Economic Survival When Markets Are Incomplete*

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ABSTRACT

We consider an infinite horizon economy with incomplete markets with two agents and one good. We begin with an example in which an agent's consumption is zero eventually with probability one even if she has correct beliefs and is marginally more patient. We then provide two general results: (a) a precise statement indicating that if markets are effectively incomplete forever then on any path on which some agent's consumption is eventually bounded away from zero, the other agent's consumption is arbitrarily close to zero infinitely often, and (b) for a robust class of economies with incomplete markets, there are equilibria in which an agent's consumption is zero eventually with probability one even though she has correct beliefs. Our results mark a sharp contrast with the case studied by Sandroni (2000) and Blume and Easley (2004) where markets are complete.

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1. INTRODUCTION

The main purpose of general equilibrium models of macroeconomic and financial phenomena is to explain the behaviour of consumption and that of the prices of goods and assets in economies with heterogeneous agents, and recent results let us claim that in the case of models with dynamically complete markets, there is a complete understanding of the asymptotic properties of these variables. Indeed, for some time now it has been known that if agents have homogeneous beliefs (even if they are not correct) and the same degree of impatience, Pareto optimality of equilibrium allocations implies that the consumption of every agent must be bounded away from zero, i.e. every agent "dominates" (this technical term has become standard in the literature and corresponds to the word "survive" in less formal parlance), regardless of attitudes towards risk; furthermore, if agents differ in their degree of impatience, then in the long run only the most patient have positive wealth, consume the entire output of the economy, and determine prices regardless of the agents' preferences towards risk. The result was conjectured by Ramsey (1928 pp. 558-559) and later proved by Becker (1980), Rader (1981) and Bewley (1982). The line of research was completed by considering the case with heterogeneous beliefs, results due to Sandroni (2000) and Blume and Easley (2004).[1] Sandroni considered a Lucas tree economy with dynamically complete markets and populated by expected utility maximizers. He showed that an agent whose belief has higher entropy accumulates more wealth and so entropy of beliefs is the appropriate measure of belief accuracy to study wealth accumulation. As a result, among agents with the same discount factor, only traders with correct beliefs, or those whose forecasts merge with the truth, dominate, and in the absence of such traders, no investor whose forecasts are persistently wrong dominates in the presence of a learner. Blume and Easley (2004) showed that Pareto optimality of the allocation guarantees the results. One concludes that in dynamically complete market economies, survival depends only on the degree of impatience and the accuracy of beliefs since the equilibrium allocation is necessarily Pareto optimal; attitudes toward risk are irrelevant. This is significant because it appears to validate the market selection hypothesis (henceforth, MSH) which, in the weak form due to Alchian (1950) and Friedman (1953), requires that only agents whose behaviour is consistent with rational and informed maximization of returns can survive and affect prices in the long run.[2]

The natural question is whether the fact that survival depends only on discount factors

[1] Sandroni (2000) and Blume and Easley (2004) respond to the earlier work of Blume and Easley (1992), a pioneering paper that studied the general equilibrium dynamics of wealth accumulation when agents use fixed savings rates and arbitrary portfolio rules. It showed that a trader with correct beliefs who uses a portfolio rule that does not lead to the maximization of the one period ahead expected value of the logarithm of wealth (the Kelly criterion) need not dominate. The principal criticism of that result is that agents do not optimally choose consumption and saving in an intertemporal framework.

[2] There is more than one view of what constitutes the MSH. Authors like Cootner (1967) and Fama (1965) offered a stronger version of the MSH which claims that markets select for investors with correct beliefs. A common implication of both versions is that rational expectations models are appropriate to describe long run outcomes. The stronger version of the MSH due to Cootner (1967) and Fama (1965) implies also that in the long run correct beliefs can be inferred from equilibrium prices.

and the accuracy of beliefs reflects an intrinsic property of competitive markets or whether it is a consequence of the assumption that markets are dynamically complete. Very little is known about this and that is the question we address by considering an economy with only one good in which two agents trade a single inside asset over an infinite horizon in dynamically incomplete markets.

We begin with a leading example where agent 1 has arbitrary CRRA preferences and a positive stochastic endowment forever, and agent 2 has logarithmic preferences and a positive endowment only at date zero. We show that even if agents are equally patient and have correct beliefs, one can find a time invariant asset structure such that the consumption of the agent with logarithmic preferences converges to zero with probability one in every equilibrium. A continuity argument shows that the same is true even if agent 2 is marginally more patient or if she holds correct beliefs and agent 1 does not.

The example shows that the factors determining survival with complete markets have little relevance when markets are dynamically incomplete. As for the MSH in dynamically incomplete markets economies, our example shows that no entropy measure can be critical to understanding survival because any properly defined entropy measure must attain its maximum when beliefs are correct and, as per the example, the mere fact that one has correct beliefs does not guarantee survival.[3]

Our leading example leads us to two rather different conjectures about the implications of market incompleteness in general infinite horizon economies: (a) that the consumption of some agent comes arbitrarily close to zero infinitely often, and (b) that the consumption of some agent is eventually close to zero. In the rest of the paper we refine and strengthen these conjectures to obtain a strong set of results.

Before stating and discussing our results let us recall the economics that drives the result when markets are complete. In such a framework, at an interior allocation, the utility gradients of the different agents point in the same direction. It follows that with preferences that are additively separable across time, the ratio of (the one-period ahead intertemporal) marginal rates of substitution—so they are probability weighted where the beliefs could be subjectively held, i.e. heterogeneous and incorrect—of the two agents weighted by the discount factors is one independent of the date and event; that is the key implication of Pareto optimality and that drives all the results. In particular, if both the agents have correct beliefs and the same discount factor then both dominate, that is their consumption is eventually uniformly positive. We write the ratio of (the one-period ahead ratio of the) marginal utilities—the ratio of the derivatives of the Bernoulli functions—of the two agents as the ratio of two stochastic processes where each is a product martingale with conditional mean one. At any Pareto optimal allocation, the ratio of the processes

[3]This resolves an open question posed by Sandroni (2004 on page 10) as the following quotation indicates: “The results in this paper can only suggest, but they do not prove, that belief accuracy measured by a properly defined entropy measure is critical for survival in dynamic incomplete market economies.” The question was posed in response to an example in Blume and Easley (2004) discussed in footnote 10 below.

that we construct is degenerate and we recover the result for Pareto optimal equilibrium allocations due to Blume and Easley (2004).[4] However, when markets are incomplete, typically, the utility gradients of different agents are not aligned and the ratio of our martingales is not degenerate. In fact since the logarithms of the processes have an additive structure, their limiting behaviour can be analyzed by using a suitable Strong Law of Large Numbers; under appropriate assumptions, the limit could even be zero or infinity from which it follows that one or the other agent vanishes.

Our first main result is very intuitive since it is based on the idea that if market incompleteness has bite then marginal rates of substitution will not be equalized and therefore one can have arbitrarily long strings of states where the ratio (across agents) of marginal utilities keeps rising; the technical tool used is Levy's conditional form of the Second Borel-Cantelli Lemma further generalized by Freedman (1973). More formally, in Theorem 1 (i) we show that if in the limit the ratio of marginal rates of substitution does not display one period ahead conditional variability, then the marginal rates of substitution are equalized in the limit. Theorem 1 (ii) shows that if, on the other hand, the ratio of marginal utilities does display one period ahead conditional variability, then some agent must consume arbitrarily close to zero infinitely often. Simply put, if market incompleteness is effective forever then either (a) one of the two agents will eventually cease to consume, or (b) the equilibrium is complicated in that the consumption of some agent will be arbitrarily close to zero infinitely often. The results hold with probability one and applies equally regardless of whether beliefs are homogeneous or heterogeneous.

Theorem 1 shows that examples of infinite horizon economies with incomplete markets that have appeared in the literature are very special. In many of those examples, after some finite date the continuation economy displays effectively complete markets.[5] In others, though markets are effectively incomplete, the asset structure is specified in a manner that ensures that trading possibilities are so narrow that the idea behind our proof of Theorem 1 (ii) has no bite. There is one further possibility that is not covered by our discussion so far, namely, that the ratio of marginal utilities does not display one period ahead variability even though the ratio of marginal rates of substitution does display such variability so that both the agents have consumption uniformly bounded away from zero. Such a case is very special and can arise only with heterogeneous and well chosen beliefs; Coury and Sciubba (2005) provide such an example.[6] All of these examples are discussed in Section 4.3.[7]

[4] The result in Sandroni (2000) is not covered by our approach since we restrict attention to short maturity assets while he considers Lucas trees.

[5] Although this feature is very useful in constructing examples, it clearly goes against the motivation for studying models with incomplete markets.

[6] Their construction appears to be special since they start with a Pareto optimal allocation that can be supported as an equilibrium with incomplete markets with homogeneous beliefs, and then they change beliefs and/or discount factors in a manner that leaves demand behaviour unchanged so that the same allocation continues to specify equilibrium consumption in the economy with heterogeneous beliefs.

[7] Duffie et al (1994) provide an existence theorem for Lucas-tree economies with incomplete markets in which consumption is uniformly bounded away from zero. For that result it is crucial that there are

Our second main result, Theorem 4, shows that for a robust family of endowment distributions, there are equilibria with homogeneous and correct beliefs in which the same agent eventually consumes zero on almost all paths, i.e. we specify a class of economies where the phenomenon exhibited by the example holds. This result is much more surprising and less intuitive. It appears to require a fairly strong restriction on the distribution of endowments across agents and little else.

In fact, our approach is constructive. We propose a method for constructing feasible consumption processes that satisfy the Euler equations, that have summable supporting prices, and that have an additional property that ensures that one of the agents vanishes. The method generates consumption processes that are uniquely specified for each value of consumption at the initial date; furthermore, the consumption processes induced are continuous and monotone in the initial value. We trivially obtain a family of "no trade" equilibria that are supported with trivial asset portfolios so that the process that specifies the value of the portfolios is uniformly bounded. The latter has been proposed as a desirable property of equilbria in infinite horizon economies and has been studied in detail by Magill and Quinzii (1994), Levine and Zame (1996), Hernandez and Santos (1996), and Florenzano and Gourdel (1996).[8] We then show that for each such no trade equilibrium, there is an open set of endowment distributions that leads to an equilibrium that is weaker in that there may be no uniform bound across paths on debt. This equilibrium concept requires maximization subject to a sequence of budget constraints and a single transversality condition at date zero, and market clearing. We prove that it does not permit Ponzi schemes.[9]

Our work has implications for the MSH which we now highlight. Blume and Easley (2004) conclude that the accuracy of beliefs is not the key that explains survival and that the MSH may fail because wrong beliefs can lead to greater savings, a point also made by Sandroni (2004). They do so on the basis of an example of an incomplete markets economy where an agent with correct beliefs is driven out of the market by traders with less accurate beliefs.[10] Our second result indicates that market incompleteness rather than wrong beliefs cause greater savings. Furthermore, our example, where agents with correct

no short sales and no one period inside assets either.

[8]Equilibria with a uniform bound on the value of debt, a condition that is often equivalent to requiring a transversality condition at every date and event, are usually justified by appealing to an unmodeled institutional device that ensures that the economy is immune to Ponzi schemes.

[9]So our equilibrium concept provides a less demanding institutional framework that achieves the purpose noted in footnote 8. Santos and Woodford (1997) propose a notion of equilibrium without uniform bounds for a much more general set-up. Blume and Easley (2004) provide an example in which the equilibrium value of an agent's debt diverges according to the agent's subjectively held incorrect belief, i.e. the one relevant for specifying the agent's budget set.

[10]In their example the economy is deterministic but some agent mistakenly believes it to be stochastic; as a consequence, completing the market in their example economy leads to nonexistence, a fact that they note. In our leading example completing the market leads to an equilibrium where the allocation is Pareto optimal and, by the result in Blume and Easley (2004), both the agents dominate.

These authors present a second example that shows that there are situations in which relative entropy is simply the wrong measure of belief accuracy because it does not match well with the asset structure.

beliefs are driven out by agents with wrong beliefs, makes very clear that even the version of the MSH due to Alchian (1950) and Friedman (1953) does not hold in general. Coury and Sciubba (2005) argue that, when markets are incomplete, agents with wrong beliefs may survive and so one cannot infer the true probability distribution by only observing asset prices; their claim is based upon assuming the existence of an equilibrium where an agent with correct beliefs has consumption that is uniformly positive infinitely often and then showing that there must exist an economy with heterogeneous beliefs with the same consumption profiles. Since prices are "as if" agents had correct beliefs, their result casts some doubt on the version of the MSH due to Cootner (1967) and Fama (1965) but it is consistent with the version due to Alchian (1950) and Friedman (1953).

To summarize, this paper contributes to various areas. It provides an almost complete characterization of limiting consumption behaviour when markets are incomplete and shows that about the only way that one can get simple limiting behaviour is if one agent is driven out of the market. It goes on to show that such a possibility is a robust outcome. It follows that the strong results regarding the validity of the MSH that have appeared depend critically on having complete markets or a Pareto optimal allocation. The paper also contributes to the general equilibrium literature by pointing out hitherto unknown properties of infinite horizon economies with incomplete markets; this is all the more important because of the widespread use of such models in the modern literature in macroeconomics. Finally, the method for constructing equilibria that we propose sheds light on the structure of the equilibrium set when markets are incomplete;[11] also, the method might be of use to researchers in the area of computational general equilibrium.

In Section 2 we introduce the model and define the relevant notions of survival. Section 3 contains the leading example. Afterwards, in Section 4 we develop the general approach to study the long run dynamics of equilibria and present Theorem 1 and our discussion of earlier examples in the literature. Finally, in Section 5 we construct the equilibria in which only one agent survives. Concluding remarks are presented in Section 6. All the proofs are gathered in the Appendix.

2. MODEL

2.1 PROBABILITY NOTATION

We consider an infinite horizon with dates $t = 0, 1, 2, \cdots$. The temporal state space is $\mathcal{S} := \{1, 2, \cdots, S\}$. $\mathcal{S}^t$ is the t-fold Cartesian product of $\mathcal{S}$ and $\Omega := \mathcal{S}^\infty$ with typical element $\omega = (s_1, s_2, \cdots)$ where s_t is the realization at date $t \geq 1$. In fact, we shall write $\omega = (s_1(\omega), s_2(\omega), \cdots)$. Also $s^t = (s_1, \cdots, s_t)$ and if we wish to make the dependence on ω explicit, we shall use $s^t(\omega) := (s_1(\omega), \cdots, s_t(\omega))$. $\Omega(s^t) := \{\omega \in \Omega : \omega = (s^t, s_{t+1}, \cdots), s^t \in \mathcal{S}^t\}$ is a *t-cylinder* and $\mathcal{F}_t$ is the σ-algebra obtained by considering finite unions of the sets $\Omega(s^t)$ for fixed t. This induces a sequence of σ-algebras on Ω denoted $\{\mathcal{F}_t\}_{t=1}^\infty$ where $\mathcal{F}_{t-1} \subset \mathcal{F}_t$ for all $t \geq 1$; we set $\mathcal{F}_0 := \{\emptyset, \Omega\}$, and we set $\sigma\Big(\cup_{t\geq 0} \mathcal{F}_t\Big) \subset \mathcal{F}$. That is our

[11] We remind the reader that very little is known about this beyond the analysis in Levine and Zame (2001) for the case of one good economies with idiosyncratic shocks and increasing patience.

filtration with $\mathcal{F}$ a σ-algebra on Ω. All statements will be made using $(\Omega, \mathcal{F})$.

Any function $X : \Omega \to R$ that is $\mathcal{F}$-measurable is a *random variable.* From here on a *process* denotes $X = \{X_t\}_{t=0}^{\infty}$ with $X_t : \Omega \to R$ and $\mathcal{F}_t$-measurable.

For $Q : \mathcal{F} \to [0,1]$ a probability measure, let dQ_t be the $\mathcal{F}_t$ measurable function defined by $dQ_t(\omega) := Q(\Omega(s^t(\omega)))$ for $t \geq 1$ and $dQ_0 := 1$, i.e. $dQ_t(\omega)$ is the probability of the cylinder $\Omega(s^t(\omega))$. We also define the one period ahead conditional probability that state s occurs by $Q_t(\omega) := \frac{dQ_t(\omega)}{dQ_{t-1}(\omega)}$. $E_Q[X|\mathcal{G}]$ denotes the expectation operator applied to the random variable $X : \Omega \to R$ restricted to the σ-algebra $\mathcal{G}$ where $\mathcal{G} \subset \mathcal{F}$ and where the expectation is taken with respect to the measure Q. $E_Q[X|\mathcal{G}]$ is a $\mathcal{G}$-measurable random variable. Recall that $\mathcal{L}^{\infty}(\Omega, \mathcal{F}, Q)$ denotes the (equivalence class of) measurable functions that are bounded in the essential sup norm with respect to the measure Q. We define[12]

$\Psi^t := \{f : \Omega \to R : f \,\text{is}\, \mathcal{F}_t - \text{measurable}\}$
$\Psi_+^{t,Q} := \{f \in \Psi^t : f(\omega) \geq 0 \;\; Q - \text{a.s.}\, \omega\}$
$\Psi^Q := \{(f_0, f_1, \cdots) \in \times_{t=0}^{\infty} \Psi^t : \sup_{t \geq 0} \operatorname{ess\,sup}_{\omega \in \Omega; Q} |f_t(\omega)| < \infty\}$
$\Psi_+^Q := \{(f_0, f_1, \cdots) \in \times_{t=0}^{\infty} \Psi_+^t : \sup_{t \geq 0} \operatorname{ess\,sup}_{\omega \in \Omega; Q} |f_t(\omega)| < \infty\}$.

2.2 THE ECONOMY

There is only one perishable good at each date. An agent is denoted $i \in \mathcal{I}$. There are two agents, so $\mathcal{I} := \{1, 2\}$, each of whom lives forever.

$\omega \in \Omega$ is chosen according to the objective probability measure P while agent i's subjective belief is denoted P_i. So we work with three probability triples: the objective triple $(\Omega, \mathcal{F}, P)$ that is relevant for economic aggregates, and the subjective triples $(\Omega, \mathcal{F}, P_i)$, $i = 1, 2$, that are the relevant spaces for the agents' decisions. For the main results we shall assume that the one period ahead conditional probability that state s occurs is uniformly positive and agents correctly believe that these probabilities are uniformly bounded away from zero.[13] So, define $\underline{p} := \inf_{t \geq 0} \text{ess.} \inf_{\omega \in \Omega; P} P_t(\omega)$.

ASSUMPTION A.1: $0 < \underline{p} \leq \inf_{t \geq 0} \text{ess.} \inf_{\omega \in \Omega; P_i} P_{i,t}(\omega)$.

The *aggregate endowment* process is denoted $Z := \{Z_t\}_{t=0}^{\infty}$ and its range is $[\underline{z}, \bar{z}]$ so that for all $t \geq 0$, $Z_t(\omega) \in [\underline{z}, \bar{z}]$ $P-$a.s. ω. The *endowment* process of i is denoted $z_i := \{z_{i,t}\}_{t=0}^{\infty}$, a nonnegative process. Of course, $z_1 + z_2 = Z$; we also assume that the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ is generated by the union of $\sigma(z_1)$ and $\sigma(z_2)$ where, for a random variable X, $\sigma(X)$ is the σ-algebra generated by X.

ASSUMPTION A.2: $[\underline{z}, \bar{z}] \subset R_{++}$. $z_i \in \times_{t=0}^{\infty} \Psi_+^{t,P_i}$.

u_i is i's state independent Bernoulli utility function. β_i is agent i's discount factor. $\beta_i = 0$ is ruled out to avoid the trivial case.

[12]For h an $\mathcal{F}$-measurable function, the notation $\operatorname{ess\,sup}_{\omega \in \Omega; Q} h$ is used to denote the essential supremum of h taken over the set Ω with respect to the measure Q.

[13]This assumption is standard in the literature (see Sandroni (2000) and Blume and Easley (2004)).

ASSUMPTION A.3: $u_i : R_+ \to R$ is strictly increasing, strictly concave, and C^2 with $\lim_{c\to 0^+} u_i'(c) = \infty$. $\beta_i \in (0,1)$.

To prove our robust existence result, Theorem 4, we need to impose a bound on the degree of relative risk aversion.

ASSUMPTION A.4: For $i = 1, 2$, $1 \geq -\frac{c \cdot u_i''(c)}{u_i'(c)}$ for all $c > 0$.

There is a single asset available in zero net supply. It pays the return r, where r is a process whose range is $[\underline{r}, \bar{r}]$ so that for all $t \geq 0$, $r_t(\omega) \in [\underline{r}, \bar{r}]$ $P-$a.s. ω. The returns are assumed to be nonnegative and nontrivial, and the asset trades at the price process q.

ASSUMPTION A.5: $[\underline{r}, \bar{r}] \subset R_{++}$.

A.5 does not allow the asset to be an Arrow security; the role of this restriction will be discussed in Section 4.3.

The next assumption will be used to prove that the consumption processes that we construct and use in Theorems 3 and 4 are supportable as equilibria. Notice that, under A.2-3 and A.5, $M < \infty$ where M is specified in A.6.

ASSUMPTION A.6: $\beta_i < 1/M$ where $M := \max \left\{ \frac{\bar{r} \cdot u_2'(\underline{z}/2)}{\underline{r} \cdot u_2'(\bar{z})}; \frac{\bar{r} \cdot u_1'(\underline{z}/2)}{\underline{r} \cdot u_1'(\bar{z})} \right\}$.

We shall impose one further assumption; it will be stated and discussed in Section 5.1.

REMARK 1: Assumptions A.4 and A.6 will be used only in Section 5. The discussion in Section 5.5 will indicate that an assumption that is weaker than A.4, but more cumbersome to state since it takes into account the characteristics of the endowment process, suffices for Theorem 4 to go through. Also, instead of Assumption A.6 we can impose a weaker condition that is appropriate when the aggregate endowment process and asset return process are not i.i.d.; once again, this is not stated formally since the gain in generality is not justified by the notational complication.

An *economy* is a list $(P, Z, P_1, P_2, \beta_1, \beta_2, u_1, u_2, r)$. A *private ownership economy* is a list $(P, z_1, z_2, P_1, P_2, \beta_1, \beta_2, u_1, u_2, r)$ and is related to an economy by the relation $Z = z_1 + z_2$.

The *consumption* process of i is denoted c_i. We require $c_i \in \Psi_+^{P_i}$ and for such a c_i, the *utility payoff* is given by $\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(c_{i,t})|\mathcal{F}_0](\omega)$. i's holding of the asset is a process denoted θ_i. $\theta_{i,-1}(\omega) = 0$ is introduced as a convenient notational convention.

The pair (c_1, c_2) is *feasible* if $c_i \in \Psi_+^{P_i}$ for $i \in \mathcal{I}$ and at every $t \geq 0$, $c_{1,t}(\omega) + c_{2,t}(\omega) = Z_t(\omega)$ $P-$a.s. ω. A *market clearing allocation* consists of $(c_1, c_2, \theta_1, \theta_2)$ such that (c_1, c_2) is feasible and, at every $t \geq 0$, $\theta_{1,t}(\omega) + \theta_{2,t}(\omega) = 0$ $P-$a.s. ω.

At each pair (ω, t), agents trade in the asset market and in the spot market for the good. Since there is only one good, given q and z_i, each c_i determines one and only one

θ_i. Given the consumption process c_i, θ_i *is a supporting portfolio process* at the prices q if

(i) $\theta_{i,t} \in \Psi^{t,P_i}\ \forall\, t \geq 0$ and
(ii) $\forall\, t \geq 0,\ c_{i,t}(\omega) + q_t(\omega) \cdot \theta_{i,t}(\omega) \leq z_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega)\ P_i - \text{a.s.}\, \omega$.

2.3 EQUILIBRIUM—NECESSARY CONDITIONS

A notion of equilibrium in our model economy requires the specification of a budget set subject to which each agent maximizes. Evidently, the budget set will incorporate a sequence of budget constraints, i.e. it will require the existence of a supporting portfolio process; additional conditions will be imposed to guarantee that a maximizer exists.

