NOTE: USING THE EMERGENT SEED TO COMPLETELY CHARACTERIZE "EVOLUTION AND INFORMATION IN A GIFT GIVING GAME."

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Abstract. Let β be the ratio of benefit to cost of an altruistic act controlling for trembles. In "Evolution and Information in a Gift Giving Game" [3] the stochastically stable strategy is found if $\beta < 2$ or $\beta > 4$. In this note the stochastically stable strategy is found for the intermediary range and the speed of evolution is bounded.

It is shown that if $\beta \leq 3$ then players are selfish in the long run, if $\beta \geq 3$ then players are altruistic in the long run and use a particularly strong strategy, called the "team" strategy.

1. Introduction

There is a fundamental underlying structure that evolution follows, called the *emergent seed*. In "Evolution and Information in a Gift Giving Game" finding the emergent seed allows a simple and complete characterization of stochastically stable strategies, and gives us the tightest known bound on the speed of evolution.

Johnson, Levine and Pesendorfer [3] analyzed one of the most complex problems in the evolutionary literature: when cooperation can evolve in a matching game. To find the solution Johnson et al. used the most advanced technique then in the literature: Ellison's [1] *radius* and *coradius* technique. Ellison [1] shows that when the radius of a limit set is greater than its coradius then it is stochastically stable. In Johnson et al. this occurs when $\beta < 2$ or $\beta > 4$.

Since, to the author's knowledge, this is the only paper other than Ellison [1] to use the radius and coradius technique; it is natural to see how the emergent seed methodology (Hasker [2]) fared in that problem. The emergent seed is the most likely path for evolution that includes all possible states. We say that something is in the *core* of the emergent seed if wherever evolution starts, it is likely to pass through that state. In Johnson et al. [3] all limit sets are in the core. This makes finding their stochastic potential simple since for limit sets in the core their stochastic potential is the cost of the emergent seed minus the radius of the limit set.

In the next section a simplified version of the model in Johnson et al. is presented along with the relevant results about the Emergent Seed. In section 3 the strategies that are potentially stochastically stable are identified, and then in section 4 the Emergent Seed is found and the stochastically stable strategy is identified.

2. The Model and the Emergent Seed

Players live for two periods with their lifetime utility being the sum of their utility when young and old. When young a player either gives a gift or not and when old the player either receives a gift or not. The game is:

$$\begin{array}{cc} & \text{young} \\ & \begin{array}{cc} 1 & 0 \end{array} \\ \text{old} & \begin{array}{|c|c|} \hline \alpha, -1 & 0, 0 \\ \hline \end{array} \end{array}.$$

There are N young people in each period and N old people who are matched by equal likelihood. Society can enable gift giving by using *social norms* (Okuno-Fujiwara and Postlewaite [4]). Every old person has a *social status*, $f \in \{r, g\}$. Given this social status the young person follows a *social standard of behavior*, $a : \{r, g\} \to \{1, 0\}$, and then the transition rule determines his social status next period $\tau : \{r, g\} \times \{1, 0\} \to \{r, g\}$. A strategy is a pair*:* $\sigma = \{a, \tau\}$.

Notice that different people can have different opinions about a person's social status. There are 16 different transition rules that represent the 16 different ways people can think about each other. In order to keep track of all of these possibilities, each person has a *social status vector*, $F \in \{r, g\}^{16}$. A given strategy uses only F_k—the k'th element of this vector $k \in \{1, 2, 3, ..., 16\}$. Write $a(F)$ for $a(F_k)$, and $\tau_k(a, F) = \tau_k(a, F_k)$ for $a \in \{0, 1\}$.

