

# Dynamic Linear Economies with Social Interactions

Onur Özgür and Alberto Bisin

April 5, 2011

# DYNAMIC LINEAR ECONOMIES WITH SOCIAL INTERACTIONS<sup>1</sup>

ONUR ÖZGÜR<sup>a</sup> AND ALBERTO BISIN<sup>b</sup>

Social interactions arguably provide a rationale for several important phenomena, from smoking and other risky behavior in teens to peer effects in school performance. We study *social interactions* in *dynamic* economies. For these economies, we provide existence (Markov Perfect Equilibrium in pure strategies), ergodicity, and welfare results. We characterize several equilibrium properties of policy functions, spatial correlations, and social multiplier effects. Most importantly, we study formally the issue of the identification of social interactions, emphasizing the restrictions imposed by dynamic equilibrium conditions with respect to economies populated by myopic agents and economies in which spatial correlation is induced by selection.

KEYWORDS: Conditional covariance stationarity, conformity, ergodicity, habits, identification, Markov perfect equilibrium, social interactions, social norms.

## 1. INTRODUCTION

Agents interact in markets as well as socially—that is, in the various socioeconomic groups they belong to. Models of social interactions are designed to capture in a simple abstract way socioeconomic environments in which markets do not mediate all of agents’ choices. In such environments agents’ choices are determined by their preferences as well as by their interactions with others—that is, their positions in a predetermined network of relationships, e.g., a family, a peer group, or more generally any socioeconomic group.<sup>1</sup>

Social interactions arguably provide a rationale for several important phenomena, Peer effects, in particular, have been indicated as one of the main empirical determinants of

---

<sup>1</sup>We are grateful to Bernard Salanié for some insightful comments early on. We also thank Massimiliano Amarante, Olivier Armantier, Jess Benhabib, Ken Binmore, Michele Boldrin, Yann Bramoullé, Pierre-André Chiappori, In-Koo Cho, Aureo De Paula, Bryan Graham, Ali Hortaçsu, Frédéric Koessler, Takashi Kunimoto, Justin Leroux, David Levine, Bob Lucas, Debraj Ray, Manuel Santos, Tom Sargent, José Scheinkman, Karl Schlag, Paolo Siconolfi, Yves Sprumont, Bruno Strulovici, Jean Tirole, and many seminar participants. Finally, we would like to thank Ben Golub and Itai Sher who read an early version of the paper and made detailed suggestions which greatly improved the presentation. Part of this research was done while Özgür was visiting the Economics Department at the Université Laval. Thanks to Yann Bramoullé and Bernard Fortin for organizing the visit. Özgür is grateful for financial support to “La Chaire du Canada en Économie des Politiques Sociales et des Ressources Humaines” at Université Laval, CIREQ, CIRPÉE, and FQRSC.

<sup>a</sup>Université de Montréal, Department of Economics, CIREQ, CIRANO; [onur.ozgur@umontreal.ca](mailto:onur.ozgur@umontreal.ca)

<sup>b</sup>New York University, Department of Economics, NBER, CIREQ; [alberto.bisin@nyu.edu](mailto:alberto.bisin@nyu.edu)

<sup>1</sup>The integration of models of social interactions within economic theory is an active and interesting area of research. See the recent *Handbook of Social Economics*, Benhabib, Bisin, and Jackson (2011).

risky behavior in adolescents.<sup>2</sup> Relatedly, peer effects have been studied in connection with education outcomes,<sup>3</sup> obesity,<sup>4</sup> friendship and sex,<sup>5</sup> labor market referrals,<sup>6</sup> neighborhood and employment segregation,<sup>7</sup> criminal activity,<sup>8</sup> and several other socioeconomic phenomena.<sup>9</sup>

The large majority of the existing models of social interactions are static; or, when dynamic models of social interactions are studied, it is typically assumed that agents are myopic and their choices are subject to particular behavioral assumptions.<sup>10</sup> In this paper, we contribute to this literature by studying social interactions in *dynamic* economies. We focus our attention on *linear economies*, in which each agent's preferences are quadratic. Dynamic linear models of course have appealing analytical properties. Hansen and Sargent (2004) study this class of models systematically, exploiting the tractability of linear control methods and matrix Riccati equations. While the class of economies we study in this paper allows for a countable number of heterogeneous agents and an infinite horizon, giving rise to infinite dimensional systems, some tractability is still maintained. Furthermore, in the class of economies we study agents display preferences for *conformity*, that is, preferences which incorporate the desire to conform to the choices of agents in a reference group.

More specifically, each agent's preferences are hit by random preference shocks over time. Each agent interacts with agents in his social reference group, in the sense that each agent's instantaneous preferences depend on the current choices of agents in his so-

---

<sup>2</sup>See e.g., Axtell et al. (2006), Cutler and Glaeser (2007), Gaviria and Raphael (2001), Krauth (2006), Kremer and Levy (2008), Nakajima (2007), Sacerdote (2001).

<sup>3</sup>Altonji et al. (2005), Ammermueller and Pischke (2009), Bénabou (1996), Bifulco et al. (2009), Borjas (1995), De Giorgi et al. (2009), Evans et al. (1992), Hoxby (2000a,b), Soetevent and Kooreman (2007).

<sup>4</sup>Burke and Heiland (2007), Christakis and Fowler (2007).

<sup>5</sup>Akerlof et al. (1996), Bearman et al. (2004), Cipollone and Rosolia (2007), Conti et al. (2009), Currarini et al. (2009), Kandel (1978), Moody (2001).

<sup>6</sup>Bayer et al. (2008), Calvo-Armengol and Jackson (2004), Conley and Topa (2002), Goldin and Katz (2002), Granovetter (1973, 1995), Ioannides and Datcher Loury (2004), and Topa (2001).

<sup>7</sup>Bénabou (1993), Case and Katz (1991), Crane (1991), Durlauf (1996, 2004), Hoff and Sen (2005), Katz et al. (2001), Ludwig et al. (2001), Mobius (2000), Schelling (1972).

<sup>8</sup>Calvo-Armengol et al. (2009), Glaeser, Sacerdote, and Scheinkman (1996), Kling et al. (2005), Ludwig et al. (2001).

<sup>9</sup>See Bisin et al. (2010), Glaeser and Scheinkman (2001), Moody (2001) for surveys.

<sup>10</sup>Exceptions include an example on female labor force participation in Glaeser and Scheinkman (2001), Binder and Pesaran (2001) on life-cycle consumption under social interactions, Blume (2003) on social stigma, Brock and Durlauf (2010) and De Paula (2009) on duration models, Ioannides and Soetevent (2007) on endogenous local and global interactions, where agents best respond to lagged decisions, and the theoretical analysis of Bisin, Horst, and Özgür (2006).

cial reference group, as a direct externality. Each agent's instantaneous preferences also depend on the agent's own previous choice, representing the inherent costs to dynamic behavioural changes due e.g., to habits. When agents' reference groups overlap, each agent's *optimal* choice depends on all the other agents's previous choices and current preference shocks, as long as they are observable. We allow for complete and incomplete information with respect to preference shocks. Requiring that the social and informational structure of each agent satisfy a symmetry condition, we restrict our analysis to *symmetric Markov perfect equilibria*. Agents' choices at equilibrium are determined by *linear* policy (best reply) functions. More specifically, e.g., in infinite-horizon economies, a symmetric Markov perfect equilibrium is represented by a symmetric policy function,  $g$ , which maps an agent's current choice at time  $t$ , linearly in each agent's past choices,  $x_{t-1}^b$ , in each agent's contemporaneous idiosyncratic preference shock,  $\theta_t^b$ , and in the mean preference shock,  $\bar{\theta}$ :

$$g(x_{t-1}, \theta_t) = \sum_{b \in \mathbb{A}} c^b x_{t-1}^b + \sum_{b \in \mathbb{A}} d^b \theta_t^b + e \bar{\theta}$$

For these economies, we provide some fundamental theoretical results: (Markov perfect) equilibria exist (for finite economies they are unique) and they induce an ergodic stochastic process over the equilibrium configuration of actions. Furthermore, a stationary ergodic distribution exists. We also derive a recursive algorithm to compute equilibria. The proof of the existence theorem, in particular, requires some subtle arguments. In fact, standard variational arguments require bounding the marginal effect of any infinitesimal change  $dx^a$  on the agent's value function. But in the class of economies we study, the envelope theorem (as e.g., in [Benveniste and Scheinkman \(1979\)](#)) is not sufficient for this purpose, as  $dx^a$  affects agent  $a$ 's value function directly and indirectly, through its effects on all agents  $b \in \mathbb{A} \setminus a$ 's choices, which in turn affect agent  $a$ 's value function. The marginal effect of any infinitesimal change  $dx^a$  is then an infinite sum of endogenous terms. In our economy, however, we can exploit the linearity of policy functions to represent a symmetric MPE by a fixed point of a recursive map which can be directly studied.

Exploiting the linear structure of our economies we can study equilibria in some detail, characterizing the parameters of the policy function as well as a fundamental statistical property of equilibrium, the cross-sectional auto-correlation of actions. Based on this,

we obtain a series of results regarding the welfare properties of equilibrium and various comparative dynamics exercises of interest. First of all, we show that, since social interactions are modelled in this paper as a preference externality, equilibria will not be efficient in general. We also characterize the form of the inefficiency: at equilibrium each agent's policy function weights too heavily the agent's own preference shock and previous action and not enough the other agents'. The comparative dynamics exercises illustrate e.g., the equilibrium effects of the strength of social interactions and of the social and informational structure of the economy.

Finally, we exploit our characterization results of the equilibria to address generally the issue of identification of social interactions in our context, with population data. While the empirical literature has often interpreted a significant high correlation of socioeconomic choices across agents, e.g., peers, as evidence of social interactions, in the form, e.g., of preferences for conformity, it is well known at least since the work of [Manski \(1993\)](#) that the empirical study of social interactions is plagued by subtle identification problems. Intuitively, in our economy for instance, the spatial correlation of actions at equilibrium can be due to social interactions or to the spatial correlation of preference shocks. More formally, take two agents, e.g., agent  $a$  and agent  $b$ . A positive correlation between  $x_t^a$  and  $x_t^b$  could be due to e.g., preference for conformity. But the positive correlation between  $x_t^a$  and  $x_t^b$  could also be due to a positive correlation between  $\theta_t^a$  and  $\theta_t^b$ . In this last case, preferences for conformity and social interactions would play no role in the correlation of actions at equilibrium. Rather, such correlation would be due to the fact that agents have correlated preferences. Correlated preferences could generally be due to some sort of assortative matching or positive selection, which induce agents with correlated preferences to interact socially.

In the context of our economy, we ask whether the restrictions implied by the dynamic equilibrium analysis help identify social interactions and distinguish them from correlated preferences. We show that the answer is in fact affirmative, but only if the economy is non-stationary, in a precise sense. To illustrate our results, consider for instance the issue of peer effects in adolescents' substance use. Suppose the econometrician observes the behavior of a population of students in a school over time (at different grades). A significant high correlation of socioeconomic choices across students in the school could be due to selection in the endogenous composition of the school in terms of unobserved (to the econometrician) correlated characteristics of the agents. Any significant variation in students' behavior through time (grades) must however be due to social interactions.

A student whose choice is affected by the choices of his school peers will in fact rationally anticipate how much longer he will interact with them. In particular, his propensity to conform to his peers' actions will tend to decrease over time (grades) and will be the lowest in the final years in the school. This non-stationarity of each student's behavior at equilibrium is the key to the identification of social interaction in our class of economies.<sup>11</sup>

The simplicity of linear models allows us to extend our analysis in several directions which are important in applications and empirical work. This is the case, for instance of general (including asymmetric) neighborhood network structures for social interactions. But our analysis extends also to general stochastic processes for preference shocks and to the addition of global interactions. One particular form of global interactions occurs when each agent's preferences depend on an average of actions of all other agents in the population, e.g. [Brock and Durlauf \(2001a\)](#), and [Glaeser and Scheinkman \(2003\)](#). This is the case, for instance, if agents have preferences for *social status*. More generally, global interactions could capture preferences to adhere to aggregate norms of behavior, such as specific group cultures, or other externalities as well as price effects. Finally, and perhaps most importantly, we extend our analysis to encompass a richer structure of dynamic dependence of agents' actions at equilibrium. In particular we study an economy in which agents' past behavior is aggregated through an accumulated stock variable which carries habit persistence, which can be directly applied e.g., to the issue of teenage substance addiction due to peer pressure at school. With respect to the addiction literature, as e.g., [Becker and Murphy \(1988\)](#), we model the dynamics of addiction considering peer effects not only in a single-person decision problem, but rather in a social equilibrium, allowing for the intertemporal feedback channel between agents across social space and through time.<sup>12</sup> In this context we show that in equilibrium each agent's choice depends on the stock of his neighbors' actions, on their long-term behavioral patterns rather than just on their previous period actions. Also, in non-stationary economies, as the final period approaches, each agent assigns higher weights to his own stock, giving rise to an initiation-addiction behavioral pattern at equilibrium which is consistent with observation, e.g., in [Cutler and Glaeser \(2007\)](#) and [DeCicca et al. \(2008\)](#).

---

<sup>11</sup>This pattern of behavior appears consistent with the peer effects study of [Hoxby \(2000a,b\)](#).

<sup>12</sup>See also [Becker et al. \(1994\)](#), [Gul and Pesendorfer \(2007\)](#), [Gruber and Koszegi \(2001\)](#); see [Rozen \(2010\)](#) for theoretical foundations for intrinsic linear habit formation; see also [Elster \(1999\)](#) and [Elster and Skog \(1999\)](#) for surveys.

## 2. DYNAMIC ECONOMIES WITH SOCIAL INTERACTIONS

While we develop most of our analysis in the context of linear models, it is useful to set up the general model first, as we do in this section, to be as clear and specific as possible regarding the assumptions we impose on the economy we study.

Time is discrete and is denoted by  $t = 1, \dots, T$ . We allow both for infinite economies ( $T = \infty$ ) and economies with an end period ( $T < \infty$ ). A typical economy is populated by a countable set of *agents*  $\mathbb{A}$ , a generic element of which being denoted by  $a$ .<sup>13</sup> Each agent lives for the duration of the economy. At the beginning of each period  $t$ , agent  $a$ 's random preference *type*  $\theta_t^a$  is drawn from  $\Theta$ , a compact subset of a finite dimensional Euclidean space  $\mathbb{R}^n$ . The random variables  $\theta_t^a$  are independently and identically distributed across time and agents with probability law  $\nu$ .<sup>14</sup> We assume, with no loss of generality, that the random variable  $\theta_t := (\theta_t^a)_{a \in \mathbb{A}}$  is defined, for all  $t$ , on the canonical probability space  $(\Theta, \mathcal{F}, \mathbb{P})$ , where  $\Theta := \{(\theta^a)_{a \in \mathbb{A}} : \theta^a \in \Theta\}$ . At each period  $t$ , agent  $a \in \mathbb{A}$  chooses an *action*  $x_t^a$  from the set  $X$ , a compact subset of a finite dimensional Euclidean space  $\mathbb{R}^p$ . Let  $\mathbf{X} := \{x = (x^a)_{a \in \mathbb{A}} : x^a \in X\}$  be the space of individual action profiles.

Each agent  $a \in \mathbb{A}$  *interacts* with agents in the set  $N(a)$ , a nonempty subset of the set of agents  $\mathbb{A}$ , which abstractly represents agent  $a$ 's social reference group. The map  $\mathbb{A} : N \rightarrow 2^{\mathbb{A}}$  is referred to as a *neighbourhood correspondence* and is assumed exogenous. Agent  $a$ 's instantaneous preferences depend on the current choices of agents in his reference group,  $\{x_t^b\}_{b \in N(a)}$ , representing social interactions as direct preference externalities. Agent  $a$ 's instantaneous preferences also depend on the agent's own previous choice,  $x_{t-1}^a$ , representing inherent costs to dynamic behavioural changes due e.g., to habits. In summary, agent  $a$ 's instantaneous preferences at time  $t$  are represented by a continuous utility function

$$(x_{t-1}^a, x_t^a, \{x_t^b\}_{b \in N(a)}, \theta_t^a) \mapsto u(x_{t-1}^a, x_t^a, \{x_t^b\}_{b \in N(a)}, \theta_t^a)$$

Agents discount expected future utilities using the common stationary discount factor

---

<sup>13</sup>We study an economy populated by a countably infinite number of agents where  $\mathbb{A} := \mathbb{Z}$ , but our analysis applies to economies with a finite number of agents.

<sup>14</sup>We use the i.i.d. assumption for methodological clarity. The focus throughout the paper until Section 6 is on understanding purely endogenously generated equilibrium correlations so that they could be cleanly compared to the alternative models of correlated preferences discussed in the identification section (Section 6). The model, of course, is flexible to accomodate correlated shocks across agents and time. We show how this is done in Section 7.2.

$\beta \in (0, 1)$ .

Let  $x^{t-1} = (x_0, x_1, \dots, x_{t-1})$  and  $\theta^{t-1} = (\theta_1, \dots, \theta_{t-1})$  be the  $(t-1)$ -period choices and type realizations, where  $x_0 \in \mathbf{X}$  is the initial configuration. Before each agent's time  $t$  choice,  $x^{t-1}$  is observed by all agents and the current value of the random variable  $\theta_t$  realizes. Agent  $a \in \mathbb{A}$  observes only the part  $I_a \theta_t := \{\theta_t^b : b \in I(a)\}$ , where  $I(a) \subset \mathbb{A}$  is his *information set*. Similarly, let  $I_a \theta^{t-1} = (I_a \theta_1, \dots, I_a \theta_{t-1})$ . We study both economies with *complete information*,  $I(a) = \mathbb{A}$ , and economies with *incomplete information*,  $I(a) \subsetneq \mathbb{A}$ . After time  $t$  choices are made,  $x_t = (x_t^b)_{b \in \mathbb{A}} \in \mathbf{X}$  becomes common knowledge and the economy moves to time  $t+1$ .

A *strategy* for an agent  $a$  is a sequence of measurable functions  $x^a = (x_t^a)$ , where for each  $t$ ,  $x_t^a : \mathbf{X}^t \times (\Theta^{I(a)})^t \rightarrow X$ . Agents' strategies along with the probability law for types induce a stochastic process over future configuration paths. Each agent  $a \in \mathbb{A}$ 's objective is to choose  $x^a$  to maximize

$$(2.1) \quad E \left[ \sum_{t=1}^T \beta^{t-1} u(x_{t-1}^a, x_t^a, \{x_t^b\}_{b \in N(a)}, \theta_t^a) \mid (x_0, \theta_1) \right]$$

given the strategies of other agents and given  $(x_0, I_a \theta_1) \in \mathbf{X} \times \Theta^{I(a)}$ .

We require that the social and informational structure satisfies the following symmetry restrictions:<sup>15</sup>

1. For all  $a, b \in \mathbb{A}$ ,  $N(b) = R^{b-a} N(a)$ , where  $R^{b-a}$  is the canonical *shift* operator in the direction  $b-a$ .<sup>16</sup>
2. For all  $a, b \in \mathbb{A}$ ,  $I(b) = R^{b-a} I(a)$ .

We restrict our analysis to *symmetric Markov perfect equilibria*. Agents' strategies are *Markovian* if after any  $t-1$ -period history  $(x^{t-1}, \theta^t)$ , they depend only on the previous period configuration  $x_{t-1}$  and the current type realizations  $\theta_t$ .

Because of symmetry, it is enough to analyze the optimization problem relative to a single reference agent, say agent  $0 \in \mathbb{A}$ . Assume that the optimal choice of any economic agent  $b \in \mathbb{A}$  is determined by a continuous choice function  $g : \mathbf{X} \times \Theta^{I(0)} \times \{1, \dots, T\} \rightarrow X$  such that for all  $t = 1, \dots, T$  and after any history  $(x^{t-1}, \theta^t) \in \mathbf{X}^t \times \Theta^t$ , his  $t$ -th period

<sup>15</sup>Heterogeneity can be incorporated into the probabilistic structure of the types  $\theta_t^a$ . Also, we can allow for heterogeneity of the network structure across agents by augmenting the strategy spaces to incorporate network structure into individual heterogeneity. We do this in Section 7.1.

<sup>16</sup>That is,  $c \in N(a)$  if and only if  $c + (b-a) \in N(b)$ . The operations of addition and subtraction are legitimate given that we typically let  $\mathbb{A} := \mathbb{Z}^d$ , the  $d$ -dimensional integer lattice.



choice is given by

$$x_t^b(g)(x^{t-1}, \theta^t) = g_{T-(t-1)}(R^b x_{t-1}, R^b I_0 \theta_t)$$

The value of the optimization problem of agent  $a$  is then given by<sup>17</sup>

$$V_g^T(R^a x_0, R^a I_0 \theta_1) = \max_{(x_t^a)_{t=1}^T} E \left[ \sum_{t=1}^T \beta^{t-1} u(x_{t-1}^a, x_t^a, \{x_t^b(g)\}_{b \in N(a)}, \theta_t^a) \right]$$

The value function associated with this dynamic choice problem can be shown to satisfy Bellman's Principle of Optimality by standard arguments (see e.g., [Stokey and Lucas \(1989\)](#)) and, hence, can be written in the following recursive form,

$$(2.2) \quad \begin{aligned} V_g^{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t) &= \max_{x_t^a \in X} E \left[ u(x_{t-1}^a, x_t^a, \{x_t^b(g)\}_{b \in N(a)}, \theta_t^a) \right. \\ &\quad \left. + \beta V_g^{T-t} \left( R^a \left( x_t^a, \{x_t^b(g)\}_{b \neq a} \right), R^a I_0 \theta_{t+1} \right) \right] \end{aligned}$$

for  $t = 1, \dots, T$  and for all  $(x^{t-1}, \theta^t) \in \mathbf{X}^t \times \Theta^t$ .<sup>18</sup> We are now ready to define our equilibrium concept.

**DEFINITION 1** *A symmetric Markov Perfect Equilibrium (MPE) of a dynamic economy with social interactions is a measurable map  $g^* : \mathbf{X} \times \Theta^{I(0)} \times \{1, \dots, T\} \rightarrow X$  such that for all  $a \in \mathbb{A}$ , for all  $t = 1, \dots, T$ , and for all  $(x^{t-1}, \theta^t) \in \mathbf{X}^t \times \Theta^t$*

$$(2.3) \quad \begin{aligned} g_{T-(t-1)}^*(R^a x_{t-1}, R^a I_0 \theta_t) &\in \operatorname{argmax}_{x_t^a \in X} E \left[ u(x_{t-1}^a, x_t^a, \{x_t^b(g^*)\}_{b \in N(a)}, \theta_t^a) \right. \\ &\quad \left. + \beta V_{g^*}^{T-t} \left( R^a \left( x_t^a, \{x_t^b(g^*)\}_{b \neq a} \right), R^a I_0 \theta_{t+1} \right) \right] \end{aligned}$$

Clearly, an MPE is necessarily a *subgame perfect equilibrium*; that is, each agent's continuation strategy is a best response to other agent's continuation strategies after any possible history. Notice also the time notation we use for the Markovian policy:  $g_{T-(t-1)}^*$  denotes the first-period equilibrium choice in a  $T - (t - 1)$ -period economy. Since economies are nested,  $g_{T-(t-1)}^*$  represents also the  $t$ -period equilibrium choice in a  $T$ -period economy.