The first condition is that asset prices satisfy the no arbitrage property. Define

$$\mathcal{P}(q;Q) := \Big\{ p \in \times_{t=0}^{\infty} \Psi_+^{t,Q}\ :\ \forall\, t \geq 0, \quad p_t(\omega) \cdot q_t(\omega) = E_Q[p_{t+1} \cdot r_{t+1} | \mathcal{F}_t](\omega) \quad Q - \text{a.s.}\ \omega \Big\},$$

where we have one degree of freedom (normalization), the set of Arrow price processes for the asset price process q and the measure Q. The no arbitrage property requires that $\mathcal{P}(q;Q) \neq \emptyset$ where $Q = P, P_i$ ($Q = P$ when beliefs are correct).

In our framework, at any interior solution to the maximization problem with a supporting portfolio process a set of first order conditions necessarily hold. Say that c_i is an *Euler process at the price process* q if

$$\forall\, t \geq 0,\ q_t(\omega) = \beta_i \cdot \frac{E_{P_i}[r_{t+1} \cdot u_i'(c_{i,t+1}) | \mathcal{F}_t](\omega)}{u_i'(c_{i,t}(\omega))} \qquad P_i - \text{a.s.}\, \omega.$$

Evidently, if c_i is an *Euler process at the price process* q then $\mathcal{P}(q;P_i) \neq \emptyset$.

Furthermore, in infinite horizon models one must also rule out Ponzi schemes, i.e. a trading plan that generates income at a date-event and rolls over debt in a manner that prevents an income loss at every other date-event, since, with monotonically increasing preferences, the existence of a Ponzi scheme in the budget set would imply that there is no maximizer and therefore no equilibrium. We follow Magill and Quinzii (1994) to define a *Ponzi scheme* at a no arbitrage price process q.

DEFINITION 1: Given i, let q be such that $\mathcal{P}(q;P_i) \neq \emptyset$. A *Ponzi scheme* is a θ and a pair (ω', t') such that (i) $\theta_t \in \Psi^{t,P_i}\ \forall\, t \geq 0$, (ii) $\theta_t(\omega) = 0$ for all $\omega \in \Omega$ if $t < t'$ and $\theta_t(\omega) = 0$ for all t if $\omega \notin \Omega(s^t(\omega'))$,
$-1 = q_{t'}(\omega') \cdot \theta_{t'}(\omega')$,
$0 = r_t(\omega) \cdot \theta_{t-1}(\omega) - q_t(\omega) \cdot \theta_t(\omega)$ for all $t \geq t' + 1$ and $P_i - \text{a.s.}\, \omega$.

2.4 IDC EQUILIBRIUM

We introduce a notion of equilibrium with uniform bounds on the value of debt. i's *IDC (implicit debt constraint) budget set* is defined as

$$\begin{aligned} BC_i(q) := \Big\{ c_i \in \Psi_+^{P_i} :\ & \text{there exists } \theta_i, \text{ with } \theta_{i,t} \in \Psi^{t,P_i}\ \forall\, t \geq 0, \text{ such that} \\ & \forall\, t \geq 0,\ c_{i,t}(\omega) + q_t(\omega) \cdot \theta_{i,t}(\omega) \leq z_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega)\ P_i - \text{a.s.}\, \omega, \\ & \sup_{t \geq 0} \operatorname{ess\,sup}_{\omega \in \Omega; P_i} |q_t(\omega) \cdot \theta_{i,t}(\omega)| < \infty \Big\}. \end{aligned}$$

The first set of conditions require that the consumption process be in i's consumption set, the second that there exists a supporting portfolio process, and the last condition is an *implicit debt constraint* that requires that the value of debt be uniformly bounded. Implicit debt constraints have been treated extensively in earlier literature on incomplete market economies with an infinite time horizon, e.g. Magill and Quinzii (1994) who provide conditions such that in any equilibrium where a transversality condition holds at every date-event, the value of debt is uniformly bounded.

For i, c_i is an *IDC maximizer* given q if (i) $c_i \in BC_i(q)$ and (ii) there is no $\tilde{c}_i \in BC_i(q)$, with supporting portfolio $\tilde{\theta}_i$, for which
$\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_{P_i}[u_i(\tilde{c}_{i,t})|\mathcal{F}_0](\omega) > \lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_{P_i}[u_i(c_{i,t})|\mathcal{F}_0](\omega)$.

DEFINITION 2: An *IDC equilibrium* is a tuple $(c_1^*, c_2^*, \theta_1^*, \theta_2^*, q^*)$ that is a market clearing allocation such that, at the prices q^*, c_i^*, with supporting portfolio θ_i^*, is an IDC maximizer for $i = 1, 2$.

In an IDC equilibrium, an agent maximizes discounted expected utility by choosing a process for consumption, i.e. $\{c_{i,t}\}_{t=0}^{+\infty}$ with the restriction that, for all t, $c_{i,t}$ is $\mathcal{F}_t$-measurable, that the spot market budget constraints are met, and an additional condition is met so as to ensure that the budget sets are appropriately bounded so that a maximizer exists. The IDC budget set does not permit Ponzi schemes (see Magill and Quinzii (1994)).

2.5 SURVIVAL

We formalize the various notions of asymptotic behaviour that we shall use by following the definitions that have been established in the literature. The term "survive" in its usual meaning corresponds to the formal "dominates".

DEFINITION 3: Fix a path ω.
Agent i dominates on ω if $\liminf_t c_{i,t}(\omega) > 0$.
Agent i survives on ω if $\liminf_t c_{i,t}(\omega) = 0$ and $\limsup_t c_{i,t}(\omega) > 0$.
Agent i vanishes on ω if $\limsup_t c_{i,t}(\omega) = 0$.

The definitions given are made operational by considering the behaviour of marginal utility. Given consumption processes for $i \in \mathcal{I}$, define the ratio of marginal utilities

$$y_t(\omega) := \frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))}.$$

The proof of the following lemma is straightforward hence omitted.

LEMMA 1: Assume A.3. Then
agent 2 vanishes on ω $\iff$ $\lim_t y_t(\omega) = \infty$;
agent 2 survives on ω $\iff$ $0 \le \liminf_t y_t(\omega) < \limsup_t y_t(\omega) = \infty$;
agent 2 dominates on ω $\iff$ $0 \le \liminf_t y_t(\omega) \le \limsup_t y_t(\omega) < \infty$.

The corresponding results for agent 1 are obtained by studying the behaviour of $1/y_t(\omega)$. Both the agents dominate on ω if and only if $0 < \liminf_t y_t(\omega) \leq \limsup_t y_t(\omega) < \infty$. Clearly, for feasible processes and strictly positive aggregate endowments, on a given path, both agents cannot vanish.

3. A LEADING EXAMPLE

We turn to our example which has five salient features. (i) $u_1(x) = (1/(1-a))x^{1-a}$ with $a > 0$ and $u_2(x) = \log x$. (ii) $z_{2,0}(\omega) = Z_0(\omega)$ and $z_{2,t}(\omega) = 0$ otherwise. (iii) The uncertainty in the model comes from 1's endowment which follows an i.i.d. process with two points in its support: $Z \in \{\underline{z}, \bar{z}\}$ with probability $p \in (0,1)$ and $(1-p)$ respectively. (iv) The asset is on the aggregate endowment so $r_t(\omega) = Z_t(\omega)$. (v) The beliefs of each agent are $(p_i, (1-p_i))$ with $p_i \in (0,1)$ and both could hold incorrect beliefs (though one or both could hold the correct belief).

It is known that 2's decision rule is

$c_{2,t}(\omega) = (1-\beta_2) \cdot w_{2,t}(\omega)$ and $\theta_{2,t}(\omega) = \beta_2 \cdot [w_{2,t}(\omega)/q_t(\omega)]$

where $w_{2,t}(\omega) = r_t(\omega) \cdot \theta_{2,t-1}(\omega) = Z_t(\omega) \cdot \theta_{2,t-1}(\omega)$ so that it is independent of p_2. It follows that at a feasible allocation where agent 2 optimizes given prices $q_t(\omega)$, in particular at equilibrium, $\theta_{2,t}(\omega) = \beta_2 \cdot [Z_t(\omega) \cdot \theta_{2,t-1}(\omega)/q_t(\omega)]$ so that such prices, in particular equilibrium prices, must satisfy

$q_t(\omega) = \beta_2 \cdot Z_t(\omega) \cdot [\theta_{2,t-1}(\omega)/\theta_{2,t}(\omega)]$.

As for 1, when agent 2 optimizes and the allocation is feasible, we must have

$c_{1,t}(\omega) = Z_t(\omega) - c_{2,t}(\omega) = Z_t(\omega) - (1-\beta_2) \cdot w_{2,t}(\omega) = Z_t(\omega)[1 - (1-\beta_2) \cdot \theta_{2,t-1}(\omega)]$.

Furthermore, the first order conditions for 1 are

$$\beta_1 \, E_{P_1}[(c_{1,t})^{-a} \cdot Z_t | \mathcal{F}_{t-1}](\omega) = q_{t-1}(\omega) \cdot (c_{1,t-1}(\omega))^{-a}$$

where we use the fact that $r_t(\omega) = Z_t(\omega)$.

By substituting for $c_{1,t}$ and q_{t-1} we obtain

$$\beta_1 \, E_{P_1}\Big[\Big(Z_t[1-(1-\beta_2) \cdot \theta_{2,t-1}]\Big)^{-a}\Big) Z_t | \mathcal{F}_{t-1}\Big](\omega) = \beta_2 \cdot Z_{t-1}(\omega) \cdot \tfrac{\theta_{2,t-2}(\omega)}{\theta_{2,t-1}(\omega)} \cdot \Big(Z_{t-1}(\omega)[1-(1-\beta_2) \cdot \theta_{2,t-2}(\omega)]\Big)^{-a}.$$

We have obtained a stochastic difference equation in $\theta_{2,t}$ such that if an allocation is feasible, if it is maximizing for 2, and if it satisfies the first order conditions for 1 then $\theta_{2,t}$ must satisfy the difference equation; therefore, a $\theta_{2,t}$ process that obtains in equilibrium will satisfy the stochastic difference equation.[14]

By simplifying the condition we obtain

$$\frac{\beta_1}{\beta_2} \cdot \frac{(1-\beta_2) \cdot \theta_{2,t-1}(\omega)}{[1-(1-\beta_2) \cdot \theta_{2,t-1}(\omega)]^a} = \frac{[Z_{t-1}(\omega)]^{1-a}}{E_{P_1}[Z^{1-a}]} \cdot \frac{(1-\beta_2) \cdot \theta_{2,t-2}(\omega)}{[1-(1-\beta_2) \cdot \theta_{2,t-2}(\omega)]^a}.$$

It follows that if $(1-\beta_2) \cdot \theta_{2,t-1}(\omega) \in (0,1)$ then $(1-\beta_2) \cdot \theta_{2,t}(\omega) \in (0,1)$ and the system

[14] Existence of an IDC equilibrium follows from our Theorem 3.

has a real valued solution. By iterating we see that

$$\Leftrightarrow \qquad \frac{(1-\beta_2)\cdot\theta_{2,T}(\omega)}{[1-(1-\beta_2)\cdot\theta_{2,T}(\omega)]^a} = \Big(\frac{\beta_2}{\beta_1}\Big)^T \cdot \frac{\Pi_{t=1}^T\Big(Z_t(\omega)]^{1-a}\Big)}{\Big(E_{P_1}[Z^{1-a}]\Big)^T} \cdot \frac{(1-\beta_2)\cdot\theta_{2,0}(\omega)}{[1-(1-\beta_2)\cdot\theta_{2,0}(\omega)]^a}$$

$$\Leftrightarrow \qquad \frac{1}{T}\cdot\log\Big(\frac{(1-\beta_2)\cdot\theta_{2,T}(\omega)}{[1-(1-\beta_2)\cdot\theta_{2,T}(\omega)]^a}\Big) = \log\Big(\frac{\beta_2}{\beta_1}\Big)+\Big(\frac{1}{T}\sum_{t=1}^{T}\log\,[Z_t(\omega)]^{1-a}\Big)-\log\Big(E_{P_1}[Z^{1-a}]\Big)$$

$$+\frac{1}{T}\cdot\log\Big(\frac{(1-\beta_2)\cdot\theta_{2,0}(\omega)}{[1-(1-\beta_2)\cdot\theta_{2,0}(\omega)]^a}\Big).$$

Since Z_t is an i.i.d. process (and obviously uniformly bounded), the Strong Law of Large Numbers guarantees that

$$\frac{1}{T}\sum_{t=1}^{T}\log\,[Z_t(\omega)]^{1-a} \to E_P[\log Z^{1-a}] \quad P-\text{a.s.}$$

with the consequence that, by Jensen's inequality,

$$\Big(\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\log\,[Z_t(\omega)]^{1-a}\Big) - \log\Big(E_P[Z^{1-a}]\Big) < 0 \quad P-\text{a.s.}$$

It follows that if $p_1 = p$, so that 1's beliefs are correct, and $\beta_1 = \beta_2 = \beta$, so that both the agents are equally impatient, then

$$\log\Big(\frac{(1-\beta)\cdot\theta_{2,T}(\omega)}{[1-(1-\beta)\cdot\theta_{2,T}(\omega)]^a}\Big) \to -\infty \quad P-\text{a.s.}$$

$$\Leftrightarrow \quad \Big(\frac{(1-\beta)\cdot\theta_{2,T}(\omega)}{[1-(1-\beta)\cdot\theta_{2,T}(\omega)]^a}\Big) \to 0 \quad \Leftrightarrow \quad \theta_{2,T}(\omega)\to 0 \quad \Leftrightarrow \quad c_{2,T}(\omega)\to 0 \quad P-\text{a.s.}$$

and so in *every* equilibrium of the example, agent 2 vanishes with probability one.

Since the application of Jensen's inequality above is strict, agent 2 could have correct beliefs and agent 1 could have incorrect ones in an open set around p and 1 could even be marginally more impatient than 2, and yet 2 vanishes almost surely in every equilibrium.

The example shows in a very clear manner that no entropy measure can be critical to understanding survival because any properly defined entropy measure must attain its maximum when beliefs are correct.

REMARK 2: We note the following feature of the example. Since $c_{2,t}(\omega) = (1-\beta_2)\cdot r_t(\omega)\cdot\theta_{2,t-1}(\omega)$, $q_t(\omega)\cdot\theta_{2,t}(\omega) = \beta_2\cdot w_{2,t}(\omega) = \beta_2\cdot(c_{2,t}(\omega)/(1-\beta_2))$ so debt is uniformly bounded in any equilibrium since consumption is nonnegative and bounded by the uniform upper bound on the aggregate endowment.

For later reference we note that $r_t(\omega)\cdot u_2'(c_{2,t}(\omega)) = r_t(\omega)/c_{2,t}(\omega) = 1/((1-\beta_2)\cdot\theta_{2,t-1}(\omega))$; so $r_t(\omega)\cdot u_2'(c_{2,t}(\omega))$ is an $\mathcal{F}_{t-1}-$measurable quantity. Also

$$\frac{r_t(\omega)\cdot u_1'(c_{1,t}(\omega))}{E_{P_1}\,[r_t\cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)} \to \frac{r_t(\omega)\cdot u_1'(Z_t(\omega))}{E_{P_1}\,[r_t\cdot u_1'(Z_t)|\mathcal{F}_{t-1}](\omega)},$$

a nondegenerate random variable; this ensures that the assumption that we introduce as A.7 in Section 5.1 holds in the example.

The analysis in this section depends heavily on the endowment structure where 2 has no endowment except in period 0. Theorem 4 will show that, in fact, the property we identify is robust to changes in the endowment process, preferences, and asset structure.

3.1 THE GENERAL LESSON

The example in Section 3 is indicative of a very interesting phenomenon that appears to be driven by the fact that markets are incomplete. In fact the phenomenon in the example leads to two rather different conjectures about the implications of market incompleteness: (a) that the consumption of some agent could be repeatedly arbitrarily close to zero and (b) that the consumption of some agent stays close to zero eventually. We would like to know the extent to which these results are a general property of economies with dynamically incomplete markets. With appropriate formalizations of the fact that markets are effectively incomplete forever, Theorem 1 in Section 4.2 will show that (a) holds while Theorem 4 in Section 5.5 will show that, in a robust class of economies, (b) holds. More precisely, Theorem 1 (ii) will show that on every path on which the ratio of the (one period ahead ratio of) marginal utilities, y_t/y_{t-1}, displays variability, the consumption of some agent gets arbitrarily close to zero infinitely often; while Theorem 4 will specify a robust class of economies with equilibria in which the consumption of an agent stays close to zero eventually on every path. We remark that one expects a version of Theorem 1 to hold in specifications of infinite horizon economies with incomplete markets that are not covered by our analysis.

4. RULING OUT DOMINANCE

In this section we prove our first main result: we shall show that market incompleteness is incompatible with both agents consuming uniformly positive quantities eventually. To be able to prove the results, we use the insights gained from the analysis of the example to formulate the problem in general terms. In Section 4.1 we use the Euler equations for the two agents to express the ratio of the derivatives of the Bernoulli utility functions of the two agents, the ratio of marginal utilities, as a stochastic process with a very convenient structure and identify some key properties that the transformed process satisfies. This reformulation is valid even when the subjective beliefs of the agents do not coincide with the truth and are not homogeneous. Then, in Section 4.2 we state and discuss Theorem 1. Section 4.3 relates our result to examples of infinite horizon economies with incomplete markets that have appeared in the literature.

4.1 FIRST ORDER CONDITIONS AND THEIR IMPLICATIONS

As Sandroni (2000) and Blume and Easley (2004) show, in the case where markets are complete, the behaviour of the variable y_t is rather simply determined by the ratio of the discount factors, the ratio of the posterior beliefs of agents, and an initial condition. In Proposition 1 we show that, when markets are incomplete, the behaviour of y_t can be captured succinctly using the ratio of two processes where each is the product of random variables with conditional mean one (taken with respect to the subjectively held belief) in addition to the ratio of the discount factors and an initial condition.

Given consumption processes for $i \in \mathcal{I}$, define

$$\hat{r}_{i,t}(\omega) := \frac{r_t(\omega) \cdot u_i'(c_{i,t}(\omega))}{E_{P_i}\left[r_t \cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}\right](\omega)}, \qquad R_{i,T}(\omega) := \Pi_{t=1}^{T} \hat{r}_{i,t}.$$

PROPOSITION 1: Assume A.2, A.3, and A.5. Then $E_{P_i}\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right](\omega) = 1$. Furthermore, if the consumption processes c_i are Euler processes at the price process q, then

(i) $R_{i,1+T}(\omega) = \beta_i^{T+1} \cdot \dfrac{u_i'(c_{i,1+T}(\omega))}{u_i'(c_{i,0}(\omega))} \cdot \Pi_{t=0}^{T} \left(\dfrac{r_{1+t}(\omega)}{q_t(\omega)} \right),$

(ii) $\dfrac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} = \dfrac{\beta_2}{\beta_1} \cdot \dfrac{y_t(\omega)}{y_{t-1}(\omega)}$ and $y_T(\omega) = \left(\dfrac{\beta_1}{\beta_2} \right)^T \cdot \dfrac{R_{2,T}(\omega)}{R_{1,T}(\omega)} \cdot y_0(\omega),$

(iii) $y_{t-1}(\omega) = \dfrac{\beta_2}{\beta_1} \cdot E_{P_2}\left[\hat{r}_{1,t} \cdot y_t|\mathcal{F}_{t-1}\right](\omega), \qquad \dfrac{1}{y_{t-1}(\omega)} = \dfrac{\beta_1}{\beta_2} \cdot E_{P_1}\left[\hat{r}_{2,t} \cdot \dfrac{1}{y_t} \middle| \mathcal{F}_{t-1} \right](\omega).$

REMARK 3: When we consider Pareto optimal allocations obtainable as competitive equilibria, $(\beta_2/\beta_1) \cdot \frac{P_{2,T}(\omega)}{P_{1,T}(\omega)} \cdot y_T(\omega) = y_{T-1}(\omega)$ and $(\beta_2/\beta_1)^T \cdot \Pi_{t=1}^{T} \left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \right) \cdot y_T(\omega) = y_0(\omega)$. From Proposition 1 (ii) it follows that $\frac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} = \frac{P_{1,t}(\omega)}{P_{2,t}(\omega)} \quad \forall\, t \geq 0,\; P - \text{a.s.}\, \omega$. In the case where beliefs are homogeneneous one obtains the result that both agents dominate if and only if $\beta_1 = \beta_2$ while i dominates and $-i$ vanishes if and only if $\beta_i > \beta_{-i}$. This turnpike result for complete market economies is well known (Becker (1980), Rader (1981), and Bewley (1982)). When beliefs are heterogeneous and $\beta_1 = \beta_2$ both agents dominate on a path if and only if $0 < \liminf \Pi_{t=1}^{T} \left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \right)$ and $\limsup \Pi_{t=1}^{T} \left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \right) < \infty$. This result is due to Sandroni (2000) and Blume and Easley (2004).

4.2 THE RESULT

In this section we restrict attention to the case where the agents are equally impatient and we study the asymptotic behavior of their consumption processes on paths where (a) the ratio of marginal rates of substitution does not display one period ahead conditional variability in the limit, and (b) the ratio of marginal utilities does display such variability infinitely often, i.e. markets are effectively incomplete forever. A third case is (c) where the ratio of marginal rates of substitution does display variability infinitely often but only because of the variability in beliefs, a case displaying perverse behaviour that we shall discuss at some length. Theorem 1 provides a very strong result when markets are effectively incomplete forever: the consumption of some agent approaches zero infinitely often and it could even happen that consumption is zero eventually.

To be more precise, we define the sets

$$V_0 := \Big\{\omega \lim_t \operatorname{var}\Big[\log\Big(\frac{P_{2,t}}{P_{1,t}}\cdot\frac{y_t}{y_{t-1}}\Big)\Big|\mathcal{F}_{t-1}\Big](\omega) = 0\Big\},$$

$$V_\epsilon := \Big\{\omega : \limsup_t \operatorname{var}\Big[\log\Big(\frac{P_{2,t}}{P_{1,t}}\cdot\frac{y_t}{y_{t-1}}\Big)\Big|\mathcal{F}_{t-1}\Big](\omega) \geq \epsilon\Big\},$$

$$V^y_\epsilon := \Big\{\omega : \limsup_t \operatorname{var}\Big[\log\Big(\frac{y_t}{y_{t-1}}\Big)\Big|\mathcal{F}_{t-1}\Big](\omega) \geq \epsilon\Big\}.[15]$$

Recall that in the case of Pareto optimal allocations, as noted in Remark 3, marginal rates of substitution are equal at every date-event. In Theorem 1 (i) we show that when one restricts attention to paths in V_0, marginal rates of substitution are equalized in the limit, $\lim_t\Big(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)}\cdot\frac{y_t(\omega)}{y_{t-1}(\omega)}\Big) = 1$; the result has an interesting implication for the behaviour of consumption in the case where beliefs are homogeneous and this is discussed later. On the other hand, for paths in $\cup_{\epsilon>0}V^y_\epsilon$ that satisfy a very weak additional property, every positive lower bound on consumption is violated infinitely often for some agent. This result, Theorem 1 (ii), can be read as showing that when markets are effectively incomplete forever, the only simple equilibria are the ones in which only one agent lives in the limit since in all the others some agent must consume arbitrarily close to zero infinitely often.

There are two cases to which Theorem 1 (ii) does not apply— (c) above where perverse behaviour is generated by choosing beliefs appropriately, and paths in $\cup_{\epsilon>0}V^y_\epsilon$ that do not satisfy an additional property—that we now discuss in detail.

$\Omega/(\cup_{\epsilon>0}V^y_\epsilon)$ is the set on which for any economy with homogeneous beliefs, markets are effectively complete in the limit, i.e. y_t/y_{t-1} does not display one period ahead variability. In an economy with heterogeneous beliefs it is possible that even though y_t/y_{t-1} converges, the ratio of marginal rates of substitution displays variability, i.e. $V^{\text{sub}} \neq \emptyset$ where $V^{\text{sub}} := \Big(\cup_{\epsilon>0}\ V_\epsilon\Big) \cap \Big(\Omega/(\cup_{\epsilon>0}V^y_\epsilon)\Big)$ and "sub" denotes the perverse behaviour induced by well chosen subjective beliefs. This case, identified as (c) at the beginning of the subsection, appears to be very special since the consumption processes in the limit must be supportable as a Pareto optimal allocation in an economy with homogeneous beliefs even though marginal rates of substitution do not converge when beliefs are heterogeneous. This is the first case in which Theorem 1 (ii) does not apply.