A state in this model is a distribution over strategies σ and social status vectors F. The distribution over F is secondary to our analysis. The final distribution over F is a perturbation of the distribution from the last period. Players will know the distribution from the last period, but do not know the final distribution. The final distribution is generated from the prior one by replacing each old person's social status vector by a random one with probability $\eta > 0$. This means players expect to meet each of the 2^{16} social status vectors with positive probability. Notice that this means that controlling for these trembles the reward for giving the gift is $\beta \equiv (1 - \eta)\alpha$ controlling for these trembles. We assume that $\beta \geq 1$ and that it is a rational number.

The distribution over σ is the focus of our analysis. As in a standard evolutionary model a player changes his strategy each period with probability $\tau \in (0, 1)$. If he changes his strategy with probability $(1 - \varepsilon)$ he chooses a best response assuming the distribution over strategies remains constant in the future and that the distribution over flags is generated from the prior as discussed above. With probability $\varepsilon > 0$ he *experiments,* or choose a strategy at random. We analyze the model as ε goes to zero.

For small enough ε what matters is the number of ε probability events it takes to transition from one distribution over strategies to another. Let z_t be the distribution over σ's in period t. Then this gives us the cost function:

$$C(z_{t+1}|z_t) = \left\{ \begin{array}{l} \text{The number of experiments needed} \\ \text{to transition from } z_t \text{ to } z_{t+1} \end{array} \right. .$$

We can normalize this by N,

$$c(z'|z) = \lim_{N\to\infty} \left\{ \frac{C(z'|z)}{N} | \{z', z\} \in Z(N) \right\}$$

where $Z(N)$ is the set of distributions that are feasible given N. Denote $\overrightarrow{c}(z'|z)$ as the least cost way to transition from z to z' in a finite number of periods.

For small ε society will be in a *limit set* (ω) almost all the time. This is a set that can not be left without experiments and if society is at some $z \in \omega$ then society will be at any other $z' \in \omega \backslash z$ at some time in the future. Let the collection of limit sets be Ω. In this model all limit sets are Nash equilibria.

Young [5] shows that ω only has positive probability in the limiting distribution if it has the least cost absorption tree. An absorption tree with base ω is a least cost graph over the Ω such that every $\omega' \in \Omega\backslash$

ω transitions to ω, and ω does not transition to any other limit set. Let $\overrightarrow{c}_\omega$ be this cost, this is called ω's *stochastic potential*. A limit set is evolutionarily successful or *stochastically stable* if $\omega \in \arg\min_{\omega' \in \Omega} \overrightarrow{c}_{\omega'}$.

Order Ω as a vector, then an arbitrary graph over Ω can be denoted as a triplet: $g = \{\omega_k, z(\omega_k), \overrightarrow{c}(z(\omega_k)|\omega_k)\}_{k=1}^{|\Omega|}$, where $z(\omega_k) \in \Omega \cup \emptyset$ and $\overrightarrow{c}(\emptyset|\omega) = 0$. Then ω' is a *direct successor* of ω if $\omega' = z(\omega)$, call ω' a *successor* of ω if it is in the transitive closure of the direct successor relationship. Denote the cost of a graph as $\overrightarrow{c}(g) = \Sigma_{k=1}^{|\Omega|} \overrightarrow{c}(z(\omega_k)|\omega_k)$.

The *emergent seed* is a least cost graph such that every ω has a unique direct successor and there is an ω which is the successor of all other $\omega' \in \Omega$. Note that there is a cycle of ω's satisfying the second part of this definition; this substructure is the *core*. In general the emergent seed is found through an iterative process. One finds the base of the emergent seed as follows:

Definition 1. *The base of the emergent seed (E_0) is $\{\omega_k, z_1(\omega_k), r(\omega_k)\}_{k=1}^{|\Omega|}$ where $r(\omega_k) = \min_{\omega' \in \Omega_9} \overrightarrow{c}(\omega'|\omega_k)$ and $z_1(\omega_k) \in \arg\min_{\omega' \in \Omega_9} \overrightarrow{c}(\omega'|\omega_k)$.*

Note that $r(\omega)$ is the *radius* (Ellison [1]). In this analysis the emergent seed is the base of the emergent seed. Furthermore there are multiple emergent seeds, which simplifies our analysis because every limit set will be in the core of one of them.

