ON CHOOSING WHICH GAME TO PLAY WHEN IGNORANT OF THE RULES

MURALI AGASTYA[1]
Economics Discipline, SEPS, University of Sydney

This version: 8 October, 2005

ABSTRACT. This paper suggests a theory of choice among strategic situations when the rules of play are not properly specified. We take the view that a "strategic situation" is adequately described by a TU game since it specifies what is feasible for each coalition but is silent on the procedures that are used to allocate the surplus. We model the choice problem facing a decision maker (DM) as having to choose from finitely many "actions". The known "consequence" of the ith action is a coalition from game f_i over a fixed set of players $N_i \cup \{d\}$ (where d stands for the DM). Axioms are imposed on her choice as the list of consequences $(f_1, \ldots, f_m)$ from the m actions varies.

We characterize choice rules that are based on marginal contributions of the DM in general and on the Shapley Value in particular.

JEL-CLASSIFICATION: C79, D89

KEYWORDS: Unstructured strategic interaction, coalition form games, Individual Decision Making, Shapley Value, Marginal Contributions

Correspondence: Murali Agastya
Economics Department
University of Sydney
Sydney NSW 2006
AUSTRALIA
m.agastya@econ.usyd.edu.au
Phone: +61 2 93513071
Fax: +61 2 93514341

[1]It is a pleasure to thank Ken Binmore, Tilman Börgers, Menahim Yaari and Uzi Segal for helpful comments on early versions of this paper.
This draft is completed during my visit to HSS, California Institute of Technology. I am grateful for their hospitality.
This project is funded under ARC Discovery Project DP0210019

1. Introduction

Consider an agent that must choose to interact with one of several groups. It may be a firm deciding which foreign market to enter or a lawyer weighing the pros and cons of entering into a partnership with different potential candidate firms. In many such situations, the eventual payoff of the agent is the outcome of a strategic interaction that unfolds following her choice. Non-cooperative game theory offers a potent framework to phrase (and answer) such questions, *provided* the rules of the game properly specified and players know them.

Knowledge of the rules is sometimes a valid assumption, especially if the conflict situation is embedded within a well defined legal framework. At other times however, players get to know the rules only when they are already in the conflict situation or perhaps learn them after repeated play. But often, as experiences with experimental subjects show, even when informed of the rules, individuals find it difficult to understand them immediately. The question is then, how are consistent choices *among* strategic situations made without a complete knowledge of the rules at the time of making the choice.

Note that it is also not enough to assume approximate knowledge of the rules. For, it is well known that equilibrium outcomes (and payoffs) in a game are very sensitive to the precise specification of the rules of play: intuitively small changes to the extensive form can lead to substantial changes in equilibrium payoffs. Therefore, if one assumes that individuals make consistent choices, it is important to understand how a decision maker might form an ex-ante evaluation of a conflict situation. This paper is a contribution towards such an understanding.

To begin, how might one formally represent a strategic situation? Indeed, is it not the rules that determine the strategic content of any conflict. If so, one cannot speak of a strategic situation under ignorance of the rules! Nevertheless, we contend that it is sometimes possible to speak of strategic interaction even without knowing the rules. Indeed, suppose that we specify a set of players and the size of the pie that the players can share if they strike a bargain. That would then be a description of a conflict situation even though the bargaining process that determines the individual allocation of the pie remains unspecified.

In fact, in this paper, we take the view that a game in *coalition form* can adequately represent the unstructured interaction between the players. A coalition form game consists of a finite set of players, say N and a real valued function, say f, whose value $f(S)$ at a coalition $S \subseteq N$ is the total surplus that players in S can avail for themselves should they come to an agreement. In other words, a coalition form specifies what it is feasible for different players but does not specify the process for arriving at the individual allocations.

In the model we consider in the next section, a decision maker (DM) must choose from m "actions". If she chooses the action $i \in \{1, \ldots, m\}$, the "consequence" is a coalition from game f_i over a fixed set of players $N_i \cup \{d\}$ (where d stands for the DM). In what follows, we refer to f_i as a "strategic situation" and not a TU game since in our interpretation, the rules are not specified. At the time of making her choice, she is fully aware of the profile of SS, denoted by $F = (f_1, \ldots, f_m)$ – which we shall refer to as a *choice situation*. Let $\mathbb{G}$ denote a set of choice situations.

Taking $\mathbb{G}$ as a given, the DM is fully described by a mapping $F \longmapsto C(F)$ where $C(F) \subseteq \{1, \ldots, m\}$. Interpret $i \in C(F)$ but $j \notin C(F)$ to mean that

> "the DM likes the prospect of being in the strategic situation f_i to the prospect of being in the strategic situation f_j".