Also, Theorem 1 (ii) does not apply when we consider the set of paths, denoted V^y_∞ below, where the ratio of marginal utilities displays one period ahead variability infinitely often but very rarely, in the sense that the maximal length of the time interval until it displays variability again diverges on each path. To formalize the notion we need some definitions. For $\epsilon > 0$ and every $\omega \in V^y_\epsilon$, define $\Delta^\epsilon_t(\omega) := \inf_{k\geq 1} \operatorname{var}\Big[\log\Big(\frac{y_{t+k}}{y_{t+k-1}}\Big)\Big|\mathcal{F}_{t+k-1}\Big](\omega) \geq$

[15] Since $\cup_{\epsilon>0}V_\epsilon = \Omega/V_0$, V_0 and $\cup_{\epsilon>0}V_\epsilon$ partition the set of paths in accordance with the limiting behaviour of the variance of the ratio of one period ahead marginal rates of substitution.

ϵ as the minimum number of periods it takes for the ratio of marginal utilities to display one period ahead variability after date t. Clearly, $\Delta_t^\epsilon(\omega)$ is finite for every ϵ, t and $\omega \in V_\epsilon^y$. However, it may diverge as t diverges. For $T \in [0, +\infty)$, define the set

$$V_{T,\epsilon}^y := \Big\{\omega : \limsup_t \operatorname{var}\Big[\log\Big(\frac{y_t}{y_{t-1}}\Big)\Big|\mathcal{F}_{t-1}\Big](\omega) \geq \epsilon \text{ and } \sup_t \Delta_t^\epsilon(\omega) = T\Big\}.$$

The set of paths where the ratio of marginal utilities displays one period ahead variability infinitely often, $\cup_{\epsilon>0} V_\epsilon^y$, can be partitioned into two sets, one containing those paths where the ratio of one period ahead marginal utilities displays variability on some bounded interval of time of length $T < \infty$, $\cup_{T,\epsilon>0} V_{T,\epsilon}^y$, and its complement, the set V_∞^y on which $\sup_t \Delta_t^\epsilon(\omega) = +\infty$. The interest of studying paths in the set V_∞^y is not evident.[16]

We turn to the implication of Theorem 1 (i) for consumption behaviour in the case where beliefs are homogeneous and correct. The fact that, in the case of Pareto optimal allocations, marginal rates of substitution are equal at every date-event implies that, when both the agents have positive wealth, both agents have consumption bounded away from zero. One might conjecture that the same is true for consumption for paths in V_0 since marginal rates of substitution are equal in the limit but this is far from obvious; we do not have an example but we believe that it is possible that an agent has consumption that is simultaneously arbitrarily close to zero infinitely often or even eventually zero.

In the example and in the results in Section 5, since $\hat{r}_{2,t}(\omega) = 1$ always, $\hat{r}_{1,t}(\omega)$ must display variability to guarantee that agent 2 vanishes and so, by Proposition 1, y_t/y_{t-1} also displays variability. So, although both parts of Theorem 1 are compatible with the consumption of an agent being arbitrarily close to zero eventually, our construction confirms the phenomenon for paths in $\cup_{\epsilon>0} V_\epsilon^y$, the case covered by Theorem 1 (ii).

We can now state our result.

THEOREM 1: Consider an IDC equilibrium. Assume that $\beta_1 = \beta_2$, that A.1, A.2, A.3, and A.5 hold. Then,
(i) $\lim_t \Big(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_t(\omega)}{y_{t-1}(\omega)}\Big) = 1$ P-a.s. $\omega \in V_0$.
(ii) for every $T < \infty$, $\epsilon > 0$, and n,
$\liminf_t c_{i,t}(\omega) < 1/n$ P-a.s. $\omega \in V_{T,\epsilon}^y \cap \{\omega : \liminf_t c_{j,t}(\omega) \geq 1/n\}$.

The idea behind the proof of Theorem 1 (i) is relatively straightforward. For the proof of Theorem 1 (ii) we use a version of the Second Borel-Cantelli Lemma that does not require independence and appears in Freedman (1973). It is easier to grasp the intuition of the proof in the case where beliefs are homogeneous. In that case we use the result that we just mentioned to show that the event "the ratio of marginal utilities never falls and rises by a pre-fixed amount each time for a pre-fixed number of times," an event that has uniformly positive conditional probability under A.1, implies that that event happens infinitely often on the set identified in the statement of Theorem 1 (ii). The event that we

[16]Results on the lack of collinearity of marginal utility vectors in generic finite horizon incomplete market economies suggest that the set V_∞^y might even be null for generic economies.

have identified can be specified so as to ensure that the ratio of marginal utilities violates any prespecified upper bound infinitely often and that clinches the result.

4.3 RELATING TO EARLIER EXAMPLES

We begin the section by relating Theorem 1 to an example in Coury and Sciubba (2005) of an infinite horizon economy with incomplete markets where both the agents dominate. They construct the example by starting with a Pareto optimal allocation supportable with incomplete markets and then changing beliefs in a manner that leaves demand unchanged. This is possible since markets are incomplete and leads to a suboptimal allocation with respect to the new beliefs. However, the construction is clearly degenerate. Their example corresponds to the set labeled V^{sub} in Section 4.2. As we indicated, even though one can study survival on that set with our tools, the interest of doing so is not evident.

We turn to an example provide by Levine and Zame (2001) in which both agents dominate. They show how a random selection from a static economy can be used to construct a sunspot equilibrium in the infinite horizon economy. This requires two goods at each date so that there is multiplicity of equilibria in the static economy. Since the sunspot realization is fixed once and for all at the first date, markets are effectively complete from then onwards and Theorem 1 (i) applies.

Kubler and Schmedders (2002) provide various examples of economies that are particular cases of our general model where both agents dominate. Theorem 1 (ii) does not apply since the main feature in all of their examples is that assets are restricted to be Arrow securities a case that we rule out by A.5. This causes trades to be restricted in a special way that, together with the fact that, on the set where assets do not pay, the range of behaviour is very limited since it must be driven by the endowment process which has finite support, makes the marginal utility process degenerate even though markets are effectively incomplete. In fact, if we carry out an analysis analogous to that in Theorem 1 we are led to the conclusion that in their equilibria once one conditions on trading the securities and continuing to a state in which some traded Arrow security has a nonzero payoff, the ratio of marginal utilities is degenerate. This lack of variability breaks the intuitive mechanism that makes Theorem 1 (ii) work.

Blume and Easley (2004) provide an example where an agent with correct beliefs vanishes, a phenomenon that is along the lines of our leading example except that their probabilistic structure is much simpler; also, as the authors note, their construction is not robust to completing the market since in that case equilibrium fails to exist.

Constantinides and Duffie (1996) and Krebs (2004) consider economies like ours but with a dividend paying asset. Since they allow endowments to grow without any upper bound, it is not clear that an analogue of Theorem 1 can be proved in their framework.

5. EQUILIBRIA WHERE SOMEONE VANISHES

In this section we turn to our second main result. We will show that the property that the example displays, namely, that some agent vanishes with probability one, is a robust implication of market incompleteness. We do so by combining the following two results: (i) for equilibria where $\hat{r}_2$ is a degenerate process, agent 2 vanishes almost surely, and (ii) there exist open sets of endowment distributions for which one can construct equilibrium consumption processes with the property that $\hat{r}_2$ is a degenerate process.

Section 5.1 develops the first result which uses the Strong Law of Large Numbers for uncorrelated random variables with uniformly bounded second moments. Section 5.2 shows that it is possible to construct aggregate feasible consumption processes that satisfy the Euler equations, that have summable supporting prices, that induce a degenerate process $\hat{r}_2$, and that display certain monotonicity properties. In Section 5.3, we define TC0 equilbrium, a weaker notion of equilibrium in our model economy and Theorem 2 in Section 5.4 provides conditions that let us identify IDC and TC0 equilibria. Finally, in Section 5.5 we provide our results. In Theorem 3 we show that for an appropriate distribution of endowments, we have equilibria without trade in which agent 2 vanishes a.s.; we also specify conditions such that our construction leads to an IDC equilibrium where agent 2 vanishes a.s. Finally, in Theorem 4 we provide conditions such that for every no trade equilibrium identified in Theorem 3, there is an open set of endowments for which there is a TC0 equilibrium in which agent 2 vanishes a.s.

For the main results in this section we shall assume that beliefs are correct so $P_1 = P_2 = P$. However, some results hold more generally; in such cases we make the more general statement.

5.1 THE STRONG LAW ARGUMENT

If we consider consumption processes for 1 and 2 that satisfy the Euler equations at the common price process q then, by Proposition 1, for the analysis of survival, it suffices to study the behaviour of an alternative process. We start with a result that puts together some properties of the alternative process, namely, that $\hat{r}_i$ is uniformly bounded from above and that $\lim_{T\to\infty} R_{i,T}(\omega)$ is P_i−a.s. finite. Define $\bar{\bar{r}}_i := \sup_{t\geq 0} \text{ess.}\sup_{\omega\in\Omega;P_i} \hat{r}_{i,t}(\omega)$.

PROPOSITION 2: Assume A.1, A.3 and A.5. Then $\bar{\bar{r}}_i < \infty$. Also, there is a random variable R_i^* that is nonnegative and a.s. finite with $E_{P_i}[R_i^*] \leq 1$ such that $R_i^*(\omega) = \lim_{T\to\infty} R_{i,T}(\omega)$ P_i−a.s.

By Lemma 1 requiring that Agent 2 vanish on ω is equivalent to requiring $\lim_t y_t(\omega) = \infty$. So from Proposition 1 (ii) we conclude that

$$\log(\beta_1/\beta_2) + \text{liminf}\, \frac{1}{T}\left(\sum_{t=1}^{T} \log \hat{r}_{2,t}(\omega) - \sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega)\right) > 0 \quad \Rightarrow \quad c_{2,t}(\omega) \to_{t\to+\infty} 0.$$

Evidently, if $\hat{r}_2$ is a degenerate process, and $\beta_1 = \beta_2$, then to show that 2 vanishes a.s. it suffices to show that $\limsup \frac{1}{T}\left(\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega)\right) < 0$ a.s. A possible line of argument is

$$\frac{1}{T}\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) \to \frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) < \frac{1}{T}\sum_{t=1}^{T} \log E_{P_1}[\hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) = 0$$

where the first result, with a.s. convergence, would follow from a suitable Strong Law of Large Numbers, the second uses Jensen's inequality, and the third uses the defining property $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$. For the inequality to be strict we need to guarantee that there is variability in the tail of the process $\{E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega)\}$.

ASSUMPTION A.7: $\{\omega : \text{limsup} \frac{1}{T} \sum_{t=1}^{T} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) < 0\} = \Omega$.

A.7 amounts to the requirement that on almost all paths, markets never become effectively complete so that complete risk sharing remains impossible. Jensen's inequality and $E_{P_1}[\hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) = 1$ lead to the weaker property where the set that appears in A.7 is defined with a weak inequality.

A.7 holds if the time average is uniformly below zero, a strong sufficient condition. Also, when $\hat{r}_{2,t}(\omega) = 1$, A.7 holds if $r_t(\omega) = 1$ and $\text{var}\,(Z_t|\mathcal{F}_{t-1}) > \epsilon > 0$ at every date t, i.e. the asset is a real bond and the endowment process has uniformly positive conditional variance forever. This is because $\hat{r}_{2,t}(\omega) = 1$ implies that $c_{2,t}$ and r_t move in the same direction, and so conditional variability in the endowment guarantees that $r_t \cdot u_1'(Z_t - c_{2,t})$ is nondegenerate. In fact, in our leading example $E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) < 0$ and A.7 holds since, by the third property noted in Remark 2 and Jensen's inequality and the fact that the random variable is nondegenerate,

$$E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) \to E_{P_1}\left[\log \frac{r_t \cdot u_1'(Z_t)}{E_{P_1}\left[r_t \cdot u_1'(Z_t)|\mathcal{F}_{t-1}\right]}\middle|\mathcal{F}_{t-1}\right](\omega) < \log 1 = 0,$$

and $\hat{r}_2$ was degenerate since $r_t(\omega) \cdot u_2'(c_{2,t}(\omega))$ was an $\mathcal{F}_{t-1}-$measurable quantity.

With A.7 we are able to obtain the result by applying the Strong Law of Large Numbers for uncorrelated random variables with uniformly bounded second moments. Define the set $\mathcal{A}_i := \{\omega \in \Omega : \text{liminf}\, \hat{r}_{i,t}(\omega) = 0\}$. We have

PROPOSITION 3: Assume A.1, A.3, A.5 and A.7. Then $R_{1,t}(\omega) \to 0$ $P_1 - \text{a.s.}\, \omega \in \Omega/\mathcal{A}_1$. Furthermore, given β_1 and $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\beta_2 \in (\delta \cdot \beta_1, \beta_1) \quad \Rightarrow \quad P_1\left(\left\{\omega : \log(\beta_2/\beta_1) + \frac{1}{T}\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) < 0\right\}\right) = P_1(\Omega/\mathcal{A}_1) - \epsilon.$$

REMARK 4: In the case where A.7 is strengthened to require

$$\left\{\omega : \text{limsup} \frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log \hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) \leq \epsilon < 0\right\} = \Omega,$$

the statement in the second part of Proposition 3 can be strengthened to:
given β_1, there exists $\delta \in (0, 1)$ such that

$$\beta_2 \in (\delta \cdot \beta_1, \beta_1) \quad \Rightarrow \quad P_1\left(\left\{\omega : \log(\beta_1/\beta_2) + \frac{1}{T}\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) < 0\right\}\right) = P_1(\Omega/\mathcal{A}_1).$$

The second part of Proposition 3 will be used to show that, at the margin, the turnpike property fails when markets are incomplete since the less patient agent can survive.

5.2 A CONSTRUCTIVE APPROACH TO EQUILIBRIUM

In this section we propose a methodology for constructing feasible consumption processes that satisfy $\hat{r}_{2,t}(\omega) = 1$ for every $t \geq 0$ $P-$a.s. ω in addition to satisfying the Euler equations and having summable supporting prices.

First, in Proposition 4, we gather together the basic properties of our construction, namely that the process $\hat{r}_2$ is degenerate, a related implication for $\hat{r}_1$, that the process constructed is uniquely defined for each initial condition, that it is monotone increasing and continuous in the initial condition, and that it has nice boundary behaviour with respect to the initial condition.

PROPOSITION 4: Assume A.2, A.3, and A.5, and that $P_1 = P_2 = P$. For Z an aggregate endowment process, consider a triple $(c, t_0, \omega) \in R_{++} \times \{0, 1, 2, \cdots\} \times \Omega$ such that $c \in (0, Z_{t_0}(\omega))$. Then there exists a unique pair of feasible consumption processes, denoted $\{C_{i,t}(c, t_0, \omega)\}_{t \geq t_0}$, defined only for $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$ and with $C_{1,t_0}(c, t_0, \omega) = c$ such that the following statements are true for $t \geq t_0 + 1$ $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$:

(i) $\hat{r}_{2,t}(\omega) = 1$;
(ii) $y_{t-1}(\omega) = (\beta_2/\beta_1) \cdot \hat{r}_{1,t}(\omega) \cdot y_t(\omega)$;
(iii) if (c, t_0, ω) and (c', t_0, ω) are such that $c > c'$ then $C_{1,t}(\tilde{\omega}; c, t_0, \omega) > C_{1,t}(\tilde{\omega}; c', t_0, \omega)$;
(iv) the processes $\{C_{i,t}(c, t_0, \omega)\}_{t \geq t_0}$ are continuous in c;
(v) given $t_0, \epsilon > 0$, and $T > t_0$, there exists $c > 0$ such that $Z_t(\tilde{\omega}) - C_{1,t}(\tilde{\omega}; c, t_0, \omega) < \epsilon$ for all t such that $T \geq t \geq t_0 + 1$.
(vi) If we also assume A.1 then, given $t_0, \epsilon > 0$, and $T > t_0$, there exists $A \in \mathcal{F}_T$ with $P(A) > 0$ and $c > 0$ such that $C_{1,t}(\tilde{\omega}; c, t_0, \omega) < \epsilon$ for all t such that $T \geq t \geq t_0 + 1$ and $P-$a.s. $\tilde{\omega} \in A$.

We now show that the personalized Arrow-Debreu prices that support the proposed allocation are summable. The proof consists in showing that the one period undiscounted intertemporal rate of substitution for agent 2 is uniformly bounded by M, the number specified in A.6, and then using A.6, which is a restriction on discount factors, and a property of our construction.

PROPOSITION 5: Assume A.2, A.3, A.5, and A.6, and that $P_1 = P_2 = P$. Then

$$0 \leq E_P\left[\sum_{t=t_0}^{T} \beta_i^t \cdot \frac{u_i'(C_{i,t}(\cdot; c, t_0, \omega))}{u_i'(C_{i,t}(\cdot; c, t_0, \omega))} \middle| \mathcal{F}_{t_0} \right](\tilde{\omega}) \leq 1/(1 - \beta_i \cdot M) \quad P-\text{a.s. } \tilde{\omega} \in \Omega(s^{t_0}(\omega)).$$

To apply Proposition 3 to conclude that in our solution agent 2 vanishes a.s. we need to show that $P(\mathcal{A}_1) = 0$ where $\mathcal{A}_i := \{\omega \in \Omega : \liminf \hat{r}_{i,t}(\omega) = 0\}$. This is done by showing that since the induced process y does not have zero as a limit point, neither does

c_1 have zero as a limit point which implies that zero cannot be a limit point of $\hat{r}_1$.

PROPOSITION 6: Assume A.2, A.3, and A.5, and $P_1 = P_2 = P$. Then, in the proposed solution $P(\mathcal{A}_1) = 0$.

By combining Propositions 3 and 6 we can conclude that $\sum_{t=0}^{T} \log\hat{r}_{1,t}(\omega) \to -\infty$.

5.3 TC0 EQUILIBRIUM

We introduce a second notion of equilibrium that does not impose a uniform bound on the value of debt; instead it imposes a transversality condition at date 0 where a system of personalized prices is used to evaluate the limiting value of debt.

Recall that $\mathcal{P}(q; Q)$ is the set of Arrow price processes compatible with a no arbitrage asset price process q and the measure Q. We shall assume that beliefs are homogeneous and correct, $P = P_i$. Define

$$\mathcal{P}^1(q; P) := \Big\{p \in \mathcal{P}(q; P) \; : \; \lim_{T\to+\infty} \textstyle\sum_{t=0}^{T} E_P\big[p_t|\mathcal{F}_0\big](\omega) < \infty\Big\}$$

the set of Arrow price processes that are summable with respect to the measure P.

For $p \in \mathcal{P}(q; P)$, i's *TC0 (date zero transversality condition) budget set given* (q, p) is

$$\begin{aligned} BC_i^{\mathrm{TC}}(q, p) := \Big\{c_i \in \Psi_+^P : \; & \text{there exists } \theta_i, \text{ with } \theta_{i,t} \in \Psi^{t,P} \; \forall\, t \geq 0, \text{ such that} \\ & \forall\, t \geq 0, \; c_{i,t}(\omega) + q_t(\omega) \cdot \theta_{i,t}(\omega) \leq z_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega) \; P - \text{a.s.}\, \omega, \\ & \liminf_{T\to+\infty} E_P\big[p_T \cdot q_T \cdot \theta_{i,T}|\mathcal{F}_0\big](\omega) \geq 0\Big\}. \end{aligned}$$

For i, c_i is a *TC0 maximizer* given (q, p) if (i) $c_i \in BC_i^{\mathrm{TC}}(q, p)$ and (ii) there is no other $\tilde{c}_i \in BC_i^{\mathrm{TC}}(q, p)$, with supporting portfolio $\tilde{\theta}_i$, for which

$$\lim_{T\to+\infty} \textstyle\sum_{t=0}^{T} \beta_i^t \, E_P[u_i(\tilde{c}_{i,t})|\mathcal{F}_0](\omega) > \lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t \, E_P[u_i(c_{i,t})|\mathcal{F}_0](\omega).$$

Also, given c, define the *personalized supporting price process for agent* i, denoted p_i^c, by $p_{i,t}^c(\omega) := \beta_i^t \cdot \big(u_i'(c_t(\omega))\big)/\big(u_i'(c_0(\omega))\big)$.

DEFINITION 5: An *TC0 equilibrium* is a tuple $(c_1^*, c_2^*, \theta_1^*, \theta_2^*, q^*)$ that is a market clearing allocation such that (i) $p_i^{c_i^*} \in \mathcal{P}^1(q^*; P)$ for $i \in \mathcal{I}$, and (ii) c_i^*, with supporting portfolio θ_i^*, is a TC0 maximizer given $(q^*, p_i^{c_i^*})$ for $i \in \mathcal{I}$.

As in an IDC equilibrium, in a TC0 equilibrium, an agent maximizes discounted expected utility by choosing a process for consumption, i.e. $\{c_{i,t}\}_{t=0}^{+\infty}$ with the restriction that, for all t, $c_{i,t}$ is $\mathcal{F}_t$-measurable, that the spot market budget constraints are met, and an additional condition is met so as to ensure that the budget sets are appropriately bounded so that a maximizer exists. In a TC0 equilibrium this additional condition takes the form of requiring that the personalized supporting price process for each agent be a summable Arrow price process, and that the limiting expected value of debt evaluated according to the agent's personalized supporting price process be zero. Lemma 17 in the Appendix shows that the TC0 budget set does not permit Ponzi schemes.

5.4 IDENTIFYING EQUILIBRIA

We turn to a result that lets us identify feasible allocations as IDC and TC0 equilibria. We make heavy use of a tool also used by Magill and Quinzii (1994), namely, Arrow-Debreu budget sets induced by personalized Arrow price processess when the no arbitrage asset price process is q. The result will be used only in the case where beliefs are homogeneous and correct; hence, in this section, we assume that $P_1 = P_2 = P$.

Define i's Arrow-Debreu budget set with prices $p_i \in \mathcal{P}^1(q; P)$ as

$$BC_i^{\mathrm{AD}}(p_i) := \Big\{c_i \in \Psi_+^P : \lim_{T\to+\infty} \sum_{t=0}^T E_P\Big[p_{i,t} \cdot c_{i,t}|\mathcal{F}_0\Big](\omega) \\ \leq \lim_{T\to+\infty} \sum_{t=0}^T E_P\Big[p_{i,t} \cdot z_{i,t}|\mathcal{F}_0\Big](\omega)\Big\}.$$

Summability of the personalized prices, $p_i \in \mathcal{P}^1(q; P)$, together with nonnegativity of i's endowment process, A.2, ensures that the value on the right is well defined and finite.

THEOREM 2: Assume A.3 and that beliefs are correct, $P_1 = P_2 = P$. Consider consumption processes $\widehat{c}_i$, $i \in \mathcal{I}$, that are feasible and an asset price process $\widehat{q}$ such that, for each $i \in \mathcal{I}$, there exists $\widehat{p}_i \in \mathcal{P}^1(\widehat{q}; P)$ such that $\widehat{c}_i$ is a maximizer on the set $BC_i^{\mathrm{AD}}(\widehat{p}_i)$, and let $\widehat{\theta}_i$ be a portfolio process that supports $\widehat{c}_i$ at the price process $\widehat{q}$. Then
(i) $(\widehat{c}_1, \widehat{c}_2, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{q})$ constitute a TC0 equilibrium;
(ii) if for $i = 1, 2$ $\lim_{T\to+\infty} E_P\Big[\widehat{p}_{i,T} \cdot \widehat{q}_T \cdot \widehat{\theta}_{i,T}|\mathcal{F}_t\Big](\omega) = 0$ for all $t \geq 1$ and P−a.s. ω, then $(\widehat{c}_1, \widehat{c}_2, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{q})$ constitute an IDC equilibrium.