Proposition 1. *If ω is in the core and $E = E_0$, then $\overrightarrow{c}_\omega = \overrightarrow{c}(E) - r(\omega)$.*

Proof. This is an absorption tree with base ω, thus $\overrightarrow{c}_\omega \leq \overrightarrow{c}(E) - r(\omega)$. Since in E_0 the direct successor of each strategy has the unconstrained minimum cost of transition $\overrightarrow{c}_\omega \geq \overrightarrow{c}(E) - r(\omega)$. ∎

There is also an elegant representation of the speed of evolution based on this representation. Along with the *coradius* Ellison [1] constructs the *modified coradius*. The modification is to subtract the radius of a strategy from every transition cost. Let $g(\omega|\omega')$ be a path from ω to ω', then

$$\widetilde{CR}(\omega) = \max_{\omega' \in \Omega \backslash \omega} \left(\min_{g(\omega|\omega')} \left[\lceil \overrightarrow{c}(g(\omega|\omega')) N \rceil - \lceil r(g(\omega|\omega')) N \rceil \right] + \lceil r(\omega') N \rceil \right)$$

and the expected waiting time to reach the stochastically stable strategy is bounded above by $\varepsilon^{-\widetilde{CR}(\omega')}$.

Lemma 2. *If ω is in the core and E has one level, then $\widetilde{CR}(\omega) = \max_{\omega' \in \Omega \backslash \omega} \lceil r(\omega') N \rceil$*

Proof. This is transparent since if ω is in the core then $\forall \omega' \in \Omega \backslash \omega \ \min_{g(\omega|\omega')} \overrightarrow{c}(g(\omega|\omega')) - r(g(\omega|\omega')) = 0$ ∎

3. The Limit sets and Nash equilibria.

There are 64 strategies in this model, however 16 of these are the *selfish strategies*—$a(r) = a(g) = 0$—and 16 more are the *generous strategies*—$a(r) = a(g) = 1$. Of the 32 remaining 16 are constructed from the other 16 by changing the "language" of the strategy. In one group green is good—$a(g) = 1$ and $a(r) = 0$—and in the other red is good ($a'(f) = 1 - a(f)$ $f \in \{r, g\}$). Looking at the "green" strategies ($a(g) = 1$ and $a(r) = 0$) an equilibrium must reward giving the gift to a green status player ($\tau(g, 1) = g$) and punish not giving a gift to a green status player ($\tau(g, 0) = r$), leaving 4 strategies:

$\tau(g,1)$	$\tau(g,0)$	$\tau(r,1)$	$\tau(r,0)$	Strategy's Name
g	r	r	g	team
g	r	g	g	weak team
g	r	r	r	insider
g	r	g	r	tit for tat

these are all Nash equilibria when $\eta = 0$ but only the first three are Nash equilibria when $\eta > 0$. We show that all of the others have a sequence of best responses that leads to the selfish strategies, thus the game is *acyclic* (Young [5]) and only Nash equilibria are limit sets.

Lemma 3. *The only limit sets are the selfish, team, weak team, and insider strategies.*

Proof. First notice that the best response to a population of players using the generous strategy is the selfish strategy, since the reward is the same and the cost is lower. Furthermore the best response to a population of selfish strategies is a selfish strategy since giving the gift is never rewarded. Thus without loss of generality assume that $a(g) = 1$ and $a(r) = 0$.

For the team, weak team, and insider strategies when these players meet a player with the green social status they should give a gift since this is rewarded and $\beta \geq 1$. They should not give a gift to a red flag player because this can never increase their payoff when old.