The methodology of this paper involves studying properties of $C(\cdot)$ under a set of axioms.

In Section 2, allowing F to vary over a sufficiently rich set, a relatively mild set of axioms are enough to show, for each i, the existence of a "utility function" $\theta_i(\cdot)$ such that $i \in C(F)$ if and only if $\theta_i(f_i) \geq \theta_j(f_j)$ for all F. This representation of $C(\cdot)$ is essentially Theorem 1. $\theta_i(f_i)$ may be regarded as the "ex-ante utility of playing the game f_i" – it is ex-ante since the DM is ignorant of the rules of the game.

It should be noted that the original motivation of the Shapley Value is that of an ex-ante evaluation of playing a game. Indeed, in the seminal contribution, Shapley (1953), it is asserted that

> "At the foundation of the theory of games is the assumption that players ... can evaluate in their own utility scales, every "prospect" that might arise as a result of play ... To apply

> the theory to any field, one would normally expect ... to include in the class of "prospects" the prospect of having to play a game ...
>
> ... "the value is best regarded as an a priori assessment of the situation based on either ignorance or disregard ... of the social organization ..."
>
> – Shapley (1953)

Yet, neither the spirit of the original construction nor any of the subsequent axiomatizations (see Roth (1977) and Agastya (1996) however) allow one to attribute the verbal motivation suggested by Shapley to the Value.

The framework outlined above on the other hand, is firmly entrenched in the standard decision theoretic methodology familiar in Economic theory: there is one DM who has to choose from a list of possibilities and axioms are placed on her choice as this list varies. Therefore our framework allows for an investigation of the Shapely Value in line with the original motivation: the term "a priori evaluation" is an evaluation of the strategic situation based on "ignorance ... of the social organization", i.e. an ignorance of the rules of conflict. A further purpose of this paper is therefore to exhibit conditions by which the function $\theta_i(\cdot)$ can be related to the Shapley Value. The importance of this exercise is underscored through the widespread use of Shapley Value as a solution concept in cooperative game theory and Economics[2].

With the above discussion in mind, in Section 3 we first introduce the "*marginality principle*". We say that $C(\cdot)$ is based on the marginality principle if $\theta_i(\cdot)$ is a linear combination of the marginal contributions of the DM to various coalitions in N_i. Recall that when $C(\cdot)$ is based the Shapley Value, then it certainly satisfies the marginality principle, but there can be many others. Theorem 2 characterizes $C(\cdot)$ that satisfy the marginality principle through an axiom that we label as "Strategic Equivalence". A further axiom "Consistency" is then shown to be necessary and sufficient for $C(\cdot)$ to based on the Shapley Value. This is Theorem 3.

Section 4 contains a brief discussion of the generality of the above results and concludes the paper.

[2]Hart and Moore (1990) use Shapley Value as the allocation rule within a firm. Stole and Zwiebel (1996) show how intrafirm bargaining can lead to this allocation rule. It is a central concept in the cost allocation literature.(See the references in Young (1988) for example, which also contains an alternative axiomatization of the value.)

Before we turn to the formal details, we draw the reader's attention to two related papers, namely Agastya (1996) and Roth (1977). This work shares its motivation with Agastya (1996) and certain insights from that construction. However, the framework of analysis is different. Here we deal with choice functions. There, n preference relations over the set of n player super-additive TU games are studied. Each of these preference relations correspond to the n roles that an individual may come to be in n player TU. Axioms are imposed that draw on the links between these preference relations and proves various representation theorems. Roth (1977) has a different motivation but can formally be regarded as the model in Agastya (1996) but one that requires lottery mixtures over games to conduct its analysis.

2. The Model

For $i = 1 \ldots, m$, N_i is a group consisting of n_i players, other than the Decision Maker (DM). The DM has to choose to play a game involving exactly one of these groups. As articulated in the introduction, the "game" involving N_i is given by a real valued function f_i defined on $2^{N_i \cup \{d\}} \setminus \emptyset$. We shall refer to f_i as a *strategic situation* (SS) involving N_i (and the DM). It is a strategic situation and not a game because no rules are specified. Let $\mathbb{G}_i$ be a set of SS over N_i and $\mathbb{G} = \mathbb{G}_1 \times \cdots \times \mathbb{G}_m$. As discussed in the Introduction, the DM is fully described by a choice correspondence $C : \mathbb{G} \longmapsto \{1, \ldots, m\}$. The theory of choice to be developed in the sequel involves characterizations of $C(\cdot)$ based on some axioms.