We conclude this section with a few remarks to justify our focus on MPEs. First of all,

<sup>17</sup>The preference shocks being serially uncorrelated, we do not need to condition on the value of past realizations. See Section 7.2 for a treatment of persistent shocks.

<sup>18</sup>We have adopted the the convention that  $V_g^0(x, I_0 \theta) := 0$  for any  $(x, \theta) \in \mathbf{X} \times \Theta$ .

Markovian strategies are not a restriction for finite-horizon economies: we prove that the unique symmetric subgame perfect equilibrium for any finite-horizon economy is necessarily Markovian. Moreover, in an infinite horizon economy ( $T = \infty$ ), a symmetric MPE is not necessarily stationary. The sequence of unique MPEs for finite horizon economies converges however to a  $g^* : \mathbf{X} \times \Theta^{I(0)} \rightarrow X$  which turns out to be a stationary MPE of the infinite-horizon economy whose properties we focus on. Finally, we refer to [Bisin, Horst, and Özgür \(2006\)](#) for a discussion of non-Markovian equilibria in a related context.

### 3. DYNAMIC LINEAR ECONOMIES WITH SOCIAL INTERACTIONS AND CONFORMITY PREFERENCES

We focus our attention on *linear economies* with *conformity preferences*. These are environments in which each agent's preferences incorporate the desire to conform to the choices of agents in his reference group.<sup>19</sup>

Preferences for conformity arguably provide a rationale for several important social phenomena. The empirical literature has for instance documented preferences for conformity as a motivation for smoking and other risky behaviour in teens. Similarly, the role of conformity is also documented by [Glaeser, Sacerdote, and Scheinkman \(1996\)](#) with regards to criminal activity and by a large literature with regards to peer effects in education outcomes.<sup>20</sup> Conformity also represents a natural environment in which to study dynamic equilibrium. In many relevant social phenomena, in fact, the effects of preferences for conformity are amplified by the presence of limits to the reversibility of dynamic choices. This is of course the case for smoking, alcohol abuse and other risky teen behaviour, which are hard to reverse because they might lead to chemical addictions. In other instances, while addiction per se is not at issue, nonetheless behavioural choices are hardly freely reversible because of various social and economic constraints, as is the case, for instance, of engaging in criminal activity. Finally, exogenous and predictable changes in the composition of groups, as e.g., in the case of school peers at the end of a school cycle, introduce important non-stationarities in the agents' choice. These non-stationarities also call for a formal analysis of dynamic social interactions.

With the objective of providing a clean and simple analysis of dynamic social interactions in a conformity economy, we impose strong(er than required) but natural

---

<sup>19</sup>While we model preferences for conformity directly as a preference externality, we intend this as a reduced form of models of behavior in groups which induce indirect preferences for conformity, as e.g., [Jones \(1984\)](#), [Cole et al. \(1992\)](#), [Bernheim \(1994\)](#), [Peski \(2007\)](#).

<sup>20</sup>See the Introduction for the relevant references.

assumptions.<sup>21</sup> In particular (i) we restrict the neighborhood correspondence to represent the minimal interaction structure allowing for overlapping groups, (ii) we restrict preferences to be quadratic, and (iii) we impose enough regularity conditions on the agents' choice problem to render it convex. Formally,

ASSUMPTION 1 *A linear conformity economy satisfies the following.*

1. Let  $\mathbb{A} := \mathbb{Z}$  represent the countable set of agents. Each agent interacts with his immediate neighbors, i.e., for all  $a \in \mathbb{A}$ ,  $N(a) := \{a - 1, a + 1\}$ .
2. The contemporaneous preferences of an agent  $a \in \mathbb{A}$  are represented by the utility function

$$(3.1) \quad u(x_{t-1}^a, x_t^a, x_t^{a-1}, x_t^{a+1}, \theta_t^a) := -\alpha_1(x_{t-1}^a - x_t^a)^2 - \alpha_2(\theta_t^a - x_t^a)^2 \\ - \alpha_3(x_t^{a-1} - x_t^a)^2 - \alpha_3(x_t^{a+1} - x_t^a)^2$$

where  $\alpha_1, \alpha_2$ , and  $\alpha_3$ , are positive constants.

3. Let  $X = \Theta = [\underline{x}, \bar{x}] \subset \mathbb{R}$ , where  $\underline{x} < \bar{x}$ . Let  $v$  be absolutely continuous with a positive density<sup>22</sup>,  $E[\theta_t^a] = \int \theta_t^a d\nu =: \bar{\theta} \in (\underline{x}, \bar{x})$ , and  $Var(\theta_t^a) = \int (\theta_t^a - \bar{\theta})^2 d\nu < \infty$ .<sup>23</sup>

Assumption 1-1 requires that the reference group of each agent  $a \in \mathbb{A}$  be composed of his immediate neighbors in the social space, namely the agents  $a - 1$  and  $a + 1$ . The utility function  $u$  defined in Assumption 1-2 describes the trade-off that agent  $a \in \mathbb{A}$  faces between matching his individual characteristics  $(x_{t-1}^a, \theta_t^a)$  and the utility he receives from conforming to the current choices of his peers  $(x_t^{a-1}, x_t^{a+1})$ . The different values of  $\alpha_i$  represent different levels of intensity of the social interaction motive relative to the own (or intrinsic) motive. Finally, Assumption 1-2 and 1-3 jointly guarantee that the agents' choice problem is convex. Finally, notice that the requirements that  $\alpha_1, \alpha_2 > 0$  anchor agents' preferences on their own private types and past choices. It is easy to see that, without such anchor, actions are driven only by social interactions and a large multiplicity of equilibria arises.

---

<sup>21</sup>See Section 7 for possible directions in which the structure and the results we obtain are easily generalized.

<sup>22</sup>We will call a measure  $\mu$  'absolutely continuous' if it is absolutely continuous with respect to the Lebesgue measure  $\lambda$ , i.e., if  $\mu(A) = 0$  for every measurable set  $A$  for which  $\lambda(A) = 0$ .

<sup>23</sup>We need absolute continuity only when we prove inefficiency. All other results are obtained without that assumption.

### 3.1. Equilibrium

We provide here the basic theoretical results regarding our dynamic linear social interaction economy with conformity. The reader only interested in the characterization can skip this section, keeping in mind that equilibria exist (for finite economies they are unique) and they induce an ergodic stochastic process over paths of action profiles. Furthermore, a stationary ergodic distribution also exists for the economy. Finally, a recursive algorithm to compute equilibria is derived. Unless otherwise mentioned specifically, the proofs of all statement and other results can be found in the Supplemental Appendix.

**THEOREM 1 (Existence - Complete Information)** *Consider an economy with conformity preferences and complete information.*

1. *If the time horizon is finite ( $T < \infty$ ), then the economy admits a unique symmetric MPE  $g^* : \mathbf{X} \times \Theta \times \{1, \dots, T\} \mapsto X$  such that for all  $t \in \{1, \dots, T\}$ , for all  $(x_{t-1}, \theta_t) \in \mathbf{X} \times \Theta$*

$$g_{T-(t-1)}^*(x_{t-1}, \theta_t) = \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a x_{t-1}^a + \sum_{a \in \mathbb{A}} d_{T-(t-1)}^a \theta_t^a + e_{T-(t-1)} \bar{\theta} \quad \mathbb{P} - a.s.$$

*where  $c_\tau^a, d_\tau^a, e_\tau \geq 0$ ,  $a \in \mathbb{A}$ , and  $e_\tau + \sum_{a \in \mathbb{A}} (c_\tau^a + d_\tau^a) = 1$ ,  $0 \leq \tau \leq T$ . Moreover, the equilibrium is also unique in the class of subgame perfect equilibria (SPE), meaning that there does not exist any non-Markovian SPE for our economy.*

2. *If the time horizon is infinite ( $T = \infty$ ), then the economy admits a symmetric stationary MPE  $g^* : \mathbf{X} \times \Theta \mapsto X$  such that*

$$g^*(x_{t-1}, \theta_t) = \sum_{a \in \mathbb{A}} c^a x_{t-1}^a + \sum_{a \in \mathbb{A}} d^a \theta_t^a + e \bar{\theta} \quad \mathbb{P} - a.s.$$

*where  $c^a, d^a, e \geq 0$ , for  $a \in \mathbb{A}$ , and  $e + \sum_{a \in \mathbb{A}} (c^a + d^a) = 1$ .<sup>24</sup>*

The theorems in this section can be extended with straightforward modifications to the case of incomplete information. We state without proof, e.g., the existence theorem for economies with incomplete information next.

---

<sup>24</sup>Several assumptions can be relaxed while guaranteeing existence. In particular, the symmetry of the neighborhood structure can be substantially relaxed. See Section 7.1 for the discussion.

**THEOREM 2 (Existence - Incomplete Information)** *Consider an economy with conformity preferences and with incomplete information.*

1. For  $T < \infty$ , the economy admits a unique symmetric MPE  $g^* : \mathbf{X} \times \Theta^{I(0)} \times \{1, \dots, T\} \mapsto X$  such that for all  $t \in \{1, \dots, T\}$ ,

$$g_{T-(t-1)}^*(x_{t-1}, I_0 \theta_t) = \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a x_{t-1}^a + \sum_{a \in I(0)} d_{T-(t-1)}^a \theta_t^a + e_{T-(t-1)} \bar{\theta} \quad \mathbb{P} - a.s.$$

where  $c_\tau^a, d_\tau^a, e_\tau \geq 0$  and  $e_\tau + \sum_{a \in \mathbb{A}} c_\tau^a + \sum_{a \in I(0)} d_\tau^a = 1$ ,  $0 \leq \tau \leq T$ .

2. For  $T = \infty$ , the economy admits a symmetric MPE  $g^* : \mathbf{X} \times \Theta^{I(0)} \mapsto X$  such that

$$g^*(x_{t-1}, I_0 \theta_t) = \sum_{a \in \mathbb{A}} c^a x_{t-1}^a + \sum_{a \in I(0)} d^a \theta_t^a + e \bar{\theta} \quad \mathbb{P} - a.s.$$

where  $c^a, d^a, e \geq 0$  and  $e + \sum_{a \in \mathbb{A}} c^a + \sum_{a \in I(0)} d^a = 1$ .

The proof of the existence theorem requires some subtle arguments. While referring to the Appendix for details, a few comments here in this respect will be useful. Consider the (infinite dimensional) choice problem of each agent  $a \in \mathbb{A}$ . To be able to apply standard variational arguments to this problem it is necessary to bound the marginal effect of any infinitesimal change  $dx^a$  on the agent's value function. To this end, the Envelope theorem (as e.g., in [Benveniste and Scheinkman \(1979\)](#)) is not enough, as  $dx^a$  affects agent  $a$ 's value function directly and indirectly, through its effects on all agents  $b \in \mathbb{A} \setminus a$ 's choices, which in turn affect agent  $a$ 's value function. The marginal effect of any infinitesimal change  $dx^a$  is then an infinite sum. Furthermore, each term in the sum contains endogenous terms from some agent  $b \in \mathbb{A} \setminus a$ 's policy function (and there is an infinite number of them), which makes it impossible to adopt the methodology used by [Santos \(1991\)](#) to prove the smoothness of the policy function in infinite dimensional recursive choice problems.

In our economy, with quadratic utility, policy functions are necessarily linear and, provided we show that equilibria are interior, symmetric MPE's in pure strategies can be represented by a policy function which is obtained as a fixed point of a recursive map which can be directly studied.<sup>25</sup> Extending the existence proof to general preferences

---

<sup>25</sup>The class of economies we study are theoretically equivalent to a class of stochastic games, with an infinite number of agents, and uncountable state spaces. Ready-to-use results for the existence of "pure strategy" Markov-perfect equilibria for these environments do not exist. For the state of the art in that

would require therefore sufficient conditions on the structural parameters to control the curvature of the policy function of each agent's decision problem. We conjecture that this can be done although sufficient conditions do not appear transparently from our proof.

### 3.2. The parameters of the policy function

By exploiting the linearity of policy functions, our method of proof is constructive, producing a direct and useful recursive computational characterization for the parameters of the symmetric policy function at equilibrium. We repeatedly exploit this characterization in the next section e.g., when performing comparative dynamics exercises. Consider the choice problem of agent 0. For any  $T$ -period economy, agent 0's dynamic program yields a FOC that takes the following form (see Lemma 3)

$$(3.2) \quad x_1^0 = \left( \frac{1}{\Delta_T} \right) \left( \alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \sum_{a \neq 0} \gamma_T^b x_1^b + \mu_T \bar{\theta} \right)$$

where the coefficients  $\Delta_T$  and  $\gamma_T^b$ , and  $\mu_T$  are the effects on agent zero's discounted expected marginal utility of changes in agents 0 and  $b$ 's first period actions and the change in the level of  $\bar{\theta}$ , respectively.

Let  $L_T$  be a map induced by (3.2) s.t.  $(\hat{c}, \hat{d}, \hat{e}) = L_T(c, d, e)$ , by matching coefficients of the policy on both sides of (3.2), i.e., for each  $a \in \mathbb{A}$

$$(3.3) \quad \begin{aligned} \hat{c}^a &= \Delta_T^{-1} \left( \alpha_1 \mathbf{1}_{\{a=0\}} + \sum_{b \neq 0} \gamma_T^b c^{a-b} \right) \\ \hat{d}^a &= \Delta_T^{-1} \left( \alpha_2 \mathbf{1}_{\{a=0\}} + \sum_{b \neq 0} \gamma_T^b d^{a-b} \right) \\ \hat{e} &= \Delta_T^{-1} \left( \mu_T + e \sum_{b \neq 0} \gamma_T^b \right) \end{aligned}$$

Let  $L_{c,d,e} := \{(c, d, e) : e \geq 0, c^a \geq 0, d^a \geq 0, \forall a \text{ and } e + \sum_a (c^a + d^a) = 1\}$  be the space of nonnegative coefficient sequences summing to 1. The existence of an equilibrium policy for the first period of a  $T$ -period economy is then equivalent to the existence of a coefficient sequence  $(c_T^*, d_T^*, e_T^*)$  which is the fixed point of the map  $L_T : L_{c,d,e} \rightarrow L_{c,d,e}$  induced by (3.2).

The parameters of the map  $L_T$ , namely  $\Delta_T, (\gamma_T^b)_{a \neq 0}, \mu_T$ , depend only on the con-

---

literature, see [Mertens and Parthasarathy \(1987\)](#) and [Duffie et al. \(1994\)](#). See also [Mertens \(2002\)](#) and [Vieille \(2002\)](#) for surveys.

tinuation equilibrium coefficients  $(c_s^*, d_s^*, e_s^*)_{s=1}^{T-1}$  in a linear fashion (see (A.10), (A.12), and (A.14) for their detailed expressions). For  $T = 1$ , the parameters of  $L_1$  are dictated directly by the underlying preferences, namely  $\Delta_1 = \alpha_1 + \alpha_2 + 2\alpha_3$ ,  $\gamma_1^1 = \gamma_1^{-1} = \alpha_3$ ,  $\gamma_1^b = 0$ , for all  $b \neq -1, 0, 1$ , and  $\mu_1 = 0$ . Thus, the map  $L_1$  defined by the system in (3.3) becomes a contraction mapping whose unique fixed point is computed as the unique root to a second-order difference equation that satisfies transversality conditions toward both infinities. Consequently, the equilibrium policy coefficients are computed as in the next Theorem.

**THEOREM 3 (Recursive algorithm)** *Consider a finite-horizon  $T$ -period economy with conformity preferences ( $\alpha_i > 0$ ,  $i = 1, 2, 3$ ) and complete information.*

- (i) *The map  $L_1$  for a one-period economy, defined in (3.3), admits a unique fixed point. We compute the (exponential) coefficient sequence in closed-form. For any  $a \in \mathbb{A}$ ,*

$$(3.4) \quad c_1^{*a} = r_1^{|a|} \left( \frac{\alpha_1}{\alpha_1 + \alpha_2} \right) \left( \frac{1 - r_1}{1 + r_1} \right) \quad \text{and} \quad d_1^{*a} = r_1^{|a|} \left( \frac{\alpha_2}{\alpha_1 + \alpha_2} \right) \left( \frac{1 - r_1}{1 + r_1} \right)$$

where  $r_1 = \left( \frac{\Delta_1}{2\alpha_3} \right) - \sqrt{\left( \frac{\Delta_1}{2\alpha_3} \right)^2 - 1}$  and  $\Delta_1 = \alpha_1 + \alpha_2 + 2\alpha_3$ .

- (ii) *The coefficients  $(c_s^*, d_s^*, e_s^*)_{s=2}^T$  are computed recursively as the unique fixed points of the contraction maps  $L_s : L_{c,d,e} \rightarrow L_{c,d,e}$ ,  $s = 2, \dots, T$ , defined in (3.3), whose parameters  $\Delta_s, (\gamma_s^a)_{a \neq 0}, \mu_s$  depend linearly only on the continuation coefficients  $(c_\tau^*, d_\tau^*, e_\tau^*)_{\tau=1}^{s-1}$ , as defined in (A.10), (A.12), and (A.14).*
- (iii) *Moreover,  $\lim_{T \rightarrow \infty} (c_T^*, d_T^*, e_T^*) = (c^*, d^*, e^*)$  exists and it is the coefficient sequence of the stationary Markovian equilibrium policy function for the infinite-horizon economy.*

Fixed point calculations take less than a few seconds on an ordinary computer, for each period. Finally, the sequence of fixed point maps that we compute at each iteration converges to a policy sequence, which turns out to be the infinite-horizon stationary MPE. The convergence is very rapid, under a few minutes.

### 3.3. Ergodicity

With such a characterization of the parameters of the policy function at hand, we are able to characterize very tightly the spatial (cross-sectional) and intertemporal behavior

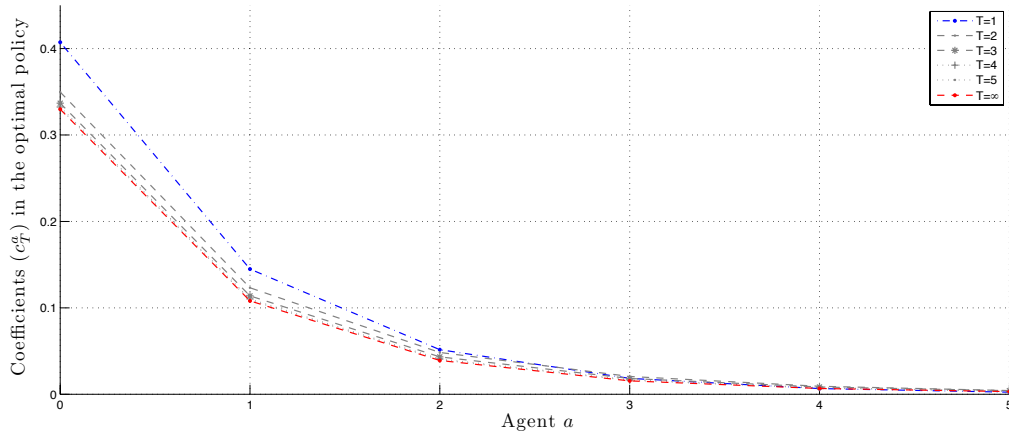


FIGURE 1.— Non-stationary Optimal Policy.

of the equilibrium process emerging from the class of dynamic models we study. Let  $\pi_0$  be an initial distribution on the configuration space  $\mathbf{X}$ . Given the initial distribution  $\pi_0$ , a stationary MPE of the economy with conformity induces an equilibrium process  $(x_t \in \mathbf{X})_{t=0}^\infty$  (via the policy function  $g^*$ ) and an associated transition function  $Q_{g^*}$ . This latter generates iteratively a sequence of distributions  $(\pi_t)_{t=1}^\infty$  on the configuration space  $\mathbf{X}$ , i.e., for  $t = 0, 1, \dots$

$$\pi_{t+1}(A) = \pi_t Q_{g^*}(A) = \int_{\mathbf{X}} Q_{g^*}(x_t, A) \pi_t(dx_{t+1})$$

We show first that, given the induced equilibrium process, the transition function  $Q_{g^*}$  admits an *invariant distribution*  $\pi$ , i.e.,  $\pi = \pi Q_{g^*}$  and that the equilibrium process starting from  $\pi$  is *ergodic*<sup>26</sup>.

Ergodicity does not necessarily imply the convergence of the equilibrium process to a unique distribution starting from an arbitrary initial distribution  $\pi_0$ . Conditions are necessary to guarantee such convergence.<sup>27</sup> We next show that, for any initial distribution  $\pi_0$  and a stationary Markovian policy function  $g^*$ , the equilibrium process  $(x_t \in \mathbf{X})_{t=0}^\infty$

<sup>26</sup>We call a Markov process  $(x_t)$  with state space  $\mathbf{X}$  under a probability measure  $P$  ergodic if  $\frac{1}{T} \sum_{t=1}^T f(x_t) \rightarrow \int f dP$   $P$ -almost surely for every bounded measurable function  $f : \mathbf{X} \rightarrow \mathbb{R}$ . See, e.g., [Duffie et al. \(1994\)](#) for a similar usage.

<sup>27</sup>The well-known Döblin conditions to that effect can be found in [Doob \(1953\)](#). See also [Tweedie \(1975\)](#) for a similar characterization.



converges in distribution to the invariant distribution  $\pi$ , independently of  $\pi_0$ .<sup>28</sup> This also implies that  $\pi$  is the *unique* invariant distribution of the equilibrium process  $(x_t \in \mathbf{X})_{t=0}^\infty$ . More specifically,

**THEOREM 4 (Ergodicity)** *Suppose the process  $((\theta_t^a)_{t=-\infty}^\infty)_{a \in \mathbb{A}}$  is i.i.d. with respect to  $a$  and  $t$  according to  $\nu$ . The equilibrium process  $(x_t \in \mathbf{X})_{t=0}^\infty$  induced by a symmetric stationary Markov perfect equilibrium of an economy with conformity via the policy function  $g^*(x_{t-1}, \theta_t)$  and the unique invariant measure  $\pi$  as the initial distribution is ergodic;  $\pi$  is the joint distribution of*

$$(3.5) \quad x_t = \left( \frac{e \bar{\theta}}{1 - C} + \sum_{s=1}^\infty \sum_{b_1 \in \mathbb{A}} \cdots \sum_{b_s \in \mathbb{A}} c^{b_1} \cdots c^{b_{s-1}} d^{b_s} \theta_{t+1-s}^{a+b_1+\cdots+b_s} \right)_{a \in \mathbb{A}}$$

where  $C := \sum_{a \in \mathbb{A}} c^a$  is the sum of coefficients in the stationary policy function that multiply corresponding agents' last period choices. Moreover, the sequence  $(\pi_t)_{t=1}^\infty$  of distributions generated by the equilibrium process  $(x_t \in \mathbf{X})_{t=0}^\infty$  converges to  $\pi$  in the topology of weak convergence for probability measures, independently of any arbitrary initial distribution  $\pi_0$ .<sup>29</sup>

#### 4. CHARACTERIZATION OF EQUILIBRIUM

Exploiting the linear structure of our economies we can study equilibria in some detail. Recall that the policy function in each period  $t = 1, \dots, T$ , for each agent  $a \in \mathbb{A}$ , is

$$(4.1) \quad x_t^a = \sum_{b \in \mathbb{A}} c_{T-(t-1)}^b x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^b \theta_t^{a+b} + e_{T-(t-1)} \bar{\theta},$$

with  $e_{T-(t-1)} + \sum_{a \in \mathbb{A}} (c_{T-(t-1)}^a + d_{T-(t-1)}^a) = 1$ , when  $T$  is finite; and

$$(4.2) \quad x_t^a = \sum_{b \in \mathbb{A}} c^b x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d^b \theta_t^{a+b} + e \bar{\theta},$$

with  $e + \sum_{a \in \mathbb{A}} (c^a + d^a) = 1$ , in the infinite-horizon case.

