, we need only consider the remaining case in which $\tau = 1/2$. In this case, $|1 - 2\tau_k| \to 0$. Consequently, for any nonzero $t \in [0,1]^m$, because $\mathbf{1} \cdot t > 0$, we have $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t)$ for k large enough and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(1/2, f, g)(t)$ for every non zero $t \in [0,1]^m$. Hence, by G.1, $h(\tau_k, f_k, g_k)(t) \to h(1/2, f, g)(t)$ for ν a.e. $t \in [0,1]^m$, and so again by Lemma B.1, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$. ■

Lemma B.3. *The metric space $(\mathcal{M}, \delta)$ is an absolute retract.*

Proof. As a matter of notation, for $f, g \in \mathcal{M}$, write $f \leq g$ if $f(t) \leq g(t)$ for ν a.e. t in $[0,1]^m$. Also, for any sequence of monotone functions $f_1, f_2, \ldots$, in $\mathcal{M}$, denote by $f_1 \vee f_2 \vee \ldots$ the monotone function taking the value $\lim_n [f_1(t) \vee f_2(t) \vee \ldots \vee f_n(t)]$ for each t in $[0,1]^m$. This is well-defined by Lemma A.3.

Let $h : [0,1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be the continuous function defined by (B.1). Since for any $g \in \mathcal{M}$, $h(\cdot, \cdot, g)$ is a contraction for $\mathcal{M}$, $(\mathcal{M}, \delta)$ is contractible. Hence, by Borsuk (1966, IV (9.1)) and Dugundji (1965), it suffices to show that for each $f' \in \mathcal{M}$ and each neighborhood U of f', there exists a neighborhood V of f' and contained in U such that the sets V^n, $n \geq 1$, defined inductively by $V^1 = h([0,1], V, V)$, $V^{n+1} = h([0,1], V, V^n)$, are all contained in U.[30]

For each V, note that if $g \in V^1$, then $g = h(\tau, f_0, f_1)$ for some $\tau \in [0,1]$ and some $f_0, f_1 \in V$. Hence, by the definition of h, we have $g \leq f_0 \vee f_1$ and either

[30] This condition, which is intimately related to the local contractibility of $\mathcal{M}$, can more easily be related to local convexity. For example, if $\mathcal{M}$ is convex, instead of merely contractible, and $h(\alpha, f, g) = \alpha f + (1-\alpha) g$ is the usual convex combination map, the condition follows immediately if $\mathcal{M}$ is, in addition, locally convex.

$f_0 \leq g$ or $f_1 \leq g$. We may choose the indices so that $f_0 \leq g \leq f_0 \vee f_1$. Inductively, it can similarly be seen that if $g \in V^n$, then there exist $f_0, f_1, ..., f_n \in V$ such that

$$f_0 \leq g \leq f_0 \vee ... \vee f_n. \tag{B.2}$$

Suppose now, by way of contradiction, that there is no open set V containing $f' \in \mathcal{M}$ and contained in the neighborhood U of f' such that all the V^n as defined above are contained in U. Then, successively for each $k = 1, 2, ..$, taking V to be $B_{1/k}(f')$, the $1/k$ ball around f', there exists n_k such that some $g_k \in V^{n_k}$ is not in U. Hence, by (B.2), there exist $f_0^k, ..., f_{n_k}^k \in V = B_{1/k}(f')$ such that

$$f_0^k \leq g_k \leq f_0^k \vee ... \vee f_{n_k}^k. \tag{B.3}$$

Consider the sequence $f_0^1, ..., f_{n_1}^1, f_0^2, ..., f_{n_2}^2, ...$. Because f_j^k is in $B_{1/k}(f')$, this sequence converges to f'. Let us reindex this sequence as $f_1, f_2, ...$. Hence, $f_j \to f'$.

Because for every n the set $\{f_n, f_{n+1}, ...\}$ contains the set $\{f_0^k, ..., f_{n_k}^k\}$ whenever k is large enough, we have

$$f_0^k \vee ... \vee f_{n_k}^k \leq \vee_{j \geq n} f_j,$$

for every n and all large enough k. Combined with (B.3), this implies that

$$f_0^k \leq g_k \leq \vee_{j \geq n} f_j \tag{B.4}$$

for every n and all large enough k.