Observe that each f_i is in fact a $2^{n_i} - 1$ dimensional vector and therefore, in general, $\mathbb{G}_i$ is some subset of $\mathbb{R}^{2^{n_i}-1}$. Here, we assume $\mathbb{G} = \mathbb{R}_+^{2^{n_i}-1}$ so that $f_i(S) \geq 0$ for all non-empty $S \subseteq N_i \cup \{d\}$ although much of the analysis to follow can be generalized when $\mathbb{G}_i$ is a convex cone (of super-additive f_i). αf_i, the scalar multiplication of $f_i \in \mathbb{G}_i$, $\alpha \in \mathbb{R}$ and the addition of $f_i, f_i' \in \mathbb{G}_i$ are well defined concepts. We also assume throughout that $C(\cdot)$ is continuous in the sense that for every infinite sequence $\{F_k\} \subseteq \mathbb{G}$,

$$\left(i \in C(F_k) \ \forall k \quad \wedge \quad \lim_{k \to \infty} F_k = F^* \right) \quad \Rightarrow \quad i \in C(F^*)$$

It is important to remark that the analysis to follow assumes that $C(F) \neq \emptyset$ for all F. In other words, the DM prefers choosing to engage in one of the strategic situations to the status quo. It will be evident to the reader that this is not an important restriction. If such an assumption were not made, the Axioms to follow can be assumed to hold *in the event the DM chooses not to remain in the status quo.* As to when it is optimal not to remain in the status quo can then be deduced separately.

Define $\delta_i^* \in \mathbb{G}_i$ as

$$\delta_i^*(S) = \begin{cases} 1 & \text{if } d \in S \\ 0 & \text{otherwise}. \end{cases} \qquad S \subseteq N_i \cup \{d\}, S \neq \emptyset \tag{1}$$

In the SS $\alpha\delta_i^*$, the DM is uniquely responsible for the surplus of α. If the consequence of the DM's action were to be $\alpha\delta_i^*$, it appears reasonable for the DM to expect to get α. Our first axiom embodies this intuition together with the implicit assumption that a unit of "utile" received in a SS interaction from N_i translates to a unit of "utile" received in a SS over N_j.

Axiom 1 (Comparability). *Suppose* $F = (\alpha_1\delta_1^*, \ldots, \alpha_n\delta_m^*)$. *for some scalars* $\alpha_i \geq 0$ *for* $i = 1, \ldots, m$. *Then* $i \in C(F)$ *if and only if* $\alpha_i \geq \alpha_j$ *for all* $j = 1, \ldots m$.

Axiom 2 (Scale Invariance). $C(\lambda F) = C(F)$ *for all* $\lambda > 0$ *and* $F \in \mathbb{G}$.

Axiom 2 may be thought of in two ways. First, just as in Nash (1950), we might think of the worth of a coalition as the combined vNM utility from some underlying outcome. Since the vNM utility functional is unique only up to an affine transformation, F and λF represent the same strategic environments. Alternatively, one may simply view the Axiom as a heuristic procedure by an agent of limited rationality that seeks to simplify the choice problem into forming simple equivalence classes.

Definition 1 (Revealed Preference and Equivalance of SS). $F_{-i} \in \mathbb{G}_{-i}$ *is said to reveal that* $f_i \in \mathbb{G}_i$ *is preferred to* $f_i' \in \mathbb{G}_i$ *if* $i \in C(f_i, F_{-i})$ *and* $i \notin C(f_i', F_{-i})$.

f_i *is equivalent to* f_i' *if one cannot be revealed preferred to the other by any* $F_i \in \mathbb{G}_{-i}$.

Note that it is plausible that f_i is revealed preferred to f_i' by F_{-i} while the opposite is true at some other F_{-i}'. This might for example happen if two different groups correspond to firms which ultimately interact in the same product market. Axiom 3 and Axiom 4 below are based on the hypothesis that choosing i is mutually exclusive of anything that might happen in from the game involving N_j and thus rule out above mentioned possibilities. To exposition, first introduce the notation $\succ_i$ and $\sim_i$ where

$$f_i \succ_i f_i' \quad \Leftrightarrow \quad \exists\, F_{-i} \text{ that reveals } f_i \text{ is preferred to } f_i' \text{ and} \tag{2}$$

$$f_i \sim_i f_i' \quad \Leftrightarrow \quad f_i \text{ is equivalent to } F_i'. \tag{3}$$

Axiom 3 (Group Independence)**.** *For each $i = 1, \dots m$ and there does not exist a sequence $f_i^1, \dots, f_i^{k+1}$ in $\mathbb{G}_i$ such that $f_i^{k+1} = f_i^1$ and $f_i^1 \succ_i f_i^2 \succeq_i \cdots \succeq_i f_i^{k+1}$.*

Axiom 4 (Equivalence)**.** *Let $F = (f_1, \dots, f_m)$ and $F' = (f_1', \dots, f_m')$ be such that $f_i \sim_i f_i'$ for all $i = 1, \dots, m$. Then $C(F) = C(F')$.*

It is worth pointing out that Axiom 4 is a separate condition and not a consequence of the definition of $\sim_i$. For example, (assuming $m = 2$), without the above Axiom, it is impossible to conclude from $f_i \sim f_i'$ that $j \in C(f_i, f_j) \Rightarrow j \in C(f_i', f_j)$ for a $j \neq i$. Axiom 4 plays a key role in all our results.