First of all, we study the *parameters of the policy function*. The coefficients  $c_{T-(t-1)}^b$

<sup>28</sup>Note however that Theorem 1 does not guarantee that the policy function  $g^*(x_{t-1}, \theta_t)$  is unique.

<sup>29</sup>A sequence of probability measures  $(\lambda_t)$  is said to converge weakly (or in the topology of weak convergence for probability measures) to  $\lambda$  if, for any bounded, measurable, continuous function  $f : \mathbf{X} \rightarrow \mathbb{R}$ ,  $\lim_{t \rightarrow \infty} \int f d\lambda_t = \int f d\lambda$  almost surely (see e.g. [Kallenberg \(2002\)](#), p.65).

and  $d_{T-(t-1)}^b$  (resp.  $c^b$  and  $d^b$  in the case of infinite-horizon economies), in particular, may be viewed as a measure for the total impact of the action  $x_{t-1}^{a+b}$  and of the preference shock  $\theta_t^{a+b}$  of agent  $a + b$ , respectively, on the optimal current choice of agent  $a$ ; where  $b$  concisely represents the *social distance* between the two agents.<sup>30</sup> Furthermore, we study a fundamental statistical property of equilibrium, *cross-sectional auto-correlation of actions*. In fact, although any agent  $a \in \mathbb{A}$  interacts directly only with a small subset of the population, at equilibrium, each agent's optimal choice is correlated with those of all the other agents. Let  $\rho_{a,T}$  denote the conditional correlation between the first-period equilibrium actions of agents  $a$ -step away from each other, in the  $T$ -period economy, given  $x_0 \in \mathbf{X}$ .<sup>31</sup>

$$(4.3) \quad \rho_{a,T} = \frac{\text{Cov} \left( x_1^0, x_1^a \mid x_0 \right)}{\text{Var} \left( x_1^0 \mid x_0 \right)}.$$

#### 4.1. Policy Function

Consider first a finite-horizon economy. Since the policy function for this economy is well-defined, the coefficients  $c_{T-(t-1)}^b$  and  $d_{T-(t-1)}^b$  satisfy

$$\lim_{|b| \rightarrow \infty} c_{T-(t-1)}^{a+b} = \lim_{|b| \rightarrow \infty} d_{T-(t-1)}^{a+b} = 0$$

The impact of an agent  $a + b$  on agent  $a$  tends to zero as  $|b| \rightarrow \infty$ . In this sense, linear conformity economies display weak social interactions.

Furthermore, as we have shown in Section 3.2,

$$\lim_{T \rightarrow \infty} c_T = c, \quad \lim_{T \rightarrow \infty} d_T = d, \quad \text{and} \quad \lim_{T \rightarrow \infty} e_T = e$$

---

<sup>30</sup>See Akerlof (1997) for richer definitions of social distance.

<sup>31</sup>The correlation between the first-period optimal choices of agents  $a$  and  $b$ , is

$$\frac{\text{Cov} \left( x_1^a, x_1^b \mid x_0 \right)}{\sqrt{\text{Var} \left( x_1^a \mid x_0 \right) \text{Var} \left( x_1^b \mid x_0 \right)}}.$$

Due to the symmetry imposed on our economy, such correlations are independent of agents' labels but depends only on  $|b - a|$ . Consequently, we can define the conditional correlation function with distances computed relative to any agent, in particular agent 0.

The finite-horizon parameters converge (uniformly) to the infinite-horizon stationary policy parameters.

Finally, equilibrium policy functions are non-stationary in the finite economy, as rational forward-looking agents change their behavior optimally through time. In the final periods, for example, social interactions lose weight relative to individual characteristics; see Figure 1.<sup>32</sup>

#### 4.2. Cross-sectional Auto-correlations

Exploiting the equilibrium characterization provided by Theorems 1 and 3, and the independence of preference shocks across agents, we can compute the covariance terms:

$$(4.4) \quad Cov \left( x_1^0, x_1^a \mid x_0 \right) = Var(\theta) \sum_{a_1 \in \mathbb{A}} d_T^{a_1} d_T^{a_1 - a}.$$

The expression  $\sum_{a_1 \in \mathbb{A}} d_T^{a_1} d_T^{a_1 - a}$  is the discrete self-convolution of the equilibrium policy sequence  $d_T = (d_T^{a_1})_{a_1 \in \mathbb{A}}$ , where  $a$  acts as the shift parameter.<sup>33</sup> In Figure 2 we show how the convolution behaves with respect to the distance  $a$ , for the same set of parameters as in Figure 1. Substituting the form in (4.4) back in (4.3) for both terms, we obtain

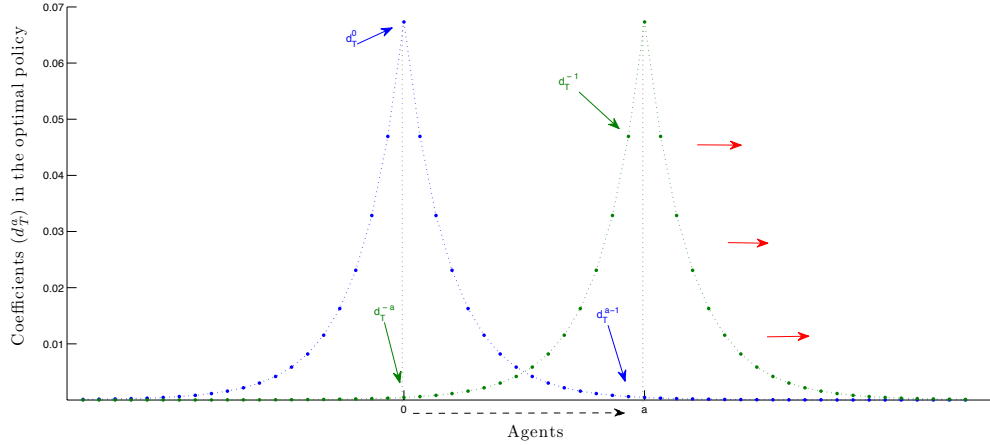


FIGURE 2.— Convolution of the Policy Coefficient Sequence.

<sup>32</sup>We plot in Figure 1 only one side of the policy coefficient sequence to get a close-up view of the change in equilibrium behavior. The left hand side is the mirror image of that due to symmetry. Parameter values for this figure are  $\frac{\alpha_1}{\alpha_3} = 1$ ,  $\frac{\alpha_3}{\alpha_2} = 10$ , and  $\beta = .95$

<sup>33</sup>See (D.1) for the derivation.

$$(4.5) \quad \rho_{a,T} = \frac{\sum_{b \in \mathbb{A}} d_T^b d_T^{b-a}}{\sum_{b \in \mathbb{A}} d_T^b d_T^b}$$

the  $a$ -step conditional cross-sectional autocorrelations for the first-period equilibrium choices of the  $T$ -period economy. Exploiting the recursive algorithm provided by Theorem 3, we can compute these autocorrelations easily for any finite economy. We can then study the behavior of the conditional correlation function  $\rho_{a,T}$  through time ( $T$ ) and across social space ( $a$ ). These correlations exhibit interesting dynamics: they are declining in  $a$ , for any  $T$ , but the rate of decline cannot be ranked in  $T$ , given  $a$ ; see Figure 3 for an example with the same parametrization we used above for the policy weights in Figure 1.

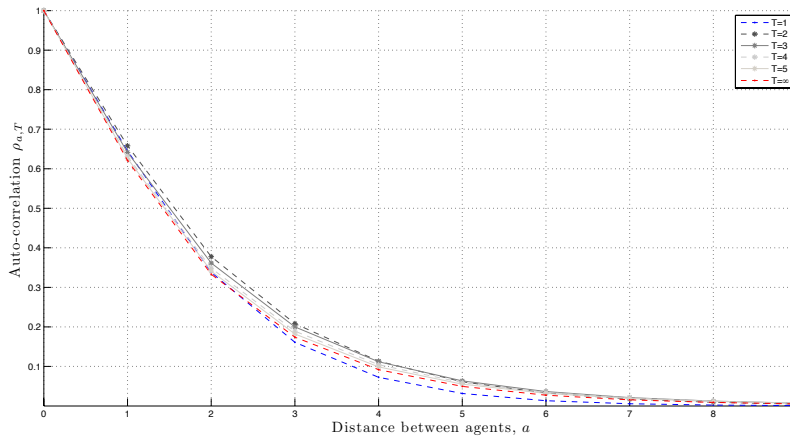


FIGURE 3.— Cross-sectional Auto-correlations.

In particular, given a  $T$ -period economy, consider the  $T$ -period rate of convergence of the spatial autocorrelations, for  $a \geq 0$ <sup>34</sup>

$$r_{a,T} = \frac{\rho_{a+1,T}}{\rho_{a,T}}.$$

We show analytically that  $r_{a,1}$  declines monotonically and becomes constant at the tail in  $a$ .<sup>35</sup> On the other hand,  $r_{a,T}$  is typically non-monotonic in  $a$ , for longer horizons, including for  $T = \infty$ ; see Figure 4. This is the case because each agent's policy function results from the composition of two distinct effects, the cross-sectional interaction and the dynamic effect of actions today in the future. The first effect vanishes exponentially

<sup>34</sup>The rate is symmetrically defined with respect to agent 0, i.e.,  $r_{a,T} = \frac{\rho_{a-1,T}}{\rho_{a,T}}$ , for any  $a \leq 0$ .

<sup>35</sup>See the proof of Proposition 1 for the argument.

with distance at equilibrium, while the rate at which the dynamic effect vanishes is not constant. In fact, the sum of the two effects becomes hyperbolic in distance, i.e., it decreases very sharply when you go beyond an agent's neighborhood and then stabilizes with social distance. The non-monotonicity of the rate of convergence  $r_{a,T}$  is the result of the self-convolution of an hyperbolic sequence of policy coefficients (see Equation (4.4) and Figure 2), which enters in the computation of the covariance. In the case of static economies (or equivalently, of economies in their last period), the dynamic effect is not present, the sequence of policy coefficients is therefore exponential (rather than hyperbolic), and the rate of convergence  $r_{a,1}$  is monotonically decreasing.

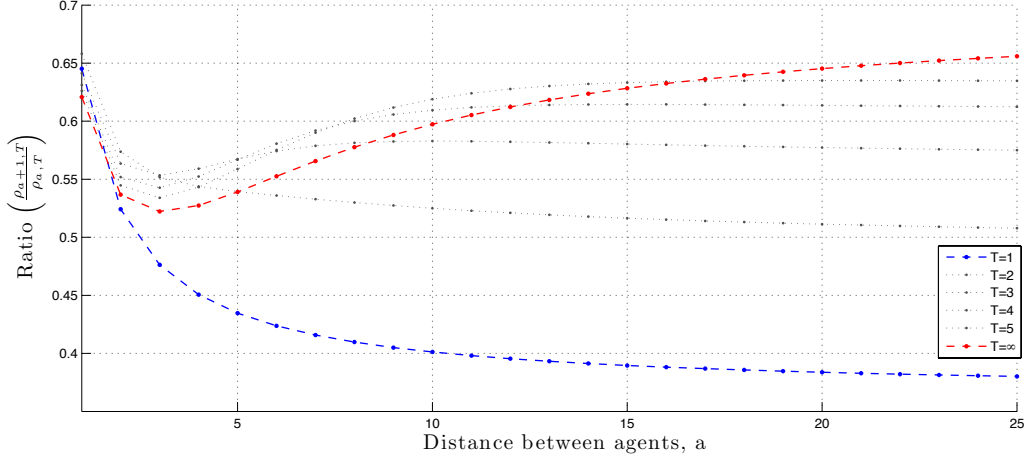


FIGURE 4.— Rate of Convergence of the Auto-correlations.

Finally, consider the  $T$ -period rate of tail convergence of the spatial autocorrelations,

$$r_T := \lim_{a \rightarrow \infty} r_{a,T} = \lim_{a \rightarrow \infty} \left( \frac{\rho_{a+1,T}}{\rho_{a,T}} \right)$$

Similarly, let the same rate for the infinite-horizon economy ( $T = \infty$ ) be represented by  $r$ .

**PROPOSITION 1 (Tail Convergence Monotonicity)** <sup>36</sup> *The rate  $r_T$  is monotone increasing with respect to the length of the economy,*

$$r_{T+1} > r_T, \text{ for finite } T \geq 1.$$

---

<sup>36</sup>The proof is in Supplemental Appendix D.

Moreover, the sequence of tail convergence rate for finite-horizon economies converges to that of the infinite-horizon economy as the horizon length gets larger and the limit rate is strictly less than 1:

$$\lim_{T \rightarrow \infty} r_T = r < 1.$$

In other words, even though the autocorrelation functions might behave non-monotonically for shorter social distances, they eventually converge (as social distance  $a \rightarrow \infty$ ) to an exponential rate in the tail. Moreover, rates of tail convergence are higher the farther is the final period of the economy (as  $T \rightarrow \infty$ ). This is because rational agents choose to correlate their actions more with their neighbors in early periods and progressively less so as they approach the end of their social interactions. Finally, as the infinite-horizon limit is approached, the rate of tail convergence becomes stationary (as to be expected since finite-horizon equilibria approximate the stationary infinite-horizon equilibrium). We use this intuition to the fullest extent when discussing identification in Section 6.

In an infinite-horizon economy social interactions manifest themselves at the stationary ergodic distribution by means of spatial autocorrelation of actions. Given  $x_0 \in \mathbf{X}$ , the conditional covariance in period  $t$  of an infinite-horizon economy, between two agents  $a$  agents away from each other is denoted by  $Cov(x_t^0, x_t^a \mid x_0)$ . Let  $Cov(x_\infty^0, x_\infty^a)$  be the  $a$ -step *unconditional* covariance at the ergodic stationary distribution. Since the stationary MPE is ergodic, it is easy to see from Lemma 2 (i) and Theorem 4 that as  $t$  gets arbitrarily large, the conditional  $t$ -period covariance between agents 0 and  $a$  converges to its unconditional counterpart at the limit distribution, i.e.,

$$Cov(x_\infty^0, x_\infty^a) = \lim_{t \rightarrow \infty} Cov(x_t^0, x_t^a \mid x_0)$$

Moreover, the limit unconditional correlation  $\rho_b$  between the actions of agents  $a$  and  $a+b$  is independent of  $x_0$  and it satisfies

$$\rho_b = \frac{Cov(x_\infty^0, x_\infty^a)}{Var(x_\infty^0)} = \lim_{t \rightarrow \infty} \frac{Cov(x_t^0, x_t^a \mid x_0)}{Var(x_t^0 \mid x_0)}$$

Finally, because of the stationarity of the policy function in (4.2), the limit covariance between two agents  $a$  agents away from each other can be written as

$$\begin{aligned}
Cov(x_\infty^0, x_\infty^a) &= \lim_{t \rightarrow \infty} Cov(x_t^0, x_t^a \mid x_0) \\
(4.6) \qquad \qquad &= \sum_{a_1 \in \mathbb{A}} \sum_{b_1 \in \mathbb{A}} c^{a_1} c^{b_1} Cov(x_\infty^{a_1}, x_\infty^{a+b_1}) + Var(\theta) \sum_{a_1 \in \mathbb{A}} d^{a_1} d^{a_1-a},
\end{aligned}$$

and hence it has a simple recursive structure. In fact, since the sum of the stationary weights multiplying covariances on the right hand side are strictly less than one, this system can be seen as a contraction operator. Hence, for each one-step conditional autocorrelation sequence, there is a unique stationary unconditional autocorrelation sequence that we can compute using the above recursive system easily. We later exploit this recursive structure further in Section 6.1 when we compare equilibrium stationary distributions induced by myopic and rational agents.

In Figure 5, we report the correlation functions in both the mild and strong conformity parameterizations as a function of social distance,  $b$ .<sup>37</sup> Two effects are worth mentioning here. Firstly, both correlation functions converge to zero as the distance between two agents become arbitrarily large. Secondly, this convergence is much faster in the case of mild interactions than in the case of strong interactions. For example, the correlation between the equilibrium choices of agent  $a$  and agent  $a+3$  (or  $a-3$  due to symmetry) is about 7% in the case of mild interactions whereas it is about 75% in the case of strong interactions. The correlation between the equilibrium choices of agent  $a$  and agent  $a+6$  are about 0% and 40% respectively. The strength of the desire to conform built in individuals' preferences determine endogenously, at equilibrium, the size of the effective neighborhood with which an individual interacts.

## 5. EQUILIBRIUM PROPERTIES AND COMPARATIVE DYNAMICS

In this section we first study the welfare properties of equilibrium and then we use the characterization of equilibria we obtained to produce several simulations illustrating various comparative dynamics exercises of interest.

### 5.1. (In)efficiency

Social interactions are modelled in this paper as a preference externality, that is, by introducing a dependence of agent  $a$ 's preferences on his/her peers' actions. Not surprisingly, therefore, equilibria will not be efficient in general. In this section we also

---

<sup>37</sup>See Section 5.2 for the parameter values for the mild and strong interaction cases.

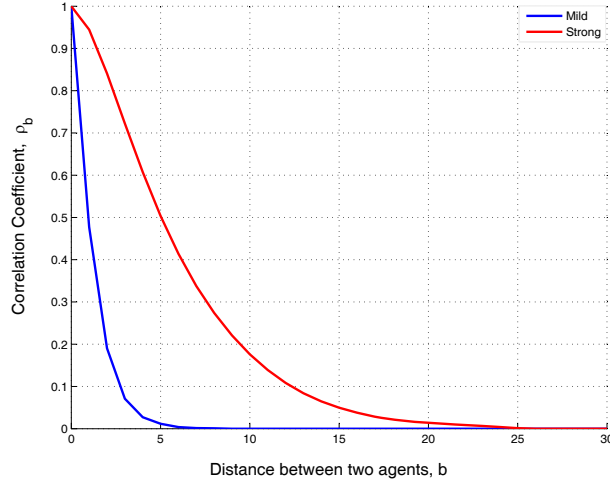


FIGURE 5.— Correlation function at the ergodic distribution for Mild and Strong Interactions.

characterize the form the inefficiency takes when social interactions are modelled as preferences for conformity.

A benevolent social planner, taking into account the preference externalities and at the same time treating all agents symmetrically, would maximize the expected discounted utility of a generic agent, say of agent  $a \in \mathbb{A}$ , by choosing a symmetric choice function  $h$  in  $CB(\mathbf{X} \times \Theta, X)$ , the space of bounded, continuous, and  $X$ -valued measurable functions. The choice of  $h$  induces, in a recursive way, a sequence of choices for any agent  $b \in \mathbb{A}$ , given  $(x_0, \theta_1)$ , by

$$(5.1) \quad x_t^b(h)(x^{t-1}, \theta^t) = h_{T-(t-1)}(R^b x_{t-1}, R^b I_0 \theta_t), \quad \text{for } t = 1, \dots, T$$

Then, the planner solves<sup>38</sup>

$$\begin{aligned} \max_{\{h \in CB(\mathbf{X} \times \Theta, X)\}} \int \sum_{t=1}^T \beta^{t-1} & \left( -\alpha_1 (x_{t-1}^a(h)(x^{t-1}, \theta^t) - x_t^a(h)(x^{t-1}, \theta^t))^2 \right. \\ & -\alpha_2 (\theta_t^a - x_t^a(h)(x^{t-1}, \theta^t))^2 \\ & -\alpha_3 (x_t^{a-1}(h)(x^{t-1}, \theta^t) - x_t^a(h)(x^{t-1}, \theta^t))^2 \\ & \left. -\alpha_3 (x_t^{a+1}(h)(x^{t-1}, \theta^t) - x_t^a(h)(x^{t-1}, \theta^t))^2 \right) \prod_{t=1}^T \mathbb{P}(d\theta_t) \pi_0(dx_0) \end{aligned}$$

<sup>38</sup>With the convention that  $h_{T-(t-2)}(R^a x_{t-2}, R^a I_0 \theta_{t-1}) = x_0^a$  when  $t = 1$ .



where  $\pi_0$  is an absolutely continuous distribution on the initial choice profiles with a positive density. This problem can be written recursively. For any agent  $a \in \mathbb{A}$ , for all  $t = 1, \dots, T$ , and all  $(x_{T-1}, \theta_T) \in \mathbf{X} \times \Theta^{I(0)}$ , let the value of using the choice rule  $h$  in the continuation be defined as

$$\begin{aligned} V^{h,T-(t-1)}(R^a x_{T-1}, R^a \theta_T) = & -\alpha_1 (x_{t-1}^a - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \\ & -\alpha_2 (\theta_t^a - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \\ & -\alpha_3 (h_{T-(t-1)}(R^{a-1} x_{t-1}, R^{a-1} I_0 \theta_t) - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \\ & -\alpha_3 (h_{T-(t-1)}(R^{a+1} x_{t-1}, R^{a+1} I_0 \theta_t) - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \\ & +\beta \int V^{h,T-t} \left( R^a \{h_t(R^b x_{t-1}, R^b I_0 \theta_t)\}_{b \in \mathbb{A}}, R^a I_0 \theta_{t+1} \right) \mathbb{P}(d\theta_{t+1}) \end{aligned}$$

which leads us to the following definition

**DEFINITION 2 (Recursive Planning Problem)** *Let a  $T$ -period linear economy with social interactions and conformity preferences be given. Let  $\pi_0$  be an absolutely continuous distribution on the initial choice profiles with a positive density. A symmetric Markovian choice function  $g : \mathbf{X} \times \Theta^{I(0)} \times \{1, \dots, T\} \rightarrow X$  is said to be **efficient** if it is a solution, for all  $a \in \mathbb{A}$ , and for all  $t = 1, \dots, T$ , to*

$$\begin{aligned} \arg \max_{\{h \in CB(\mathbf{X} \times \Theta, X)\}} \int & \left[ -\alpha_1 (x_{t-1}^0 - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \right. \\ & -\alpha_2 (\theta_t^a - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \\ & -\alpha_3 (h_{T-(t-1)}(R^{a-1} x_{t-1}, R^{a-1} I_0 \theta_t) - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \\ & -\alpha_3 (h_{T-(t-1)}(R^{a+1} x_{t-1}, R^{a+1} I_0 \theta_t) - h_{T-(t-1)}(R^a x_{t-1}, R^a I_0 \theta_t))^2 \\ & \left. +\beta V^{h,T-t} \left( R^a \{h(R^b x_{t-1}, R^b I_0 \theta_t)\}_{b \in \mathbb{A}}, R^a I_0 \theta_{t+1} \right) \right] \mathbb{P}(d\theta_t) \mathbb{P}(d\theta_{t+1}) \pi_t(dx_{t-1}) \end{aligned}$$

where  $\pi_t$  is the distribution on  $t$ -th period choice profiles induced by  $\pi_0$  and the planner's choice rule  $h$ .