Now, $f_0^k \to f'$ as $k \to \infty$. Hence, by Lemma B.1, $f_0^k(t) \to f'(t)$ for ν a.e. t in $[0,1]^m$. Consequently, if for ν a.e. t in $[0,1]^m$, $\vee_{j \geq n} f_j(t) \to f'(t)$ as $n \to \infty$, then (B.4) and Lemma A.2 would imply that for ν a.e. t in $[0,1]^m$, $g_k(t) \to f'(t)$. Then, Lemma B.1 would imply that $g_k \to f'$ contradicting the fact that no g_k is in U, and completing the proof that $(\mathcal{M}, \delta)$ is an absolute retract.

It therefore remains only to establish that for ν a.e. $t \in [0,1]^m$, $\vee_{j \geq n} f_j(t) \to f'(t)$ as $n \to \infty$. But, by Lemma C.2, because $(A, \geq)$ is locally complete this will follow if $f_j(t) \to_j f'(t)$ for ν a.e. t, which follows from Lemma B.1 because $f_j \to f'$. ■

C. Completeness and Local Completeness

In each of the lemmas below, it is assumed that $(A, \geq)$ is a compact metrizable semilattice.

Lemma C.1. *$(A, \geq)$ is a complete semilattice.*

Proof. Because A is compact and metrizable, S has a countable dense subset, $\{a_1, a_2, ...\}$. Let $a^* = \lim_n a_1 \vee ... \vee a_n$, where the limit exists by Lemma A.3. Now, suppose that b is an upper bound for S and that a is an arbitrary element of S. Then, some sequence, a_{n_k}, converges to a. Moreover, $a_{n_k} \leq a_1 \vee ... \vee a_{n_k} \leq b$ for every k. Taking the limit as $k \to \infty$ yields $a \leq a^* \leq b$. Hence, $a^* = \vee S$. ■

Lemma C.2. *$(A, \geq)$ is locally complete if and only if for every $a \in A$ and every sequence a_n converging to a, $\lim_n(\vee_{k \geq n} a_k) = a$.*

Proof. For each n, let $b_n = \lim_{k \geq n}(a_n \vee ... \vee a_k)$. This is well-defined by Lemma A.3 since $a_n \vee ... \vee a_k$ is nondecreasing in k. Consequently, $\lim_n b_n = \lim_n(\vee_{k \geq n} a_k)$ exists by Lemma A.3 since $\{b_n\}$ is nonincreasing.

We first demonstrate the "only if" direction. Suppose $(A, \geq)$ is locally complete and that U is a neighborhood of a. Then, there exists a neighborhood W of a contained in U such that every subset of W has a least upper bound in U. In particular, because for n large enough $\{a_n, a_{n+1}, ...\}$ is a subset of W, the least upper bound of $\{a_n, a_{n+1}, ...\}$, namely $\vee_{k \geq n} a_k$, is in U for n large enough. Since U was arbitrary, this implies $\lim_n(\vee_{k \geq n} a_k) = a$.

We now turn to the "if" direction. Fix any $a \in A$, and let $B_{1/n}(a)$ denote the open ball around a with diameter $1/n$. For each n, $\vee B_{1/n}(a)$ is well-defined by Lemma C.1. Moreover, because $\vee B_{1/n}(a)$ is nondecreasing in n, $\lim_n \vee B_{1/n}(a)$ exists. We first argue that $\lim_n \vee B_{1/n}(a) = a$. For each n, we may construct, as in the proof of Lemma C.1, a sequence $\{a_{n,m}\}$ of points in $B_{1/n}(a)$ such that $\lim_m(a_{n,1} \vee ... \vee a_{n,m}) = \vee B_{1/n}(a)$. We may therefore choose m_n sufficiently large so that the distance between $a_{n,1} \vee ... \vee a_{n.m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$. Consider now the sequence $\{a_{1,1}, ..., a_{1,m_1}, a_{2,1}, ..., a_{2,m_2}, a_{3,1}, ..., a_{3,m_3}, ...\}$. Because $a_{n,m}$ is in $B_{1/n}(a)$, this sequence converges to a. Consequently, by hypothesis,

$$\lim_n(a_{n,1} \vee ... \vee a_{n,m_n} \vee a_{(n+1),1} \vee ... \vee a_{(n+1),m_{(n+1)}} \vee ...) = a.$$