In contrast the best response to the tit-for-tat is a generous strategy, since one is always rewarded for giving a gift, and as mentioned above the best response to this is the selfish strategy. This leaves three possible non-Nash transition rules. Either $\tau(g,1) = \tau(g,0) \in \{r, g\}$ or $\tau(g,1) = r$ and $\tau(g,0) = g$. In both cases a selfish strategy is a best response because the transition rule either ignores or punishes giving gifts to green status players.

Thus the team, weak team, insider, and selfish strategies are all limit sets. Since at any other distribution a sequence of best responses leads to the selfish strategies, the game is acyclic and these are all of the limits sets. ∎

4. The Emergent Seed of "Evolution and Information in a Gift Giving Game."

To find the emergent seed we will use the *gain function* from Johnson et al. [3]. Letting p_F be the distribution over social status vectors and p_σ be the distribution over strategies the *gain function* is the difference between using some strategy σ_i and some alternative strategy σ_a:

$$\begin{aligned} G(\sigma_i, \sigma_a, p_F, p_\sigma) &= \Sigma_{F'} \left[a_i(F') - a_a(F')\right] p_F(F') \\ &\quad + \Sigma_{F'} \Sigma_{\sigma' = \{a', \tau'\}} \beta \left[\left(a'\left(\tau'\left(a_a(F'), F'\right)\right) - a'\left(\tau'\left(a_i(F'), F'\right)\right)\right)\right] p_\sigma(\sigma') p_F(F') \end{aligned}$$

The first term, $a_i(F') - a_a(F')$ is the first period difference in payoff and the expected consequence of this action is the second term. To find the radius we want to find the distribution over strategies that is closest to $p_\sigma(\sigma_i) = 1$ such that $G(\sigma_i, \sigma_a, p_F, p_\sigma) \leq 0$.

Lemma 4. *In the emergent seeds the direct successor of the team, weak team, and insider strategies is the selfish strategies, the direct successor of the selfish strategies is either the team, weak team, or insider. The radius of the selfish strategies is $\frac{1}{\beta}$, of the team is $\frac{\beta-1}{2\beta}$, and of the weak team or insider is $\min\left\{\frac{1}{\beta}, \frac{\beta-1}{2\beta}\right\}$.*

Proof. In the proof assume that green is the good social status, or $a_i(g) \geq a_i(r)$. Let $G(\sigma_i, \sigma_a, p_F, p_\sigma | F'_k = f)$ be the gain given $F'_k = f \in \{r, g\}$, then clearly:

$$G(\sigma_i, \sigma_a, p_F, p_\sigma) \geq \min\left\{G(\sigma_i, \sigma_a, p_F, p_\sigma | F'_k = g), G(\sigma_i, \sigma_a, p_F, p_\sigma | F'_k = r)\right\}$$

and when $F'_k = F'_i$ σ_i is not a best response if the right hand side is weakly negative for some σ_a. The distribution we will analyze will have $p(\sigma_i) = 1 - \rho$ and $p(\tilde{\sigma}) = \rho$, $\tilde{\sigma}$ will be the invading strategy and will not necessarily be the same as σ_a.

If σ_i is a selfish strategy, let both σ_a and $\tilde{\sigma}$ be either the team, weak team, or insider strategy. Then without loss of generality we can let $F'_k = F'_a$ since the selfish strategy is independent of social status. Now clearly $G(\sigma_i, \sigma_a, p_F, p_\sigma | F'_a = r) = 0$ since both strategies call for the same action if the social status is r. Then $G(\sigma_i, \sigma_a, p_F, p_\sigma | F'_a = g) = 1 - \beta\rho$ and the selfish strategy is a worse response than that cooperative strategy when $\rho \geq \rho^*_g = \frac{1}{\beta}$.