Definition 2 (Null Player)**.** *The DM is said to a null player in f_i if $f_i(S \cup d) = f(S)$ for all $S \subseteq N_i$, $S \neq \emptyset$ and $f_i(d) = 0$.*

Let $\mathbb{N}_i$ denote the set of all $f_i \in \mathbb{G}_i$ in which the DM is a null player.

As a null player, the DM does not contribute to any coalition nor does she achieve anything on her own. The next axiom is (a mild form of) the assertion that a DM would prefer being in a SS in which she is essential for generation of surplus in at least one coalition to being a null player.

Axiom 5 (Nullity)**.** *Suppose $F = (f_1, \dots, f_n)$ is such that $f_i \in \mathbb{N}_i$ and $i \in C(F)$, then $f_j \in \mathbb{N}_j$ for all j and $C(F) = \{1, \dots, m\}$.*

It is worth pointing out Axiom 5 does not play an essential role in Theorem 1 although it is important for the results of Section 3 and Section 4. For Theorem 1 below, it would suffice to assume that for each i there exist F and F' such that $i \in C(F)$ but $i \notin C(F')$. It does play a crucial role for our remaining results and therefore we impose it at the outset.

Theorem 1. *The following statements are equivalent:*

(1) $C(\cdot)$ *satisfies Axiom 1 – Axiom5.*
(2) *For* $i = 1, \ldots, m$, *there exist a continuous* $\theta_i : \mathbb{G}_i \longrightarrow \mathbb{R}$ *that is a).* $\theta_i(\delta_i^*) = 1$, *b).* $\theta_i(f_i) = 0$ *if* $f_i \in \mathbb{N}_i$, *c). homogenous of degree one and c). for all* $F \in \mathbb{G}$,

$$i \in C(F) \quad \Leftrightarrow \quad \theta_i(f_i) \geq \theta_j(f_j) \qquad \forall j \tag{4}$$

For the sake of clarity, we collect some of the arguments used in the proof Theorem 1 in the following Lemma.

Lemma 1. *Under the hypotheses of Theorem 1, the following hold:*

(1) $\succeq_i$ *is complete, transitive and continuous.*
(2) *If the DM is a null player in* f_i' *but not in* f_i, *then* $f_i \succ_i f_i'$.
(3) *If the DM is a null player in* f_i *and* f_i', *then* $f_i \sim_i f_i'$.
(4) $\succeq_i$ *is homothetic, i.e.* $f_i \succeq f_i' \quad \Leftrightarrow \quad \lambda f_i \succeq_i \lambda f_i'$ *for all* $\lambda > 0$.

Proof of the above Lemma is given after the proof of Theorem 1.

Proof of Theorem 1. That (2) $\Rightarrow$ (1) is clear. We will not prove the converse. By Lemma 1, $\succeq_i$ is a continuous, complete and transitive preference relation on the non-negative orthant of an Euclidean space. Therefore it admits a continuous utility representation $U_i(\cdot)$. Due to Part 3, Lemma 1 we can choose the normalization that $U_i(f_i) = 0$ for any $f_i \in \mathbb{N}_i$. By Part 2, $U_i(f_i) > 0$ whenever $f_i \in \mathbb{G}_i \setminus \mathbb{N}_i$ and in particular, $U_i(\delta_i^*) > 0$.

By Part 4 of Lemma 1, $U_i(\cdot)$ may be assumed to be homogeneous of degree one. Setting $\theta_i(f_i) = U_i(f_i)/U_i(\delta_i^*)$, we have

$$\begin{aligned} U_i(f_i) &= \theta_i(f_i)U_i(\delta_i^*) \\ &= U_i(\theta_i(f_i)\,\delta_i^*) \end{aligned}$$

and therefore

$$f_i \sim_i \theta_i(f_i)\delta_i^* \tag{5}$$

Given $F = (f_1, \ldots, f_m)$ let $F^* = (\theta_1(f_1)\delta_1^*, \ldots, \theta_n(f_n)\delta_n^*)$. By Axiom 4, $C(F) = C(F^*)$. Conclude by Axiom 1 that $i \in C(F)$ if and only if $\theta_i(f_i) \geq \theta_j(f_j)$ for all j. □