As noted, preferences for conformity introduce an externality in each agent  $a \in \mathbb{A}$ 's decision problem, which depends directly on the actions of agents in neighbourhood  $N(a)$  and, indirectly, on the actions of all agents in the economy. In equilibrium, agents do not internalize the impact of their choices on other agents today and in the future. More precisely,

**THEOREM 5 (Inefficiency of equilibrium)** *A symmetric MPE of a conformity economy is inefficient.*

Furthermore, an efficient policy function will tend to weight less heavily the agent's own-effect and more heavily other agents' effects, relative to the equilibrium policy. This effect, hence the inefficiency, are neatly exhibited by comparing the equations determining policy weights in the planner (E.3) and equilibrium (A.2) scenarios. The (absolute value of the) weights the planner's equation associates on neighbors' choices is twice as large as the weights associated to neighbors in the equilibrium equation  $\left(\frac{\alpha_3}{\alpha_1 + \alpha_2 + 2\alpha_3}\right)$ . As a consequence, the relative weights that the planner assigns to neighbors' choices are always higher than the ones that each agent uses in equilibrium.<sup>39</sup>

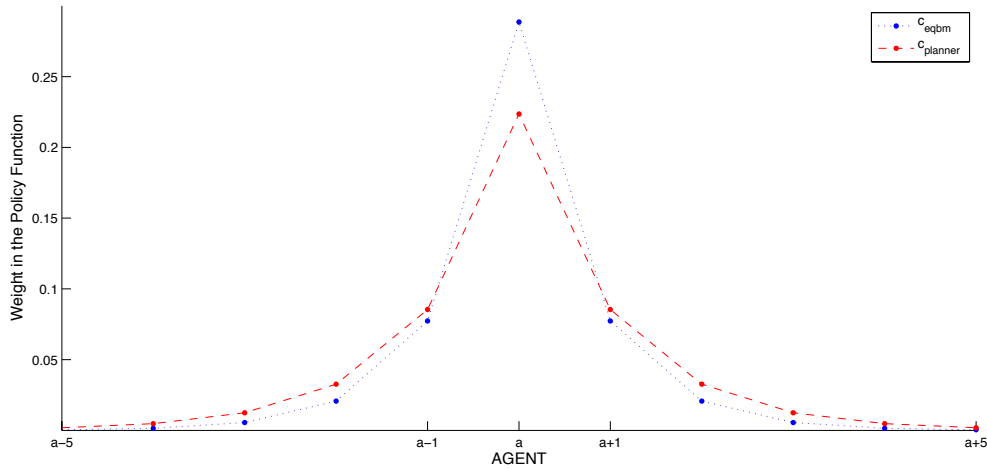


FIGURE 6.— Inefficiency of equilibrium.

A graphic representation of the inefficiency is obtained in Figure 6, which presents the coefficient plot for the equilibrium policy of a one-period economy (equivalently the final period of any finite-horizon economy):  $c_{eqbm}$  (blue dots), and for the planner's solution,  $c_{planner}$  (red dots), respectively, for a given agent  $a \in \mathbb{A}$ , and for a given set of parameter values ( $\frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = 1$ , and  $\beta = .95$ ).<sup>40</sup>

<sup>39</sup>Normalizing the relative coefficients to form a probability measure (see the argument in the proof of Lemma 2 (iv)), we have that the measure obtained from the planner's policy is a *mean-preserving spread* of the measure obtained at equilibrium.

<sup>40</sup>We call this parametrization the *mild-interaction* case in Section 5.2.

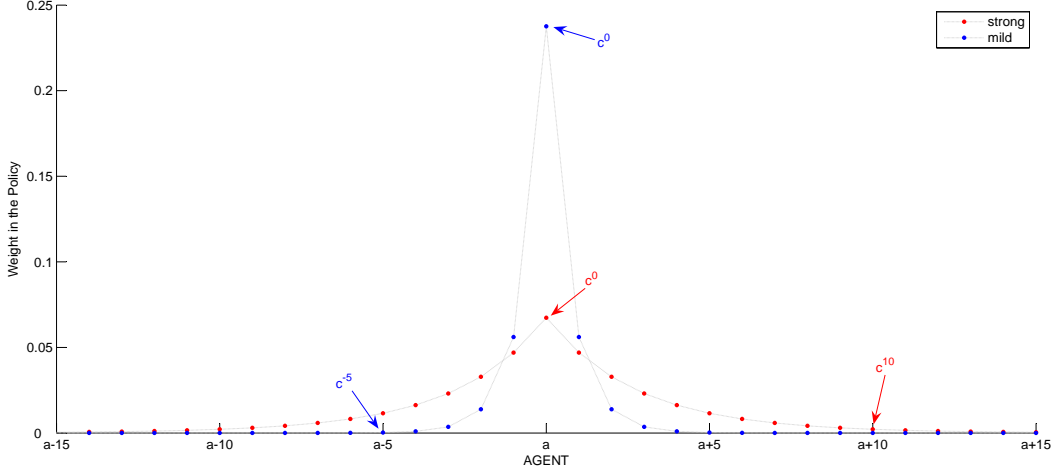


FIGURE 7.— Weights on past history in the stationary policy function.

### 5.2. Comparative Dynamics: Peer Effects

The strength of the agents' preferences for conformity depends on the size of the preference parameter  $\alpha_3$  relatively to  $\alpha_1$  and  $\alpha_2$ . A policy function is represented in Figure 7, which compares a case with *mild* preferences for conformity (with parametrization  $\frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = 1$ )<sup>41</sup> with one with *strong* preferences for conformity (with parametrization  $\frac{\alpha_1}{\alpha_2} = 1$ ,  $\frac{\alpha_2}{\alpha_3} = \frac{1}{20}$ ). On the x-axis, we plot agent  $a$  and his neighbors, while on the y-axis, we plot the weights  $(c^b)_{b \in \mathbb{A}}$  that the symmetric policy function  $g$  associates with the last period actions of agents  $(a + b)_{b \in \mathbb{A}}$ . While each agent's interaction neighborhood is only composed of two agents, in effect local interactions involve indirectly larger groups. How large are the groups depends endogenously on the strength of the agents' preferences for conformity. Notice e.g., that in Figure 7, local interactions involve effectively a group of about ten neighbors when preferences for conformity are mild and involve a group of about thirty neighbors when preferences for conformity are strong. Furthermore, for the same cases of mild and strong conformity, we compare in Figure 8 the case in which neighborhoods are overlapping,  $N(a) = \{a - 1, a + 1\}$ , with the case of non-overlapping one-sided neighborhoods,  $N(a) = \{a + 1\}$ .<sup>42</sup> Two effects are present here. Firstly, as in

<sup>41</sup>The discount rate is fixed at  $\beta = .95$  in all the simulations unless mentioned otherwise.

<sup>42</sup>In this case, the policy function is

$$x_t^a = g(R^a x_{t-1}, R^a \theta_t) = \sum_{b \geq 0} c^b x_{t-1}^{a+b} + \sum_{b \geq 0} d^b \theta_t^{a+b} + e \bar{\theta}.$$

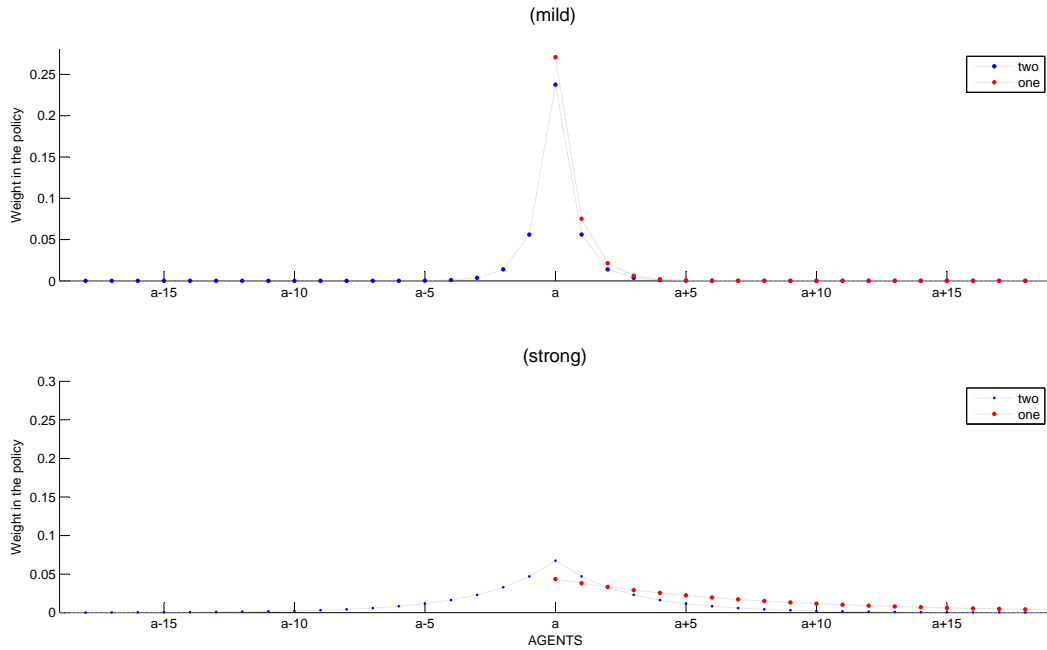


FIGURE 8.— One-sided vs. two-sided interactions.

Figure 7, an increase in the strength of the interaction parameter spreads the interaction effects over a larger social geography. Secondly, this spread is observed most significantly in the case of non-overlapping neighborhoods due to the uni-directional nature of the interactions.

At the ergodic stationary distribution, when the dependence of the agents' actions in equilibrium are independent of the initial configuration of actions  $x_0$ , such correlations in endogenously formed groups is manifested in a phenomenon which we refer to as *local norms of behavior* (see Figure 9).<sup>43</sup> In Figure 9, we plot 100 neighboring agents on the x-axis and their optimal choices drawn from the limit distribution at the same future date, on the y-axis. In the top panel, clearly the optimal actions are more spread and do not follow a significant pattern. In the bottom panel though, the optimal choices are more concentrated and follow a clear path. This is due to the fact that, in equilibrium agents conform to the actions of neighboring agents, leading the way to the creation of similar local behavior. In the bottom panel of Figure 9, we observe groups of agents (e.g., in the neighborhood of agent 20) choosing relatively low actions, and other groups

<sup>43</sup>See Appendix G for details about how we simulate the ergodic stationary distribution of actions of the economy.

(e.g., in the neighborhood of agent 70) choosing instead high actions. Two interesting aspects of this phenomenon are firstly that every individual uses the same symmetric policy function to make his choices and all heterogeneity is captured by random types and we still have high spatial correlation and high spatial variation. Secondly, the initial configuration of actions is irrelevant since the limit distribution of individual actions in this economy is ergodic.

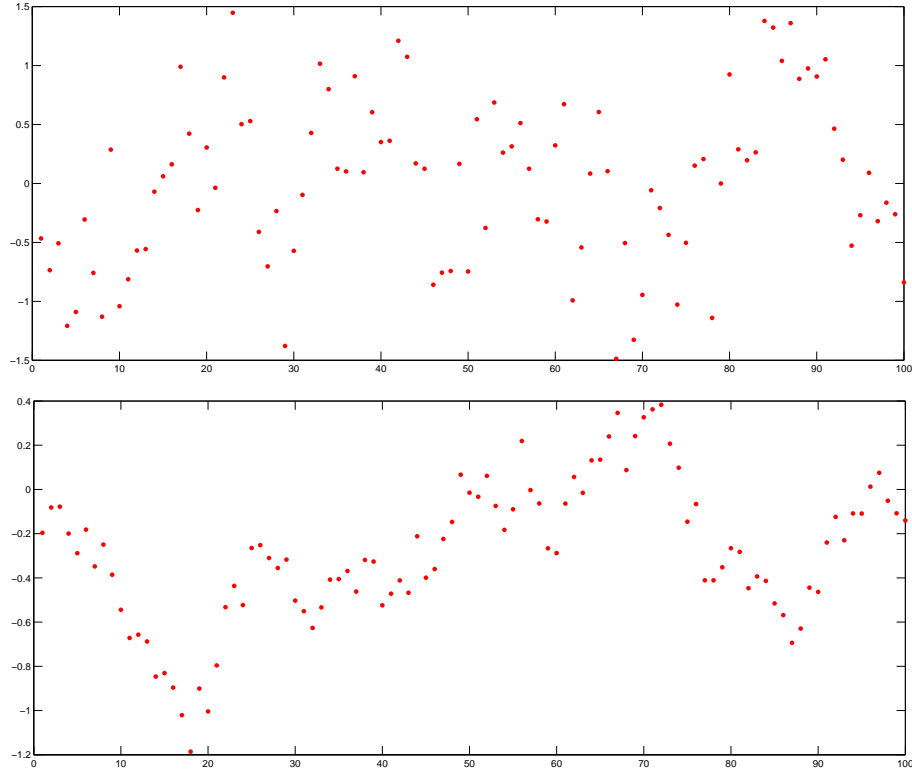


FIGURE 9.— Ergodic Limit of *Mild* (top) and *Strong* (bottom) Interactions for 100 neighboring agents.

### 5.3. Comparative Dynamics: Information

In Figure 10, we compare the case in which agents have complete information with the case in which they have incomplete information. In this last case, the policy function is

$$x_t^a = g(R^a x_{t-1}, R^a I_0 \theta_t) = \sum_{b \in \mathbb{A}} c^b x_{t-1}^{a+b} + \sum_{b \in I(0)} d^a \theta_t^{a+b} + e \bar{\theta}.$$

In particular, we record the effect of an expansion of the information set  $R^a I_0 \theta_t$  (individuals whose types are observed by agent  $a$ ) on best responses. We start with an information structure in which each agent observes his own type only. We then increase the number of types observed by each agent  $a$  (maintaining the symmetry of two-sided interactions) up to the complete information limit. The red dots represent the optimal

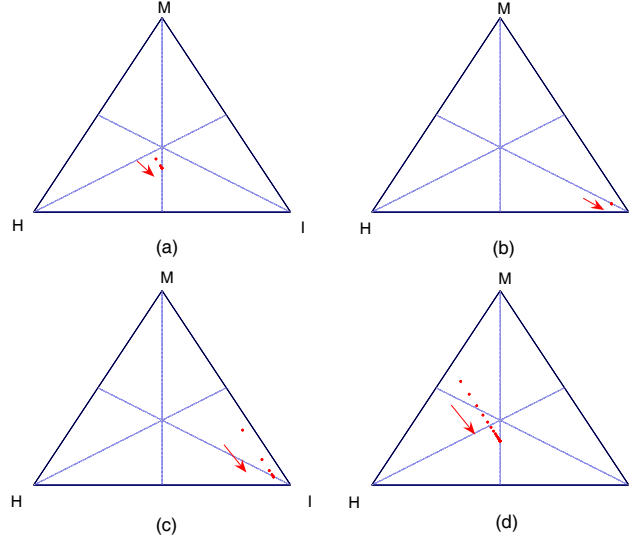


FIGURE 10.— Effect of Information on Interactions.

weights in the policy function of an of agent  $a$  as a response to the informational structure. The lower left vertex represents **(H)**istory, the total sum of weights assigned to last period's choices. The lower right vertex represents **(I)**nformation, the sum of weights on current types observed. Finally, the upper vertex represents average information, **(M)**ean type,  $\bar{\theta}$ . In part (a), we have *mild preferences* for conformity once again. The dots are concentrated near the middle of the triangle (equal weights on history, information, and mean type) and they do not move much as a response to changes in the amount of current information. Part (d) is the counterpart with *strong interactions*. Hence the significant change from almost no weight on current information to almost equal weights. Individuals use the information in the best possible way by putting more weight on it in their policy functions. This is due to the fact that forming expectations more precisely how the neighbors will behave becomes more important for each agent, due to the increased strength of interactions. Part (c) is *mild interactions* but *strong own-type effect* ( $\frac{\alpha_1}{\alpha_2} = \frac{1}{20}$ ,  $\frac{\alpha_2}{\alpha_3} = 20$ ) and part (b) is *strong interactions* and *strong own-type effect*

( $\frac{\alpha_1}{\alpha_2} = \frac{1}{20}$ ,  $\frac{\alpha_2}{\alpha_3} = 1$ ). We do not see much change in (b), although most of the total weight is put on information. This is mainly due to the fact that any agent  $a$  cares so much about his current type that, he neglects the other effects. In (c), although the own-effect is still strong, due to the strength of interactions, each agent uses the average information to form the best expectations regarding the behavior of the other agents. As the amount of information increases, each agent forms better expectations by transferring the policy weight from average information to precise information on close neighbors.

## 6. IDENTIFICATION

We study here the identification of social interactions in the context of our linear dynamic economy with conformity. Identification obtains when the restrictions imposed on actions at equilibrium by preferences for conformity are distinct from those imposed by other relevant structural models.<sup>44</sup> Consider in particular an alternative structural model characterized by (cross-sectionally) correlated preferences across agents. This specific alternative model is focal because correlated preferences could be generally due to some sort of assortative matching or positive selection in social interaction, which induces agents with correlated preferences to interact socially. Suppose an econometrician observes panel data of individual actions over time displaying spatial correlation of individual actions at each time. Such correlation can generally be due to social interactions, as our analysis has shown. Such correlation could also ensue, however, from the spatial correlation of preference types, which we have excluded by assumption in our analysis to this point. But is there any structure in the spatial correlation which is implied by preference for conformity and not by correlated preferences? An affirmative answer to this question implies that the social interaction model is identified with respect to the correlated preferences model.

The structural analysis of identification in linear economies with social interactions starts with [Manski \(1993\)](#).<sup>45</sup> Manski restricts his analysis to static linear models, or, more specifically, linear economies in which the social interactions operate through the mean

---

<sup>44</sup>The question of identification in economics has been clearly defined by [Koopmans \(1949\)](#) and [Koopmans and Reiersøl \(1950\)](#). The issue of identification goes back to [Pigou \(1910\)](#), [Schultz \(1938\)](#), [Frisch \(1934, 1938\)](#), [Marschak \(1942\)](#), [Wald \(1950\)](#), [Hurwicz \(1950\)](#). By *identification* we mean *identification in population* (Sometimes *identification in population* is called *identifiability*; see e.g., [Chiappori and Ekeland \(2009\)](#)). More recent surveys on the topic exist of course; see [Rothenberg \(1971\)](#), [Hausman and Taylor \(1983\)](#), [Hsiao \(1983\)](#), [Matzkin \(2007\)](#), and [Dufour and Hsiao \(2008\)](#).

<sup>45</sup>[Blume et al. \(2011\)](#), [Blume and Durlauf \(2005\)](#), [Brock and Durlauf \(2007\)](#), [Graham \(2011\)](#), and [Manski \(1993, 2000, 2007\)](#) survey the main questions pertaining to identification in this social context.

action in a pre-specified group, (*linear in means models*). In this context, identification is problematic due to the colinearity problem introduced by the mean action, the so-called *reflection problem*, and due to the possible correlation of unobservables. In the context of linear in means models, a recent literature has studied identification under the condition that the population of agents could be partitioned into a sequence of finitely-populated non-overlapping groups; see e.g., [Graham and Hahn \(2005\)](#).<sup>46</sup>

The economies we study in this paper are related to those studied by [Manski \(1993\)](#) and others in that we maintain linearity, an assumption which renders identification harder (see [Blume et al. \(2011\)](#)). On the other hand, we introduce several fundamental distinguishing features: In particular, we allow for more general forms of social interactions across agents and for dynamic economies. More precisely, in the class of economies we study, the equilibrium action of agent  $a$  in an infinite horizon economy satisfies

$$x_t^a = \beta_1 x_{t-1}^a + \beta_2 \theta_t^a + \sum_{b \neq 0} \beta_{3,b} x_t^{a+b};$$

while in a linear in means economy the corresponding equation is:<sup>47</sup>

$$x^a = \beta \theta^a + \gamma \sum_{b \in N(a)} x^b.$$

By studying populations composed of an infinite number of overlapping neighborhoods our analysis sheds some light on the nature of identification results which exploit an infinite number of non-overlapping groups, as in [Graham and Hahn \(2005\)](#) and in the literature discussed in footnote 46. The overlapping structure of our neighborhoods, in fact, breaks the independence which is required when non-overlapping groups are considered.<sup>48</sup> Furthermore, by studying dynamic models we are able to exploit the theoretical implications deriving from the optimality of the dynamic choices of agents on time series autocorrelations of actions, over and above the implications regarding the cross-sectional

---

<sup>46</sup>Also: [Davezies et al. \(2009\)](#) extends these results exploiting variation over the size of the populations; [Graham \(2008\)](#) uses excess variance across groups; [Bramoullé et al. \(2009\)](#) uses reference group heterogeneity for identification. Other recent contributions include [Glaeser and Scheinkman \(2001\)](#), [De Paula \(2009\)](#), [Evans et al. \(1992\)](#), [Ioannides and Zabel \(2008\)](#), and [Zanella \(2007\)](#).

<sup>47</sup>Note that, to ease the comparison we adopt here the best-reply representation of equilibrium actions; see equation (3.2).

<sup>48</sup>We maintain however the assumption of symmetric neighborhoods, an assumption which, as is the case for linearity, renders identification harder: See [Bramoullé et al. \(2009\)](#) for a study of the identification power of observable asymmetric neighborhoods.



(spatial) correlations. In a related context, [De Paula \(2009\)](#) and [Brock and Durlauf \(2010\)](#) also exploit the properties of dynamic equilibrium, the discontinuity in adoption curves in their continuous time model, to identify social interactions. Finally, [Patacchini, Rainone, and Zenou \(2011\)](#) exploits the intertemporal variation in individual choices relative to individual and peers' characteristics to identify social interactions in education from contextual and correlated effects.<sup>49</sup>

We turn to our main identification results. The first series of results regards the identification of the *dynamic structure*—that is, distinguishing the properties of dynamic social interaction economies from those of myopic (hence static) economies. The second series of results regards instead the identification of *social interactions*, that is, distinguishing preference for conformity from correlated preferences.