The theorem is proved by showing that since $\widehat{c}_i$ is a maximizer on the set $BC_i^{\mathrm{AD}}(\widehat{p}_i)$ and it satisfies the sequence constraints in the set $BC_i(\widehat{q})$ with supporting asset portfolio $\widehat{\theta}_i$, the transversality condition $\lim_{T\to+\infty} E_P\Big[\widehat{p}_{i,T} \cdot \widehat{q}_T \cdot \widehat{\theta}_{i,T}|\mathcal{F}_0\Big](\omega) = 0$ holds. So $\widehat{c}_i \in BC_i^{\mathrm{TC}}(\widehat{q}, \widehat{p}_i)$. Also, for $p_i \in \mathcal{P}(q; P)$, $BC_i^{\mathrm{TC}}(q, p_i)$ is contained $BC_i^{\mathrm{AD}}(p_i)$. So $\widehat{c}_i$ is a maximizer on the set $BC_i^{\mathrm{TC}}(\widehat{q}, \widehat{p}_i)$ Theorem 2 (i) follows as a direct consequence. As for Theorem 2 (ii), one shows easily that the transversality condition holds at every $t \geq 0$, and the result follows from Theorem 5.2 in Magill and Quinzii (1994); their result applies since, as they note, preferences with discounted additively separable expected utility representations satisfy the assumption of uniform impatience.[17]

Lemma 20 in the Appendix provides sufficient conditions for verifying that a c_i is a maximizer on $BC_i^{\mathrm{AD}}(\widehat{p}_i)$.

5.5 THE RESULT

We turn to our second main result which shows that the phenomenon exhibited in the leading example, wherein an agent with correct beliefs vanished almost surely, is a robust possibility.

Theorem 3 invokes Theorem 2 (ii) to conclude that there is an IDC equilibrium in which agent 2 vanishes a.s. It shows that quite generally an economy has a continuum of

[17]Conversely, as Magill and Quinzii (1994) note, if we consider an IDC equilibrium and summable supporting Arrow price processes then, necessarily, the transversality condition holds at every node.

endowment distributions at each of which there is a no trade equilibrium in which agent 2 vanishes a.s. It also provides conditions, that include the special case where agent 2 has a logarithmic Bernoulli function and an endowment at only date 0, such that there is an IDC equilibrium in which agent 2 vanishes a.s.

The only element that is new here is a proof of the fact that under the conditions specified in Theorem 3 (i), a transversality condition can be shown to hold at every date and event; that result is proved in Lemma 21.

THEOREM 3: Assume A.1-3, A.5-7, $\beta_1 \geq \beta_2$, and $P_1 = P_2 = P$. Also assume that either one of the following two conditions holds:
(i) For some $c > 0$, $\forall\, t \geq 1$, and $P-$a.s. $\tilde{\omega}$
$u_2'(C_{2,t}(\tilde{\omega}; c, 0, \omega)) \cdot \Big(z_{2,t}(\tilde{\omega}) - C_{2,t}(\tilde{\omega}; c, 0, \omega)\Big) = \bar{c}_{2,t}$ and
$u_2'(C_{2,0}(\tilde{\omega}; c, 0, \omega)) \cdot \Big(z_{2,0}(\tilde{\omega}) - C_{2,0}(\tilde{\omega}; c, 0, \omega))\Big) = -\text{Lim}_{T\to+\infty} \sum_{\tau=1}^{T} \beta_2^{\tau} \cdot \bar{c}_{2,\tau}$,
(ii) $z_1^c = \{C_{1,t}(c, 0, \omega)\}_{t\geq 0}$ for some $c \in (0, Z_0(\omega))$ so that the proposed solution is supported as a no trade equilibrium.
Then the economy has an IDC equilibrium in which agent 2 vanishes almost surely.

Case (ii) guarantees that our construction is not vacuous. The condition in case (i) holds if $u_2(x) = \log x$ and $z_{2,t}(\omega) = 0$ for $t \geq 1$. So the example in Section 3 generalizes to arbitrary nonnegative asset payoffs and arbitrary characteristics for agent 1.

Theorem 4 shows that for every endowment distribution in some neighbourhood of an endowment distribution that is supported as a no trade IDC equilibrium, there exists a TC0 equilibrium. The proof uses A.4, which imposes a bound on the coefficient of relative risk aversion, to show that for the allocation identified in Theorem 3, the value of excess demand evaluated using the personalized Arrow-Debreu price process of each agent is monotone in a single parameter; furthermore, the value is continuous and has the right boundary behaviour.[18] The rest of the proof consists in manipulating the allocation by starting at date 1 and using the fact that markets are incomplete to conclude that one can choose consumption at date 0 in a manner that is consistent with feasibility and the Euler equations and thereby reduce the problem to that of a fixed point problem in two dimensions which has a solution for endowments in a neighbourhood of the no trade endowments by continuity since no trade is a solution by Theorem 3.

Define the space of endowment distributions compatible with the aggregate endowment process Z as $Z_1(Z) := \{(z_{1,0}, z_{1,1}, \cdots) \in \Psi_+ : (Z_0 - z_{1,0}, Z_1 - z_{1,1}, \cdots) \in \Psi_+\}$.

THEOREM 4: Assume A.1-7, $\beta_1 \geq \beta_2$, and $P_1 = P_2 = P$. Let $(z_1^*, z_2^*) = (\{C_{1,t}(c^*, 0, \omega)\}_{t\geq 0}, \{C_{2,t}(c^*, 0, \omega)\}_{t\geq 0})$ for some $c^* \in (0, Z_0(\omega))$. There exists $\mathcal{N}(z_1^*)$ an open subset of $Z_1(Z)$ such that for every (z_1, z_2), where $z_1 \in \mathcal{N}(z_1^*)$ and $z_2 := Z - z_1$, there exists a TC0 equilibrium in which agent 2 vanishes with probability one.

[18]Under A.4 the proof of Theorem 4 goes through even when an agent has a zero endowment at every date and event; this shows quite clearly that in general A.4 can be weakened as we noted in Remark 1.

REMARK 5: It follows from Remark 4 that a continuity argument can be used to provide analogues of Theorems 3 and 4 in the case where $\beta_1 < \beta_2$ but sufficiently close; this generalizes a property that the example in Section 3 displayed.

6. CONCLUDING REMARKS

We considered an infinite horizon economy with incomplete markets with two agents and one good and provided two general results on the asymptotic properties of consumption. Our first result is a precise statement indicating that if markets are effectively incomplete then on any path some agent's consumption is arbitrarily close to zero infinitely often. From this result we draw the conclusion that if market incompleteness is effective and forever then either one of the two agents will eventually cease to consume, or the equilibrium is complicated in the sense that the consumption of some agent will be arbitrarily close to zero infinitely often. Our second result shows that, for a robust class of economies with incomplete markets, there are equilibria in which an agent's consumption is zero eventually with probability one even though she has correct beliefs and is marginally more patient. It follows that the strong results regarding the validity of the MSH and the Ramsey conjecture that have appeared in the literature depend critically on having complete markets or a Pareto optimal allocation. In addition, our result helps to disentangle the role played by the heterogeneity of beliefs from that played by the market structure in determining the survival prospects of an agent. It suggests that over saving is a phenomena associated with incompleteness rather than with differences of opinions.

When utility is unbounded below, Theorem 1 (ii) implies that the continuation utility is arbitrarily low infinitely often. This can be interpreted as showing that the implicit punishment required to ensure that an agent continues to participate in the market is the confiscation of her entire endowment, i.e. the maximal possible punishment.

Although we develop our results in a one good, one asset, and two agent model, we believe the main lesson from Theorem 1 holds in a much wider class of model economies. In particular, since Theorem 1 is based on pairwise comparisons of the agents' marginal rates of substitution, we conjecture that it holds with any finite number of agents, goods and numeraire assets provided some asset has strictly positive return in every state of nature. More precisely, if markets are effectively incomplete forever, the consumption of at most one agent can be bounded away from zero. On the other hand, the proofs of Theorems 3 and 4 rely heavily on the assumption that there are only two agents. Therefore, whether in more general set-ups there exist endowment distributions such that one agent vanishes remains an open question.

Models where the first order conditions hold with inequalities, i.e. situations where nonnegativity or bounding constraints bind, are not covered by the model in this paper and, therefore, our results do not apply to them. Given the prevalance of such models in the modern literature on computational general equilibrium and macroeconomics, it would be very interesting to study the asymptotic properties of consumption in such models; perhaps our techniques can be adapted to such situations.

APPENDIX

PROOF OF PROPOSITION 1

That $E_{P_i}\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right](\omega) = 1$ follows from the definition of the process $\hat{r}_i$.

(i) Since, by hypothesis, c_i satisfies the Euler equations for i at q, we have

$$q_{t-1}(\omega) = \beta_i \cdot \frac{E_{P_i}[r_t \cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}](\omega)}{u_i'(c_{i,t-1}(\omega))} \quad \Leftrightarrow \quad \hat{r}_{i,t}(\omega) = \frac{\beta_i \cdot r_t(\omega) \cdot u_i'(c_{i,t}(\omega))}{q_{t-1}(\omega) \cdot u_i'(c_{i,t-1}(\omega))}$$

$$\Rightarrow \qquad R_{i,1+T}(\omega) = \Pi_{t=0}^{T} \hat{r}_{i,1+t}(\omega) = \beta_i^{T+1} \cdot \frac{u_i'(c_{i,1+T}(\omega))}{u_i'(c_{i,0}(\omega))} \cdot \Pi_{t=0}^{T}\left(\frac{r_{1+t}(\omega)}{q_t(\omega)}\right).$$

(ii) Under A.2 and A.3 $u_i'(c_{i,t}(\omega))$ is uniformly positive. So, invoking A.5, we have $\hat{r}_{i,t}(\omega) > 0$. Since

$$\frac{\hat{r}_{1,t}(\omega)}{\hat{r}_{2,t}(\omega)} = \frac{\frac{\beta_1 \cdot r_t(\omega) \cdot u_1'(c_{1,t}(\omega))}{q_{t-1}(\omega) \cdot u_1'(c_{1,t-1}(\omega))}}{\frac{\beta_2 \cdot r_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{q_{t-1}(\omega) \cdot u_2'(c_{2,t-1}(\omega))}} = \frac{\beta_1}{\beta_2} \cdot \frac{\frac{u_1'(c_{1,t}(\omega))}{u_2'(c_{2,t}(\omega))}}{\frac{u_1'(c_{1,t-1}(\omega))}{u_2'(c_{2,t-1}(\omega))}} = \frac{\beta_1}{\beta_2} \cdot \frac{y_{t-1}(\omega)}{y_t(\omega)},$$

so that the ratio y_{t-1}/y_t, adjusted by the discount factors, equals the ratio between the intertemporal marginal rate of substitution for agent 1 and agent 2, and

$$\Rightarrow \qquad y_T(\omega) = \frac{\left(\frac{\beta_1}{\beta_2}\right)^T}{\Pi_{t=1}^{T}\left(\frac{\hat{r}_{1,t}(\omega)}{\hat{r}_{2,t}(\omega)}\right)} \cdot y_0(\omega) = \left(\frac{\beta_1}{\beta_2}\right)^T \cdot \frac{R_{2,T}(\omega)}{R_{1,T}(\omega)} \cdot y_0(\omega).$$

(iii) Finally, by rewriting the first property in (ii) we have

$$\hat{r}_{2,t}(\omega) \cdot y_{t-1}(\omega) = \frac{\beta_2}{\beta_1} \cdot \hat{r}_{1,t}(\omega) \cdot y_t(\omega) \quad \Leftrightarrow \quad E_{P_2}\left[\hat{r}_{2,t} \cdot y_{t-1}|\mathcal{F}_{t-1}\right](\omega) = \frac{\beta_2}{\beta_1} \cdot E_{P_2}\left[\hat{r}_{1,t} \cdot y_t|\mathcal{F}_{t-1}\right](\omega)$$

and the first result in (iii) follows by using the fact that $E_{P_i}\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right](\omega) = 1$. The second result in (iii) is proved in a similar manner. ∎

PROOF OF THEOREM 1

(i) By definition, on the set V_0

$$\lim_t \left[\log\left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_t(\omega)}{y_{t-1}(\omega)}\right) - E\left[\log\left(\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}\right)\middle|\mathcal{F}_{t-1}\right](\omega)\right] = 0.$$

Equivalently, using Proposition 1 (ii),

$$\lim_t \left[\log\left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)}\right) - E\left[\log\left(\frac{P_{2,t}}{P_{1,t}} \cdot \frac{\hat{r}_{2,t}}{\hat{r}_{1,t}}\right)\middle|\mathcal{F}_{t-1}\right](\omega)\right] = 0.$$

So there exists a process $\{\lambda_t\}_{t\geq 0}$ such that λ_t is $\mathcal{F}_t-$measurable and for every $\epsilon > 0$ there exists $t(\epsilon,\omega)$ such that $t > t(\epsilon,\omega)$ implies $\left|\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} - \lambda_{t-1}(\omega)\right| < \epsilon$. It follows that $t > t(\epsilon,\omega) \Rightarrow (\lambda_{t-1}(\omega)-\epsilon) \cdot P_{1,t}(\omega) \cdot \hat{r}_{1,t}(\omega) < P_{2,t}(\omega) \cdot \hat{r}_{2,t}(\omega) < (\lambda_{t-1}(\omega)+\epsilon) \cdot P_{1,t}(\omega) \cdot \hat{r}_{1,t}(\omega)$.

Since λ_{t-1} is $\mathcal{F}_{t-1}-$measurable, we have $t > t(\epsilon,\omega)$ implies $(\lambda_{t-1}(\omega)-\epsilon)\cdot E_{P_1}[\hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega) < E_{P_2}[\hat{r}_{2,t}|\mathcal{F}_{t-1}](\omega) < (\lambda_{t-1}(\omega)+\epsilon)\cdot E_{P_1}[\hat{r}_{1,t}|\mathcal{F}_{t-1}](\omega)$. Since $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$ and $\epsilon > 0$ is arbitrary, we have $\lambda_{t-1} = 1$ $P-$a.s. $\omega \in V_0$. It follows from an application of Proposition 1 (ii) that $\lim_t \left(\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)}\cdot\frac{y_t(\omega)}{y_{t-1}(\omega)}\right) = 1$.

(ii) We start with three results that we will need. The first is Levy's conditional form of the Second Borel-Cantelli Lemma which follows from a more general result due to Freedman (1973 Proposition 39). The second result puts bounds on the conditional probability with which there is variability in y_t/y_{t-1}. The third, shows that on any path on which some event occurs infinitely often, the event consisting of the first event followed by any finite string of realizations of y_t such that $y_t/y_{t-1} \geq 1$ also occurs infinitely often.

For $E \in \mathcal{F}$ an event, let 1_E denote the indicator function. Recall that $\Big\{\omega : \Omega_t \text{ i.o.}\Big\} = \Big\{\omega : \sum_{t=1}^{\infty} 1_{\Omega_t}(\omega) = +\infty\Big\}$.

LEMMA 2: Let $\{\Omega_t\}_{t=0}^{\infty}$ be a sequence of events adapted to the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Then

$$\sum_{t=1}^{\infty} 1_{\Omega_t}(\omega) = +\infty \quad P-\text{a.s. } \omega \in \Big\{\omega : \sum_{t=1}^{\infty} E\left[1_{\Omega_t}|\mathcal{F}_{t-1}\right](\omega) = +\infty\Big\}.$$

LEMMA 3: Assume A.1. Then $\forall\, t \geq 1 \quad P\Big[\frac{y_t}{y_{t-1}} \geq 1 \,\Big|\, \mathcal{F}_{t-1}\Big](\omega) \geq \underline{p} > 0 \quad P-\text{a.s. } \omega \in \Omega$. Furthermore, $\text{var}\Big[\log\left(\frac{y_t}{y_{t-1}}\right)\Big|\mathcal{F}_{t-1}\Big](\omega) \geq \epsilon > 0$ implies that there exists $\gamma > 0$ such that

$$P\Big[1-\gamma \geq \frac{y_t}{y_{t-1}}\,\Big|\,\mathcal{F}_{t-1}\Big](\omega) \geq \underline{p} > 0 \quad \text{and} \quad P\Big[\frac{y_t}{y_{t-1}} \geq 1+\gamma\,\Big|\,\mathcal{F}_{t-1}\Big](\omega) \geq \underline{p} > 0.$$

PROOF: By Proposition 1 (ii), $\Big(y_t(\omega)/y_{t-1}(\omega)\Big) = \Big(\hat{r}_{2,t}(\omega)/\hat{r}_{1,t}(\omega)\Big)$.

Since for all $t \geq 1$ and $P-$a.s. $\omega \in \Omega$, $E_{P_i}[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$, $i = 1,2$, under A.1 the first result follows.

Also, the second result follows because if for some pair (t,ω)
$\forall\,\gamma > 0 \qquad P\Big[1-\gamma < \frac{\hat{r}_{2,t}}{\hat{r}_{1,t}} < 1+\gamma\,\Big|\,\mathcal{F}_{t-1}\Big](\omega) = 1 \quad \Rightarrow \quad \text{var}\Big[\frac{\hat{r}_{2,t}}{\hat{r}_{1,t}}\,\Big|\,\mathcal{F}_{t-1}\Big](\omega) = 0.$ ■

Define the set $\quad \Omega_{1,t}^N = \Big\{\omega : \frac{y_{t'}(\omega)}{y_{t'-1}(\omega)} \geq 1,\ \forall\, t' = t+1-N,\cdots,t\Big\}$.

LEMMA 4: Let $\{\Omega_t\}_{t=0}^{\infty}$ be a sequence of events adapted to the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Then

$$\forall N \geq 1 \quad \sum_{t=1}^{\infty} 1_{\Omega_{t-N}\cap\Omega_{1,t}^N}(\omega) = +\infty \quad P-\text{a.s. } \omega \in \{\omega : \Omega_t \text{ i.o.}\}.$$

PROOF: As an implication of Lemma 3 we have

$$\omega \in \Omega_{t-N}\cap\Omega_{1,t-1}^{N-1} \quad \Rightarrow \quad E\Big[1_{\Omega_{t-N}\cap\Omega_{1,t}^N}\Big|\mathcal{F}_{t-1}\Big](\omega) = P\Big[\frac{y_t}{y_{t-1}} \geq 1\Big|\mathcal{F}_{t-1}\Big](\omega) \geq \underline{p} > 0,$$

where we use the convention that $\Omega_{1,t}^0 = \Omega$ to handle the case where $N = 1$, and $E\Big[1_{\Omega_{t-N}\cap\Omega_{1,t}^N}\Big|\mathcal{F}_{t-1}\Big](\omega)$ is non-negative otherwise.

For $\widetilde{\omega} \in \{\omega : \Omega_t \text{ i.o.}\}$ arbitrarily chosen, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $\widetilde{\omega} \in \Omega_{t_k}$ for every $k = 1, 2, \cdots$. Since $\Omega_{1,t}^0 = \Omega$, $\widetilde{\omega} \in \Omega_{(t_k+1)-1} \cap \Omega_{1,(t_k+1)-1}^{1-1}$ and therefore, by the implication of Lemma 3,

$$\sum_{t=1}^{\infty} E\Big[1_{\Omega_{t-1} \cap \Omega_{1,t}^1} \Big| \mathcal{F}_{t-1}\Big](\widetilde{\omega}) \geq \sum_{k=1}^{\infty} P\Big[\frac{y_{t_k+1}}{y_{t_k}} \geq 1 \Big| \mathcal{F}_{t_k}\Big](\widetilde{\omega}) = +\infty$$

and it follows by Lemma 2 that $\sum_{t=1}^{\infty} 1_{\Omega_{t-1} \cap \Omega_{1,t}^1}(\omega) = +\infty \quad P-\text{a.s. } \omega \in \{\omega : \Omega_t \text{ i.o.}\}$.

Consider $\widetilde{\omega} \in \{\omega : \Omega_t \text{ i.o.}\}$ arbitrarily chosen and suppose that the result holds for $N-1$. So there exists $\{t_k\}_{k=1}^{\infty}$ such that $\widetilde{\omega} \in \Omega_{t_k-(N-1)} \cap \Omega_{1,t_k}^{N-1} = \Omega_{(t_k+1)-N} \cap \Omega_{1,(t_k+1)-1}^{N-1}$ so that, by the implication of Lemma 3,

$$\sum_{t=1}^{\infty} E\Big[1_{\Omega_{t-N} \cap \Omega_{1,t}^N} \Big| \mathcal{F}_{t-1}\Big](\widetilde{\omega}) \geq \sum_{k=1}^{\infty} P\Big[\frac{y_{t_k+1}}{y_{t_k}} \geq 1 \Big| \mathcal{F}_{t_k}\Big](\widetilde{\omega}) = +\infty$$

and it follows by Lemma 2 that $\sum_{t=1}^{\infty} 1_{\Omega_{t-N} \cap \Omega_{1,t}^N}(\omega) = +\infty \quad P-\text{a.s. } \omega \in \{\omega : \Omega_t \text{ i.o.}\}$. That completes the induction argument and the proof. ∎

Set $\underline{y}_n := \big(u_2'(\overline{z}-1/n)/u_1'(1/n)\big)$ and $\overline{y}_n := \big(u_2'(1/n)/u_1'(\underline{z}-1/n)\big)$. For $\gamma > 0$ identified in Lemma 3, let $T_n(\gamma)$ satisfy $\underline{y}_n \cdot (1+\gamma)^{T_n(\gamma)} > \overline{y}_n$. For the rest of the proof, the values of ϵ, T, n, and the value of γ induced by ϵ, will be considered to be fixed.

Define $\{T_t^s(\omega)\}_{s=1}^{T_n(\gamma)}$ by the rule $T_t^1(\omega) := t + \Delta_t^{\epsilon}(\omega)$, $T_t^{s+1}(\omega) := T_t^s(\omega) + \Delta_{T_t^s(\omega)}^{\epsilon}(\omega)$. This procedure produces a function T_t^s that is strictly increasing in s for every ω. Therefore for any (ω, t, τ) such that $\frac{y_t(\omega)}{y_{t-1}(\omega)} \geq 1+\gamma$ and there exists $t' \geq 0$ such that $T_{t'}^{\tau}(\omega) \leq t$, one can define $b_t^{\tau}(\omega) := \sup_{t' \geq 0}\{t' : T_{t'}^{\tau}(\omega) \leq t\}$; set $b_t^{\tau}(\omega) = -1$ otherwise.

For $1 \leq \tau \leq t$ and $1 \leq N \leq t$, define the set

$$\Omega_{2,t}^{\tau} = \left\{ \omega : \begin{array}{l} b_t^{\tau}(\omega) \neq -1, \ \frac{y_t(\omega)}{y_{t-1}(\omega)} \geq 1+\gamma, \ \frac{y_{b_t^{\tau}(\omega)}}{y_{b_t^{\tau}(\omega)-1}} \geq 1+\gamma, \text{ and} \\ \frac{y_{t'}(\omega)}{y_{t'-1}(\omega)} \geq 1 \ \forall\, t' = b_t^{\tau}(\omega)+1, \cdots, t-1 \end{array} \right\}.$$

Notice that if $\omega \in \Omega_{2,t}^{\tau}$ then, necessarily, $\#\left\{\frac{y_{t'}(\omega)}{y_{t'-1}(\omega)} \geq 1+\gamma,\ t' = b_t^{\tau}(\omega), \cdots, t\right\} = \tau$; this follows from the definition of the function b_t^{τ}. It follows that

$$\left\{\Omega_{2,t}^{T_n(\gamma)} \text{ i.o.}\right\} \cap \{\omega : \liminf_t c_{j,t}(\omega) \geq 1/n\} \subset \left\{\left\{\omega : y_t(\omega) \geq \underline{y}_n \cdot (1+\gamma)^{T_n(\gamma)} > \overline{y}_n\right\} \text{ i.o.}\right\}.$$

We will show that $\left\{\Omega_{2,t}^{T_n(\gamma)} \text{ i.o.}\right\}$ P-a.s. $\omega \in V_{T,\epsilon}^y$. It follows that the event $\left\{\{\omega : y_t(\omega) > \overline{y}_n\} \text{ i.o.}\right\}$ occurs P-a.s. $\omega \in V_{T,\epsilon}^y \cap \{\omega : \liminf_t c_{j,t}(\omega) \geq 1/n\}$ letting us conclude that, P a.s., on the set $V_{T,\varepsilon}^y$, if agent j consumes $1/n$ infinitely often, then agent i cannot consume $1/n$ inifinitely often.