Proof of Lemma 1. Proof of Part 1: Continuity of $\succeq_i$ follows directly from the continuity of $C(\cdot)$. Completeness is also immediate. To complete the proof Part 1, it remains to verify that $\succeq_i$ is transitive. There are essentially three cases to consider.

$$f_i \sim_i f_i' \quad \wedge \quad f_i' \sim f_i'' \quad \Rightarrow \quad f_i \sim_i f_i'' \tag{6}$$

$$f_i \succ_i f_i' \quad \wedge \quad f_i' \sim f_i'' \quad \Rightarrow \quad f_i \succ_i f_i'' \tag{7}$$

$$f_i \succ_i f_i' \quad \wedge \quad f_i' \succ f_i'' \quad \Rightarrow \quad f_i \succ_i f_i'' \tag{8}$$

Suppose (6). Then $C(f_i, F_{-i}) = C(f_i', F_{-i}) = C(f_i'', F_{-i})$ for all F_{-i}. Therefore $f_i \sim_i f_i''$ follows immediately. Next suppose (7). By definition, there exists a F_{-i} such that $i \in C(f_i, F_i)$ and $i \notin C(f_i', F_{-i})$. But since $f_i' \sim_i f_i''$, $C(f_i'', F_{-i}) = C(f_i', F_{-i})$ and as a consequence $i \notin C(f_i'', F_{-i})$. Consequently, $f_i \succ_i f_i''$. Finally, assume (8). Assume by way of contradiction that either $f_i \sim_i f_i''$ or $f_i'' \succ_i f_i$. If the former is the case, this is as in (7) and leads to the contradictory implication that $f_i' \succ_i f_i$. On the other hand if the latter is the case, the cyclical sequence f_i, f_i', f_i'', f_i violates Axiom 3.

Proof of Part 2: Let f_j^ε be a SS over N_j in which the DM is not a null player for every $\varepsilon > 0$ but $\lim_{\varepsilon \to 0} f_j^\varepsilon = f_j^*$ is a SS in which the DM is indeed a null player. By Axiom 5, we have the following:

$$\begin{aligned} C\left(f_i, f_j^*, F_{-i}\right) &= \{i\} \\ C\left(f_i, f_j^\varepsilon, F_{-i}\right) &\subseteq \{i, j\} \qquad \forall\, \varepsilon > 0 \\ C\left(f_i', f_j^\varepsilon, F_{-i}\right) &= \{j\} \qquad \forall\, \varepsilon > 0 \end{aligned}$$

It is clear from the above, that in order to satisfy continuity, for all ε sufficiently small, $C\left(f_i, f_j^\varepsilon, F_{-i}\right) = \{i\}$. Therefore, $f_i \succ_i f_i'$.

Proof of Part 3: If F_{-i} is such that d is null in each f_j, then $C(f_i, F_{-i}) = C(f_i', F_{-i}) = \{1, \ldots, m\}$ for all $i \in \{1, \ldots, m\}$. Suppose f_i, f_i' are both null for d. Suppose $i \in C(F)$. Then by Axiom 5, $C(F) = \{1, \ldots, m\}$ and d is null in every f_j for all $j = 1, \ldots, m$. At $F' = (f_i', F_{-i})$ it is still the case, by Axiom 5, that $C(F') = \{1, \ldots, m\}$. Therefore, $f_i \sim f_i'$.

Proof of Part 4: This is an immediate implication of Axiom 2 and Axiom 3.

□

The *marginal contribution* of the DM to a coalition $S \subseteq N_i \cup \{d\}$ in f_i is $[f_i(S) - f_i(S \setminus \{d\})]$. In Theorem 1, we have shown that axioms placed on $C(\cdot)$ are enough for the DM to calculate the "utility" of each of the SSs in a given CS and then pick the one that offers the highest "utility". Our objective in this section is to suggest conditions under which the "utility" is based only on the marginal contributions.

Definition 3 (Marginality Principle)**.** *The choice rule $C(\cdot)$ is said to be based on the marginality principle if for each $i \in \{1, \ldots, m\}$, there exists a $p_i : 2^{N_i \setminus \emptyset} \longrightarrow \mathbb{R}$ such that $i \in C(F) \Leftrightarrow \theta_i(f_i) \geq \theta_j(f_j)$ where*

$$\theta_i(f_i) = \sum_{S \subseteq N_i} p_i(S)\left[f_i(S \cup \{d\}) - f_i(S)\right] \tag{9}$$

and $\theta_i(\delta_i^*) = 1$.