### 6.1. *Dynamic Rationality vs. Myopia*

In this section we compare equilibrium configurations of dynamic economies with rational agents with those of economies with myopic agents. When agents are myopic, even economies with a dynamic structure, e.g., when agents' actions at time  $t$  depend on their previous actions, are effectively static. These economies have been extensively studied in the theoretical and empirical literature on social interactions, following the mathematical physics literature in statistical mechanics on interacting particle systems. Suppose that myopic agents, when called to make a choice, act as if they expect never to be called to act again.<sup>50</sup> Given initial history  $x_{t-1}$  and realization  $\theta_t$ , each myopic agent  $a \in \mathbb{A}$  chooses  $x_t^a \in X$  to maximize

$$u(x_{t-1}^a, x_t^a, x_t^{a-1}, x_t^{a+1}, \theta_t^a) := -\alpha_1(x_{t-1}^a - x_t^a)^2 - \alpha_2(\theta_t^a - x_t^a)^2 - \alpha_3(x_t^{a-1} - x_t^a)^2 - \alpha_3(x_t^{a+1} - x_t^a)^2$$

There exists then a unique symmetric policy function for any agent  $a$ ,  $g^{a,m}$  ( $m$  for myopic) such that

$$g^{a,m}(x_{t-1}, \theta_t; \alpha) := \sum_{b \in \mathbb{A}} c^{b,m} x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d^{b,m} \theta_t^{a+b}$$

---

<sup>49</sup>See also [Cabral \(1990\)](#) for an early discussion of these issues and [Young \(2009\)](#); see [Blume et al. \(2011\)](#) for an up to date survey.

<sup>50</sup>See e.g., [Blume and Durlauf \(2001\)](#), [Brock and Durlauf \(2001b\)](#) and [Glaeser and Scheinkman \(2003\)](#) for a comprehensive survey. [Liggett \(1985\)](#) is the standard reference for the mathematical literature.

where we make explicit the dependence of the policy function on the preference parameters  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ .<sup>51</sup> The coefficients of the policy function  $g^{a,m}$  are equal to the ones of the unique MPE policy function of a one-period ( $T = 1$ ) social interactions economy:  $c^{b,m} = c_1^b$ ,  $d^{b,m} = d_1^b$ , for  $b \in \mathbb{A}$ . In this sense, myopic models are nested within the class of dynamic models we study.

In the following we ask whether the spatial correlations generated by the long-run stationary distribution of an infinite-horizon model can be distinguished from those obtained as the limit distribution of a myopic model. Let  $g^a(x_{t-1}, \theta_t^a; \alpha)$  denote agent  $a$ 's policy function from the dynamic social interaction model, where we make once again explicit the dependence of the policy function on  $\alpha$ . We say that  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  is a stochastic process induced by the dynamic economy with parameters  $\alpha$  if it satisfies

$$x_t^a = g^a(x_{t-1}, \theta_t^a; \alpha), \text{ for any } a \in \mathbb{A} \text{ and any } t \geq 1$$

We instead say that  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  is a stochastic process induced by the myopic economy with parameters  $\alpha$  if it satisfies

$$x_t^a = g^{a,m}(x_{t-1}, \theta_t; \alpha), \text{ for any } a \in \mathbb{A} \text{ and any } t \geq 1$$

We are now ready to introduce our definition of identification of social interactions.

**DEFINITION 3** *Let  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  denote a stochastic process induced by the dynamic economy with parameters  $\alpha$ . We say that the dynamic economy with parameters  $\alpha$  is identified with respect to myopic economies if there does not exist an  $\hat{\alpha}$ , such that the process  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  is also induced by a myopic economy with parameter  $\hat{\alpha}$ .*

---

<sup>51</sup>In some of the literature, myopic agents are modelled not only as assuming that all agents in the economy only interact once, but also that their neighbors are not changing their previous period actions. In this case an agent  $a$  solves

$$\max_{x_t^a \in X} -\alpha_1(x_{t-1}^a - x_t^a)^2 - \alpha_2(\theta_t^a - x_t^a)^2 - \alpha_3(x_{t-1}^{a-1} - x_t^a)^2 - \alpha_3(x_{t-1}^{a+1} - x_t^a)^2.$$

and his policy function is

$$x_t^a = \beta_1 x_{t-1}^a + \beta_2 \theta_t^0 + \beta_3 x_{t-1}^{-1} + \beta_3 x_{t-1}^1.$$

It can be shown, see [Glaeser and Scheinkman \(2003\)](#), that the ergodic stationary distribution of actions in this economy coincides with that of *myopic* agents as defined in the text. As a consequence, our identification results extend to this economy as well.

The characterization of the spatial correlation of actions at equilibrium for different time-horizons  $T$ , which we provided in Section 4.2, gives us a straightforward answer to the identification question. Recall in fact that the coefficients of the policy function  $g^{a,m}$  are equal to the ones of the unique MPE policy function of a one-period ( $T = 1$ ) social interactions economy. Recall also that the covariances between agents's choices obtained from data generated by a typical model of infinite-horizon stationary social interactions are fundamentally different from those generated by a myopic model. In particular, we have shown in Section 4.2 that, for a typical choice of  $\alpha$ ,

$$r_{a,T} = \frac{\rho_{a+1,T}}{\rho_{a,T}}$$

is non-monotonic in  $a$ , for longer horizon economies; and so is  $r_a$ , the ratio of the limit economy with  $T = \infty$ ); while  $r_{a,1}$  declines monotonically in  $a$ , for any  $\alpha$ ; see Figure 4. Moreover, the limit unconditional covariances inherit the (non)-monotonicity features of their one-step conditional counterparts. Finally, by continuity, the non-monotonicity property necessarily holds for an open set of the parameter space, and is hence *robust*. Summarizing, then, we have the following.<sup>52</sup>

**PROPOSITION 2 (Rationality vs. Myopia)** *A dynamic economy with parameter  $\alpha$  is identified with respect to myopic economies, for a robust subset  $\alpha$ .*

Finally, consider an econometrician fitting a static (myopic) model through data generated by the dynamic equilibrium of an economy with parameter  $\alpha$ . From Proposition 1,  $r_1(\alpha) < r(\alpha)$  for any possible  $\alpha$ . As a consequence, the parameter  $\hat{\alpha}$  estimated by the econometrician imposing the static (myopic) structure on the data, will satisfy  $r_1(\hat{\alpha}) = r(\alpha) > r_1(\alpha)$ . From (B.1), however,  $r_1$  is monotonically decreasing in  $\left(\frac{\Delta_1}{2\alpha_3}\right)$ .<sup>53</sup> As a consequence,

$$\left(\frac{\hat{\alpha}_3}{\hat{\alpha}_1 + \hat{\alpha}_2 + 2\hat{\alpha}_3}\right) > \left(\frac{\alpha_3}{\alpha_1 + \alpha_2 + 2\alpha_3}\right),$$

and the econometrician **overestimates** the social interaction effects.

---

<sup>52</sup>We present in Appendix F.1 the results of a simulation where we report  $r_a$  as a function of  $a$ , at the stationary distribution, for different levels of strength of interaction proxied by the ratio  $\left(\frac{2\alpha_3}{\Delta_1}\right)$ .

<sup>53</sup>Remember that  $\Delta_1 := \alpha_1 + \alpha_2 + 2\alpha_3$ .

## 6.2. Social Interactions vs. Selection

In our dynamic economies, spatial correlation of individual actions at each time is induced by social interactions and preference for conformity. But spatial correlation of actions could be induced in principle also by spatial correlation of preference types, with no social interaction. Take two agents, e.g., agent  $a$  and agent  $b$ . A positive correlation between  $x_t^a$  and  $x_t^b$  could be due to a positive correlation between  $\theta_t^a$  and  $\theta_t^b$ . In this last case, preferences for conformity and social interactions would play no role in the correlation of actions at equilibrium. Rather, such correlation would be due to the fact that agents have correlated preferences. As we already noted, correlated preferences could be generally due to some sort of assortative matching or positive selection in social interaction, which induces agents with correlated preferences to interact socially.

In our economy, at a symmetric Markov perfect equilibrium, each agent  $a \in \mathbb{A}$  acts according to the policy function  $g_{T-(t-1)}^a(x_{t-1}, \theta_t^a; \alpha)$ , where we make once again explicit the dependence of the policy function on the preference parameters  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ . If  $T = \infty$ , the policy function is stationary  $g^a(x_{t-1}, \theta_t^a; \alpha)$ . Recall that the parameter  $\alpha_3$  represents the weight of conformity in each agent's preferences. It follows that  $\alpha_3 = 0$  corresponds to an economy with no social interactions. We say that  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  is a stochastic process induced by  $\alpha$  and  $(\theta_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  if it satisfies

$$x_t^a = g_{T-(t-1)}^a(x_{t-1}, \theta_t^a; \alpha), \text{ for any } a \in \mathbb{A} \text{ and any } t = 1, \dots, T$$

We are now ready to construct our definition of identification of social interactions.

**DEFINITION 4** Let  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  denote a stochastic process induced by  $\alpha$  and  $(\theta_t^a)_{t \geq 1}^{a \in \mathbb{A}}$ , where  $(\theta_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  is i.i.d. across agents and serially uncorrelated, that is, where  $\text{Cov}(\theta_t^a, \theta_t^b) = \text{Cov}(\theta_t^a, \theta_{t+1}^a) = 0$  for any  $a \neq b \in \mathbb{A}$  and any  $t \geq 1$ . We say that  $\alpha$  is identified if there does not exist an  $\hat{\alpha}$ , with  $\hat{\alpha}_3 = 0$ , such that the process  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$  is also induced by  $\hat{\alpha}$  and some stochastic process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$ . We say that social interactions are identified if some  $\alpha$ , with  $\alpha_3 > 0$ , is identified.

The conditions for identification of social interactions can be weakened by restricting the stochastic process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$ . We say that  $\alpha$  is (resp. social interactions are) *identified relative to a set of preference shocks* if  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  in Definition 4 is required to belong to a set of preference shocks which satisfies some specific restriction.

Finally, the conditions for identification of social interactions can be strengthened by limiting the observable properties of the process  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$ .

We first consider the case of an infinite horizon economy: policy functions are stationary and an ergodic distribution exists. In this context, we study first the possibility of obtaining identification by observing the properties of the stationary distribution of actions rather than the whole panel  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$ . We then pass on to identification tout court, that is exploiting the whole dynamic restrictions imposed by the model on  $(x_t^a)_{t \geq 1}^{a \in \mathbb{A}}$ , not just the restrictions on the stationary distribution. We shall see that results are negative in both cases, that is, identification is not obtained in general. Secondarily, we study identification relative to a series of relevant restrictions on the stochastic process for preference shocks  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$ . These restrictions are meant to capture natural properties of the selection mechanism which induces agents to display spatially correlated preferences.

### 6.2.1. Infinite horizon (stationary) economies

Consider first the stationary distribution of actions as identified by its implied spatial correlation function  $\rho_b$ .

**PROPOSITION 3** *Social interactions are not identified by the properties of the spatial correlation function  $\rho_b$  of the stationary distribution of actions in infinite horizon economies.*

The proof is simple and instructive hence is given below.

**PROOF:** We have shown in Section 3.3 that the stationary distribution of our dynamic economy with social interactions, that is,  $\alpha_3 > 0$ , and i.i.d. preference shock process  $\{\theta_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$ , is given by the ergodic measure  $\pi$  in (3.5), i.e.  $\pi$  is the joint distribution of

$$x_t = \left( \frac{e(\alpha) \bar{\theta}}{1 - C(\alpha)} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c(\alpha)^{b_1} \cdots c(\alpha)^{b_{s-1}} \left( d(\alpha)^{b_s} \theta_{t+1-s}^{a+b_1+\cdots+b_s} \right) \right)_{a \in \mathbb{A}}$$

Consider now an alternative specification of our economy with *no interactions* between agents ( $\hat{\alpha}_3 = 0$ ) and *no habits* ( $\hat{\alpha}_1 = 0$ ) but simply a preference shock process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  and *own type effects* with  $\hat{\alpha}_2 > 0$ . For this economy, equilibrium choice of agent  $a$  at

time  $t$  is given by  $x_t^a = \hat{\theta}_t^a$ . As long as the process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  is the one where

$$\hat{\theta}_t^a := \frac{e(\alpha) \bar{\theta}}{1 - C(\alpha)} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c(\alpha)^{b_1} \cdots c(\alpha)^{b_{s-1}} \left( d(\alpha)^{b_s} \theta_{t+1-s}^{a+b_1+\cdots+b_s} \right)$$

the probability distributions that the two specifications (with and without interactions)) generate on the observables of interest,  $\{x_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$ , are identical. Hence, one cannot identify from the stationary distribution of choices which specification generates the data. *Q.E.D.*

More generally, we investigate if the dynamic equilibrium restrictions of our model are sufficient to identify social interactions.

**PROPOSITION 4** *Social interactions are not identified in infinite horizon economies.*<sup>54</sup>

An intuition about this result can be obtained by loosely reducing the identification of social interactions in infinite horizon economies to the well known problem of distinguishing a VAR from an MA( $\infty$ ) process. Stacking in a vector  $\mathbf{x}_t$  (resp.  $\theta_t$ ) the actions  $x_t^a$  over the index  $a \in \mathbb{A}$  (resp. the preference shocks  $\theta_t^a$ ), policy functions can be loosely written as a VAR:

$$\mathbf{x}_t = \Phi \mathbf{x}_{t-1} + \delta_t, \quad \text{with } \delta_t = \Gamma \theta_t + e \bar{\theta}$$

where  $E(\delta_t \delta_{t-\tau}) = 0$  for all  $\tau > 0$ . Under standard stationarity assumptions, the VAR has an MA( $\infty$ ) representation

$$\mathbf{x}_t = (I_A - \Phi \mathbf{L})^{-1} \delta_t = \delta_t + \Psi_1 \delta_{t-1} + \Psi_2 \delta_{t-2} + \dots$$

for a sequence  $\Psi_1, \Psi_2 \dots$  such that  $(I_A - \Phi \mathbf{L})(I_A + \Psi_1 \mathbf{L} + \Psi_2 \mathbf{L}^2 + \dots) = I_A$ . The argument in the proof of Proposition 4 therefore amounts to picking

$$\mathbf{x}_t = \hat{\theta}_t = \delta_t + \Psi_1 \delta_{t-1} + \Psi_2 \delta_{t-2} + \dots$$

---

<sup>54</sup>The proof is similar to the proof of Proposition 3 with the extra requirement that observational-equivalence should also hold for processes with an initial condition and for their intertemporal transitions. We put the proof in the Supplemental Appendix F.2 for interested readers.

### 6.2.2. Finite-horizon (non-stationary) economies

Consider now the case of a finite horizon economy. In this case the unique policy function and the distribution of actions are *non-stationary*, as we have shown, and hence identification might obtain in those environments where correlated effects satisfy a weak stationary law through time. Consider then a restriction to the class of admissible preference shock processes  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  which satisfy the following *conditional covariance stationarity* restriction:

**DEFINITION 5** (Conditional Covariance Stationarity) *A process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  is said to be conditional covariance stationary if  $\text{Cov}(\hat{\theta}_t^a, \hat{\theta}_t^b \mid \hat{\theta}_{t-1}, \dots, \hat{\theta}_1) = Z(a, b, \hat{\theta}_{t-1}, \dots, \hat{\theta}_{t-n}) \in \mathbb{R}$ , for  $a, b \in \mathbb{A}$ ,  $t = n + 1, \dots, T$ .<sup>55</sup>*

This condition defines a large class of stochastic processes for which the covariance between the preference shocks of any two agents  $a$  and  $b$ , depends on at most a finite memory (represented by  $n$ ) of past realizations of the process, and possibly the relative positions of agents  $a$  and  $b$  in the social group. It is a relatively weak and natural condition in that it allows for the tailoring of the intertemporal dependence of agents' types to their relative positions in the network; what it excludes is events in the distant past from having a significant effect on the *joint* determination of agents' types today.<sup>56</sup> Conditional covariance stationarity of preference shocks is in fact sufficient for identification of social interactions.

**PROPOSITION 5** *Social interactions are identified relative to processes satisfying the conditional covariance stationarity restriction with  $n \leq T - 2$ .*

While the spatial autocorrelations between agents' choices have limited memory across

---

<sup>55</sup>The original definition is due to Mandelbrot (1967) who provides the conditional spectral analysis of *sporadically varying* random functions in the mathematical theory of information transmission with noise. In his environment, he requires  $E[\hat{\theta}_t \hat{\theta}_{t+n} \mid 1 \leq t < t+n \leq T]$  to be independent of  $t$ . Ours is a slightly weaker condition since it uses fixed finite memory. For more recent usage of conditional covariance restrictions see the Times Series literature studying persistence of conditional variances, especially Bollerslev and Engle (1993), Bollerslev et al. (1994), and Engle and Bollerslev (1986).

<sup>56</sup>All existing social interaction models we can think of have stochastic structures that are special cases of this class. More specifically, they typically assume either time-independent or finite memory Markov structures to model exogenous effects; see e.g., Brock and Durlauf (2001b), Conley and Topa (2003), De Paula (2009), Glaeser and Scheinkman (2001), Nakajima (2007), Topa (2001), and Young (2009).

periods in the absence of interaction effects, they vary in presence of social interactions.<sup>57</sup> The condition  $n \leq T - 2$  guarantees that the economy lasts longer than the memory of the spatial auto-correlation process, so that the non-stationarity of the equilibrium process can manifest itself in the dynamics of spatial auto-correlations. This is at the heart of the proof, whose details follow.

PROOF: Consider a finite-horizon,  $T$ -period economy with  $T \geq 2$  and  $n \leq T - 2$ . In the absence of interactions ( $\hat{\alpha}_3 = 0$ ), agent  $a$ 's final period optimal choice is<sup>58</sup>

$$(6.1) \quad x_T^a = c_1(\hat{\alpha}) x_{T-1}^a + d_1(\hat{\alpha}) \hat{\theta}_T^a$$

Thanks to the linearity of the policy functions across periods with  $\hat{\alpha}_3 = 0$ , any path of shock realizations  $(\hat{\theta}_1, \dots, \hat{\theta}_{T-1})$ , given  $x_0$ , generates a path of configurations  $(x_0, x_1, \dots, x_{T-1})$ . Thus, conditioning on all imaginable choice paths spans all imaginable preference shock paths, given that the observables are generated by the above-mentioned policy functions. The  $a$ -step covariance between equilibrium choices of agent 0 and  $a$  in case of *interactions* is then given by

$$(6.2) \quad \begin{aligned} Cov \left( x_T^0, x_T^a \mid x_{T-1}, \dots, x_0 \right) &= Cov \left( x_T^0, x_T^a \mid x_{T-1} \right), \quad \forall (x_0, \dots, x_{T-1}) \\ &= Cov \left( \sum_{b_1 \in \mathbb{A}} d_1^{b_1} \theta_T^{b_1}, \sum_{b_2 \in \mathbb{A}} d_1^{b_2} \theta_T^{a+b_2} \right) \\ &= Var(\theta) \sum_{b_1 \in \mathbb{A}} d_1^{b_1} d_1^{b_1-a} \end{aligned}$$

thus the covariance term is independent of the conditioned upon path. So, in order the specification with no interactions to be observationally indistinguishable from the interactions case, the  $a$ -step conditional covariances, computed using (6.1) should satisfy, for all  $(x_0, \hat{\theta}_1, \dots, \hat{\theta}_{T-1})$

$$(6.3) \quad \begin{aligned} Cov \left( x_T^0, x_T^a \mid x_{T-1} \right) &= d_1(\hat{\alpha})^2 Cov \left( \hat{\theta}_T^0, \hat{\theta}_T^a \mid \hat{\theta}_{T-1}, \dots, \hat{\theta}_1, x_0 \right) \\ &= d_1(\hat{\alpha})^2 Z(0, a, \hat{\theta}_{T-1}, \dots, \hat{\theta}_{T-n}) \\ &= d_1(\hat{\alpha})^2 \bar{Z}(0, a) \end{aligned}$$

---

<sup>57</sup>More specifically, they can be ordered with respect to their spatial rate of tail convergence; see Proposition 1.

<sup>58</sup>In the absence of interactions, agents solve (dynamic) individual maximization problems. Elementary dynamic programming techniques, as in Stokey and Lucas (1989), yield the policy functions we use here.



The function  $\bar{Z}$  is implicitly defined to capture the fact that to match the covariance term in (6.2), the covariance in (6.3) can depend only on the relative positions of agents and on nothing else. This is not an assumption but an observational-equivalence restriction. The same observational-equivalence idea should hold for all  $a \in \mathbb{A}$ , in period  $T - 1$ . Similar calculations in period  $T - 1$  yield

$$\begin{aligned}
 \text{Cov} \left( x_{T-1}^0, x_{T-1}^a \middle| x_{T-2} \right) &= \text{Var}(\theta) \sum_{b_1 \in \mathbb{A}} d_2^{b_1} d_2^{b_1-a} \\
 &= d_2(\hat{\alpha})^2 Z(0, a, \hat{\theta}_{T-2}, \dots, \hat{\theta}_{T-1-n}), \quad \forall (\hat{\theta}_{T-2}, \dots, \hat{\theta}_{T-1-n}) \\
 (6.4) \quad &= d_2(\hat{\alpha})^2 \bar{Z}(0, a)
 \end{aligned}$$

where the first equality is as in (6.2); second is the restriction imposed by observable indistinguishability combined with conditional covariance stationarity; finally third is by conditional covariance stationarity across periods using (6.3). Putting the equilibrium restrictions in periods  $T - 1$  and  $T$  together, using (6.3) and (6.4), we can write

$$\frac{\text{Cov} \left( x_{T-1}^0, x_{T-1}^{a+1} \middle| x_{T-2} \right)}{\text{Cov} \left( x_{T-1}^0, x_{T-1}^a \middle| x_{T-2} \right)} = \frac{\bar{Z}(0, a+1)}{\bar{Z}(0, a)} = \frac{\text{Cov} \left( x_T^0, x_T^{a+1} \middle| x_{T-1} \right)}{\text{Cov} \left( x_T^0, x_T^a \middle| x_{T-1} \right)}$$

Since the choice of  $a$  is arbitrary, we can look at the same expression as  $a$  becomes progressively larger. So, as  $a \rightarrow \infty$ , the expression should give

$$(6.5) \quad \lim_{a \rightarrow \infty} \frac{\text{Cov} \left( x_{T-1}^0, x_{T-1}^{a+1} \middle| x_{T-2} \right)}{\text{Cov} \left( x_{T-1}^0, x_{T-1}^a \middle| x_{T-2} \right)} = r_2 = r_1 = \lim_{a \rightarrow \infty} \frac{\text{Cov} \left( x_T^0, x_T^{a+1} \middle| x_{T-1} \right)}{\text{Cov} \left( x_T^0, x_T^a \middle| x_{T-1} \right)}$$

which is a contradiction to Proposition 1. Therefore, there does not exist a conditional covariance stationary preference shock process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  that generates an equilibrium choice process  $\{x_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  under the no interactions specification ( $\hat{\alpha}_3 = 0$ ) that is observationally equivalent to the process generated by the local interactions ( $\alpha_3 \neq 0$ ) process. This concludes the proof. *Q.E.D.*

## 7. EXTENSIONS

The class of social interaction economies we studied in this paper has been restricted along several dimensions to better provide a stark theoretical analysis. Some of these restrictions, however, turn out to be important in applications and empirical work. In

this section, therefore, we illustrate how our analysis can be extended to study more general neighborhood network structures for social interactions, more general stochastic processes for preference shocks, the addition of global interactions, that is, interactions at the population level, and the effects of stock variables which carry habit effects.