But because every $a_{k,j}$ in the join in parentheses on the left-hand side above (denote this join by b_n) is in $B_{1/n}(a)$, we have

$$a_{n,1} \vee ... \vee a_{n,m_n} \leq b_n \leq \vee B_{1/n}(a).$$

Therefore, because for every n the distance between $a_{n,1} \vee ... \vee a_{1,m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$, Lemma A.2 implies that $\lim_n \vee B_{1/n}(a) = \lim_n b_n$. But since $\lim_n b_n = a$, we have $\lim_n \vee B_{1/n}(a) = a$, as desired. Next, for each n, let S_n be an arbitrary non empty subset of $B_{1/n}(a)$, and choose any $s_n \in S_n$. Then $s_n \leq \vee S_n \leq \vee B_{1/n}(a)$. Because $s_n \in B_{1/n}(a)$, Lemma A.2 implies that $\lim_n \vee S_n = a$. Consequently, for every neighborhood U of a, there exists n large enough such that $\vee S$ (well-defined by Lemma C.1) is in U for every subset S of $B_{1/n}(a)$. Since a was arbitrary, $(A, \geq)$ is locally complete. ■

Lemma C.3. *If A is a subset of $\mathbb{R}^K$ and $\geq$ is the coordinatewise partial order, then $(A, \geq)$ is locally complete.*

Proof. Suppose that $a_n \to a$. By Lemma C.2, it suffices to show that $\lim_n(\vee_{k \geq n} a_k) = a$. By Lemma A.3, $\lim_n(\vee_{k \geq n} a_k)$ exists and is equal to $\lim_n \lim_m(a_n \vee ... \vee a_m)$ since $a_n \vee ... \vee a_m$ is nondecreasing in m, and $\lim_m(a_n \vee ... \vee a_m)$ is nonincreasing in n. For each dimension $k = 1, ..., K$, let $a^k_{n,m}$ denote the first among $a_n, a_{n+1}, ..., a_m$ with the largest kth coordinate. Hence, $a_n \vee ... \vee a_m = a^1_{n,m} \vee ... \vee a^K_{n,m}$, where the right-hand side consists of K terms. Because $a_n \to a$, $\lim_m a^k_{n,m}$ exists for each k and n, and $\lim_n \lim_m a^k_{n,m} = a$ for each k. Consequently, $\lim_n \lim_m(a_n \vee ... \vee a_m) = \lim_n \lim_m(a^1_{n,m} \vee ... \vee a^K_{n,m}) = a \vee ... \vee a = a$, as desired. ■

Lemma C.4. *If for all $a \in A$, every neighborhood of a contains a' such that $b' \leq a'$ for all b' close enough to a, then $(A, \geq)$ is locally complete.*

Proof. Suppose that $a_n \to a$. By Lemma C.2, it suffices to show that $\lim_n(\vee_{k \geq n} a_k) = a$. For every n and m, $a_m \leq a_m \vee a_{m+1} \vee ... \vee a_{m+n}$, and so taking the limit first as $n \to \infty$ and then as $m \to \infty$ gives $a \leq \lim_m \vee_{k \geq m} a_k$, where the limit in the center exists by Lemma A.3 because the sequence is monotone. Hence, to show that $\text{limsup}_m a_m = a$, it suffices to show that $\lim_m \vee_{k \geq m} a_k \leq a$.

Let U be a neighborhood of a and let a' be chosen as in the statement of the lemma. Hence, for m large enough, $a_m \in U$ and so $a_m \leq a'$. Consequently, for m large enough and for all n, $a_m \vee a_{m+1} \vee ... \vee a_{m+n} \leq a'$. Taking the limit first in n and then in m yields $\lim_m \vee_{k \geq m} a_k \leq a'$. Because for every neighborhood U of a this holds for some a' in U, $\lim_m \vee_{k \geq m} a_k \leq a$, as desired. ■

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