Consider $1_{\Omega_{2,t}^1}$. Since

$$E\Big[1_{\Omega_{2,t}^1} \Big| \mathcal{F}_{t-1}\Big](\omega) = P\Big[\frac{y_t}{y_{t-1}} \geq 1+\gamma \Big| \mathcal{F}_{t-1}\Big](\omega),$$

by Lemma 3 we know that

$$\omega \in \Big\{\omega : \mathrm{var}\Big[\log\Big(\frac{y_t}{y_{t-1}}\Big)\Big|\mathcal{F}_{t-1}\Big] \geq \epsilon\Big\} \quad \Rightarrow \quad E\Big[1_{\Omega^1_{2,t}}\Big|\mathcal{F}_{t-1}\Big](\omega) \geq \underline{p} > 0;$$

also, $E\Big[1_{\Omega^1_{2,t}}\Big|\mathcal{F}_{t-1}\Big](\omega)$ is non-negative otherwise. It follows that on $V^y_{T,\epsilon}$, $\sum_{t=1}^{\infty} E\Big[1_{\Omega^1_{2,t}}\Big|\mathcal{F}_{t-1}\Big](\omega) = +\infty$ and, therefore, by Lemma 2, $\sum_{t=1}^{\infty} 1_{\Omega^1_{2,t}}(\omega) = +\infty$ P-a.s. $\omega \in V^y_{T,\epsilon}$.

Similarly, again by Lemma 3, if $\omega \in \Omega^{\tau}_{2,t'} \cap \Omega^{t-1-t'}_{1,t-1} \cap \Big\{\omega : \mathrm{var}\Big[\log\Big(\frac{y_t}{y_{t-1}}\Big)\Big|\mathcal{F}_{t-1}\Big] \geq \epsilon\Big\}$, where we use the convention that $\Omega^0_{1,t} = \Omega$ to handle the case in which $t = t' + 1$, then

$$E\Big[1_{\Omega^{\tau+1}_{2,t}}\Big|\mathcal{F}_{t-1}\Big](\omega) = P\Big[\frac{y_t}{y_{t-1}} \geq 1 + \gamma\Big|\mathcal{F}_{t-1}\Big](\omega) \geq \underline{p} > 0.$$

Now suppose it is true that $\sum_{t=1}^{\infty} 1_{\Omega^{\tau}_{2,t}}(\omega) = +\infty$ P-a.s. $\omega \in V^y_{T,\epsilon}$ for some τ. By Lemma 4 $\sum_{t=1}^{\infty} 1_{\Omega^{\tau}_{t-T} \cap \Omega^{T}_{1,t}}(\omega) = +\infty$ P-a.s. $\omega \in V^y_{T,\epsilon}$ and $\widetilde{\omega} \in V^y_{T,\epsilon}$ implies that there exists a sequence $\{t'_k\}_{k=1}^{\infty}$ such that $\widetilde{\omega} \in \Omega^{\tau}_{2,t'_k} \cap \Omega^{(t'_k+T)-t'_k}_{1,t'_k+T}$. Furthermore, for every k there necessarily exists $t_k \in \{t'_k+1, t'_k+2, \cdots, t'_k+T\}$ such that $\widetilde{\omega} \in \Big\{\omega : \mathrm{var}\Big[\log\Big(\frac{y_{t_k}}{y_{t_k-1}}\Big)\Big|\mathcal{F}_{t_k-1}\Big] \geq \epsilon\Big\}$; this follows from the fact that T is a uniform upper bound on the number of periods with no variability. It follows that $\widetilde{\omega} \in \Omega^{\tau}_{2,t'_k} \cap \Omega^{t_k-1-t'_k}_{1,t_k-1} \cap \Big\{\omega : \mathrm{var}\Big[\log\Big(\frac{y_{t_k}}{y_{t_k-1}}\Big)\Big|\mathcal{F}_{t_k-1}\Big] \geq \epsilon\Big\}$, where we use the fact that $\widetilde{\omega} \in \Omega^{(t'_k+T)-t'_k}_{1,t'_k+T}$ implies that $\widetilde{\omega} \in \Omega^{t_k-1-t'_k}_{1,t_k-1}$ also for $t'_k + 1 \leq t_k \leq t'_k + T$. Hence

$$\sum_{t=1}^{\infty} E\Big[1_{\Omega^{\tau+1}_{2,t}}\Big|\mathcal{F}_{t-1}\Big](\widetilde{\omega}) \geq \sum_{k=1}^{\infty} E\Big[1_{\Omega^{\tau+1}_{2,t_k}}\Big|\mathcal{F}_{t_k-1}\Big](\widetilde{\omega}) = \sum_{k=1}^{\infty} P\Big[\frac{y_{t_k}}{y_{t_k-1}} \geq 1 + \gamma\Big|\mathcal{F}_{t_k-1}\Big](\widetilde{\omega}) = +\infty$$

and it follows from Lemma 2 that $\sum_{t=1}^{\infty} 1_{\Omega^{\tau+1}_{2,t}}(\omega) = +\infty$ P-a.s. $\omega \in V^y_{T,\epsilon}$. This completes the induction on τ. Hence, for every $\tau \geq 0$, $\sum_{t=1}^{\infty} 1_{\Omega^{\tau+1}_{2,t}}(\omega) = +\infty$ P-a.s. $\omega \in V^y_{T,\epsilon}$; in particular, $\sum_{t=1}^{\infty} 1_{\Omega^{T_n(\gamma)}_{2,t}}(\omega) = +\infty$ P-a.s. $\omega \in V^y_{T,\epsilon}$. We have shown that $\Big\{\Omega^{T_n(\gamma)}_{2,t} \text{ i.o.}\Big\}$ P-a.s. $\omega \in V^y_{T,\epsilon}$ as required. ■

PROOF OF PROPOSITION 2

The proof follows from Lemma 5 and 6. Lemma 5 shows that if asset returns are non-negative and the one period ahead conditional probability that state s occurs is uniformly positive, A.1, then $\hat{r}_{i,t}(\omega)$ is nonnegative and uniformly bounded above. Lemma 6 uses the martingale convergence theorem to show that $\lim_{T\to\infty} R_{i;0,T}(\omega)$ is P_i-a.s. finite.

LEMMA 5: Assume A.3 and $\underline{r} \geq 0$. Then $0 \leq \hat{r}_{i,t}(\omega) \leq 1/P_{i,t}(\omega)$. Hence, under A.1, A.3, and A.5, $\bar{\hat{r}}_i < \infty$.
PROOF: Since u_i is strictly increasing and $\underline{r} \geq 0$,
$P_{i,t}(\omega) \leq P_{i,t}(\omega) + \frac{E_{P_i}[r_t \cdot u'_i(c_{i,t})|\mathcal{F}_{t-1}, \Omega/\Omega(s^t(\omega))](\omega)}{r_t(\omega)\cdot u'_i(c_{i,t}(\omega))} = \frac{E_{P_i}[r_t \cdot u'_i(c_{i,t})|\mathcal{F}_{t-1}](\omega)}{r_t(\omega)\cdot u'_i(c_{i,t}(\omega))} = \frac{1}{\hat{r}_{i,t}(\omega)}$. ■

LEMMA 6: Assume A.3 and $\underline{r} \geq 0$. Then there is a random variable R^*_i that is nonnegative and a.s. finite with $E_{P_i}[R^*_i] \leq 1$ such that $R^*_i(\omega) = \lim_{T\to\infty} R_{i,T}(\omega)$ P_i-a.s.

PROOF: Under the stated condition, $\{R_{i,t}\}$ is a nonnegative martingale since $E_{P_i}\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right] = 1$. Since $\sup_{t\geq 1} E_{P_i}\left[R_{i,t}\right] = 1 < +\infty$, the Martingale Convergence Theorem applies. ∎

PROOF OF PROPOSITION 3

Let us define a sequence of truncated processes parametrized by $\epsilon > 0$ by setting $g^{\epsilon}_{1,t}(\omega) := \log\left(\max\left\{\hat{r}_{1,t}(\omega), \epsilon\right\}\right)$ and $\mathcal{B}_{1,\epsilon} := \{\omega : \text{limsup}\,\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[g^{\epsilon}_{1,t}|\mathcal{F}_{t-1}](\omega) < 0\}$.

Ω can be partitioned into three sets: $\cup_{n\geq 1}\mathcal{B}_{1,1/n}$, $\mathcal{A}_1$, and $\Omega/(\mathcal{A}_1 \cup (\cup_{n\geq 1}\mathcal{B}_{1,1/n}))$, where $\mathcal{A}_1 := \{\omega \in \Omega : \text{liminf}\,\hat{r}_{1,t}(\omega) = 0\}$. We first show that under A.7 the third set is null.

LEMMA 7: Assume A.7. Then $\Omega/\mathcal{A}_1 \subset \cup_{n\geq 1}\mathcal{B}_{1,1/n}$, where $\mathcal{A}_1 := \{\omega : \text{liminf}\,\hat{r}_{1,t}(\omega) = 0\}$, so that for all $\omega \in \Omega/\mathcal{A}_1$ there exists $\epsilon(\omega)$ such that $\omega \in \mathcal{B}_{1,\epsilon(\omega)}$.
PROOF: Consider $\tilde{\omega} \in \Omega/\mathcal{A}_1$. So $\text{liminf}\,\hat{r}_{1,t}(\tilde{\omega}) = 2\cdot\epsilon(\tilde{\omega}) > 0$ and there exists $t(\tilde{\omega})$ such that $t \geq t(\tilde{\omega}) \quad \Rightarrow \quad \hat{r}_{1,t}(\tilde{\omega}) \geq \epsilon(\tilde{\omega})$. Furthermore, by A.7,

$$\text{limsup}\left(\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right) = s(\tilde{\omega}) < 0.$$

Since

$$0 = \text{limsup}\left(\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) - \frac{1}{T}\sum_{t=t(\tilde{\omega})+1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right)$$

$$\leq \text{limsup}\left(\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right) - \text{limsup}\left(\frac{1}{T}\sum_{t=t(\tilde{\omega})+1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right)$$

we must have

$$\text{limsup}\left(\frac{1}{T}\sum_{t=t(\tilde{\omega})+1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right) \leq \text{limsup}\left(\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega})\right) = s(\tilde{\omega}) < 0$$

$$\Rightarrow \quad \text{limsup}\,\frac{1}{T}\sum_{t=t(\tilde{\omega})+1}^{T} E_{P_1}[\log\hat{r}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) < 0$$

$$\Rightarrow \quad \text{limsup}\,\frac{1}{T}\sum_{t=t(\tilde{\omega})+1}^{T} E_{P_1}[\log\left(\max\left\{\hat{r}_{1,t}, \epsilon(\tilde{\omega})\right\}\right)|\mathcal{F}_{t-1}](\tilde{\omega}) < 0.$$

Since $\text{limsup}\,\frac{1}{T}\sum_{t=1}^{t(\tilde{\omega})} E_{P_1}[\log\left(\max\left\{\hat{r}_{1,t}, \epsilon(\tilde{\omega})\right\}\right)|\mathcal{F}_{t-1}](\tilde{\omega}) = 0$,

$$\text{limsup}\,\frac{1}{T}\sum_{t=1}^{T} E_{P_1}[g^{\epsilon(\tilde{\omega})}_{1,t}|\mathcal{F}_{t-1}](\tilde{\omega}) < 0$$

so that $\tilde{\omega} \in \mathcal{B}_{1,\epsilon(\tilde{\omega})}$ as required. ∎

We continue with the proof of Proposition 3.

Since $\epsilon < \epsilon' \quad \Rightarrow \quad g^{\epsilon}_{1,t}(\omega) \leq g^{\epsilon'}_{1,t}(\omega) \quad \forall\, t, \quad \forall\, \omega$, it follows that $\epsilon < \epsilon' \quad \Rightarrow \quad \mathcal{B}_{1,\epsilon'} \subset \mathcal{B}_{1,\epsilon}$. So $\mathcal{B}_{1,1/n} \subset \mathcal{B}_{1,1/(n+1)} \subset \cdots$, and we set $\mathcal{B}_{1,0} := \cup_{n\geq 1}\mathcal{B}_{1,1/n}$. It follows that

$P_1(\mathcal{B}_{1,1/n}/\mathcal{A}_1)$ increases monotonically to $P_1(\mathcal{B}_{1,0}/\mathcal{A}_1)$. So for all $p > 0$, there exists $\epsilon(p)$ such that $P_1(\mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1) \geq P_1(\mathcal{B}_{1,0}/\mathcal{A}_1) - p$.

For fixed p and corresponding $\epsilon(p)$, consider the truncated process $\{g_{1,t}^{\epsilon(p)}\}_{t=0}^{+\infty}$ defined earlier. It is uniformly bounded below and, under A.1, A.3, and A.5, by Lemma 5, it is also uniformly bounded above. Hence the process $\{\, E_{P_1}[g_{1,t}^{\epsilon(p)}|\mathcal{F}_{t-1}]\}_{t=0}^{+\infty}$ is uniformly bounded below and above.

Define

$$\bar{g}_{1,t}^{\epsilon(p)}(\omega) := g_{1,t}^{\epsilon(p)}(\omega) - E_{P_1}[g_{1,t}^{\epsilon(p)}|\mathcal{F}_{t-1}](\omega).$$

It follows that the process $\{\bar{g}_{1,t}^{\epsilon(p)}\}_{t=0}^{+\infty}$ is uniformly bounded above and below. Furthermore, $E_{P_1}[\,\bar{g}_{1,t}^{\epsilon(p)}\bar{g}_{1,t+k}^{\epsilon(p)}|\mathcal{F}_{t-1}] = 0$ for all $k \geq 1$, for all $t \geq 0$. Therefore, by the Strong Law of Large Numbers for uncorrelated random variables with uniformly bounded second moments (Chung 1974, page 103),

$$\lim_{T\to+\infty} \frac{1}{T}\sum_{t=1}^{T} \bar{g}_{1,t}^{\epsilon(p)}(\omega) = 0 \quad P_1 - \text{a.s.}$$

$$\Rightarrow \qquad \text{limsup}\, \frac{1}{T}\sum_{t=1}^{T} g_{1,t}^{\epsilon(p)}(\omega) \leq \text{limsup}\, \frac{1}{T}\sum_{t=1}^{T} E_{P_1}[g_{1,t}^{\epsilon(p)}|\mathcal{F}_{t-1}](\omega).$$

Since $\omega \in \mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1$ implies $\text{limsup}\, \frac{1}{T}\sum_{t=1}^{T} E_{P_1}[g_{1,t}^{\epsilon(p)}|\mathcal{F}_{t-1}](\omega) < 0$, it follows that $\forall\, \omega \in \mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1$, $\sum_{t=1}^{T} g_{1,t}^{\epsilon(p)}(\omega) \to -\infty$ so that $\forall\, \omega \in \mathcal{B}_{1,\epsilon(p)}/\mathcal{A}_1$, $\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) =\to -\infty$ since $\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) = \sum_{t=1}^{T} g_{1,t}^{0}(\omega) \leq \sum_{t=1}^{T} g_{1,t}^{\epsilon(p)}(\omega) \to -\infty$. The proof of the first part is completed by noting that as p goes to zero, we approximate the set $\mathcal{B}_{i,0}/\mathcal{A}_i$ and, by Lemma 7, that set coincides with $\Omega/\mathcal{A}_1$.

For the second part we set $\mathcal{C}_{1,\delta} := \{\omega \in \Omega : \text{limsup}\, \frac{1}{T}\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) < \log\delta\} \cap (\Omega/\mathcal{A}_1)$. Clearly, $\delta' < \delta''$ implies that $\mathcal{C}_{1,\delta'} \subset \mathcal{C}_{1,\delta''}$. It follows that $\cup_{n\geq 1}\mathcal{C}_{1,1-1/n} = \Omega/\mathcal{A}_1$ and hence that $P_1(\mathcal{C}_{1,1-1/n})$ increases monotonically to $P_1(\Omega/\mathcal{A}_1)$ so that for all $\epsilon > 0$, there exists $\delta = 1 - 1/n$ such that $P_1(\mathcal{C}_{1,\delta}) \geq P_1(\Omega/\mathcal{A}_1) - \epsilon$. ■

PROOF OF PROPOSITION 4

We give an outline of the proof. In Lemma 8 we show that one can work with the process c_1 and the process y interchangeably. Lemma 9 is the crucial step in which we study the parameterized fixed point of a special one dimensional map. Lemma 10 takes the fixed point found in Lemma 9 and deduces properties induced by it on consumption, marginal utility, Euler equations, etc. A recursive application of Lemma 10 going forward leads us to most of the properties in Proposition 4 including monotonicity and continuity in the initial value. Lemma 11 provides the boundary behaviour properties.

Throughout we write $E[X]$ instead of $E_P[X]$.

For $Z > 0$, let the function $\mathcal{Y}_Z : (0, Z) \to (0, \infty)$ be defined by $\mathcal{Y}_Z(c_1) = \frac{u_2'(Z-c_1)}{u_1'(c_1)}$.

LEMMA 8: Assume A.3. $\mathcal{Y}_Z$ is increasing in c_1, it is onto, and continuous with a continuous inverse.

PROOF: The result is a consequence of A.3; in particular, we use the fact that u_i are strictly concave, continuously differentiable, and satisfy the Inada condition at $c = 0$. ■

Given Z and feasible consumption processes, by Lemma 8, for any (t, ω) we have $y_t(\omega) = \mathcal{Y}_{Z_t(\omega)}(c_{1,t}(\omega))$. The inverse of $\mathcal{Y}_Z$ is denoted $(\mathcal{Y}_Z)^{-1}(y)$; by Lemma 8, it is well defined and continuous.

Proposition 4 is proved by using a recursive construction in the variable $y_t(\omega)$ which, by Lemma 8, is equivalent to using the variable $c_{1,t}(\omega)$. However, to establish the basic properties of the construction, it is easier to work with the variable $\lambda := r \cdot u_2'(c_2)/y$. Lemma 9 studies the existence and monotonicity properties of the fixed point in λ of a special function.

LEMMA 9: Assume A.2, A.3, and A.5. For $t = 1, 2, \cdots$ and $\omega \in \Omega$, and $y > 0$, define $\underline{\lambda}(t, \omega, y) := \frac{r_t(\omega) \cdot u_2'(Z_t(\omega))}{y}$ and consider the function $f_{t,\omega,y} : [\underline{\lambda}(t, \omega, y), +\infty) \to [(\beta_1/\beta_2) \cdot \underline{r} \cdot u_1'(\bar{z}), +\infty)$ in the variable λ defined by

$$f_{t,\omega,y}(\lambda) := (\beta_1/\beta_2) \cdot E\Big[r_t \cdot u_1'\Big(Z_t - (u_2')^{-1}\Big(\frac{y \cdot \lambda}{r_t}\Big)\Big)\Big|\mathcal{F}_{t-1}\Big](\omega).$$

Then (i) $f_{t,\omega,y}$ has a unique fixed point denoted $\lambda^*(t, \omega, y)$,
(ii) $\lambda^*(t, \omega, y) > \max_{\omega' \in \Omega(s^{t-1}(\omega))} \frac{r_t(\omega') \cdot u_2'(Z_t(\omega'))}{y}$ and $\lambda^*(t, \omega, y) > (\beta_1/\beta_2) \cdot \underline{r} \cdot u_1'(\bar{z})$,
(iii) $y \cdot \lambda^*(t, \omega, y) > y' \cdot \lambda^*(t, \omega, y')$ if and only if $\lambda^*(t, \omega, y) < \lambda^*(t, \omega, y')$, in particular $y > y'$ if and only if $\lambda^*(t, \omega, y) < \lambda^*(t, \omega, y')$,
(iv) $\lambda^*(t, \omega, y)$ is continuous in y,
(v) $\lambda^*(t, \omega, y) \to_{y \to 0} \infty$, and
(vi) $\lambda^*(t, \omega, y) \cdot y \to_{y \to \infty} \infty$.

PROOF: Notice that even though the domain of the function $f_{t,\omega,y}$ is $\mathcal{F}_t-$measurable, the function is defined in a manner that makes it $\mathcal{F}_{t-1}-$measurable. This is important.
(i) Under A.5 $\underline{r} > 0$ so $\underline{\lambda}(t, \omega, y) \geq 0$. It can be verified that $f_{t,\omega,y}(\underline{\lambda}(t, \omega, y)) = (\beta_1/\beta_2) \cdot E\Big[r_t \cdot u_1'(0)\Big|\mathcal{F}_{t-1}\Big](\omega) = \infty$, where we use the Inada condition; furthermore, $f_{t,\omega,y}$ is continuous and strictly decreasing. Under A.2 and A.3 $(\beta_1/\beta_2) \cdot \bar{r} \cdot u_1'(\bar{z}) < \infty$; therefore, $\mathrm{Lim}_{\lambda \to \infty} f_{t,\omega,y}(\lambda) < \infty$. It follows that $f_{t,\omega,y}$ has a unique fixed point.
(ii) As noted at the beginning of the proof, $f_{t,\omega,y}$ is $\mathcal{F}_{t-1}-$measurable and, therefore, the fixed point $\lambda^*(t, \omega, y)$ is also $\mathcal{F}_{t-1}-$measurable. Since $f_{t,\omega,y}(\underline{\lambda}(t, \omega, y)) = \infty$, we must have $\lambda^*(t, \omega, y) > \max_{\omega' \in \Omega(s^{t-1}(\omega))} \frac{r_t(\omega') \cdot u_2'(Z_t(\omega'))}{y}$, the highest possible value for $\underline{\lambda}(t, \omega, y)$. The second part follows from the fact that $f_{t,\omega,y}$ is strictly decreasing.
(iii) Suppose that $y \cdot \lambda^*(t, \omega, y) > y' \cdot \lambda^*(t, \omega, y')$. Since $f_{t,\omega,y}$ is strictly decreasing, and from the particular way in which y and λ enter the expression,

$$f_{t,y,\omega}(\lambda^*(t, \omega, y)) < f_{t,y',\omega}(\lambda^*(t, \omega, y'))$$

so that by the fixed point property we have $\lambda^*(t, \omega, y) < \lambda^*(t, \omega, y')$. We have shown that

$$y \cdot \lambda^*(t, \omega, y) > y' \cdot \lambda^*(t, \omega, y') \qquad \Leftrightarrow \qquad \lambda^*(t, \omega, y) < \lambda^*(t, \omega, y').$$

(iv) Notice that by (i), $\lambda^*(t,\omega,y)$ exists for all $y > 0$, and by the monotonicity result in (iii), the only sorts of discontinuities that are possible are of the first kind. So if $\lambda^*(t,\omega,\cdot)$ is discontinuous at $\tilde{y}$ then, introducing notation for right-hand and left-hand limits, $\lambda^*(t,\omega,\tilde{y}^-) > \lambda^*(t,\omega,\tilde{y}^+)$. So, by (iii), $\tilde{y}^- \cdot \lambda^*(t,\omega,\tilde{y}^-) < \tilde{y}^+ \cdot \lambda^*(t,\omega,\tilde{y}^+)$ and therefore $\lambda^*(t,\omega,\tilde{y}^-) < \lambda^*(t,\omega,\tilde{y}^+)$ since $\tilde{y}^- = \tilde{y}^+$. The contradiction that results shows that such discontinuities are not present.
(v) Since $\underline{\lambda}(t,\omega,y) \rightarrow_{y\rightarrow 0} \infty$, we can use (ii) to conclude that $\lambda^*(t,\omega,y) \rightarrow_{y\rightarrow 0} \infty$.
(vi) Notice that $\lambda^*(t,\omega,y) \rightarrow_{y\rightarrow\infty} 0$ requires that $f_{t,\omega,y}(\lambda^*(t,\omega,y)) \rightarrow 0$ which cannot hold under A.2, since $\underline{r} > 0$, A.3, since u_1 is strictly increasing and strictly concave, and A.5, since $\bar{z} < \infty$. It follows that $\lambda^*(t,\omega,y) \rightarrow_{y\rightarrow\infty} \epsilon > 0$ so that $\lambda^*(t,\omega,y)\cdot y \rightarrow_{y\rightarrow\infty} \infty$.

■

The next result induces values for consumption at the fixed point identified in Lemma 9 and specifies the implications on intertemporal marginal utilities induced by those values.