### 7.1. General Neighborhood Network Structures

Throughout the paper, we studied symmetric neighborhood structures. This is generalized easily. Consider an arbitrary neighborhood network structure (not necessarily translation invariant),  $N : \mathbb{A} \rightarrow 2^{\mathbb{A}}$ . Suppose also that a generic agent  $a$ 's preferences are represented by the utility function  $u^a$  defined as

$$\begin{aligned} u^a(x_{t-1}^a, x_t^a, \{x_t^b\}_{b \in N(a)}, \theta_t^a) &:= -\alpha_{a,1}(x_{t-1}^a - x_t^a)^2 - \alpha_{a,2}(\theta_t^a - x_t^a)^2 \\ &\quad - \sum_{b \in N(a)} \alpha_{a,b}(x_t^b - x_t^a)^2 \end{aligned}$$

Notice that we allow for the preferences of any two agents  $a$  and  $b$  to be arbitrarily different in their parametrization, provided either  $\alpha_{a,1} > 0$  or  $\alpha_{a,2} > 0$  and  $\sum_{b \in N(a)} \alpha_{a,b} < \infty$  so that peer effects are bounded. Under this specification, best-responses are well defined, interior, and well-behaved. An MPE exists and the policy function of an arbitrary agent  $a \in \mathbb{A}$  at equilibrium has the following form

$$g_{T-(t-1)}^a(x_{t-1}, \theta_t) = \sum_{b \in \mathbb{A}} c_{T-(t-1)}^{a,b} x_t^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^{a,b} \theta_t^b + e_{T-(t-1)}^a \bar{\theta}$$

where, as before, all coefficients are non-negative and  $\sum_{b \in \mathbb{A}} (c_t^{a,b} + d_t^{a,b}) + e_t^a = 1$ . For uniqueness of equilibrium, it is sufficient that the relative composition of the peer effects within the determinants of individual choice be uniformly bounded, i.e., that there exists a positive constant  $K$  such that for each individual  $a \in \mathbb{A}$

$$\frac{\sum_{b \in N(a)} \alpha_{a,b}}{\alpha_{a,1} + \alpha_{a,2} + \sum_{b \in N(a)} \alpha_{a,b}} < K.$$

Under this condition,<sup>59</sup> best responses induce a contraction operator and we obtain a

---

<sup>59</sup>A related condition is referred to, in the literature, as the *Moderate Social Influence* condition; see e.g. Glaeser and Scheinkman (2003), Horst and Scheinkman (2006), and Ballester, Calvó-Armengol, and Zenou (2006) for restrictions in a similar spirit.

unique equilibrium for any finite-horizon economy.

Ergodicity (relative to a given MPE) and welfare results extend straightforwardly, as do identification results. Notably, our positive identification result for non-stationary economies, Proposition 5, also extends: since preference parameters of any agent  $a$  are stationary, in a finite-horizon economy, correlations of equilibrium actions between agents vary only due to interactions for preference processes that satisfy a Conditional Covariance Stationarity restriction.

## 7.2. General Stochastic Processes for Preference Shocks

The agents in our model make their decisions based on past behavior and current shocks. Our analysis however extends straightforwardly to economies where shocks are persistent across time as long as the economy is one of complete information.<sup>60</sup> We give here, as an illustration, an example of Markov dependence, where at any given period the probability of next period shocks depends on current realizations.

Consider any  $T$ -period economy with  $T \leq \infty$ . Recall from Section 2 that preference shocks  $\theta_t := (\theta_t^a)_{a \in \mathbb{A}}$  are defined on the canonical probability space  $(\Theta, \mathcal{F}, \mathbb{P})$ , where  $\Theta := \{(\theta^a)_{a \in \mathbb{A}} : \theta^a \in \Theta\}$ . Let  $Q : \Theta \times \mathcal{F} \rightarrow \mathbb{R}_+$  be a *transition function* such that

- (i) for any period  $t$  and any  $\theta \in \Theta$ ,  $Q(\theta, A) = \Pr\{\theta_{t+1} \in A \mid \theta_t = \theta\}$ , for all  $A \in \mathcal{F}$ .
- (ii) for each  $A \in \mathcal{F}$ ,  $Q(\cdot, A)$  is  $\mathcal{F}$ -measurable.

Any agent  $a \in \mathbb{A}$  solves the problem in (2.1) with persistent shocks where the expectation operator acts on the distribution induced by  $Q$  and other agents' strategies. We can write the problem recursively. The policy function of an arbitrary agent  $a \in \mathbb{A}$  at equilibrium, in this economy, is

$$g_{T-(t-1)}^a(x_{t-1}, \theta_t) = \sum_{b \in \mathbb{A}} c_{T-(t-1)}^b x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^{a+b} \theta_t^a + e_{T-(t-1)}(\theta_t, a)$$

for some positive coefficients  $(c_{T-(t-1)}^b)_{b \in \mathbb{A}}$ ,  $(d_{T-(t-1)}^{a+b})_{b \in \mathbb{A}}$ , and some constant  $e_{T-(t-1)}(\theta_t, a)$  that depends only on the current type profile and on the agent's name,  $a$ .

Once again, existence, ergodicity (relative to a given MPE), and welfare results extend straightforwardly, as well as our identification results.

---

<sup>60</sup>The mathematical issues arising in dynamic models with incomplete information are both well-known and outside the scope of the present paper. See Mailath and Samuelson (2006) for an extensive survey.

### 7.3. Global Interactions

Introducing global determinants of individual behavior into our framework is also relatively straightforward.<sup>61</sup> In particular, consider an economy in which the preferences of each agent  $a \in \mathbb{A}$  depend also on the average action of the agents in the economy. Let the average action given a choice profile  $x$  be defined as

$$p(x) := \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{a=-n}^n x^a,$$

when the limit exists. Let  $\mathbf{X}_e$  denote the set of all configurations such that the associated average action exists:

$$\mathbf{X}_e := \left\{ x \in \mathbf{X} : \exists p(x) := \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{a=-n}^n x^a \right\}$$

The preferences of the agent  $a \in \mathbb{A}$  in period  $t$  are described by the instantaneous utility function  $u : \mathbf{X}_e \times \Theta \rightarrow \mathbb{R}$  of the conformity class

$$\begin{aligned} u^a(x_{t-1}^a, x_t^a, \{x_t^b\}_{b \in N(a)}, \theta_t^a, p(x_t)) := \\ -\alpha_1 (x_{t-1}^a - x_t^a)^2 - \alpha_2 (\theta_t^a - x_t^a)^2 - \sum_{b \in N(a)} \alpha_{3,b} (x_t^b - x_t^a)^2 - \alpha_4 (p(x_t) - x_t^a)^2 \end{aligned}$$

Given  $x \in \mathbf{X}_e$ , the initial configuration of actions, a symmetric stationary MPE of a dynamic economy with local and global interactions is a map  $g : \mathbf{X} \times \Theta \times X \rightarrow X$  and a map  $F : X \rightarrow X$  such that, for each  $a \in \mathbb{A}$ :

$$\begin{aligned} g(R^a x_{t-1}, R^a \theta_t, p_t) = \arg \max_{x_t^a \in X} E \left[ u \left( x_{t-1}^a, x_t^a, \{g(R^b x_{t-1}, R^b \theta_t, p_t)\}_{b \in N(a)}, \theta_t^a, p_t \right) \right. \\ \left. + \beta V_g \left( R^a \left( x_t^a, \{g(R^b x_{t-1}, R^b \theta_t, p_t)\}_{b \neq a} \right), R^a \theta_{t+1}, p_{t+1} \right) \mid (x_{t-1}, \theta_t) \right] \end{aligned}$$

with

$$p_{t+1} = F(p_t), \quad p_1 = p(x) \quad \text{and} \quad p_t = p(x_t) \quad \text{almost surely.}^{62}$$

<sup>61</sup>With respect to the analysis of MPE with local and global interactions in finite economies (as e.g., in [Blume and Durlauf \(2001\)](#) and in [Glaeser and Scheinkman \(2003\)](#)), a few technical subtleties arise in our economy due to the infinite number of agents. The techniques we use are extensions of the ones we used in a previous paper, [Bisin, Horst, and Özgür \(2006\)](#). We refer the reader to this paper for details. Some of the needed mathematical analysis is developed in [Föllmer and Horst \(2001\)](#) and [Horst and Scheinkman \(2006\)](#).

At a symmetric MPE, any agent rationally anticipates that all others play according to the policy function  $g$  and also anticipates the sequence of average actions  $\{p(x_t)\}_{t \in \mathbb{N}}$  to be determined recursively via the map  $F$ .

For this economy, we can show that the endogenous sequence of average actions  $\{p(x_t)\}_{t \in \mathbb{N}}$  exists almost surely if the initial configuration  $x$  belongs to  $\mathbf{X}_e$ , and that it follows a deterministic recursive relation.<sup>63</sup> As a consequence, our main results extend and the policy function of an arbitrary agent  $a \in \mathbb{A}$  at equilibrium is

$$g^a(x_{t-1}, \theta_t) = \sum_{b \in \mathbb{A}} c_b x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d_b \theta_t^{a+b} + e \bar{\theta} + B^*(p(x))$$

for some positive coefficients  $(c^b)_{b \in \mathbb{A}}$ ,  $(d^b)_{b \in \mathbb{A}}$ ,  $e$ , and some constant  $B^*(p(x))$  that depends only on the initial average action,  $p(x)$ .

#### 7.4. Social Accumulation of Habits

In this section, we generalize the class of the economies we have studied to encompass a richer structure of dynamic dependence of agents' actions at equilibrium. Consider an economy where preferences of agent  $a \in \mathbb{A}$  are represented by a utility function

$$u(S_t^a, x_t^a, \{x_t^b\}_{b \in N(a)}, \theta_t^a) := -\alpha_1 (S_t^a - x_t^a)^2 - \alpha_2 (\theta_t^a - x_t^a)^2 - \sum_{b \in N(a)} \alpha_{3,b} (x_t^b - x_t^a)^2$$

where  $S_t^a$  represents an accumulated stock variable,

$$S_{t+1}^a = (1 - \delta) S_t^a + x_t^a$$

For instance,  $S_t^a$  captures what the addiction literature calls a “*reinforcement effect*” on agent  $a$ 's substance consumption. In this economy the policy function at equilibrium is

$$g_{T-(t-1)}^a(S_t, \theta_t) = \sum_{b \in \mathbb{A}} c_{T-(t-1)}^{a,b} S_t^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^{a,b} \theta_t^b + e_{T-(t-1)}^a \bar{\theta}$$

Note that in equilibrium each agent's choice depends on the stock of his neighbors' actions, that is, on their long-term behavioral patterns rather than just their previous

---

<sup>63</sup>Linearity is crucial for these results. Only in this case, in fact, can the dynamics of average actions  $\{p(x_t)\}_{t \in \mathbb{N}}$  be described recursively. In models with more general local interactions, the average action typically is not a sufficient statistic for the aggregate behavior of the configuration  $x$ ; hence a recursive relation typically fails to hold. In such more general cases, the analysis must be pursued in terms of empirical fields. Interested reader should consult [Föllmer and Horst \(2001\)](#).

period actions. Also, as the final period approaches, agent  $a$  assigns *uniformly* higher weights to his own stock.

## 8. CONCLUSION

Social interactions provide a rationale for several important phenomena at the intersection of economics and sociology. The theoretical and empirical study of economies with social interactions, however, has been hindered by several obstacles. Theoretically, the analysis of equilibria in these economies induces generally intractable mathematical problems: equilibria are represented formally by a fixed point in configuration of actions, typically an infinite dimensional object; and embedding equilibria in a full dynamic economy adds a second infinite dimensional element to the analysis. Computationally, these economies are also generally plagued by a curse of dimensionality associated to their large state space. Finally, in applications and empirical work, social interactions are typically identified, even with population data, only under heroic assumptions.

In this paper we have attempted to show how some of these obstacles to the study of economies with social interactions can be overcome. Admittedly, we have restricted our analysis to linear economies, but in this context we have been able i) to obtain several desirable theoretical properties, like existence, uniqueness, ergodicity; ii) to develop simple recursive methods to rapidly compute equilibria; and iii) to characterize several general properties of dynamic equilibria. Furthermore, while linearity in principle renders the identification problem in static economies with social interaction almost insurmountable, we have been able to exploit the properties of dynamic equilibria in non-stationary economies to produce a positive identification result.

In conclusion, we believe that the class of dynamic linear economies with social interactions we have studied in this paper can be fruitfully and easily employed in applied and empirical work.

## REFERENCES

- AKERLOF, G. (1997), ‘Social Distance and Social Decisions,’ *Econometrica*, **65**, 1005-1027.
- AKERLOF, G. A., KATZ, M. L., AND YELLEN, J. L. (1996), “An Analysis of Out-of-Wedlock Childbearing in the United States,” *Quarterly Journal of Economics*, **111**, 277-317.
- ALIPRANTIS, D., AND BORDER, K. C. (2006), *Infinite Dimensional Analysis*, third edition. Springer-Verlag, Berlin.
- ALTONJI, J. G., ELDER, T. E., AND TABER, C. R. (2005), “Selection on Observed and Unobserved

- Variables: Assessing the Effectiveness of Catholic Schools,” *Journal of Political Economy*, **113**(1), 151-183.
- AMMERMUELLER, A. AND PISCHKE J.S., (2009), “Peer Effects in European Primary Schools: Evidence from PIRLS,” *Journal of Labor Economics*, **27**, 3, 315-348.
- AXTELL, R., S. DURLAUF, J.M. EPSTEIN, R. HAMMOND, B. KLEMENS, J. PARKER, Z. SONG, T. VALENTE, AND H.P. YOUNG (2006): “Social Influences and Smoking Behavior: Final Report to the American Legacy Foundation,” mimeo, Center on Social and Economic Dynamics, The Brookings Institution.
- BALLESTER, C., CALVÓ-ARMENGOL, A., ZENOU, Y. (2006), “Who’s Who in Networks. Wanted: The Key Player,” *Econometrica*, **74**, (5), 1403-1417.
- BAYER, P., ROSS, S. AND TOPA, G. (2008), “Place of Work and Place of Residence: Informal Hiring Networks and Labor Market Outcomes,” *Journal of Political Economy*, **116** (6), 1150-1196.
- BEARMAN, P. S., MOODY, J., AND STOVEL, K. (2004), “Chains of Affection: The Structure of Adolescent Romantic and Sexual Networks,” *American Journal of Sociology*, **110**(1), 44-91.
- BECKER, G.S., GROSSMAN, M., AND MURPHY, K. M., (1994), “An Empirical Analysis of Cigarette Addiction,” *American Economic Review*, **84**, 3, 396-418.
- BECKER, G. S. AND MURPHY, K. M. (1988), “A Theory of Rational Addiction,” *Journal of Political Economy*, **96**, 4, 675-700.
- BÉNABOU, R. (1993), “Workings of a City: Location, Education and Production,” *Quarterly Journal of Economics*, **108**, 619-52.
- (1996), “Equity and Efficiency in Human Capital Investment: The Local Connection,” *Review of Economic Studies*, **62**, 237-264.
- BENHABIB, J., BISIN, A., AND JACKSON, M. O. (eds.) (2011): *Handbook of Social Economics*, Vol. 1A, The Netherlands: North-Holland, 339-416.
- BENVENISTE, L. M. AND SCHEINKMAN, J. A. (1979): “On the Differentiability of the Value Function in Dynamic Models of Economics,” *Econometrica*, **47**, 727-32.
- BERGE, C. (1963): *Topological Spaces*. New York, Macmillan.
- BERNHEIM, B. D. (1994), ‘A Theory of Conformity,’ *Journal of Political Economy*, **102**, 841-877.
- BIFULCO, R., FLETCHER, J. AND S. ROSS (2009), “The Effect of Classmate Characteristics on Post-secondary Outcomes: Evidence from the Add Health”, *American Economic Journal: Economic Policy*, **3**(1): 2553.
- BINDER, M. AND PESARAN, M. H. (2001), “Life-Cycle Consumption under Social Interactions,” *Journal of Economic Dynamics and Control*, **25**, 35-83.
- BISIN, A., HORST, U. AND ÖZGÜR, O. (2006), “Rational Expectations Equilibria of Economies with Local Interactions,” *Journal of Economic Theory*, **127**, 74-116.
- BISIN, A., A. MORO, AND G. TOPA, (2010), “The Empirical Content of Models with Multiple Equilibria in Economies with Social Interactions,” mimeo, New York University.
- BLUME, L. (2003), “Stigma and Social Control,” mimeo Cornell University.
- BLUME, L., BROCK, W., DURLAUF, S., AND IOANNIDES, Y. (2011), “Identification of Social Interactions,” in Benhabib, J., Bisin, A., and Jackson, M. O. (eds.): *Handbook of Social Economics*, Vol. 1B,

- The Netherlands: North-Holland, 855-966.
- BLUME, L. AND DURLAUF, S., (2001), "The Interactions-Based Approach to Socioeconomic Behavior," in S. Durlauf and P. Young (Eds.), *Social Dynamics*, Brookings Institution Press and MIT Press, Cambridge, MA.
- (2005), "Identifying Social Interactions: A Review," in *Methods in Social Epidemiology*, J. M. Oakes and J. Kaufman, eds., San Francisco: Jossey-Bass.
- BOLLERSLEV, T. AND ENGLE, R. F. (1993), "Common Persistence in Conditional Variances," *Econometrica*, **61**, 1, 167-186.
- BOLLERSLEV, T., ENGLE, R. F., NELSON, D. B. (1994), "ARCH models," In *Handbook of Econometrics*, Engle, R. F. and McFadden, D. (eds.), North Holland Press: Amsterdam.
- BORJAS, G. (1995), "Ethnicity, neighbourhoods, and human capital externalities," *American Economic Review*, **85**, 365-390.
- BRAMOULLÉ, Y., DJEBBARI, H., AND FORTIN, B. (2009), "Identification of Peer Effects through Social Networks," *Journal of Econometrics*, **150**, 41-55.
- BROCK, W. AND DURLAUF, S. (2001a), "Discrete Choice with Social Interactions", *The Review of Economic Studies*, **68**, 235-260.
- BROCK, W. AND DURLAUF, S. (2001b), "Interactions-Based Models," in J. Heckman and E. Leamer (Eds.), *Handbook of Econometrics*, Vol. V, North-Holland, Amsterdam.
- BROCK, W. AND DURLAUF, S. (2007), "Identification of Binary Choice Models with Social Interactions," *Journal of Econometrics*, **140**, 1, 52-75.
- BROCK, W. AND DURLAUF, S. (2010), "Adoption Curves and Social Interactions," *Journal of the European Economic Association*, **8**, 1, 232-251.
- BURKE, M. AND HEILAND, F. (2007), "Social Dynamics of Obesity", *Economic Inquiry*, **45**, 3, 571-591.
- CABRAL, L. (1990): "On the Adoption of Innovations with 'Network' Externalities," *Mathematical Social Sciences*, **19**, 299-308.
- CALVO-ARMENGOL, T. AND JACKSON, M. (2004), "The Effects of Social Networks on Employment and Inequality," *American Economic Review*, **94**, 426-454.
- CALVO-ARMENGOL, T., E. PATACCHINI AND Y. ZENOU (2009), "Peer Effects and Social Networks in Education", *Review of Economic Studies*, **76**, 4, 1239-1267.
- CASE, A., AND L. KATZ (1991), "The Company You Keep: The Effect of Family and Neighborhood on Disadvantaged Families," *NBER Working Paper* 3705.
- CHIAPPORI, P. A. AND EKELAND, I. (2009), "The Micro Economics of Efficient Group Behavior: Identification," *Econometrica*, **77**, 763-99.
- CHRISTAKIS, N.A. AND J.H. FOWLER (2007), "The Spread of Obesity in a Large Social Network over 32 Years," *New England Journal of Medicine*, **357**, 370-379.
- CIPOLLONE, P. AND ROSOLIA, A. (2007), "Social Interactions in High School: Lessons from an Earthquake," *American Economic Review*, **97**(3), 948-965.
- COLE, H. L., MAILATH, G. J. AND POSTLEWAITE, A. (1992), "Social Norms, Savings Behaviour, and Growth," *Journal of Political Economy*, **100**, 1092-1125.
- CONLEY, T. AND TOPA, G. (2002), "Socio-economic Distance and Spatial Patterns in Unemployment,"



- Journal of Applied Econometrics*, **17**(4), 303-327.
- CONLEY, T. AND TOPA, G. (2003), "Identification of Local Interaction Models with Imperfect Location Data," *Journal of Applied Econometrics*, **18**, 5, 605-618.
- CONTI, G., GALEOTTI, A., MUELLER, G. AND S. PUDNEY (2009), "Popularity," ISER Working Paper 2009-03.
- CRANE, J. (1991), "The Epidemic Theory of Ghettos and Neighborhood Effects of Dropping Out and Teenage Childbearing," *American Journal of Sociology*, 96, 1226-1259.
- CURRARINI, S., M.O. JACKSON, AND P. PIN (2009), "An Economic Model of Friendship: Homophily, Minorities, and Segregation," *Econometrica*, **77**(4), 1003-45.
- CUTLER, D. M. AND GLAESER, E. L. (2007), "Social Interactions and Smoking," mimeo Harvard University and NBER working paper No. 13477.
- DAVEZIES, L., D'HAULTFOEUILLE, X., AND FOUGÈRE, D., (2009), "Identification of Peer Effects Using Group Size Variation," *The Econometrics Journal*, **12**, 3, 397-413.
- DECICCA, P., KENKEL, D. S., AND MATHIOS, A. D. (2008), "Cigarette Taxes and the Transition from Youth to Adult Smoking: Smoking Initiation, Cessation, and participation," *Journal of Health Economics*, **27**(4), 904-917.
- DE GIORGI, G., PELLIZZARI, M., AND REDAELLI, S. (2009), "Be as Careful of the Company You Keep as of the Books You Read: Peer Effects in Education and on the Labor Market," NBER Working Paper No. 14948.
- DE PAULA, A. (2009), "Inference in a Synchronization Game with Social Interactions," *Journal of Econometrics*, **148**, 5671.
- DOOB, J. L. (1953), *Stochastic Processes*, New York: Wiley.
- DUFFIE, D., GEANAKOPOLOS, J., MAS-COLELL, A., AND MCLENNAN, A. (1994), "Stationary Markov Equilibria," *Econometrica*, **62**, 745-781.
- DUFOUR, J-M. AND HSIAO, C. (2008), "Identification," *The New Palgrave Dictionary of Economics*, Second Edition, Eds. Steven N. Durlauf and Lawrence E. Blume.
- DURLAUF, S. (1996), "Neighborhood Feedbacks, Endogenous Stratification, and Income Inequality," in Barnett, W., G. Gandolfo, and C. Hillinger (Eds.), *Dynamic Disequilibrium Modelling: Proceedings of the Ninth International Symposium on Economic Theory and Econometrics*, Cambridge University Press, New York.
- (2004), "Neighborhood Effects," in *Handbook of Urban and Regional Economics*, vol. 4, ed. J. V. Henderson and J. F. Thisse. Amsterdam: North Holland.
- ELSTER, J. (1999), *Addiction: Entries and Exits*, edited by Jon Elster, New York: Russell Sage Foundation.
- ELSTER, J. AND SKOG, O. (1999), *Getting Hooked: Rationality and the Addictions*, edited by Jon Elster and Ole-Jørgen Skog, Cambridge: Cambridge University Press.
- ENGLE, R. F. AND BOLLERSLEV, T. (1986), "Modelling the Persistence of Conditional Variances," *Econometric Reviews*, **5**, 1-50; 81-87.
- EVANS, W., W. OATES, AND R. SCHWAB (1992), "Measuring Peer Group Effects: A Study of Teenage Behavior," *Journal of Political Economy*, **100**, 51, 966-991.