From this case we can develop the key insights for the rest of the proof. First notice that if $a_a(F') = a_i(F')$ then this does not impact the sign of $G(\sigma_i, \sigma_a, p_F, p_\sigma)$. Thus from now on we look at σ_a where $a_a(f) = 1 - a_i(f)$ for either $f = g$ or $f = r$. We want the invading strategy to reward σ_a and punish σ_g

thus $\tilde{a}(F') = 1 - a_i(F')$, notice this implies that the invading strategy uses the same transition rule as σ_i in order to be certain that $\tilde{a}(F') = 1 - a_i(F')$.

Given these insights if σ_i is the team, weak team, or insider strategy then $G(\sigma_i, \sigma_a, p_F, p_\sigma | F_i' = g) = -1 + \beta(1-\rho) - \beta\rho$ and this is negative when $\rho \geq \rho_g^{**} = \frac{\beta-1}{2\beta}$. If $a_a(g) = a_a(r) = 0$ then σ_a is a selfish strategy, and letting $P(F_i' = g)$ be the probability that F_i' is g then $G(\sigma_i, \sigma_a, p_F, p_\sigma) = P(F_i' = g)\, G(\sigma_i, \sigma_a, p_F, p_\sigma | F_i' = g)$. This is non-negative when $\rho \geq \rho_g^{**}$ and a selfish strategy is a best response.

We can use the same method when $F_i' = r$, in this case $G(\sigma_i, \sigma_a, p_F, p_\sigma | F_i' = r) = 1 + \beta(1-\rho)V - \beta\rho$. In this expression $V = a_i(\tau(0,r)) - a_i(\tau(1,r))$, or it is the reaction of σ_i to players taking the wrong action at social status r. Again $G(\sigma_i, \sigma_a, p_F, p_\sigma | F_i' = r)$ is negative if $\rho \geq \rho_r^*(V) = \frac{1+\beta V}{(V+1)\beta}$. Consider the strategy with $a_a(g) = a_a(r) = 1$ (a generous strategy). If $P(F_i' = r)$ is the probability that F_i' is r then $G(\sigma_i, \sigma_a, p_F, p_\sigma) = P(F_i' = r)\, G(\sigma_i, \sigma_a, p_F, p_\sigma | F_i' = r)$ which is non-negative if $\rho \geq \rho_r^*$ and thus a generous strategy is a best response. If all players choose a generous strategy a selfish strategy is the unique best response and we are in the basin of attraction of the selfish strategies.

For the team strategy $V = 1$ and $\frac{1+\beta}{2\beta} > \frac{1-\beta}{2\beta}$, for the weak team or insider strategy $V = 0$ and the radius is $\min\left\{\frac{\beta-1}{2\beta}, \frac{1}{\beta}\right\}$. ∎

In total there are six emergent seeds, the difference being which cooperative strategy is in the core; the selfish strategies are always in the core and must be paired with one of the six other limit sets. Since all seven limit sets are in a core the strategy which is stochastically stable is simply the one with the highest radius.

Proposition 5. *If $\beta \leq 3$ selfish strategies are stochastically stable and $\widetilde{CR}(selfish) = \left\lceil \frac{\beta-1}{2\beta} N \right\rceil$. If $\beta \geq 3$ then the team strategies are stochastically stable and $\widetilde{CR}(team) = \left\lceil \frac{\beta-1}{2\beta} N \right\rceil$. In the latter case players are equally likely to use the green team and the red team strategy. If $\beta = 3$ then the weak team and insider strategies are also stochastically stable.*

Proof. This is immediate from Lemma 4, given Proposition 1 and Lemma 2 Note that the modified coradius of the green team strategy is the red team strategy; thus the radius and modified coradius are the same. ∎

In comparison Johnson, Levine, and Pesendorfer [3] finds results only if $\beta < 2$ or $\beta > 4$. One explanation for this difference is that Johnson et al. use the radius and coradius, while our methodology naturally results in finding the *modified coradius*. Ellison [1] shows that if the radius is greater than the modified coradius then a strategy is stochastically stable. In this game this completely characterizes the result; but without the emergent seed methodology this is not transparent.

References

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