To introduce the next axiom, we define a special class of SS in $\mathbb{G}_i$. Given $S \subseteq N_i \cup \{d\}$, let $\delta_S \in \mathbb{G}_i$ be such that for each $T \subseteq N_i \cup \{d\}$,

$$\delta_S(T) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise,} \end{cases}$$

Given $S \subseteq N_i$, $f_i + \alpha\delta_S$ is the SS in which the coalition S is uniquely responsible for an additional α units of output relative to f_i. When $f_i \sim_i f_i'$, it means the DM is unable to distinguish between them. The next axiom says that an identical improvement the "productivity" of a coalition S in both f_i and f_i' will not enable the DM to distinguish between them.

Axiom 6 (Strategic Equivalence)**.** *Let $f_i, f_i' \in \mathbb{G}_i$ be such that $i \in C(f_i, F_{-i}) \Leftrightarrow i \in C(f_i', F_{-i})$ for all $F_{-i} \in \mathbb{G}_{-i}$. Then, $i \in C(f_i, F_{-i}) \Leftrightarrow i \in C(f_i', F_{-i})$ for every $S \subseteq N_i \cup \{d\}$ for all $F_{-i} \in \mathbb{G}_{-i}$.*

Theorem 2. *The following statements are equivalent.*

(1) *$C(\cdot)$ satisfies Axiom 1- Axiom 6.*
(2) *$C(\cdot)$ is based on the Marginality Principle.*

Proof. That (2) implies (1) is easy to see. We shall prove the converse. Part 1, Part 4 of Lemma 1 and Axiom 6 mean that $\succeq_i$ satisfies allows us to apply Lemma 6, Agastya (1996), which shows that it admits a utility representation of the form

$$U_i(f_i) = \sum_{S \subseteq N_i \cup \{d\}, S \neq \emptyset} q(S) f_i(S)$$

for some set of weights $\{q(S)\}_{S \subseteq N_i \cup \{d\}, S \neq \emptyset}$. Rewrite the above as

$$\begin{aligned} U_i(f_i) &= q(\{d\}) f_i(\{d\}) + \sum_{S \subseteq N_i, S \neq \emptyset} q(S \cup \{d\}) f_i(S \cup \{d\}) \\ &\quad + \sum_{S \subseteq N_i, S \neq \emptyset} q(S) f_i(S) \end{aligned}$$

Given a f_i, define f_i^* to be

$$f_i^*(S) = \begin{cases} 0 & \text{if } S = d \\ f(S) & \text{if } d \notin S \\ f(S \setminus d) & \text{if } d \in S \end{cases} \quad \text{for } S \neq \emptyset, S \subseteq N_i \cup \{d\}$$

and note that

$$\begin{aligned} U_i(f_i^*) &= \sum_{S \subseteq N_i, S \neq \emptyset} q(S \cup \{d\}) f_i(S) \\ &\quad + \sum_{S \subseteq N_i, S \neq \emptyset} q(S) f_i(S) \end{aligned}$$

Use Part 3, Lemma 1 to choose a normalization that $U_i(\cdot)$ is identically zero on $\mathbb{N}_i$. Since $f_i^* \in \mathbb{N}_i$,

$$\begin{aligned} U_i(f_i) &= U_i(f_i) - U_i(f_i^*) \\ &= q(\{d\}) f_i(\{d\}) \\ &\quad + \sum_{S \subseteq N_i, S \neq \emptyset} q(S \cup \{d\}) [f_i(S \cup \{d\}) - f_i(S)] \end{aligned}$$

Take $p_i(S) = q(S \cup \{d\}) / U_i(\delta_i^*)$ and

$$\theta_i(f_i) = \sum_{S \subseteq N_i} p_i(S) [f_i(S \cup \{d\}) - f_i(S)].$$

Apply Theorem 1 to complete the proof. □

3.1. Shapley Value As the Decision Criterion

The Shapley value of the DM in a SS f_i is given by

$$\text{Sh}_i(f_i, N_i) = \sum_{S \subseteq N_i} \frac{s!(n_i - s + 1)!}{(n_i + 1)!}[f_i(S \cup d) - f_i(S)] \tag{10}$$

where s denotes the number of players in a coalition S. A particular instance of a choice rule that obeys the marginality principle occurs when $\theta_i(\cdot)$ obtained in Theorem 2 is in fact $Sh_i(\cdot)$. We now introduce a further axiom by which it will become unique.