LEMMA 10: Assume A.2, A.3, and A.5. Let $y_{t-1} : \Omega \rightarrow R_+$ be an $\mathcal{F}_{t-1}-$measurable function. Set

$$c_{2,t}(\omega) := (u_2')^{-1}\left(\frac{y_{t-1}\cdot\lambda^*(t,\omega,y_{t-1}(\omega))}{r_t}\right), \quad c_{1,t}(\omega) := Z_t(\omega) - c_{2,t}(\omega), \quad y_t(\omega) = \mathcal{Y}_{Z_t(\omega)}(c_{1,t}(\omega)).$$

Then (i) $c_{i,t}(\omega) \geq 0$ and is $\mathcal{F}_t-$measurable, (ii) if $y_{t-1}(\omega) > y'_{t-1}(\omega)$ then the induced values satisfy $y_t(\omega) > y'_t(\omega)$, (iii) $y_t(\omega)$ is a continuous function of $y_{t-1}(\omega)$, (iv) $\frac{r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)} = (\beta_1/\beta_2)\cdot E[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)$ so $r_t(\omega)\cdot u_2'(c_{2,t}(\omega))$ is $\mathcal{F}_{t-1}-$measurable and $\hat{r}_{2,t}(\omega) = 1 \quad P-\text{a.s.}\,\omega$, and (v) $y_t(\omega) = \frac{\beta_1}{\beta_2}\cdot\frac{1}{\hat{r}_{1,t}(\omega)}\cdot y_{t-1}(\omega)$.
PROOF: (i) As per the definition in the hypothesis $\lambda^*(t,\omega,y_{t-1}(\omega)) = \frac{r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)}$. So using Lemma 9 (ii) we have $\lambda^*(t,\omega,y_{t-1}(\omega)) \geq \underline{\lambda}(t,\omega,y_{t-1}(\omega))$

$$\Leftrightarrow \quad \frac{r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)} \geq \frac{r_t(\omega)\cdot u_2'(Z_t(\omega))}{y_{t-1}(\omega)} \quad \Leftrightarrow \quad u_2'(c_{2,t}(\omega)) \geq u_2'(Z_t(\omega))$$

so that using the fact that u_2 is concave we can conclude that $c_{2,t}(\omega) \leq Z_t(\omega)$ so that $c_{1,t}(\omega) \geq 0$. The Inada condition guarantees that $c_{2,t}(\omega) \geq 0$. Since the measurability property is evident, the proof of (i) is complete.
(ii) We can invoke Lemma 9 (iii) and the fixed point property to conclude that

$$y_{t-1}(\omega) > y'_{t-1}(\omega) \quad \Leftrightarrow \quad f_{t,y_{t-1}(\omega),\omega}(\lambda^*(t,\omega,y_{t-1}(\omega))) < f_{t,y'_{t-1}(\omega),\omega}(\lambda^*(t,\omega,y'_{t-1}(\omega))).$$

From the specification of $f_{t,y,\omega}$ and the fact that u_1 is strictly concave, it is easy to see that, necessarily, $c_{1,t}(\omega) > c'_{1,t}(\omega)$. An application of Lemma 8 completes the proof.
(iii) Follows form Lemma 9 (iv), the fact that u_i are twice continuously differentiable, and Lemma 8.
(iv) Follows from the fixed point property since

$$\frac{r_t(\omega)\cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)} = \lambda^*(t,\omega,y_{t-1}(\omega)) = f_{t,\omega,y_{t-1}(\omega)}(\lambda^*(t,\omega,y_{t-1}(\omega)))$$

$$= (\beta_1/\beta_2) \cdot E[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega).$$

This shows that $r_t(\omega) \cdot u_2'(c_{2,t}(\omega))$ is $\mathcal{F}_{t-1}-$measurable and so $\hat{r}_{2,t}(\omega) = 1 \quad P-\text{a.s.}\, \omega$.

(v) By manipulating the fixed point condition, we obtain

$$\frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))} = y_{t-1}(\omega) \cdot \frac{\beta_1}{\beta_2} \cdot \frac{E[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)}{r_t(\omega) \cdot u_1'(c_{1,t}(\omega))} \quad \Leftrightarrow \quad y_t(\omega) = \frac{\beta_1}{\beta_2} \cdot \frac{1}{\hat{r}_{1,t}(\omega)} \cdot y_{t-1}(\omega)$$

proving (v). ∎

Proposition 4 is proved by recursively applying Lemma 10. For existence we assume that we are given a triple $(y, t_0, \omega) \in R_{++} \times \{0, 1, 2, \cdots\} \times \Omega$, we set $y_{t_0}(\omega) := y$ and treat it as a parameter and apply Lemma 10 (i) to induce a unique process for $\{y_t(\tilde{\omega})\}_{t \geq t_0}$ and $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$. By Lemma 8 this is equivalent to starting with a triple $(c, t_0, \omega) \in R_{++} \times \{0, 1, 2, \cdots\} \times \Omega$ with the additional condition that $c \in (0, Z_{t_0}(\omega))$, setting $c_{1,t_0}(\omega) := c$ and treating it as a parameter and generating a unique pair of processes c_i that are feasible and solve the fixed point problem at each date $t \geq t_0 + 1$ and $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$.

The notation $\{C_{i,t}(c, t_0, \omega)\}_{t \geq t_0}$, where the process is defined $P-$a.s. only for $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$, was introduced in the statement of Proposition 4. For monotonicity, we consider two triples (c, t_0, ω) and (c', t_0, ω) such that $c > c'$. By Lemma 8 the induced values satisfy $y_{t_0}(\omega) > y_{t_0}'(\omega)$ so that by an iterative application of Lemma 10 (ii) $y_t(\tilde{\omega}) > y_t'(\tilde{\omega})$ for all $t \geq t_0 + 1$ and $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$. Another application of Lemma 8 establishes that $C_{1,t}(\tilde{\omega}; c, t_0, \omega) > C_{1,t}(\tilde{\omega}; c', t_0, \omega)$ for all $t \geq t_0 + 1$ and $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$.

By a direct argument, for all $t \geq t_0 + 1$ and $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$, $C_{1,t}(\tilde{\omega}; c, t_0, \omega)$ is continuous in c.

Lemma 11 establishes some boundary properties of the consumption processes that we construct and completes the proof of Proposition 4.

LEMMA 11: Assume A.1, A.2, A.3, and A.5. (i) Given t_0, ϵ, and T, where $\epsilon > 0$ and small, and $T > t_0$, there exists $A \in \mathcal{F}_T$ with $P(A) > 0$ and $c > 0$ such that $C_{1,t}(\tilde{\omega}; c, t_0, \omega) < \epsilon$ for all t such that $T \geq t \geq t_0 + 1$ and $P-$a.s. $\tilde{\omega} \in A$. (ii) Given t_0, ϵ, and T, where $\epsilon > 0$ and small, and $T > t_0$, there exists $c > 0$ such that $Z_t(\tilde{\omega}) - C_{1,t}(\tilde{\omega}; c, t_0, \omega) < \epsilon$ for all t such that $T \geq t \geq t_0 + 1$ and $P-$a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$.

PROOF: (i) By Lemma 9 (v), $\lambda^*(t, \omega, y) \rightarrow_{y \rightarrow 0} \infty$ so that, by the fixed point property, $f_{t,\omega,y}(\lambda^*(t, \omega, y)) \rightarrow_{y \rightarrow 0} \infty$. But then, under A.2, A.3, and A.5, we must have $E[c_{1,t}|\mathcal{F}_{\tilde{t}-1}](\omega) \rightarrow_{y_{t-1}(\omega) \rightarrow 0} 0$. So we have shown that for some $\tilde{\omega} \in \Omega(s^{t-1}(\omega))$, $c_{1,t}(\tilde{\omega}) \rightarrow_{y_{t-1}(\omega) \rightarrow 0} 0$, and, by Lemma 8, $y_t(\tilde{\omega}) \rightarrow_{y_{t-1}(\omega) \rightarrow 0} 0$. By recursively using the monotonicity and continuity properties, Lemma 10 (ii) and (iii), we can conclude that for any $t > t_0$, there is a $\tilde{\omega}(t)$ such that for all t' where $t \geq t' > t_0$, $y_{t'}(\tilde{\omega}(t)) \rightarrow_{y_{t_0}(\omega) \rightarrow 0} 0$, and, by Lemma 8, $c_{1,t'}(\tilde{\omega}(t)) \rightarrow_{y_{t_0}(\omega) \rightarrow 0} 0$. It follows that given t_0, ϵ, and T, where $\epsilon > 0$ and small, and $T > t_0$, there exists $\tilde{\omega} \in \Omega(s^{t-1}(\omega))$ and $c > 0$ such that $C_{1,t}(\tilde{\omega}; c, t_0, \omega) < \epsilon$ for all t such that $T \geq t \geq t_0 + 1$. Since $T < \infty$ and A.1 holds, the same is true for all $\tilde{\omega} \in A$ where $P(A) > 0$ and $A \in \mathcal{F}_T$.

(ii) By Lemma 9 (vi), the rule defining $c_{2,t}(\omega)$ in Lemma 10, and the concavity of u_2, we conclude that $c_{2,t}(\omega) \to_{y_{t-1}(\omega)\to\infty} 0$; by Lemma 8, $y_t(\omega) \to_{y_{t-1}(\omega)\to\infty} \infty$. By recursively using the monotonicity and continuity properties, Lemma 10 (ii) and (iii), we can conclude that for any $t > t_0$, $y_t(\omega) \to_{y_{t_0}(\omega)\to\infty} \infty$, and, by Lemma 8, $y_{t_0}(\omega) \to_{c\to Z_{t_0}(\omega)} \infty$. It follows that given t_0, ϵ, and T, where $\epsilon > 0$ and small, and $T > t_0$, there exists $c > 0$ such that $Z_t(\tilde{\omega}) - C_{1,t}(\tilde{\omega}; c, t_0, \omega) < \epsilon$ for all t such that $T \geq t \geq t_0 + 1$ and P-a.s. $\tilde{\omega} \in \Omega(s^{t_0}(\omega))$. ■

PROOF OF PROPOSITION 5

The proof follows from Lemma 12 and Lemma 13. To simply the notation we use $c_{i,t}(\omega)$ for consumption and state and prove the results for the case where $t_0 = 0$ and the processes are defined on Ω. Throughout we write $E[X]$ instead of $E_P[X]$.

LEMMA 12: Assume A.2, A.3, A.5. Then for the solution proposed

$$\text{ess. sup}_{\omega\in\Omega;P} \sup_{t\geq 0} \frac{u_2'(c_{2,t+1}(\omega))}{u_2'(c_{2,t}(\omega))} \leq M := \max\left\{\frac{\bar{r}\cdot u_2'(\underline{z}/2)}{\underline{r}\cdot u_2'(\bar{z})}; \frac{\bar{r}\cdot u_1'(\underline{z}/2)}{\underline{r}\cdot u_1'(\bar{z})}\right\}.$$

PROOF: If not then there is an A with $P(A) > 0$, such that for every $\omega \in A$ there exists a $t(\omega)$ such that

$$\frac{u_2'(c_{2,t(\omega)+1}(\omega))}{u_2'(c_{2,t(\omega)}(\omega))} > M \qquad \Rightarrow \qquad \frac{u_2'(c_{2,t(\omega)+1}(\omega))}{u_2'(c_{2,t(\omega)}(\omega))} > \frac{\bar{r}\cdot u_2'(\underline{z}/2)}{\underline{r}\cdot u_2'(\bar{z})}$$

$$\Rightarrow \qquad \frac{r_{t(\omega)+1}(\omega)\cdot u_2'(c_{2,t(\omega)+1}(\omega))}{u_2'(c_{2,t(\omega)}(\omega))} > \frac{\bar{r}\cdot u_2'(\underline{z}/2)}{u_2'(\bar{z})}.$$

As shown in the proof of Lemma 10 (v),

$$r_{t+1}(\omega)\cdot\frac{u_2'(c_{2,t+1}(\omega))}{u_2'(c_{2,t}(\omega))} = \frac{E[r_{t+1}\cdot u_1'(c_{1,t+1})|\mathcal{F}_t](\omega)}{u_1'(c_{1,t}(\omega))},$$

so we must also have

$$\frac{E[r_{t(\omega)+1}\cdot u_1'(c_{1,t(\omega)+1})|\mathcal{F}_{t(\omega)}](\omega)}{r_{t(\omega)+1}(\omega)\cdot u_1'(c_{1,t(\omega)}(\omega))} > M \quad \Rightarrow \quad \frac{E[r_{t(\omega)+1}\cdot u_1'(c_{1,t(\omega)+1})|\mathcal{F}_{t(\omega)}](\omega)}{r_{t(\omega)+1}(\omega)\cdot u_1'(c_{1,t(\omega)}(\omega))} > \frac{\bar{r}\cdot u_1'(\underline{z}/2)}{\underline{r}\cdot u_1'(\bar{z})}$$

so that, since $c_{1,t}(\omega) \leq \bar{z}$ and $u_1'' < 0$,

$$\Rightarrow \qquad \frac{E[r_{t(\omega)+1}\cdot u_1'(c_{1,t(\omega)+1})|\mathcal{F}_{t(\omega)}](\omega)}{r_{t(\omega)+1}(\omega)\cdot u_1'(\bar{z})} > \frac{\bar{r}\cdot u_1'(\underline{z}/2)}{\underline{r}\cdot u_1'(\bar{z})}$$

$$\Rightarrow \qquad E[r_{t(\omega)+1}\cdot u_1'(c_{1,t(\omega)+1})|\mathcal{F}_{t(\omega)}](\omega) > \bar{r}\cdot u_1'(\underline{z}/2)$$

since $\underline{r} \leq r_{t(\omega)+1}$. It follows that for some $\tilde{\omega} \in \Omega(s^{t(\omega)}(\omega))$,

$$u_1'(c_{1,t(\omega)+1}(\tilde{\omega})) > u_1'(\underline{z}/2) \qquad \Leftrightarrow \qquad c_{1,t(\omega)+1}(\tilde{\omega}) < \underline{z}/2 \leq Z_t/2$$

$$\Leftrightarrow \quad c_{2,t(\omega)+1}(\tilde{\omega}) > Z_t/2 \geq \underline{z}/2 \quad \Rightarrow \quad r_{t(\omega)+1}(\tilde{\omega})\cdot u_2'(c_{2,t(\omega)+1}(\tilde{\omega})) < \bar{r}\cdot u_2'(\underline{z}/2)$$

$$\Leftrightarrow \qquad \frac{r_{t(\omega)+1}(\tilde{\omega}) \cdot u_2'(c_{2,t(\omega)+1}(\tilde{\omega}))}{u_2'(\bar{z})} < \frac{\bar{r} \cdot u_2'(\underline{z}/2)}{u_2'(\bar{z})}.$$

But that contradicts the fact that $r_t(\omega) \cdot u_2'(c_{2,t}(\omega))$ is always $\mathcal{F}_{t-1}$-measurable since we started by saying that $\frac{r_{t(\omega)+1}(\omega) \cdot u_2'(c_{2,t(\omega)+1}(\omega))}{u_2'(c_{2,t(\omega)}(\omega))} > \frac{\bar{r} \cdot u_2'(\underline{z}/2)}{u_2'(\bar{z})}$. ■

LEMMA 13: Assume A.2, A.3, A.5, and A.6. Then

$$0 \leq E\Big[\sum_{t=0}^{T} \beta_i^t \cdot \frac{u_i'(c_{i,t})}{u_i'(c_{i,0})}\Big|\mathcal{F}_0\Big](\omega) \leq 1/(1-\beta_i \cdot M).$$

PROOF: We prove the result for $i = 1$ since it is trivial for $i = 2$.

Since, by Proposition 4, in the proposed solution

$$y_t(\omega) = \frac{1}{\Pi_{\tau=1}^t[\hat{r}_{1,\tau}(\omega)]} \cdot y_0(\omega) \qquad \Leftrightarrow \qquad \frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))} = \frac{1}{\Pi_{\tau=1}^t[\hat{r}_{1,\tau}(\omega)]} \cdot \frac{u_2'(c_{2,0}(\omega))}{u_1'(c_{1,0}(\omega))}$$

$$\Leftrightarrow \qquad \beta_1^t \cdot \frac{u_1'(c_{1,t}(\omega))}{u_1'(c_{1,0}(\omega))} = \beta_1^t \cdot \Pi_{\tau=1}^t[\hat{r}_{1,\tau}(\omega)] \cdot \frac{u_2'(c_{2,t}(\omega))}{u_2'(c_{2,0}(\omega))}$$

$$\Rightarrow \qquad 0 \leq E\Big[\sum_{t=0}^{T} \beta_1^t \cdot \frac{u_1'(c_{1,t})}{u_1'(c_{1,0})}\Big|\mathcal{F}_0\Big](\omega) = E\Big[\sum_{t=0}^{T} \beta_1^t \cdot \Pi_{\tau=1}^t[\hat{r}_{1,\tau}] \cdot \frac{u_2'(c_{2,t})}{u_2'(c_{2,0})}\Big|\mathcal{F}_0\Big](\omega)$$

$$\leq \sum_{t=0}^{T} \beta_1^t \cdot (M)^t \cdot E\Big[\Pi_{\tau=1}^t[\hat{r}_{1,\tau}]\Big|\mathcal{F}_0\Big](\omega) = \sum_{t=0}^{T} \beta_1^t \cdot (M)^t$$

where we use the fact that $E\left[\hat{r}_{i,t}|\mathcal{F}_{t-1}\right](\omega) = 1$ together with the law of iterated expectations. The result follows by taking the limit. ■

PROOF OF PROPOSITION 6

The proof follows from Lemma 14-16. Throughout we write $E[X]$ instead of $E_P[X]$.

LEMMA 14: Assume A.3 and $\underline{r} \geq 0$. In the proposed solution, $P\{\omega : \liminf y_t(\omega) \to_{t\to\infty} 0\} = 0$.

PROOF: Since $y_T(\omega) = \frac{1}{\Pi_{t=1}^T\left[\hat{r}_{1,t}(\omega)\right]} \cdot y_0(\omega)$ and since, by Lemma 5, we know that $R_{1,t}(\tilde{\omega})$ is a.s. bounded, we conclude that $\liminf y_T(\omega) > 0$ a.s. ■

LEMMA 15: Assume $\underline{z} > 0$, $\underline{r} \geq 0$, and A.3. In the proposed solution, $P(\underline{\mathcal{C}}_1) = 0$ where $\underline{\mathcal{C}}_i := \{\omega \in \Omega : \liminf c_{i,t}(\omega) = 0\}$.

PROOF: Given y_0, choose $K > 0$. For any such K let $c_K > 0$ solve the equation $u_2'(\underline{z} - c_K) = u_1'(c_K) \cdot y_0(\tilde{\omega})/K$.

For any $\tilde{\omega} \in \underline{\mathcal{C}}_1$ and such a K there exists a sequence $\{t_\tau^K\}$ of periods such that $c_{1,t_\tau^K} \leq c_K$ so $y_{t_\tau^K}(\tilde{\omega}) \leq y_0(\tilde{\omega})/K$. Then Lemma 14 implies that $P(\underline{\mathcal{C}}_1) = 0$. ■

LEMMA 16: Assume A.2, A.3, and $\underline{r} \geq 0$. In the proposed solution $P(\mathcal{A}_1) = 0$.

PROOF: Since $\bar{z} < \infty$, if, for some $\tilde{\omega}$, $\liminf \hat{r}_{1,t}(\tilde{\omega}) = 0$ then $\limsup E[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\tilde{\omega}) = \infty$. We shall argue that in such an event c_1 must also approach zero, a zero probability event by Lemma 15.

So suppose $\tilde{\omega}$ is such that $\limsup E[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\tilde{\omega}) = \infty$ and $\liminf c_{1,t}(\tilde{\omega}) = 2\epsilon$ for some $\epsilon > 0$. It follows that there exists $\tilde{t}$ such that for $t \geq \tilde{t}$, $c_{1,\tilde{t}}(\tilde{\omega}) \geq \epsilon$. Choose $\delta(\epsilon)$ to satisfy $u_1'(\underline{z} - \delta(\epsilon)) < \Big(u_1'(\epsilon)/u_2'(\bar{z})\Big) \cdot u_2'(\delta(\epsilon))$. Necessarily, for some $t' \geq \tilde{t}$,

$$E[r_{t'} \cdot u_1'(c_{1,t'})|\mathcal{F}_{t'-1}](\tilde{\omega}) > \bar{r} \cdot \frac{u_1'(\epsilon)}{u_2'(\bar{z})} \cdot u_2'(\delta(\epsilon)),$$

and in the solution proposed

$$r_t(\omega) \cdot u_2'(c_{2,t}(\omega)) = \frac{u_2'(c_{2,t-1}(\omega))}{u_1'(c_{1,t-1}(\omega))} \cdot E[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)$$

so that for $(\tilde{\omega}, t')$

$$r_{t'}(\tilde{\omega}) \cdot u_2'(c_{2,t'}(\tilde{\omega})) \geq \frac{u_2'(Z_{t'-1}(\tilde{\omega}) - \epsilon)}{u_1'(\epsilon)} \cdot E[r_{t'} \cdot u_1'(c_{1,t'})|\mathcal{F}_{t'-1}](\tilde{\omega})$$

$$> \frac{u_2'(\bar{z})}{u_1'(\epsilon)} \cdot \bar{r} \cdot \frac{u_1'(\epsilon)}{u_2'(\bar{z})} \cdot u_2'(\delta(\epsilon)) = \bar{r} \cdot u_2'(\delta(\epsilon)).$$

Since $r_t(\omega) \cdot u_2'(c_{2,t}(\omega))$ is $\mathcal{F}_{t-1}$-measurable,

$$r_{t'}(\omega') \cdot u_2'(c_{2,t'}(\omega')) > \bar{r} \cdot u_2'(\delta(\epsilon)) \qquad \omega' \in \Omega\Big((s^{t'-1}(\tilde{\omega})\Big).$$

So $c_{2,t'}(\omega') < \delta(\epsilon)$ for all $\omega' \in \Omega\Big((s^{t'-1}(\tilde{\omega})\Big)$ and therefore, by feasibility, $c_{1,t'}(\omega') > Z_{t'}(\omega') - \delta(\epsilon)$ for all $\omega' \in \Omega\Big((s^{t'-1}(\tilde{\omega})\Big)$. It follows that

$$E[r_{t'} \cdot u_1'(c_{1,t'})|\mathcal{F}_{t'-1}](\tilde{\omega}) \leq \bar{r} \cdot u_1'(\underline{z} - \delta(\epsilon))$$

which, using the definition of $\delta(\epsilon)$, is a contradiction. We have shown that $\liminf \hat{r}_{1,t}(\tilde{\omega}) = 0$ implies that $\tilde{\omega} \in \underline{\mathcal{C}}_i$, a set that has measure zero according to Lemma 15. ∎

STATEMENT AND PROOF OF LEMMA 17

LEMMA 17: Assume A.1 and A.5. The TC0 budget set does not allow Ponzi schemes.

PROOF: It is easy to show that if θ is a Ponzi scheme at q and $p \in \mathcal{P}(q; P)$, then $-p_{t'}(\omega') = \lim_{T\to+\infty} E_P\Big[p_T \cdot q_T \cdot \theta_{i,T}\Big|\mathcal{F}_{t'}\Big](\omega')$ while $\lim_{T\to+\infty} E_P\Big[p_T \cdot q_T \cdot \theta_{i,T}\Big|\mathcal{F}_{t'}\Big](\omega) = 0$ for $\omega \notin \Omega(s^t(\omega'))$. By ruling out trivial Arrow price processes and assuming A.1, so that $dP_t(\omega') > 0$, we have $\lim_{T\to+\infty} E_P\Big[p_T \cdot q_T \cdot \theta_{i,T}\Big|\mathcal{F}_0\Big](\omega) < 0$ and the proposed Ponzi scheme entails a plan that is not an element of the budget set $BC_i^{\mathrm{TC}}(q, p)$ with $p \in \mathcal{P}(q; P)$. It follows that there can be no Ponzi scheme that is TC0 budget feasible. ∎

The same proof, with P_i instead of P, can be used to see that the IDC budget set does not allow Ponzi schemes. This follows from the fact that with the IDC budget set, the uniform bound on debt values implies that a transversality condition holds at date 0 and therefore the argument given for TC0 budget sets applies.

PROOF OF THEOREM 2

First we state and prove Lemma 18 and Lemma 19.