- FÖLLMER, H. AND U. HORST (2001), "Convergence of Locally and Globally Interacting Markov Chains," *Stochastic Processes and Their Applications*, **96**, 99–121.
- FRISCH, R. (1934), *Statistical Confluence Analysis by Means of Complete Regression Systems*, Publ. No. 5, Universitetets Økonomiske Institutt, Oslo.
- (1938), *Statistical Versus Theoretical Relations in Economic Macrodynamics*, mimeographed document for League of Nations conference.
- GAVIRIA, A., AND RAPHAEL, S. (2001), "School-based Peer Effects and Juvenile Behavior," *Review of Economics and Statistics*, **83**(2), 257–268.
- GLAESER, E., B. SACERDOTE, AND J. SCHEINKMAN (2003), "The Social Multiplier," *Journal of the European Economic Association*, **1**, 345–353.
- (1996), "Crime and Social Interactions," *Quarterly Journal of Economics*, CXI, 507–548.
- GLAESER, E. AND J. SCHEINKMAN (2003), "Non-Market Interactions," in *Advances in Economics and Econometrics: Theory and Applications, Eight World Congress, Vol. I*, (M. Dewatripont, L.P. Hansen, and S. Turnovsky, eds.), Cambridge University Press, 339–369.
- (2001), "Measuring Social Interactions," in S. Durlauf and P. Young (Eds.), *Social Dynamics*, Brookings Institution Press and MIT Press, Cambridge, MA.
- GOLDIN, C. AND KATZ, L. F. (2002), "The Power of the Pill: Oral Contraceptives and Women's Career and Marriage Decisions," *Journal of Political Economy*, **110**, 4, 730–770.
- GRAHAM, B. S., AND HAHN, J. (2005), "Identification and Estimation of the Linear-in-Means Model of Social Interactions," *Economics Letters*, **88**, 1, 1–6.
- GRAHAM, B. S. (2008), "Identifying Social Interactions through Excess Variance Contrasts," *Econometrica*, **76**, 643–660.
- (2011), "Econometric Methods for the Analysis of Assignment Problems in the Presence of Complimentarity and Social Spillovers," in In Jess Benhabib, Matthew O. Jackson and Alberto Bisin editors: *Handbook of Social Economics*, Vol. 1B, The Netherlands: North-Holland.
- GRANOVETTER, M. (1973), "The Strength of Weak Ties," *American Journal of Sociology*, 78(6), 1360–1380.
- (1995), *Getting a Job: A Study in Contacts and Careers*, University of Chicago Press.
- GRUBER, J. AND KOSZEGI, B. (2001), "Is addiction "rational"? Theory and evidence," *Quarterly Journal of Economics*, **116**, 4, 1261–1303.
- GUL, F. AND PESENDORFER, W. (2007), "Harmful Addiction," *Review of Economic Studies*, **74**, 147–172.
- HANSEN, L. AND SARGENT, T. (2004), *Recursive Models of Dynamic Linear Economies*, University of Chicago, mimeo.
- HAUSMAN, J. A. AND TAYLOR, W. E. (1983), "Identification in Linear Simultaneous Equations Models with Covariance Restrictions: An Instrumental Variables Interpretation," *Econometrica*, **51** (5), 1527–1549.
- HOFF, K. AND SEN, A. (2005), "Homeownership, Community Interactions, and Segregation," *American Economic Review*, 95, 1167–1189.
- HORST, U. AND SCHEINKMAN, J. (2006), "Equilibria in Systems of Social Interactions," *Journal of*

- Economic Theory*, **130**, 44-77.
- HOXBY, C. M. (2000a), "The Effects Of Class Size On Student Achievement: New Evidence From Population Variation," *The Quarterly Journal of Economics*, **115**, 4, 1239-1285.
- (2000b), "Peer Effects in the Classroom: Learning from Gender and Race Variation," NBER Working Paper 7867.
- HSIAO, C. (1983), "Identification," In *Handbook of Econometrics*, vol. 1, ed. Z. Griliches and M.D. Intriligator. Amsterdam: North-Holland.
- HURWICZ, L. (1950), "Generalization of the Concept of Identification," *Statistical Inference in Dynamic Economic Models*, Cowles Commission Monograph 10, New York: John Wiley and Sons.
- IOANNIDES, Y. M. AND L. DATCHER LOURY (2004), "Job Information Networks, Neighborhood Effects, and Inequality," *Journal of Economic Literature*, **42**, 1056-93.
- IOANNIDES, Y. M. AND SOETEVENT, A. R. (2007), "Social Networking and Individual Outcomes Beyond the Mean Field Case," *Journal of Economic Behavior and Organization*, **64**, 369-390.
- IOANNIDES, Y. M. AND ZABEL, J. (2008), "Interactions, Neighborhood Selection, and Housing Demand," *Journal of Urban Economics*, **63**, 229-252.
- JONES, S. R. G. (1984), *The Economics of Conformism*, Oxford: Basil Blackwell.
- KALLENBERG, O. (2002), *Foundations of Modern Probability*, second edition. Springer-Verlag, Berlin.
- KANDEL, D. B. (1978), "Homophily, Selection, and Socialization in Adolescent Friendships," *American Journal of Sociology*, **84**, 427-436.
- KATZ, L., KLING, A. AND LIEBMAN, J. (2001), "Moving to Opportunity in Boston: Early Results of a Randomized Mobility Experiment," *Quarterly Journal of Economics*, **CXVI**, 607-654.
- KLING, J., LUDWIG, J. AND KATZ, L. (2005), "Neighborhood Effects on Crime for Female and Male Youth: Evidence from a Randomized Housing Voucher Experiment," *Quarterly Journal of Economics*, **120**(1), 87-130.
- KOOPMANS, T. C. (1949), "Identification Problems in Economic Model Construction," *Econometrica*, **17**, 125-144.
- KOOPMANS, T. C. AND REIERSØL, O. (1950), "The Identification of Structural Characteristics," *Annals of Mathematical Statistics*, **21**, 165-181.
- KRAUTH, B. V. (2006), "Simulation-based estimation of peer effects," *Journal of Econometrics*, **133**, 243-271.
- KREMER, M. AND LEVY, D. (2008), "Peer Effects and Alcohol Use among College Students," *The Journal of Economic Perspectives*, **22**,3, 189-206.
- LIGGETT, T.M. (1985) *Interacting Particle Systems*, Springer Verlag, Berlin.
- LUDWIG, J., HIRSCHFELD, P. AND DUNCAN, G. (2001), "Urban Poverty and Juvenile Crime: Evidence from a Randomized Housing-Mobility Experiment," *Quarterly Journal of Economics*, **CXVI** (2): 665-679.
- MAILATH, G. J. AND SAMUELSON, L. (2006), *Repeated Games and Reputations: Long-run Relationships*, Oxford University Press, New York.
- MANDELBROT, B. (1967), "Some Noises with  $1/f$  Spectrum, a Bridge between Direct Current and White Noise," *IEEE Transactions on Information Theory*, vol. IT-13, 2, 289-298.

- MANSKI, C. (1993), "Identification of Endogenous Social Effects: The Reflection Problem," *Review of Economic Studies*, **60**, 531-42.
- (2000), "Economic Analysis of Social Interactions," *Journal of Economic Perspectives*, **14**, 115-136.
- (2007), *Identification for Prediction and Decision*, Harvard University Press, Cambridge, MA.
- MARSCHAK, J. (1942), "Economic Interdependence and Statistical Analysis," *Studies in Mathematical Economics and Econometrics*, University of Chicago Press, 135-150.
- MATZKIN, R. (2007), "Nonparametric Identification," In *Handbook of Econometrics*, vol. **6**, ed. J. Heckman and E. Leamer. Amsterdam: North-Holland.
- MERTENS, J.-F. (2002), "Stochastic Games," In R.J. Aumann and S. Hart, eds., *Handbook of Game Theory*, Vol. **3** (North-Holland, Amsterdam), Chapter 47, 1809-1832.
- MERTENS, J.-F. AND PARTHASARATHY, T. (1987), "Equilibria for Discounted Stochastic Games," Research paper 8750, CORE, Université Catholique de Louvain.
- MOBIUS, M. (2000), "The Formation of Ghettos as a Local Interaction Phenomenon," mimeo, Harvard University.
- MOFFITT, R. A. (2001), "Policy Interventions, Low-Level equilibria, and Social Interactions," in S. Durlauf and P. Young (Eds.), *Social Dynamics*, Brookings Institution Press and MIT Press, Cambridge, MA.
- MOODY, J., (2001), "Race, School Integration, and Friendship Segregation in America", *American Journal of Sociology*, **107**, 3, 679-716.
- NAKAJIMA, R. (2007), "Measuring Peer Effects on Youth Smoking Behavior", *Review of Economic Studies*, **74**, 897-935.
- PATACCHINI, E., RAINONE, E., AND ZENOU, Y. (2011), "Dynamic Aspects of Teenage Friendships and Educational Attainment," CEPR working paper DP8223.
- PESKI, M. (2007), "Complementarities, Group Formation, and Preferences for Similarity," mimeo.
- PIGOU, A. C. (1910), "A Method of Determining the Numerical Values of Elasticities of Demand," *Economic Journal*, **20**, 636-640.
- ROTHENBERG, T. J. (1971), "Identification in Parametric Models," *Econometrica*, **39** (3), 577-591.
- ROZEN, K. (2010), "Foundations of Intrinsic Habit Formation," *Econometrica*, **78** (4), 1341-1373.
- SACERDOTE, B. (2001), "Peer Effects with Random Assignments: Results from Dartmouth Roommates," *Quarterly Journal of Economics*, **116**, 681-704.
- SANTOS, M. S. (1991), "Smoothness of the Policy Function in Discrete Time Economic Models," *Econometrica*, **59**, 1365-82.
- SCHELLING, T. (1972), "A Process of Residential Segregation: Neighborhood Tipping," in *Racial Discrimination in Economic Life*, ed. by A. Pascal, Lexington Books, Lexington, MA.
- SCHULTZ, H. (1938), *Theory and Measurement of Demand*, University of Chicago Press.
- SOETEVEENT, A. AND KOOREMAN, P. (2007), "A Discrete Choice Model with Social Interactions; With an Application to High School Teen Behavior," *Journal of Applied Econometrics*, **22**, 3, 599-624.
- STOKEY, N., LUCAS, R. E. WITH PRESCOTT, E. (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, MA.

- TOPA, G. (2001), "Social Interactions, Local Spillover and Unemployment," *The Review of Economic Studies*, **68**, 261-95.
- TWEEDIE, R. L. (1975), "Sufficient Conditions for Ergodicity and Recurrence of Markov Chains in a General State Space," *Stochastic Processes and Their Applications*, **3**, 385-403.
- VIEILLE, N. (2002), "Stochastic Games: Recent Results," In R.J. Aumann and S. Hart, eds., *Handbook of Game Theory*, Vol. **3** (North-Holland, Amsterdam), Chapter 48, 1833-1850.
- WALD, A. (1950), "Note on the Identification of Economic Relations," *Statistical Inference in Dynamic Economic Models*, Cowles Commission Monograph 10, New York: John Wiley and Sons.
- YOUNG, P. (2009), "Innovation Diffusion in Heterogeneous Populations: Contagion, Social Influence, and Social Learning," *American Economic Review*, **99**, 1899-1924.
- ZANELLA, G. (2007), "Discrete Choice with Social Interactions and Endogenous Membership," *Journal of the European Economic Association*, **5**, 122-153.

# Supplement to “Dynamic Linear Economies with Social Interactions”

Onur Özgür and Alberto Bisin

April 5, 2011

# SUPPLEMENT TO “DYNAMIC LINEAR ECONOMIES WITH SOCIAL INTERACTIONS”

ONUR ÖZGÜR<sup>a</sup> AND ALBERTO BISIN<sup>b</sup>

This supplement contains proofs of the results in the main text and the details about the simulations we used to generate the Figures. All notation is as defined in the main text unless explicitly noted otherwise. To ease cross-referencing, equation and result numbering continues in sequence with that established in the main paper.

## APPENDIX A: THE EXISTENCE PROOF

We prove here Theorem 1. The proof is constructive and works in three steps, by induction on the length of the economy.<sup>1</sup>

**Step 1: Existence, uniqueness and the Markov property for  $T = 1$ .** In this symmetric environment, it is enough to analyze the optimization problem of a single agent, say of agent  $0 \in \mathbb{A}$ . We will allow for arbitrary initial histories so that one can interpret the current step either as a one-period economy or the last period of a finite-horizon economy. We will show that, agents will use only the information contained in the previous period choices  $x_0$  and current type realizations  $\theta_1$ . Let any  $t$ -length history  $(x^{t-1}, \theta^t) = (x_{-(t-1)}, \theta_{-(t-2)}, \dots, x_{-1}, \theta_0, x_0, \theta_1)$  of previous choices and preference shock realizations be given. Agent 0 solves

$$(A.1) \quad \max_{x_1^0 \in X} \left\{ -\alpha_1 (x_0^0 - x_1^0)^2 - \alpha_2 (\theta_1^0 - x_1^0)^2 - \alpha_3 (x_1^{-1} - x_1^0)^2 - \alpha_3 (x_1^1 - x_1^0)^2 \right\}$$

The first order condition

$$2 [\alpha_1 (x_0^0 - x_1^0) + \alpha_2 (\theta_1^0 - x_1^0) + \alpha_3 (x_1^{-1} - x_1^0) + \alpha_3 (x_1^1 - x_1^0)] = 0$$

implies that

$$(A.2) \quad x_1^0 = \Delta_1^{-1} (\alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \alpha_3 x_1^{-1} + \alpha_3 x_1^1)$$

---

<sup>a</sup>Université de Montréal, Department of Economics, CIREQ, CIRANO; [onur.ozgur@umontreal.ca](mailto:onur.ozgur@umontreal.ca)

<sup>b</sup>New York University, Department of Economics, NBER, CIREQ; [alberto.bisin@nyu.edu](mailto:alberto.bisin@nyu.edu)

<sup>1</sup>We laid out the problem in its recursive form for clarity in Section 2. Our method of proof attacks the sequence problem directly.

where  $\Delta_1 := \alpha_1 + \alpha_2 + 2\alpha_3 > 0$ . This choice is feasible (in  $X$ ) since it is a convex combination of elements of  $X$ , a convex set by assumption. The objective function (A.1) is strictly concave in  $x_1^0$ , thus  $x_1^0$  in (A.2) is the unique optimizer. The form in (A.2) suggests that showing the existence of a symmetric equilibrium in the continuation given history  $(x^{t-1}, \theta^t)$  is equivalent to finding the fixed point of an operator  $L_1 : B((\mathbf{X} \times \Theta)^t, X) \rightarrow B((\mathbf{X} \times \Theta)^t, X)$  that acts on the class of bounded measurable functions  $x_1 : (\mathbf{X} \times \Theta)^t \rightarrow X$  according to

$$(L_1 x_1)(x^{t-1}, \theta^t) = \Delta_1^{-1} (\alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \alpha_3 x_1(R^{-1} x^{t-1}, R^{-1} \theta^t) + \alpha_3 x_1(R x^{t-1}, R \theta^t))$$

Clearly,  $L_1$  is a self-map. We show next that it is a contraction. Endow  $B((\mathbf{X} \times \Theta)^t, X)$  with the sup norm which makes  $(B((\mathbf{X} \times \Theta)^t, X), \|\cdot\|_\infty)$  a Banach space. Pick  $x_1, \hat{x}_1 \in B((\mathbf{X} \times \Theta)^t, X)$ . We have for all  $(x^{t-1}, \theta^t)$

$$\begin{aligned} & \left| (L_1 x_1)(x^{t-1}, \theta^t) - (L_1 \hat{x}_1)(x^{t-1}, \theta^t) \right| \\ &= \Delta_1^{-1} \left| \alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \alpha_3 x_1(R^{-1} x^{t-1}, R^{-1} \theta^t) + \alpha_3 x_1(R x^{t-1}, R \theta^t) \right. \\ &\quad \left. - \alpha_1 x_0^0 - \alpha_2 \theta_1^0 - \alpha_3 \hat{x}_1(R^{-1} x^{t-1}, R^{-1} \theta^t) - \alpha_3 \hat{x}_1(R x^{t-1}, R \theta^t) \right| \\ &= \Delta_1^{-1} \left| \alpha_3 (x_1(R^{-1} x^{t-1}, R^{-1} \theta^t) - \hat{x}_1(R^{-1} x^{t-1}, R^{-1} \theta^t)) \right. \\ &\quad \left. + \alpha_3 (x_1(R x^{t-1}, R \theta^t) - \hat{x}_1(R x^{t-1}, R \theta^t)) \right| \\ &\leq \left( \frac{\alpha_3}{\Delta_1} \right) \left| x_1(R^{-1} x^{t-1}, R^{-1} \theta^t) - \hat{x}_1(R^{-1} x^{t-1}, R^{-1} \theta^t) \right| \\ &\quad + \left( \frac{\alpha_3}{\Delta_1} \right) \left| x_1(R x^{t-1}, R \theta^t) - \hat{x}_1(R x^{t-1}, R \theta^t) \right| \\ &\leq \left( \frac{2\alpha_3}{\Delta_1} \right) \|x_1 - \hat{x}_1\|_\infty \end{aligned}$$

The coefficient  $2\alpha_3 \Delta_1^{-1} < 1$  since  $\alpha_i > 0$ , for  $i = 1, 2, 3$ . Hence  $L_1$  is a contraction on  $B((\mathbf{X} \times \Theta)^t, X)$ . Thus, by Banach Fixed Point Theorem (see e.g., [Aliprantis and Border \(2006\)](#), p.95)  $L_1$  has a unique fixed point  $x_1^*$  in  $B((\mathbf{X} \times \Theta)^t, X)$ . Next, we argue that this equilibrium strategy must be Markovian and should take the convex combination form as in the statement of Theorem 1.

**LEMMA 1** (Markov Property and the Convex Combination Form) *Unique symmetric equilibrium strategy  $x_1^*$  is Markovian: For any  $t$ -length history  $(x^{t-1}, \theta^t)$ , it depends*



solely on last period equilibrium choices and current preference shock realizations, i.e.  $x_1^*(x^{t-1}, \theta^t) = g_1(x_0, \theta_1)$ , for some  $g_1 : \mathbf{X} \times \Theta \rightarrow X$ . Moreover, the Markovian policy function  $g_1$  has the convex combination form as in the statement of the theorem.