Given $S_i \subseteq N_i$ and α, let F_{α,S_i} denote the CS where $f_j = \delta_j^*$ for $j \neq i$ and $f_i = \alpha\delta_{S_i \cup \{d\}}$. When presented with F_{α,S_i}, the DM can ensure a unit of utility by choosing a $j \neq i$. If $\alpha < |S + 1|$ and the DM were to choose i, there is an average of less than one unit to be split among any coalition that might form in the SS f_i. Despite this fact, if $i \in C(F)$, then the DM must expect get more than one "utile" by choosing i. We would then think of her as an "optimist". For similar considerations, we would regard her a pessimist if she $i \notin F^*_{\alpha,S}$ when $\alpha \geq |S + 1$. For a DM that expresses neither pessimism nor optimism, Axiom 1 would then imply $C(F^*_{\alpha,S}) = \{1, \ldots, m\}$ when $\alpha = (s + 1)$.

The following Axiom, which we shall use as a replacement for Axiom 1 combines the statement that the DM is neither an optimist nor a pessimist together with the intuition expressed for Axiom 1.

Axiom 7 (Consistency)**.** *Given scalars $\alpha_j \geq 0$ for $j = 1, \ldots, m$, let F be a CS where $f_j = \alpha_i\delta_j$ if $j \neq i$ and $f_i = \alpha_i(s+1)\delta_{S \cup d}$ for some $S \subseteq N_i$. Then $j \in C(F)$ if and only if $\alpha_j \geq \alpha_{j'}$ for all $j, j' \in \{1, \ldots, m\}$*

Note that Axiom 7 implies Axiom 1 as a special case. This axiom gives the following Shapley Value theorem.

Theorem 3. *The following statements are equivalent*

(1) *$C(\cdot)$ satisfies Axiom 7, Axiom 2 - Axiom 6.*
(2) *For every $F \in \mathbb{G}$, $i \in C(F)$ if and only if $\text{Sh}_i(f_i) \geq \text{Sh}_j(f_j)$.*

Proof. That (2) implies (1) is easy to see. We shall prove the converse. Apply Theorem 2 and let $\theta_i(\cdot)$ be as its proof. It is well known that for any

$S \subseteq N_i \cup d$, $S \neq \emptyset$,

$$\mathrm{Sh}_i(\delta_S) \;=\; \begin{cases} \frac{1}{s} & \text{if } d \in S \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Moreover, it is well known that $\mathbb{B}_i = \{\delta_S : S \subseteq N_i \cup \{d\}\}$ is a basis for $R^{2^{N_i}\setminus\{d\}}$. Since $\theta_i(\cdot)$ is linear, it suffices to show that $\theta_i(\cdot)$ agrees with $\mathrm{Sh}_i(\cdot, N_i \cup \{d\})$ on $\mathbb{B}_i$. To see this, note that $\theta_i(\cdot)$ satisfies the marginality principle and therefore $\theta_i(\delta_S) = 0$ for $S \subseteq N_i$ is immediate. That $\theta_i(\delta_{S\cup d}) = 1/(s+1)$ is also immediate since $(s+1)\delta_{S\cup d} \sim_i \delta_i^*$ and $\theta_i(\delta_i^*) = 1$ by Axiom 7. □

4. Discussion

It is possible to obtain some analogue of Theorem 1 under milder conditions. We shall not pursue this here. Theorem 2 and Theorem 3 on the other hand depend on very specific implications of Axiom 6 and Axiom 7 for $\succeq_i$. It should not be surprising that the marginality principle does not hold by altering them. The remainder of this section is to suggest a slight generalization of the marginality principle and how it might obtain if $\succeq_i$ can reasonably be argued to be separable[3] across coalitions.

Definition 4 (Marginality Principle*). *The choice rule $C(\cdot)$ is said to be based on the marginality principle if for each $i \in \{1, \ldots, m\}$, there exists a sequence of functions $\{v_S(\cdot) : S \subseteq N_i, S \neq \emptyset\}$ such that $i \in C(F) \Leftrightarrow \theta_i(f_i) \geq \theta_j(f_j)$ where*

$$\theta_i(f_i) = \sum_{S \subseteq N_i} v_S(f_i(S \cup \{d\})) - v_S(f_i(S)) \tag{12}$$

and $\theta_i(\cdot)$ is homogenous of degree one.