LEMMA 18: Given q and any $p_i \in \mathcal{P}^1(q;P)$, if c_i is a maximizer on the set $BC_i^{\mathrm{AD}}(p_i)$, then $\lim_{T\to+\infty} E_P\Big[p_{i,T}\cdot q_T\cdot \theta_{i,T}\Big|\mathcal{F}_0\Big](\omega)=0$, where θ_i supports c_i at the prices q.
PROOF: Since c_i is a maximizer on the set $BC_i^{\mathrm{AD}}(p_i)$, we have $c_i \in BC_i^{\mathrm{AD}}(p_i)$; furthermore, the value of the endowment is finite, $\lim_{T\to+\infty}\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot z_{i,t}\Big|\mathcal{F}_0\Big](\omega) < \infty$, and the value of the endowment is exhausted so that $\lim_{T\to+\infty}\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot (c_{i,t}-z_{i,t})\Big|\mathcal{F}_0\Big](\omega)=0$. In addition, since θ_i supports c_i at the prices q, we can write

$$\lim_{T\to+\infty}\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot(c_{i,t}-z_{i,t})\Big|\mathcal{F}_0\Big](\omega)=\lim_{T\to+\infty}\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot(r_t\cdot\theta_{i,t-1}-q_t\cdot\theta_{i,t})\Big|\mathcal{F}_0\Big](\omega)$$

$$=\lim_{T\to+\infty} E_P\Big\{[-p_{i,0}\cdot q_0\cdot\theta_{i,0}+p_{i,1}\cdot r_1\cdot\theta_{i,0}]+\sum_{t=2}^{T}[-p_{i,t-1}\cdot q_{t-1}\cdot\theta_{i,t-1}+p_{i,t}\cdot r_t\cdot\theta_{i,t-1}]$$
$$-p_{i,T}\cdot q_T\cdot\theta_{i,T}\Big|\mathcal{F}_0\Big\}(\omega)$$

where we use the convention that $\theta_{i,-1}(\omega)=0$. By using the fact that $p_i\in\mathcal{P}^1(q;P)$, the set of summable Arrow prices with respect to P, we see that in fact we have
$0=\lim_{T\to+\infty}\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot(c_{i,t}-z_{i,t})\Big|\mathcal{F}_0\Big](\omega)=\lim_{T\to+\infty} E_P\Big[-p_{i,T}\cdot q_T\cdot\theta_{i,T}\Big|\mathcal{F}_0\Big](\omega)$. ∎

LEMMA 19: Given q and any $p_i\in\mathcal{P}^1(q;P)$, $BC_i^{\mathrm{TC}}(q,p_i)\subset BC_i^{\mathrm{AD}}(p_i)$.
PROOF: Consider $c_i\in BC_i^{\mathrm{TC}}(q,p_i)$ and let θ_i denote the corresponding asset holding process. We would like to show that

$$\lim_{T\to+\infty}\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot c_{i,t}\Big|\mathcal{F}_0\Big](\omega)\le\lim_{T\to+\infty}\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot z_{i,t}\Big|\mathcal{F}_0\Big](\omega).$$

Using the sequence of budget constraints in the definition of the set $BC_i^{\mathrm{TC}}(q,p_i)$, we have

$$\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot(c_{i,t}-z_{i,t})\Big|\mathcal{F}_0\Big](\omega)\le\sum_{t=0}^{T} E_P\Big[p_{i,t}\cdot(r_t\cdot\theta_{i,t-1}-q_t\cdot\theta_{i,t})\Big|\mathcal{F}_0\Big](\omega).$$

By an argument similar to that in Lemma 18 we conclude that for all $T\ge 0$ we have

$$\sum_{t=0}^{T} E_P\Big[p_t\cdot(c_{i,t}-z_{i,t})\Big|\mathcal{F}_0\Big](\omega)\le\ E_P\Big[-p_{i,T}\cdot\widehat{q}_T\cdot\theta_{i,T}\Big|\mathcal{F}_0\Big](\omega).$$

Since $c_i\in BC_i^{\mathrm{TC}}(q,p_i)$ implies that $\liminf_{T\to+\infty} E_P\Big[p_{i,T}\cdot\widehat{q}_T\cdot\theta_{i,T}\Big|\mathcal{F}_0\Big](\omega)\ge 0 \quad P-\text{a.s.}\,\omega$, and $p_i\in\mathcal{P}^1(q;P)$ implies that p_i is summable while (c_i-z_i) is uniformly bounded, we can conclude that $c_i\in BC_i^{\mathrm{AD}}(p_i)$. ∎

PROOF OF THEOREM 2: Recall that $\widehat{\theta}_i$ is the portfolio that supports $\widehat{c}_i$ at the price process $\widehat{q}$. By Lemma 18 $\widehat{c}_i\in BC_i^{\mathrm{TC}}(q,p_i)$ and, by Lemma 19, $BC_i^{\mathrm{TC}}(q,p_i)\subset BC_i^{\mathrm{AD}}(\widehat{p}_i)$

so that $\widehat{c}_i$ is a maximizer on $BC_i^{\text{TC}}(q, p_i)$. Since the consumption processes are aggregate feasible and, at every $t \geq 0$, $\theta_{1,t}(\omega) + \theta_{2,t}(\omega) = 0$ P-a.s. ω, which follows from the fact that the spot market budget constraints are satisfied with equality, it follows that $(\widehat{c}_1, \widehat{c}_2, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{q})$ constitute a TC0 equilibrium proving Theorem 1 (i).

To complete the proof of Theorem 1 (ii), notice that we can use Theorem 5.2 in Magill and Quinzii (1994) to conclude that since a transversality condition holds at every t for P-a.s. ω, and preferences are uniformly impatient, there is a uniform bound on the value of debt where we use the supporting asset portfolio. It follows that $\widehat{c}_i$ is a maximizer on $BC_i(\widehat{q})$ and we have an IDC equilibrium. ∎

LEMMA 20: Assume A.2 and A.3 and that $P_1 = P_2 = P$. Consider a consumption process $\widehat{c}_i$ and assume that $p_i^{\widehat{c}}$ satisfies $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\left[p_{i,t}^{\widehat{c}} \middle| \mathcal{F}_0\right](\omega) < \infty$. If $\lim_{T\to+\infty} E_P\left[\sum_{t=0}^{T} p_{i,t}^{\widehat{c}} \cdot \left(\widehat{c}_{i,t} - z_{i,t}\right) \middle| \mathcal{F}_0\right](\omega) = 0$, then $\widehat{c}_i$ is a maximizer on the set $BC_i^{\text{AD}}(p_i^{\widehat{c}})$.
PROOF: Since $\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\left[p_{i,t}^{\widehat{c}} \middle| \mathcal{F}_0\right](\omega) < \infty$ and $\bar{z} < \infty$ and $z_i \in \times_{t=0}^{\infty} \Psi_{+}^{t,P}$,

$$\lim_{T\to+\infty} \sum_{t=0}^{T} E_P\left[p_{i,t}^{\widehat{c}} \cdot z_{i,t} \middle| \mathcal{F}_0\right](\omega) < \infty.$$

Furthermore, since $\lim_{T\to+\infty} E_P\left[\sum_{t=0}^{T} p_{i,t}^{\widehat{c}} \cdot \left(\widehat{c}_{i,t} - z_{i,t}\right) \middle| \mathcal{F}_0\right](\omega) = 0$ we have $\widehat{c}_i \in BC_i^{\text{AD}}(p_i^{\widehat{c}})$.

Define $\mu_i := u_i'(\widehat{c}_{i,0}(\omega))$. $\mu_i > 0$. Clearly, $\widehat{c}_i$ is the unique solution to the system of first order conditions $\beta_i^t \cdot u_i'(\widehat{c}_{i,t}(\omega)) = \mu_i \cdot p_{i,t}^{\widehat{c}}(\omega)$. Also, the Lagrangean function

$$\lim_{T\to+\infty} \left\{ \sum_{t=0}^{T} E_P\left[\beta_i^t \cdot u_i(c_{i,t}) \middle| \mathcal{F}_0\right](\omega) + \mu_i \cdot \sum_{t=0}^{T} E_P\left[p_{i,t}^{\widehat{c}} \cdot (c_{i,t} - z_{i,t}) \middle| \mathcal{F}_0\right](\omega) \right\}$$

is strictly concave in c_i. It follows (e.g. Luenberger (1969) Theorem 1 in Section 8.5 and Lemma 1 in Section 8.7) that the first order conditions are sufficient to identify a global maximizer and $\widehat{c}_i$ maximizes the Lagrangean function. Therefore $\widehat{c}_i$ solves the constrained optimization problem. ∎

PROOF OF THEOREM 3

Throughout we write $E[X]$ instead of $E_P[X]$.

In the proposed solution, for all $t \geq 1$

$$\beta_1 \cdot \frac{E[r_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)}{u_1'(c_{1,t-1}(\omega))} = \beta_2 \cdot \frac{E[r_t \cdot u_2'(c_{2,t})|\mathcal{F}_{t-1}](\omega)}{u_2'(c_{2,t-1}(\omega))} \quad P-\text{a.s. } \omega.$$

Define an asset price process q and personalized price processes p_i by

$$q_{t-1}(\omega) := \beta_i \cdot \frac{E[r_t \cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}](\omega)}{u_i'(c_{i,t-1}(\omega))} \qquad p_{i,t}(\omega) := \beta_i^t \cdot \frac{u_i'(c_{i,t}(\omega))}{u_i'(c_{i,0}(\omega))}.$$

It follows that the consumption processes satisfy the Euler equations with the price process q and that also p_i are such that the no arbitrage condition holds and hence, since by

Proposition 5 they are summable, $p_i \in \mathcal{P}^1(q;P)$ for $i \in \mathcal{I}$. Also, using the spot market budget constraints with asset prices q and consumption process c_i, we can construct the supporting portfolio θ_i.

As in the proof of Lemma 18, if $\lim_{T\to+\infty} E\Big[p_{i,T} \cdot q_T \cdot \theta_{i,T}|\mathcal{F}_0\Big](\omega) = 0 \quad P-$a.s. holds, then $c_i \in BC_i^{\mathrm{AD}}(p_i)$.

An application of Lemma 20 shows that the consumption processes proposed are maximal for each i in $BC_i^{\mathrm{AD}}(p_i)$. To complete the proof of Theorem 3 we shall apply Theorem 2 and for that we need to verify that the transversality conditions also hold.

We continue the proof with Lemma 21 and 22.

LEMMA 21: If c_i is an Euler process at q and θ_i is a supporting portfolio then

$$\beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot q_T(\omega) \cdot \theta_{i,T}(\omega) = \beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot \Big(z_{i,T}(\omega) - c_{i,T}(\omega)\Big)$$
$$+ \sum_{\tau=0}^{T-1} \beta_i^\tau \cdot u_i'(c_{i,\tau}(\omega)) \cdot \Bigg(\Pi_{s=\tau}^{T-1} \hat{r}_{i,s+1}(\omega)\Bigg) \cdot \Big(z_{i,\tau}(\omega) - c_{i,\tau}(\omega)\Big)$$

where $\hat{r}_i$ is the process induced by c_i.

PROOF: Given any process c_i that is an Euler process at the price process q and the induced process $\hat{r}_i$, we have

$$q_{t-1}(\omega) = \beta_i \cdot \frac{E[r_t \cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}](\omega)}{u_i'(c_{i,t-1}(\omega))} \qquad \hat{r}_{i,t}(\omega) := \frac{r_t(\omega) \cdot u_i'(c_{i,t}(\omega))}{E\left[r_t \cdot u_i'(c_{i,t})|\mathcal{F}_{t-1}\right](\omega)}.$$

It follows that

$$\hat{r}_{i,t}(\omega) = \frac{\beta_i \cdot r_t(\omega) \cdot u_i'(c_{i,t}(\omega))}{q_{t-1}(\omega) \cdot u_i'(c_{i,t-1}(\omega))} \qquad \Leftrightarrow \qquad \frac{r_t(\omega)}{q_{t-1}(\omega)} = \frac{\hat{r}_{i,t}(\omega)}{\beta_i} \cdot \frac{u_i'(c_{i,t-1}(\omega))}{u_i'(c_{i,t}(\omega))}$$

$$\Rightarrow \qquad \Pi_{s=\tau}^{T-1} \frac{r_{s+1}(\omega)}{q_s(\omega)} = \Pi_{s=\tau}^{T-1} \Bigg(\frac{\hat{r}_{i,s+1}(\omega)}{\beta_i} \cdot \frac{u_i'(c_{i,s}(\omega))}{u_i'(c_{i,s+1}(\omega))}\Bigg) = \frac{1}{\beta_i^{T-\tau}} \cdot \Big(\Pi_{s=\tau}^{T-1} \hat{r}_{i,s+1}(\omega)\Big) \cdot \frac{u_i'(c_{i,\tau}(\omega))}{u_i'(c_{i,T}(\omega))}.$$

Using the spot market budget constraints

$$c_{i,t}(\omega) + q_t(\omega) \cdot \theta_{i,t}(\omega) \leq z_{i,t}(\omega) + r_t(\omega) \cdot \theta_{i,t-1}(\omega)$$

which, by monotonicity, hold as equalities, and iterating we obtain

$$q_T(\omega) \cdot \theta_{i,T}(\omega) = z_{i,T}(\omega) - c_{i,T}(\omega) + \sum_{\tau=0}^{T-1} \Bigg(\Pi_{s=\tau}^{T-1} \frac{r_{s+1}(\omega)}{q_s(\omega)}\Bigg) \cdot \Big(z_{i,\tau}(\omega) - c_{i,\tau}(\omega)\Big).$$

After carrying out the substitution we can evaluate

$$\beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot q_T(\omega) \cdot \theta_{i,T}(\omega) = \beta_i^T \cdot u_i'(c_{i,T}(\omega)) \cdot \Big(z_{i,T}(\omega) - c_{i,T}(\omega)\Big)$$
$$+ \sum_{\tau=0}^{T-1} \beta_i^\tau \cdot u_i'(c_{i,\tau}(\omega)) \cdot \Bigg(\Pi_{s=\tau}^{T-1} \hat{r}_{i,s+1}(\omega)\Bigg) \cdot \Big(z_{i,\tau}(\omega) - c_{i,\tau}(\omega)\Big). \qquad \blacksquare$$

LEMMA 22: Assume that the economy is such that in the proposed solution, $\forall\ t \geq 1$, $u_2'(c_{2,t}(\omega)) \cdot \Big(z_{2,t}(\omega) - c_{2,t}(\omega)\Big) = \bar{c}_{2,t} \quad P-\text{a.s.}\,\omega$. If there exists $\widehat{c}_{2,0}(\omega)$ that solves

$$u_2'(\widehat{c}_{2,0}(\omega)) \cdot \Big(z_{2,0}(\omega) - \widehat{c}_{2,0}(\omega)\Big) = -\text{Lim}_{T\to+\infty} \sum_{\tau=1}^{T} \beta_2^{\tau} \cdot \bar{c}_{2,\tau},$$

then for every $t \geq 1$ $\text{Lim}_{T\to+\infty} E[\beta_i^T \cdot u_i'(\widehat{c}_{i,T}) \cdot \widehat{q}_T \cdot \widehat{\theta}_{i,T}|\mathcal{F}_t](\omega) = 0 \quad P-\text{a.s. } \omega$ and the transversality conditions for both the agents is satisfied when we consider the proposed solution induced by the initial value given by $\widehat{c}_{2,0}(\omega)$.

PROOF: Consider $i = 2$. Since $\hat{r}_{2,t}(\omega) = 1 \quad \forall t \geq 0\ P-\text{a.s.}\,\omega$, the expression obtained in Lemma 21 takes the form

$$\beta_2^T \cdot u_2'(c_{2,T}(\omega)) \cdot q_T(\omega) \cdot \theta_{2,T}(\omega) = \sum_{\tau=0}^{T} \beta_2^{\tau} \cdot u_2'(c_{2,\tau}(\omega)) \cdot \Big(z_{2,\tau}(\omega) - c_{2,\tau}(\omega)\Big)$$

$$= \sum_{\tau=1}^{T} \beta_2^{\tau} \cdot \bar{c}_{2,\tau} + u_2'(c_{2,0}(\omega)) \cdot \Big(z_{2,0}(\omega) - c_{2,0}(\omega)\Big).$$

Notice that $\beta_2^T \cdot u_2'(c_{2,T}) \cdot q_T \cdot \theta_{2,T}$ is a deterministic quantity. So

$$\text{Lim}_{T\to+\infty} E[\beta_2^T \cdot u_2'(c_{2,T}) \cdot q_T \cdot \theta_{2,T}|\mathcal{F}_t](\omega) = \text{Lim}_{T\to+\infty} \sum_{\tau=1}^{T} \beta_2^{\tau} \cdot \bar{c}_{2,\tau} + u_2'(c_{2,0}(\omega)) \cdot \Big(z_{2,0}(\omega) - c_{2,0}(\omega)\Big)$$

and the limit is independent of t and will be equal to zero if $\Big(z_{2,0}(\omega) - c_{2,0}(\omega)\Big)$, equivalently $c_{2,0}(\omega)$ or $\theta_{2,0}(\omega)$, the initial asset holding for agent 2, satisfies the condition

$$u_2'(c_{2,0}(\omega)) \cdot \Big(z_{2,0}(\omega) - c_{2,0}(\omega)\Big) = -\text{Lim}_{T\to+\infty} \sum_{\tau=1}^{T} \beta_2^{\tau} \cdot \bar{c}_{2,\tau}.$$

Denote such a value $\widehat{c}_{2,0}(\omega)$ and note that $\beta_2^T \cdot u_2'(\widehat{c}_{2,T}) \cdot \widehat{q}_T \cdot \widehat{\theta}_{2,T} = -\sum_{\tau=T+1}^{\infty} \beta_2^{\tau} \cdot \bar{c}_{2,\tau}$ a deterministic quantity.

We turn to agent 1. Since the regardless of the value of $c_{2,0}$, the proposed solution does not waste resources, the asset holdings are the ones that support the consumption allocation, and the asset is in zero net supply, it follows that $\theta_{1,t}(\omega) = -\theta_{2,t}(\omega)$ for all $t \geq 0$ and P-a.s ω. So we have

$$\text{Lim}_{T\to+\infty} E[\beta_1^T \cdot u_1'(c_{1,T}) \cdot q_T \cdot \theta_{1,T}|\mathcal{F}_t](\omega) = -\text{Lim}_{T\to+\infty} E[\beta_1^T \cdot u_1'(c_{1,T}) \cdot q_T \cdot \theta_{2,T}|\mathcal{F}_t](\omega).$$

Since $\hat{r}_{2,t}(\omega) = 1$,

$$\frac{\hat{r}_{1,t}(\omega)}{\hat{r}_{2,t}(\omega)} = \frac{y_{t-1}(\omega)}{y_t(\omega)} \quad \Rightarrow \quad \hat{r}_{1,t}(\omega) = \frac{\frac{u_1'(c_{1,t}(\omega))}{u_2'(c_{2,t}(\omega))}}{\frac{u_1'(c_{1,t-1}(\omega))}{u_2'(c_{2,t-1}(\omega))}} \quad \Rightarrow \quad \frac{u_1'(c_{1,\tau}(\omega))}{u_2'(c_{2,\tau}(\omega))} = \Pi_{s=1}^{\tau}[\hat{r}_{1,s}(\omega)] \cdot \frac{u_1'(c_{1,0}(\omega))}{u_2'(c_{2,0}(\omega))}.$$

It follows that

$$\text{Lim}_{T\to+\infty} E[\beta_1^T \cdot u_1'(c_{1,T}) \cdot q_T \cdot \theta_{1,T}|\mathcal{F}_t](\omega)$$

$$= -\text{Lim}_{T\to+\infty} E[\beta_1^T \cdot \Pi_{s=1}^T[\hat{r}_{1,s}] \cdot \frac{u_1'(c_{1,0})}{u_2'(c_{2,0})} \cdot u_2'(c_{2,T}) \cdot q_T \cdot \theta_{2,T}|\mathcal{F}_t](\omega)$$

$$= -\text{Lim}_{T\to+\infty} \frac{u_1'(c_{1,0})}{u_2'(c_{2,0})} E[\Pi_{s=1}^T[\hat{r}_{1,s}] \cdot \beta_1^T \cdot u_2'(c_{2,T}) \cdot q_T \cdot \theta_{2,T}|\mathcal{F}_t](\omega).$$

But with the value $\widehat{c}_{2,0}$ and the induced consumption processes we have

$$\beta_2^T \cdot u_2'(\widehat{c}_{2,T}) \cdot \widehat{q}_T \cdot \widehat{\theta}_{2,T} = -\sum_{\tau=T+1}^{\infty} \beta_2^\tau \cdot \bar{c}_{2,\tau}$$

so that

$$\text{Lim}_{T\to+\infty} E[\beta_1^T \cdot u_1'(\widehat{c}_{1,T}) \cdot \widehat{q}_T \cdot \widehat{\theta}_{1,T}|\mathcal{F}_t](\omega)$$

$$= -\text{Lim}_{T\to+\infty} \frac{u_1'(\widehat{c}_{1,0})}{u_2'(\widehat{c}_{2,0})} \Big(-\sum_{\tau=T+1}^{\infty} \beta_2^\tau \cdot \bar{c}_{2,\tau} \Big) \cdot E[\Pi_{s=1}^T[\hat{r}_{1,s}]|\mathcal{F}_t](\omega) = 0$$

where we use the fact that $E\,[\hat{r}_{i,t}|\mathcal{F}_{t-1}](\omega) = 1$ together with the law of iterated expectations and the fact that $\text{Lim}_{T\to\infty} \sum_{\tau=T+1}^{\infty} \beta_2^\tau \cdot \bar{c}_{2,\tau} = 0$. ■

PROOF OF THEOREM 4

The proof uses A.4, which imposes a bound on the coefficient of relative risk aversion. It is based on showing first, Lemma 23, that for the allocation identified in Theorem 3, the value of excess demand evaluated using the personalized Arrow-Debreu price process of each agent is monotone in a single parameter; furthermore, the value is continuous and has the right boundary behaviour. We then show how one can start our construction from date 1, choose consumption at date 0 so as to be compatible with feasibility and the date 0 Euler equation for each agent, and yet preserve the monotonicity and continuity properties, Lemma 24. Lemma 25 provides a very simple sufficient condition for a fixed point property to hold. Finally, in Lemma 26 we show that if we start with a no trade equilibrium then there is a robust method for perturbing the endowment distribution that leads to the satisfaction of the sufficient condition specified in Lemma 25.

Throughout we write $E[X]$ instead of $E_P[X]$.

Consider a value for c^0, where $0 < c^0 < Z_0$ so that $c_{0,2} := Z_0 - c^0$ satisfies nonnegativity, and consider c^1, where $0 < c^1(\omega) < Z_1(\omega)$, a nonnegative $\mathcal{F}_1-$measurable function. By Proposition 4 we can induce a consumption process $\{C_{i,t}(c^1(\omega), 1, \omega)\}_{t\geq 1}$ for agent i where the process is defined $P-$a.s. only for $\tilde{\omega} \in \Omega(s^1(\omega))$. By varying ω, one obtains an aggegate feasible consumption process on the full state space.

For $\omega \in \Omega(s^1)$ define

$$V_{1,s^1}(c^1; z_1) := \text{lim}_{T\to+\infty} E\Big[\sum_{\tau=1}^{T} \beta_1^\tau \cdot u_1'(C_{1,\tau}(c^1(\omega), 1, \omega)) \cdot \Big(C_{1,\tau}(c^1(\omega), 1, \omega) - z_{1,\tau}\Big)\Big|\Omega(s^1)\Big](\omega),$$

$$V_{2,s^1}(c^1; z_1) := \text{lim}_{T\to+\infty} E\Big[\sum_{\tau=1}^{T} \beta_2^\tau \cdot u_2'(Z_\tau - C_{1,\tau}(c^1(\omega), 1, \omega)) \cdot \Big(C_{1,\tau}(c^1(\omega), 1, \omega) - z_{1,\tau}\Big)\Big|\Omega(s^1)\Big](\omega).$$

LEMMA 23: Assume A.1-6. Then, for $i = 1,2$ and all $s^1 = 1, \cdots, S$, $V_{i,s^1}(c^1; z_1)$ is (i) well defined, (ii) it is continuous in c^1 for every value of z_1, (iii) it is continuous in z_1 for every value of c^1, (iv) it is increasing in $c^1(\omega)$ where $\omega \in \Omega(s^1)$, and (v) for $\omega \in \Omega(s^1)$,
(a) $V_{1,s^1}(c^1; z_1) \to_{c^1(\omega)\to 0} -\infty$,
(b) $V_{1,s^1}(c^1; z_1) \to_{c^1(\omega)\to Z_1(\omega)} V_{1,s^1}(Z_1; z_1)$ where $V_{1,s^1}(Z_1; z_1) \in (0, \infty)$,
(c) $V_{2,s^1}(c^1; z_1) \to_{c^1(\omega)\to 0} V_{2,s^1}(0; z_1)$ where $V_{2,s^1}(0; z_1) \in (-\infty, +\infty)$, and
(d) $V_{2,s^1}(c^1; z_1) \to_{c^1(\omega)\to Z_1(\omega)} \infty$.
PROOF: Define

$$f^T_{1,s^1}(c^1; z_1) := E\Big[\sum_{\tau=1}^{T} \frac{\beta_1^\tau \cdot u_1'(C_{1,\tau}(c^1(\omega), 1, \omega))}{\beta_1 \cdot u_1'(c^1(\omega))} \cdot \Big(C_{1,\tau}(c^1(\omega), 1, \omega) - z_{1,\tau}\Big)\Big|\Omega(s^1)\Big](\omega).$$

We shall use the fact that $V_{1,s^1}(c^1; z_1)$ can be written as

$$V_{1,s^1}(c^1; z_1) = \beta_1 \cdot u_1'(c^1(\omega)) \cdot \lim_{T\to+\infty} f^T_{1,s^1}(c^1; z_1),$$

where $\beta_1 \cdot u_1'(c^1(\omega))$ is finite since $c^1(\omega) > 0$; a similar result holds for $V_{2,s^1}(c^1; z_1)$ since $c^1(\omega) < Z_1(\omega)$.
(i) By Proposition 5, the support price process is summable. By A.2, the individual endowent process is uniformly bounded. It follows that

$$0 \le \lim_{T\to+\infty} E\Big[\sum_{\tau=1}^{T} \frac{\beta_1^\tau \cdot u_1'(C_{1,\tau}(c^1(\omega), 1, \omega))}{\beta_1 \cdot u_1'(c^1(\omega))} \cdot z_{1,\tau}\Big|\Omega(s^1)\Big](\omega) < \infty.$$

Since the consumption process induced is aggregate feasible, we also have

$$0 < \lim_{T\to+\infty} E\Big[\sum_{\tau=1}^{T} \frac{\beta_1^\tau \cdot u_1'(C_{1,\tau}(c^1(\omega), 1, \omega))}{\beta_1 \cdot u_1'(c^1(\omega))} \cdot C_{1,\tau}(c^1(\omega), 1, \omega)\,\Big|\Omega(s^1)\Big](\omega) < \infty.$$

It follows that the difference between the two quantities is finite. By using the fact that $V_{1,s^1}(c^1; z_1) = \beta_1 \cdot u_1'(c^1(\omega)) \cdot \lim_{T\to+\infty} f^T_{1,s^1}(c^1; z_1)$, and the fact that $\beta_1 \cdot u_1'(c^1(\omega))$ is finite, since $c^1(\omega) > 0$, we conclude that $V_{1,s^1}(c^1; z_1)$ is finite. An analogous proof shows that $V_{2,s^1}(c^1; z_1)$ is finite.
(ii) We shall show that $f^T_{1,s^1}(c^1; z_1)$ is a continuous function of c^1 for every T, and that $f^T_{1,s^1}(c^1; z_1) \to V_{1,s^1}(c^1; z_1)$ uniformly. It follows that $V_{1,s^1}(c^1; z_1)$ is continuous in c^1. An analogous argument works for $V_{2,s^1}(c^1; z_1)$.

By the continuity result in Proposition 4 (iv), for every T, $f^T_{1,s^1}(c^1; z_1)$ is continuous in c^1. Furthermore

$$\sup_{c^1(\omega)\in(0,Z_1(\omega))} \Big|f^T_{1,s^1}(c^1; z_1) - \lim_{T\to+\infty} f^T_{1,s^1}(c^1; z_1)\Big|$$

$$= \sup_{c^1(\omega)\in(0,Z_1(\omega))} \Big| - \lim_{t\to+\infty} E\Big[\sum_{\tau=1}^{t} \frac{\beta_1^{T+\tau} \cdot u_1'(C_{1,T+\tau}(\cdot)}{\beta_1 \cdot u_1'(c^1(\omega))} \cdot \Big(C_{1,T+\tau}(\cdot) - z_{1,T+\tau}\Big)\Big|\Omega(s^1)\Big](\omega)\Big|$$

$$= \sup_{c^1(\omega)\in(0,Z_1(\omega))} \beta^T \cdot \Big| - \lim_{t\to+\infty} E\Big[\sum_{\tau=1}^{t} \frac{\beta_1^{\tau} \cdot u_1'(C_{1,T+\tau}(\cdot)}{\beta_1 \cdot u_1'(c^1(\omega))} \cdot \Big(C_{1,T+\tau}(\cdot) - z_{1,T+\tau}\Big)\Big|\Omega(s^1)\Big](\omega)\Big|$$

$$\leq \frac{\beta_1^T \cdot 2\bar{z}}{1-\beta_1 M}$$

where we use the fact that the supporting price process is summable, Proposition 5, and the fact that the net trade process is uniformly bounded by 0 and $2\bar{z}$. It follows that

$$\lim_{T\to+\infty} \sup_{c^1(\omega)\in(0,Z_1(\omega))} \left| f_{1,s^1}^T(c^1;z_1) - \lim_{T\to+\infty} f_{1,s^1}^T(c^1;z_1)\right| \leq \lim_{T\to+\infty} \frac{\beta_1^T \cdot 2\bar{z}}{1-\beta_1 M} = 0.$$

Now we use the fact that $V_{1,s^1}(c^1;z_1) = \beta_1 \cdot u_1'(c^1(\omega)) \cdot \lim_{T\to+\infty} f_{1,s^1}^T(c^1;z_1)$, where $\beta_1 \cdot u_1'(c^1(\omega))$ is continuous since u_i is continuously differentiable. It follows that $f_{1,s^1}^T(c^1;z_1) \to V_{1,s^1}(c^1;z_1)$ uniformly.
(iii) Given c^1, $V_{i,s^1}(c^1;\cdot)$ is linear in z_1; by Proposition 5, A.2, and the fact noted at the beginning of the proof, it is bounded. It follows that it is continuous in z_1.
(iv) We note two facts. First, each term in each sum is increasing in the value $C_{1,\tau}(c^1(\omega),1,\omega)(\tilde{\omega})$. To see this, notice that by A.4, $c \cdot u_i''(c) + u_i'(c) > 0$ for all $c > 0$ so that, using concavity, we have $c \cdot u_i''(c) + u_i'(c) - Z \cdot u_i''(c) > 0$ for $Z > 0$. It follows that $(c - Z) \cdot u_1'(c)$ is increasing in c. Similarly, $(Z-c)\cdot u_i''(Z-c) + u_i'(Z-c) > 0$ for all $0 < c < Z$ so that, using concavity, we have $-(c-Z)\cdot u_i''(Z-c) + u_i'(Z-c) + (-Z+z_1)\cdot u_i''(Z-c) > 0$ for all $0 < c < Z$ and $0 < z_1 \leq Z$. Therefore, $-(c-z_1)\cdot u_i''(Z-c) + u_i'(Z-c) > 0$ and, for $0 < c < Z$, $(c-z_1)\cdot u_1'(Z-c)$ is increasing. Evidently, $Z_t \geq z_{1,t} > 0$ since the individual endowment is always nonnegative.

Now recall that the construction in Proposition 4 has the property that $C_{1,\tau}(c^1(\omega),1,\omega)(\tilde{\omega})$ is increasing in $c^1(\omega)$. Invoking the monotonicity property of each term that we just established, we can conclude that V_{1,s^1} is increasing. By the same argument, V_{2,s^1} is increasing.
(v) Since we have already established monotonicity, the limits are well defined though they could be $+\infty$ or $-\infty$. Using the fact at the beginning of the proof, Proposition 5, and A.2, we conclude that a truncation argument can be used to establish the limiting values. Such a truncation argument allows us to use the boundary properties of the construction established in Proposition 4 (v) and (vi).

For (a) notice that for a fixed T we can find $\epsilon > 0$ such that $z_{1,t}(\tilde{\omega}) > \epsilon$ for all $1 \leq t \leq T$ and $P-$a.s. $\tilde{\omega} \in \Omega(s^1(\omega))$. The result follows by applying Proposition 4 (vi) using the Inada condition for $i = 1$, and A.1. For (b) we use the fact at the beginning of the proof and the fact that $u_1'(Z_1(\omega)) < \infty$ to conclude that the limit is positive and finite. For (c) we use the fact at the beginning of the proof and the fact that $u_2'(Z_1(\omega)) < \infty$ to conclude that the limit is finite without being able to assign a sign to it. For (d) we use Proposition 4 (v) and the Inada condition for $i = 2$. ∎

If the processes constructed with c^0, where $0 < c^0 < Z_0$, and $\{C_{i,t}(c^1(\omega),1,\omega)\}_{t\geq 1}$, where $0 < c^1(\omega) < Z_1(\omega)$ an $\mathcal{F}_1-$measurable function, also satisfy (i) the Euler equation at date 0 for both the agents, and (ii) the Arrow-Debreu budget constraint for both the agents, then we have a TC0 equilibrium. This follows from the fact that the processes constructed in Proposition 4 are feasible and satisfy the Euler equations at every date

$t \geq 1$. So the allocation chosen is an equilibrium if the following equations hold

$$\beta_1 \cdot \frac{E[r_1 \cdot u_1'(c^1)|\mathcal{F}_0](\omega)}{u_1'(c^0(\omega))} = \beta_2 \cdot \frac{E[r_1 \cdot u_2'(Z_1 - c^1)|\mathcal{F}_0](\omega)}{u_2'(Z_0(\omega) - c^0(\omega))} \qquad P-\text{a.s.}\,\omega,$$

$$u_1'(c^0(\omega)) \cdot \Big(c^0(\omega) - z_{1,0}(\omega)\Big) + \sum_{s^1 \in \mathcal{S}} P(\Omega(s^1)) \cdot V_{1,s^1}(c^1; z_1) = 0,$$

$$u_2'(Z_0(\omega) - c^0(\omega)) \cdot \Big(c^0(\omega) - z_{1,0}(\omega)\Big) + \sum_{s^1 \in \mathcal{S}} P(\Omega(s^1)) \cdot V_{2,s^1}(c^1; z_1) = 0.$$

Evidently, all three equations hold at the no trade equilibrium when the endowment distribution is given by (z_1^*, z_2^*).

Let us first consider the Euler equations at date 0.

LEMMA 24: Assume A.3 and A.5. Let $Z_0(\omega) > 0$ and $Z_1 : \Omega \to R_{++}$ be an $\mathcal{F}_1-$measurable function. Then for any $c^1 : \Omega \to R_{++}$, an $\mathcal{F}_1-$measurable function such that $c^1(\omega) < Z_1(\omega)$ for all $\omega \in \Omega$, there is a real number $f(c^1)$, with $0 < f(c^1) < Z_0(\omega)$ such that

$$\beta_1 \cdot \frac{u_2'(Z_0(\omega) - f(c^1))}{u_1'(f(c^1))} = \beta_2 \cdot \frac{E[r_1 \cdot u_2'(Z_1 - c^1)|\mathcal{F}_0](\omega)}{E[r_1 \cdot u_1'(c^1)|\mathcal{F}_0](\omega)} \qquad P-\text{a.s.}\,\omega.$$

Furthermore, the function f is strictly increasing in all of its components.

PROOF: The result follows easily from the intermediate value theorem. The right hand side of the equation is always well defined and positive, while Lemma 8 guarantees that the left hand side is continuous and has $(0, \infty)$ as its image; a solution necessarily exists.

The monotonicity property of the function f follows from the fact that asset returns are strictly positive, and the u_is are strictly increasing and strictly concave. ■

It follows that it suffices to consider a reduced system where the Euler equation is considered in implicit form. So define

$$F_1(c^1; z_1) := u_1'(f(c^1)) \cdot \Big(f(c^1) - z_{1,0}(\omega)\Big) + \sum_{s^1 \in \mathcal{S}} P(\Omega(s^1)) \cdot V_{1,s^1}(c^1; z_1),$$

$$F_2(c^1; z_1) := u_2'(Z_0(\omega) - f(c^1)) \cdot \Big(f(c^1) - z_{1,0}(\omega)\Big) + \sum_{s^1 \in \mathcal{S}} P(\Omega(s^1)) \cdot V_{2,s^1}(c^1; z_1).$$

We have shown that a TC0 equilibrium is induced at the endowment distribution (z_1, z_2) if c^{1*} is such that $F_i(c^{1*}; z_1) = 0$ for $i = 1, 2$.

LEMMA 25: Assume A.1-6. Let the endowment distribution (z_1, z_2) and $\widehat{c}^1$ be such that $F_1(\widehat{c}^1; z_1) \geq 0$ and $F_2(\widehat{c}^1; z_1) \leq 0$. Then there exists c^{1*}, an $\mathcal{F}_1-$measurable function such that $0 < c^{1*}(\omega) < Z_1(\omega)$, that satisfies $F_i(c^{1*}; z_1) = 0$ for $i = 1, 2$.

PROOF: The range of the function $\widehat{c}^1$ has at most S values that correspond to the sets $\Omega(s^1)$. Fix all but those that correspond to $s^1 = 1, 2$, and denote those two $\widehat{c}^1_a$ and $\widehat{c}^1_b$.

By Lemma 24 and Lemma 23 (iv), the first term in the expression for F_1 is increasing in each component of the function c^1; it follows that it is also bounded above. By Lemma 23 (iv), the second term in the expression for F_1 is increasing in the corresponding component of c^1. So F_1 is increasing in each component of the function c^1 and $F_1 \to -\infty$ as $c^1_a \to 0$. By an analogous argument, F_2 is increasing in each component of the function c^1 and satisfies the following boundary properties: $F_2 \to \infty$ as $c^1_a \to Z_{1,a}$ and in the vicinity of $(Z_{1,a}, 0)$, $F_2(c^1; z_1) > 0$.

In what follows, c_1 will always be a vector of the form $(c^1_a, c^1_b, \cdots, \hat{c}^1_S)$.

If $F_1(c^1; z_1) \geq 0$, then, by the monotonicity and boundary properties noted earlier, there exists a unique $\tilde{c}^1$, where $\tilde{c}^1_a = c^1_a$ and $\tilde{c}^1_b < c^1_b$, such that $F_1(\tilde{c}^1; z_1) = 0$. We introduce the notation $h_1(c^1_a)$ to denote the value $\tilde{c}^1_b$; the monotonicity property of F_1 guarantees that the function h_1 with domain $[\hat{c}^1_a, Z_{1,a}]$, where $Z_{1,a}$ denotes the aggregate endowment at date 1 in the event that corresponds to the label a, is well defined and strictly decreasing and, by the continuity property, h_1 is continuous. Furthermore, by the boundary property of F_1 we have $h_1(c^1_a) \to_{c^1_a \to Z_{1,a}} \underline{h}_1 > 0$.

By a related argument the symmetric result holds for any c^1 at which $F_2(c^1; z_1) \leq 0$. Since $F_2(\hat{c}^1; z_1) \leq 0$ and F_2 is monotone, there exists $\hat{\hat{c}}^1_a > \hat{c}^1_a$ such that $F_2((\hat{\hat{c}}^1_a, \hat{c}^1_b, \cdots, \hat{c}^1_S); z_1) = 0$. It follows that we can define a continuous function h_2 with domain $[\hat{\hat{c}}^1_a, \bar{c}^1_a]$, where $\bar{c}^1_a < Z_{1,a}$, that is strictly decreasing and satisfies the boundary property $h_2(c^1_a) \to_{c^1_a \to \bar{c}^1_a} 0$. Also, $h_2(\hat{\hat{c}}^1_a) > h_1(\hat{\hat{c}}^1_a)$.

It is evident that there is a value of c^{1*}_a at which $h_1(c^{1*}_a) = h_2(c^{1*}_a)$ so that $F_i(c^{1*}; z_1) = 0$ for $i = 1, 2$. ∎

Lemma 25 together with A.7 provide a sufficient condition under which a TC0 equilibrium exists in which agent 2 vanishes with probability one. We now show that the sufficient condition holds for an open set of endowment distributions near a no trade equilibrium at the endowment distribution (z^*_1, z^*_2).

LEMMA 26: Assume A.1-7. There exists $\mathcal{N}(z^*_1)$ an open subset of $Z_1(Z)$ such that for every (z_1, z_2), where $z_1 \in \mathcal{N}(z^*_1)$ and $z_2 := Z - z_1$, there exists a TC0 equilibrium.
PROOF: Fix $\tilde{s} \in \mathcal{S}$ and define $\tilde{s}^1 := (s_0, \tilde{s})$. Given $(\eta_1, \eta_2) \in R^2$, define

$$\epsilon(\eta_1, \eta_2; \omega) := \frac{\eta_1 \cdot u'_2(z^*_{2,0}(\omega)) - \eta_2 \cdot u'_1(z^*_{1,0}(\omega))}{P(\Omega(\tilde{s}^1))[\beta_1 \cdot u'_1(z^*_{1,1}(\omega)) \cdot u'_2(z^*_{2,0}(\omega)) - \beta_2 \cdot u'_2(z^*_{2,1}(\omega)) \cdot u'_1(z^*_{1,0}(\omega))]}$$

$$\epsilon'(\eta_1, \eta_2; \omega) := \frac{\eta_1}{u'_1(z^*_{1,0}(\omega))} - \frac{P(\Omega(\tilde{s}^1)) \cdot \beta_1 \cdot u'_1(z^*_{1,1}(\omega))}{u'_1(z^*_{1,0}(\omega))} \cdot \epsilon(\eta_1, \eta_2; \omega).$$

It is easy to check that

$$\eta_i = u'_i(z^*_{i,0}(\omega)) \cdot \epsilon'(\eta_1, \eta_2; \omega) + P(\Omega(\tilde{s}^1)) \cdot \beta_i \cdot u'_i(z^*_{i,1}(\omega)) \cdot \epsilon(\eta_1, \eta_2; \omega) \quad \text{for } i = 1, 2.$$

Now define a new endowment process $(\tilde{z}^*_1, \tilde{z}^*_2)$ by the rule

$$\tilde{z}^*_{1,0}(\omega) = z^*_{1,0}(\omega) - \epsilon'(\eta_1, \eta_2; \omega) \qquad \text{for } \omega \in \Omega(\tilde{s}^1)$$

$$\tilde{z}^*_{1,1}(\omega) = z^*_{1,1}(\omega) - \epsilon(\eta_1, \eta_2; \omega) \qquad \text{for } \omega \in \Omega(\tilde{s}^1)$$

$$\tilde{z}^*_{1,t}(\omega) = z^*_{1,t}(\omega) \qquad \text{otherwise.}$$

$\tilde{z}^*_2$ is obtained through the condition $\tilde{z}^*_1 + \tilde{z}^*_2 = Z$ so that $\tilde{z}^*_1 + \tilde{z}^*_2 = z^*_1 + z^*_2 = Z$. By choosing $\eta_1 > 0$ and $\eta_2 < 0$ appropriately we can induce values of $\epsilon(\eta_1, \eta_2; \omega)$ and $\epsilon'(\eta_1, \eta_2; \omega)$ that are sufficiently small so that $\tilde{z}^*_{i,t}(\omega) \geq 0$ for both the agents at every t and ω.

It follows that $F_1(z^*_1; \tilde{z}^*_1) = \eta_1 > 0$ and $F_2(z^*_1; \tilde{z}^*_1) = \eta_2 < 0$. So the condition in Lemma 25 is satisfied and the economy has a TC0 equilibrium where agent 2 vanishes with probability one since A.7 also holds. By Lemma 23 (iii) $F_i(c^1; \cdot)$ is continuous in z_1. It follows that there exists $\mathcal{N}$, where $\tilde{z}^*_1 \in \mathcal{N}$, an open subset of $Z_1(Z)$, such that for every (z_1, z_2), where $z_1 \in \mathcal{N}$ and $z_2 := Z - z_1$, there exists a TC0 equilibrium in which agent 2 dies with probability one. The proof is completed by setting $\mathcal{N}(z^*_1) := \mathcal{N}$. ∎

That completes the proof of Theorem 4. ∎

REFERENCES

ALCHIAN, A. (1950) "Uncertainty, Evolution, and Economic Theory," *Journal of Political Economy* 58, 211-221.

BECKER, R. (1980) "On the Long Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," *Quarterly Journal of Economics*, 45, 375-382.

BEWLEY, T. (1982) "An Integration of Equilibrium Theory and Turnpike Theory," *Journal of Mathematical Economics*, 10, 233-267.

BLUME, L., AND D. EASLEY (1992) "Evolution and Market Behaviour," *Journal of Economic Theory*, 58, 9-40.

BLUME, L., AND D. EASLEY (2004) "If You Are So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets," Working Paper, Cornell University.

CHUNG, K. L. (1974) "A Course in Probability Theory," Academic Press, London.

CONSTANTINIDES, G., AND D. DUFFIE (1996) "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104, 219-240.

COOTNER, P. (1967) "The Random Character of Stock Prices," MIT Press, Cambridge, MA.

COURY, T., AND E. SCIUBBA (2005) "Belief Heterogeneity and Survival in Incomplete Markets, " mimeo (March).

DUFFIE, D., J.D. GEANAKOPLOS, A. MAS-COLELL, AND A. MCLENNAN (1994) "Stationary Markov Equilibria," *Econometrica*, 62, 745-781.

FAMA, E. (1965) "The Behavior of Stock Market Prices," *Journal of Business*, 38 (1), 34-105.

FLORENZANO, M., AND P. GOURDEL (1996) "Equilibrium in Incomplete Markets in Infinite Horizon: Debt Constraints versus Node Constraints," *Mathematical Finance* 6, 167-196.

FREEDMAN, D. (1973) "Another Note on the Borel-Cantelli Lemma and the Strong Law, with the Poisson approximation a by-product ," *Annals of Probability*, 1, 910-925.

FRIEDMAN, M. (1953) "Three Essays on the State of Economic Science," University of Chicago Press, Chicago.
HERNANDEZ, A., AND M. SANTOS (1996) "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets," *Journal of Economic Theory*, 71, 102-130.
KREBS, T. (2004) "Testable Implications of Consumption-Based Asset Pricing Models with Incomplete Markets," *Journal of Mathematical Economics*, 40, 191-206.
KUBLER, F., AND K. SCHMEDDERS (2002) "Recursive Equilibria in Economies with Incomplete Markets," *Macroeconomic Dynamics*, 6, 284-306.
LEVINE, D., AND W. ZAME (1996) "Debt Constraint and Equilibrium in Infinite Horizon Economies with Incomplete Markets," *Journal of Mathematical Economics*, 26, 103-131.
LEVINE, D., AND W. ZAME (2001) "Does Market Incompleteness Matter?," *Econometrica*, 70, 1805-1840.
LUENBERGER, D. (1969) "Optimization by Vector Space Methods," New York: Wiley.
MAGILL, M., AND M. QUINZII (1994) "Infinite Horizon Incomplete Markets," *Econometrica*, 62, 853-880.
RADER, T. (1981) "Utility over Time: The Homothetic Case," *Journal of Economic Theory*, 25, 219-236.
RAMSEY, F. (1928) "A Mathematical Theory of Saving," *Economic Journal*, 38, 543-559.
SANDRONI, A. (2000) "Do Markets Favour Agents Able to Make Accurate Predictions," *Econometrica*, 68, 1303-1341.
SANDRONI, A. (2005) "Market Selection when Markets Are Incomplete," *Journal of Mathematical Economics*, 41, 91-104.
SANTOS, M., AND M. WOODFORD (1997) "Rational Asset Pricing Bubbles," *Econometrica*, 65, 19-57.