One could consider a wide variety of conditions that would render $\succeq_i$ separable across coalitions. One way to think of such (equivalent) conditions is as follows. Suppose at f_i we change the productivity of exactly one coalition $S \subseteq N_i \cup \{d\}$ by α_S to get a SS $f_{i,\alpha}$. That is $f_{i,\alpha_S}(T) = f_i(T)$ if $T \neq S$ and $f_{i,\alpha_S}(S) = f_i(S) + \alpha_S$ for a scalar α_S. If $f_i \sim_i f_{i,\alpha_S}$ then it must be that the increase is not sufficient to improve the attractiveness of the group i in response any F_{-i}. That one coalition is independent of another can be

[3]A binary relation $\succeq$ on $\mathbb{R}^n$ is said to be separable if it admits a utility representation of the form $U(x_1, \ldots, x_n) = \sum_{i=1}^{n} v_i(x_i)$ where $(x_1, \ldots, x_n) \in \mathbb{R}^n$ where $v_i : \mathbb{R} \longrightarrow \mathbb{R}$.

captured by requiring that changing the productivity of coalitions other S at f_i and f_{i,α_S} has not effect on this equivalence. The following makes this precise.

Axiom 8 (Separability)**.** *Suppose $f_i \sim_i f_{i,\alpha_S}$ for some α_S. Then $f_i' \sim_i f_{i,\alpha_S}'$ for any other f_i' such that $f_i(S) = f_i'(S)$*

Proposition 1. *Suppose Axiom 1-Axiom 5 and Axiom 8 hold. Then $C(\cdot)$ is based on the Marginality Principle*.*

Proof. Label the coalitions in $N_i \cup \{d\}$ from 1 to $k = 2^{n_i+1} - 1$. Fix any set of indices $J \subseteq \{1, \ldots, k\}$ and let $(x, z) \in \mathbb{R}_+^k$ be such that $x = (x_\kappa)_{\kappa \in J}$ and $z = (z_\kappa)_{\kappa \in J^c}$. Any (x, z) and (x, z') correspond to some $f_i, f_i' \in \mathbb{G}_i$ such that $f_i(S) = f_i;'(S)$ for all S that correspond to the labels in J. Further, upon repeated application, Axiom 7 is essentially the condition

$$(x,z) + (w,0) \sim_i (x,z) \quad \Leftrightarrow \quad \left(x,z'\right) + (w,0) \sim_i \left(x,z'\right) \tag{13}$$

where $(x,z), (x,z'), (x+w,z), (x+w,z') \in \mathbb{R}_+^k$.

Recall $\succeq_i$ is said to satisfy order independence in the sense of Debreu (1960) if for any $J \subseteq \{1, \ldots, k\}$ and for all $(x,z), (y,z), (x,z')$ and (y,z') in $\mathbb{R}_+^k$,

$$(y,z) \sim_i (x,z) \quad \Leftrightarrow \quad \left(y,z'\right) \sim_i \left(x,z'\right) \tag{14}$$

Since Debreu (1960) it is well known that $\succeq_i$ admits a separable representation if and only if it is order independent. Therefore, it suffices to show that (13) implies (14). But this can readily seen to be the case by expressing $(y,z) = (x,z) + (w,0)$ and $(y,z') = (x,z') + (w,0)$ and choosing $w = y - x$. Therefore, there exists a utility representation $U(\cdot)$ of $\succeq_i$ of the form

$$U(x_1, \ldots, x_k) \quad = \quad \sum_{\kappa=1}^{k} v_\kappa(x_\kappa)$$

Now mimic the proof of Theorem 2 to show that the Marginality Principle* is satisfied. □

Conclusion. This paper is motivated by a desire to understand how people make choices among strategic situations when the rules of play are not precise. Theorem 1 showed how standard "utility theory" can be brought to bear on the problem. Through a further examination of the "utility of playing a game", we have offerred additional insights for using the Shapley Value as a decision rule for making choices in such environments.

References

AGASTYA, M. (1996): "Multiplayer Bargaining Situations: A Decision Theoretic Approach," *Games and Economic Behavior*, 12(1), 1–20.

DEBREU, G. (1960): "Topological methods in cardinal utility theory," in *'Mathematical methods in the social sciences'*, ed. by K. J. Arrow, S. Karlin, and P. Suppes. Stanford University Press, Palo Alto.

HART, O., AND J. MOORE (1990): "Property Rights and the Nature of the Firm," *Journal of Political Economy*, v98(n6), 1119–58.

ROTH, A. E. (1977): "The Shapley Value as a von Neumann-Morgenstern Utility," *Econometrica*, 45, 657–664.

SHAPLEY, L. S. (1953): "A Value for n-person games," in *Reprinted in The Shapley value: Essays in honor of Lloyd S. Shapley*. Cambridge; New York and Melbourne: Cambridge University Press, 1988.

STOLE, L. A., AND J. ZWIEBEL (1996): "Organizational Design and Technology Choice under Intrafirm Bargaining," *American Economic Review*, 86, 195–222.

YOUNG, P. H. (1988): "Individual Contribution and Just Compensation," in *The Shapley value: Essays in honor of Lloyd S. Shapley*. Cambridge; New York and Melbourne: Cambridge University Press,.