PROOF: Let

$$(A.3) \quad G := \left\{ \begin{array}{l} g : \mathbf{X} \times \Theta \rightarrow X \text{ s.t. } g(x, \theta) = \sum_{a \in \mathbb{A}} c^a x^a + \sum_{a \in \mathbb{A}} d^a \theta^a + e \bar{\theta} \\ \text{with} \\ \text{(i) } c^a, d^a, e \geq 0 \text{ and } e + \sum_{a \in \mathbb{A}} (c^a + d^a) = 1 \\ \text{(ii) } (\frac{1}{2})c^{a+1} + (\frac{1}{2})c^{a-1} \geq c^a, \forall a \neq 0 \\ \text{(iii) } c^b \leq c^a, \forall a, b \in \mathbb{A} \text{ with } |b| > |a|. \\ \text{(iv) } c^a = c^{-a}, \forall a \in \mathbb{A} \\ \text{and properties (ii), (iii), and (iv) also holding for the } d = (d^a)_{a \in \mathbb{A}} \text{ sequence.} \end{array} \right\}$$

be the class of functions that are convex combinations (i) of one-period before history, current types and average type, having the (ii) ‘convexity’, (iii) ‘monotonicity’, and (iv) ‘symmetry’ properties. Property (ii) states that the rate of ‘spatial’ (cross-sectional) convergence of the policy weights is non-increasing in both directions, relative to the origin. Monotonicity property, (iii), has a very natural interpretation: agent  $b$ ’s effect on agent 0’s marginal utility is smaller than agent  $a$ ’s effect on it, if  $a$  is closer to 0 than  $b$  is. Finally, (iv) says that the policy weights are symmetric around 0. Let  $g \in G$  be such that after any history  $(x^{t-1}, \theta^t) = (x_{-(t-1)}, \theta_{-(t-2)}, \dots, x_{-1}, \theta_0, x_0, \theta_1)$

$$x_1(x^{t-1}, \theta^t) = g(x_0, \theta_1)$$

and let  $(c, d, e)$  be the coefficient sequence associated with  $g$ . Applying  $L_1$  to  $x_1$  (hence to  $g$ ), we get

$$(A.4) \quad \begin{aligned} (L_1 x_1)(x^{t-1}, \theta^t) &= \Delta_1^{-1} (\alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \alpha_3 g(R^{-1}x_0, R^{-1}\theta_1) + \alpha_3 g(Rx_0, R\theta_1)) \\ &= \Delta_1^{-1} \left[ \alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \alpha_3 \left( \sum_{a \in \mathbb{A}} c^a x_0^{a-1} + \sum_{a \in \mathbb{A}} d^a \theta_1^{a-1} + e \bar{\theta} \right) \right. \\ &\quad \left. + \alpha_3 \left( \sum_{a \in \mathbb{A}} c^a x_0^{a+1} + \sum_{a \in \mathbb{A}} d^a \theta_1^{a+1} + e \bar{\theta} \right) \right] \end{aligned}$$

Rearranging terms gives

$$\begin{aligned}
 &= \Delta_1^{-1} \left( x_0^0 \underbrace{(\alpha_1 + \alpha_3 c^{-1} + \alpha_3 c^1)}_{\Delta_1 \hat{c}^0} + \theta_1^0 \underbrace{(\alpha_2 + \alpha_3 d^{-1} + \alpha_3 d^1)}_{\Delta_1 \hat{d}^0} + 2\alpha_3 e \bar{\theta} \right. \\
 (A.5) \quad &\left. + \sum_{a \neq 0} \underbrace{(\alpha_3 c^{a-1} + \alpha_3 c^{a+1})}_{\Delta_1 \hat{c}^a} x_0^a + \sum_{a \neq 0} \underbrace{(\alpha_3 d^{a-1} + \alpha_3 d^{a+1})}_{\Delta_1 \hat{d}^a} \theta_1^a \right)
 \end{aligned}$$

The function after the last equality sign is linear in  $x_0, \theta_1$  and  $\bar{\theta}$ . So,  $L_1 x_1$  preserves the same linear form. By definition of the new coefficient sequence  $(\hat{c}, \hat{d}, \hat{e})$  in (A.5), each element of the sequence is nonnegative since each element of the original one was so. New coefficients sum up to 1 since convex combination form of  $g$  makes the sum of the coefficients inside the two parentheses on the right hand side of (A.4) equal to 1. Thus, the total sum of coefficients on the right hand side of (A.4) is  $\Delta_1^{-1}(\alpha_1 + \alpha_2 + 2\alpha_3) = 1$ , which proves property (i). The final form in (A.5) is just a regrouping of elements in (A.4). Let  $(\hat{c}^a)_{a \in \mathbb{A}}$  be the new coefficient sequence associated with  $L_1 x_T$  as defined in equation (A.5). Pick  $a \neq 0$  in  $\mathbb{A}$ ,

$$\begin{aligned}
 \hat{c}^{a+1} + \hat{c}^{a-1} &\geq \left( \frac{\alpha_3}{\Delta_1} \right) (c^a + c^{a+2}) + \left( \frac{\alpha_3}{\Delta_1} \right) (c^{a-2} + c^a) \\
 &\geq \left( \frac{\alpha_3}{\Delta_1} \right) (2c^{a+1} + 2c^{a-1}) \\
 &= 2 \left( \frac{\alpha_3}{\Delta_1} \right) (c^{a+1} + c^{a-1}) \\
 &= 2\hat{c}^a
 \end{aligned}$$

By definition of  $\hat{c}$  in (A.5), first inequality is strict if  $|a| = 1$ , is an equality otherwise; second inequality is by property (ii) on  $c$ ; last equality is once again by definition of  $\hat{c}$  in (A.5). Therefore, for any  $a \neq 0$  in  $\mathbb{A}$ ,  $\hat{c}^{a+1} + \hat{c}^{a-1} \geq 2\hat{c}^a$ , which is property (ii). Pick any  $a, b \in \mathbb{A}$  with  $|a| < |b|$ .

$$\begin{aligned}
 \hat{c}^a &= \left( \frac{\alpha_3}{\Delta_1} \right) c^{a-1} + \left( \frac{\alpha_3}{\Delta_1} \right) c^{a+1} = \left( \frac{\alpha_3}{\Delta_1} \right) c^{|a|-1} + \left( \frac{\alpha_3}{\Delta_1} \right) c^{|a|+1} \\
 &\geq \left( \frac{\alpha_3}{\Delta_1} \right) c^{|b|-1} + \left( \frac{\alpha_3}{\Delta_1} \right) c^{|b|+1} = \left( \frac{\alpha_3}{\Delta_1} \right) c^{b-1} + \left( \frac{\alpha_3}{\Delta_1} \right) c^{b+1} \\
 &= \hat{c}^b
 \end{aligned}$$

First equality is from (A.5); second by property (iv) of  $G$  in (A.3); the inequality is

property (iii) of  $G$  in (A.3); next equality is due to property (iv) of  $G$  again; and finally the last equality is by (A.5). Hence, property (iii) in (A.3) holds for the new sequence. We next show that  $\hat{c}$  satisfies (iv) in (A.3).

$$\begin{aligned}\hat{c}^a &= \left(\frac{\alpha_3}{\Delta_1}\right) c^{a-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{a+1} \\ &= \left(\frac{\alpha_3}{\Delta_1}\right) c^{-a-1} + \left(\frac{\alpha_3}{\Delta_1}\right) c^{-a+1} \\ &= \hat{c}^{-a}\end{aligned}$$

where first equality is by (A.5); the second is due to (iv) of  $G$  in (A.3); finally the last is again by (A.5).

Thus, the restriction of  $L_1$  to the subspace (call it  $B_G$ ) of bounded measurable functions that agree with an element of  $G$  after any history, maps elements of  $B_G$  into itself. Moreover, endowed with the sup norm,  $B_G$  is a closed subset of  $B((\mathbf{X} \times \Theta)^t, X)$  since it is defined by equality and inequality constraints, hence a complete metric space in its own right. Since  $L_1$  is a contraction on this latter as we just showed, it is so on  $B_G$  too and the unique fixed point  $x_1^*$  in  $B((\mathbf{X} \times \Theta)^t, X)$  must lie in  $B_G$ . Since the choice of  $t$  was arbitrary, the unique symmetric equilibrium in a one-period (continuation) economy, after any length history must be Markovian and should assume the convex combination form stated in the theorem. This concludes the proof of Lemma 1. Q.E.D.

This proves Step 1, namely that the statement of the Theorem is true for 1-period economies. Next, we prove that this result generalizes to any finite-horizon economy.

**Step 2: Induction, T-1 implies T.** Let  $T \geq 2$ . Assume that the statement of Theorem 1 is true up to  $T-1$ -period. The  $T$ -period economy can be *separated* into a first period and a  $T-1$ -period continuation economy. By hypothesis, there exists a unique symmetric MPE,  $g : \mathbf{X} \times \Theta \times \{1, \dots, T-1\} \mapsto X$ , for the  $T-1$ -period continuation economy. Agent 0 believes that all other agents, including his own reincarnations, will use that unique symmetric equilibrium map from period 2 on, i.e., for any agent  $b \in \mathbb{A}$ ,

$$x_t^b(x^{t-1}, \theta^t) = g_{T-(t-1)}(R^b x_{t-1}, R^b I_0 \theta_t), \quad \text{for all } t = 2, \dots, T$$

Given any  $t$ -length history  $(x^{t-1}, \theta^t)$ , the current strategies of all other agents  $(x_1^b)_{b \neq 0}$ , and the fact that  $(x_t^b)_{t \geq 2}^{b \in \mathbb{A}}$  are induced by  $g$ , agent 0 solves

$$(A.6) \quad \max_{x_1^0 \in X} \left\{ -\alpha_1 (x_0^0 - x_1^0)^2 - \alpha_2 (\theta_1^0 - x_1^0)^2 - \alpha_3 (x_1^{-1} - x_1^0)^2 - \alpha_3 (x_1^1 - x_1^0)^2 \right. \\ \left. + E \left[ \sum_{\tau=2}^T \beta^{\tau-1} \left( -\alpha_1 (x_{\tau-1}^0 - x_\tau^0)^2 - \alpha_2 (\theta_\tau^0 - x_\tau^0)^2 \right. \right. \right. \\ \left. \left. \left. - \alpha_3 (x_\tau^{-1} - x_\tau^0)^2 - \alpha_3 (x_\tau^1 - x_\tau^0)^2 \right) \middle| (x^{t-1}, \theta^t) \right] \right\}$$

The form of the optimal choices on the equilibrium path can be characterized as in the following Lemma.

LEMMA 2 (Convexity and Monotonicity) *Given a  $T$ -period economy, equilibrium choices satisfy the following properties:*

(i) For any period  $t \geq 2$ ,  $x_t^a$  can be written as <sup>2</sup>

$$x_t^a = \sum_{b_1 \in \mathbb{A}} \cdots \sum_{b_{t-1} \in \mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-1}^{b_{t-1}} x_1^{a+b_1+\cdots+b_{t-1}} \\ + \sum_{s=1}^{t-1} \sum_{b_1 \in \mathbb{A}} \cdots \sum_{b_{s-1} \in \mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-(t-(s-1))}^{b_{s-1}} \left( \sum_{b_s \in \mathbb{A}} d_{T-(t-s)}^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e_{T-(t-s)} \bar{\theta} \right)$$

(ii) For any  $a, b \in \mathbb{A}$ , any  $t \geq 2$ ,

$$|a| \leq |b| \implies \frac{\partial x_t^0}{\partial x_1^a} \leq \frac{\partial x_t^0}{\partial x_1^b}$$

$$(iii) \quad \frac{\partial}{\partial x_1^a} (2x_t^0 - x_t^1 - x_t^{-1}) \leq 0$$

$$(iv) \quad \frac{\partial}{\partial x_1^0} x_t^0 \leq \left( \frac{\alpha_1}{\alpha_1 + \alpha_2} \right) \frac{\partial}{\partial x_1^0} x_{t-1}^0$$

PROOF: (i) This part is simply by iterated application of the policy maps, i.e.,

$$x_t^a = g_{T-(t-1)}(R^a x_{t-1}, R^a \theta_t) \\ = \sum_{b_1 \in \mathbb{A}} c_{T-(t-1)}^{b_1} x_{t-1}^{a+b_1} + \sum_{b_1 \in \mathbb{A}} d_{T-(t-1)}^{b_1} \theta_t^{a+b_1} + e_{T-(t-1)} \bar{\theta} \\ = \sum_{b_1 \in \mathbb{A}} c_{T-(t-1)}^{b_1} \underbrace{g_{T-t}(R^{a+b_1} x_{t-2}, R^{a+b_1} \theta_{t-1})}_{x_{t-1}^{a+b_1}} + \sum_{b_1 \in \mathbb{A}} d_{T-(t-1)}^{b_1} \theta_t^{a+b_1} + e_{T-(t-1)} \bar{\theta}$$

---

<sup>2</sup>We use in expression (A.7) the convention that in the sum after the plus sign, for  $s = 1$ , the summand becomes  $\sum_{b_s \in \mathbb{A}} d_{T-(t-s)}^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e_{T-(t-s)} \bar{\theta}$ .

$$\begin{aligned}
&= \sum_{b_1 \in \mathbb{A}} c_{T-(t-1)}^{b_1} \left( \sum_{b_2 \in \mathbb{A}} c_{T-(t-2)}^{b_2} x_{t-2}^{a+b_1+b_2} + \sum_{b_2 \in \mathbb{A}} d_{T-(t-2)}^{b_2} \theta_{t-1}^{a+b_1+b_2} + e_{T-(t-2)} \bar{\theta} \right) \\
&\quad + \sum_{b_1 \in \mathbb{A}} d_{T-(t-1)}^{b_1} \theta_t^{a+b_1} + e_{T-(t-1)} \bar{\theta} \\
&\quad \vdots \\
&= \sum_{b_1 \in \mathbb{A}} \cdots \sum_{b_{t-1} \in \mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-1}^{b_{t-1}} x_1^{a+b_1+\cdots+b_{t-1}} \\
&\quad + \sum_{s=1}^{t-1} \sum_{b_1 \in \mathbb{A}} \cdots \sum_{b_{s-1} \in \mathbb{A}} c_{T-(t-1)}^{b_1} \cdots c_{T-(t-s)+1}^{b_{s-1}} \left( \sum_{b_s \in \mathbb{A}} d_{T-(t-s)}^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e_{T-(t-s)} \bar{\theta} \right)
\end{aligned}$$

(ii) For  $t = 2$ ,  $\frac{\partial}{\partial x_1^a} x_2^0 = c_1^a \geq c_1^b = \frac{\partial}{\partial x_1^b} x_2^0$  by (A.3) (iii). Suppose the claim is true for  $t \leq k$  and let  $t = k + 1$ . Assume w.l.o.g that  $a < b$ . Let  $\underline{s} := \max\{s \in \mathbb{A} : s \leq \frac{a+b}{2}\}$  and  $\bar{s} := \min\{s \in \mathbb{A} : s \geq \frac{a+b}{2}\}$ . This implies that  $\frac{\partial}{\partial x_1^a} x_k^s - \frac{\partial}{\partial x_1^b} x_k^s \geq 0$  ( $\leq 0$ ) for  $s \leq \underline{s}$  ( $s \geq \bar{s}$ ). Due to the assumed symmetry,  $\left[ \frac{\partial}{\partial x_1^a} x_k^{s-s} - \frac{\partial}{\partial x_1^b} x_k^{s-s} \right] = \left[ \frac{\partial}{\partial x_1^{a-\underline{s}+s}} x_k^0 - \frac{\partial}{\partial x_1^{b-\underline{s}+s}} x_k^0 \right]$  and  $\left[ \frac{\partial}{\partial x_1^{a-\bar{s}-s}} x_k^0 - \frac{\partial}{\partial x_1^{b-\bar{s}-s}} x_k^0 \right] = \left[ \frac{\partial}{\partial x_1^a} x_k^{\bar{s}} - \frac{\partial}{\partial x_1^b} x_k^{\bar{s}} \right]$ . This implies that for any  $s > 0$

$$\left[ \frac{\partial}{\partial x_1^{a-\underline{s}+s}} x_k^0 - \frac{\partial}{\partial x_1^{b-\underline{s}+s}} x_k^0 \right] = - \left[ \frac{\partial}{\partial x_1^{a-\bar{s}-s}} x_k^0 - \frac{\partial}{\partial x_1^{b-\bar{s}-s}} x_k^0 \right]$$

Thus, we can use this to separate  $\mathbb{A}$  into  $\{s \in \mathbb{A} : s \leq \underline{s}\}$   $\{s \in \mathbb{A} : s \geq \bar{s}\}$  and rearrange the sum

$$\begin{aligned}
\frac{\partial}{\partial x_1^a} x_t^0 - \frac{\partial}{\partial x_1^b} x_t^0 &= \sum_{s \in \mathbb{A}} c_{T-k}^s \left[ \frac{\partial}{\partial x_1^a} x_k^s - \frac{\partial}{\partial x_1^b} x_k^s \right] \\
&= \sum_{s \in \mathbb{A}} c_{T-k}^s \left[ \frac{\partial}{\partial x_1^{a-s}} x_k^0 - \frac{\partial}{\partial x_1^{b-s}} x_k^0 \right] \\
&= \sum_{\tau \geq 0} (c_{T-k}^{s-s} - c_{T-k}^{\bar{s}+s}) \left[ \frac{\partial}{\partial x_1^{a-\underline{s}+s}} x_k^0 - \frac{\partial}{\partial x_1^{b-\underline{s}+s}} x_k^0 \right] \\
&\geq 0
\end{aligned}$$

The term in the brackets is nonnegative by hypothesis. Since  $a < b$ ,  $\underline{s} \geq 0$  which implies that  $c_{T-k}^s \geq c_{T-k}^{\bar{s}}$ . But this implies that  $c_{T-k}^{s-s} \geq c_{T-k}^{\bar{s}+s}$  for any  $s \geq 0$  which means that the argument in the parenthesis is nonnegative too. So, the claim is true. The analysis for the case  $a > b$  is a straightforward modification of the same argument.

(iii) Using the  $t$ -th period equilibrium policy

$$\begin{aligned}
\frac{\partial}{\partial x_1^a} (2x_t^0 - x_t^1 - x_t^{-1}) &= \frac{\partial}{\partial x_1^a} \left[ 2 \sum_{b \in \mathbb{A}} c_{T-(t-1)}^b x_{t-1}^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^b \theta_t^b + e_{T-(t-1)} \bar{\theta} \right. \\
&\quad - \sum_{b \in \mathbb{A}} c_{T-(t-1)}^{b-1} x_{t-1}^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^{b-1} \theta_t^b + e_{T-(t-1)} \bar{\theta} \\
&\quad \left. - \sum_{b \in \mathbb{A}} c_{T-(t-1)}^{b+1} x_{t-1}^b + \sum_{b \in \mathbb{A}} d_{T-(t-1)}^{b+1} \theta_t^b + e_{T-(t-1)} \bar{\theta} \right] \\
\text{(A.7)} \quad &= \sum_{b \in \mathbb{A}} \left( 2c_{T-(t-1)}^b - c_{T-(t-1)}^{b-1} - c_{T-(t-1)}^{b+1} \right) \frac{\partial}{\partial x_1^a} x_{t-1}^b \leq 0
\end{aligned}$$

The weights in the last parenthesis are negative by property (ii) in (A.3). By iteratively applying the policy functions from period  $t$  backwards, at each iteration the weights on one-period before choices would all be positive and one preserves the convex combination form. This process ends after  $t - 1$  iteration, the end result being a convex combination of  $(x_1^b)_{b \in \mathbb{A}}$ ,  $\theta^t$  and  $\bar{\theta}$ . Thus, the weight on  $x_1^a$  is positive, which makes the last term in the last line positive. Therefore the claim is true.

(iv) Let  $t \geq 2$ .

$$\begin{aligned}
\frac{\partial}{\partial x_1^0} x_t^0 &= \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a \frac{\partial x_{t-1}^a}{\partial x_1^0} = \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a \frac{\partial x_{t-1}^0}{\partial x_1^a} \\
&\leq \sum_{a \in \mathbb{A}} c_{T-(t-1)}^a \frac{\partial x_{t-1}^0}{\partial x_1^0} = C_{T-(t-1)} \frac{\partial x_{t-1}^0}{\partial x_1^0}
\end{aligned}$$

First and second equalities and the first inequality are by the definition of the policy mapping and (i) of Lemma 2;  $C_{T-(t-1)}$  is the sum of coefficients on the past history in the period  $t$  policy. Since  $g_{T-(t-1)}$  satisfies (A.11), coefficients should match and we should have

$$\begin{aligned}
0 &= c_{T-(t-1)}^a \Delta_{T-(t-1)} - \alpha_1 I_{\{a=0\}} - \sum_{b \neq 0} \gamma_{T-(t-1)}^b c_{T-(t-1)}^{a-b} \\
\text{summing over } a, \quad 0 &= C_{T-(t-1)} \Delta_{T-(t-1)} - \alpha_1 - \sum_{b \neq 0} \gamma_{T-(t-1)}^b C_{T-(t-1)}
\end{aligned}$$

But  $\Delta_{T-(t-1)} = \alpha_1 + \alpha_2 + \sum_{b \neq 0} \gamma_{T-(t-1)}^b + \mu_{T-(t-1)}$  by definition. So,

$$\text{(A.8)} \quad C_{T-(t-1)} = \frac{\alpha_1}{\Delta_{T-(t-1)} - \sum_{b \neq 0} \gamma_{T-(t-1)}^b} = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \mu_{T-(t-1)}} \leq \frac{\alpha_1}{\alpha_1 + \alpha_2}$$

Thus,

$$\frac{\partial}{\partial x_1^0} x_t^0 \leq C_{T-(t-1)} \frac{\partial}{\partial x_1^0} x_{t-1}^0 \leq \left( \frac{\alpha_1}{\alpha_1 + \alpha_2} \right) \frac{\partial}{\partial x_1^0} x_{t-1}^0$$

which concludes the proof of Lemma 2.

*Q.E.D.*

Thanks to the linearity of the optimal future choices as shown in Lemma 2, agent 0's problem (A.6) is differentiable with respect to  $x_1^0$  and the unconstrained ( $x_1^0 \in \mathbb{R}$ ) first order condition is

$$\begin{aligned} (A.9) \quad 0 = & \alpha_1 (x_0^0 - x_1^0) + \alpha_2 (\theta_1^0 - x_1^0) + \alpha_3 (x_1^{-1} - x_1^0) + \alpha_3 (x_1^1 - x_1^0) \\ & + E \left[ \sum_{\tau=2}^T \beta^{\tau-1} \left( -\alpha_1 (x_{\tau-1}^0 - x_\tau^0) \frac{\partial}{\partial x_1^0} (x_{\tau-1}^0 - x_\tau^0) + \alpha_2 (\theta_\tau^0 - x_\tau^0) \frac{\partial}{\partial x_1^0} x_\tau^0 \right. \right. \\ & \left. \left. - \alpha_3 (x_\tau^{-1} - x_\tau^0) \frac{\partial}{\partial x_t^0} (x_\tau^{-1} - x_\tau^0) - \alpha_3 (x_\tau^1 - x_\tau^0) \frac{\partial}{\partial x_1^0} (x_\tau^1 - x_\tau^0) \right) \right] (x^{t-1}, \theta^t) \end{aligned}$$

Agent 0's problem (A.6) is strictly concave in his choice  $x_1^0$  since the second partial of the objective function with respect to  $x_1^0$ ,  $-\Delta_T$  by definition, is negative, or

$$\begin{aligned} (A.10) \quad \Delta_T := & \alpha_1 + \alpha_2 + 2\alpha_3 + \sum_{\tau=2}^T \beta^{\tau-t} \left( \alpha_1 \left( \frac{\partial}{\partial x_1^0} (x_{\tau-1}^0 - x_\tau^0) \right)^2 + \alpha_2 \left( \frac{\partial}{\partial x_1^0} x_\tau^0 \right)^2 \right. \\ & \left. + \alpha_3 \left( \frac{\partial}{\partial x_1^0} (x_\tau^{-1} - x_\tau^0) \right)^2 + \alpha_3 \left( \frac{\partial}{\partial x_1^0} (x_\tau^1 - x_\tau^0) \right)^2 \right) > 0 \end{aligned}$$

Consequently, the FOC characterizes the *unique maximizer* of the unconstrained problem ( $x_1^0 \in \mathbb{R}$ ). The following Lemma shows that equation (A.9) has a much simpler representation.

LEMMA 3 (Interiority) *Equation (A.9) can be written alternatively as*

$$(A.11) \quad 0 = -x_1^0 \Delta_T + \alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \sum_{a \neq 0} \gamma_T^a x_1^a + \mu_T \bar{\theta}$$

where  $\Delta_T := \alpha_1 + \alpha_2 + \sum_{a \neq 0} \gamma_T^a + \mu_T$  and the coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $(\gamma_T^a)_{a \neq 0}$ , and  $\mu_T$  are non-negative.

PROOF: The coefficient of  $x_1^a$  in (A.11),  $\gamma_T^a$ , is the total effect of a change in  $x_1^a$  ( $a \neq 0$ ) on the expected discounted marginal utility of agent 0 (the right hand side of (A.9)),

i.e.,

$$\begin{aligned}
\gamma_T^a &:= \alpha_3 I_{\{a \in \{-1,1\}\}} \\
(A.12) \quad & - \sum_{\tau=2}^T \beta^{\tau-1} \left( \alpha_1 \frac{\partial}{\partial x_1^a} (x_{\tau-1}^0 - x_\tau^0) \frac{\partial}{\partial x_1^0} (x_{\tau-1}^0 - x_\tau^0) + \alpha_2 \frac{\partial}{\partial x_1^a} x_\tau^0 \frac{\partial}{\partial x_1^0} x_\tau^0 \right. \\
& \quad \left. + \alpha_3 \frac{\partial}{\partial x_1^a} (x_\tau^{-1} - x_\tau^0) \frac{\partial}{\partial x_1^0} (x_\tau^{-1} - x_\tau^0) + \alpha_3 \frac{\partial}{\partial x_1^a} (x_\tau^1 - x_\tau^0) \frac{\partial}{\partial x_1^0} (x_\tau^1 - x_\tau^0) \right)
\end{aligned}$$

For any  $\tau \geq 2$ , the last two term in the summand for each period in equation (A.12) can be written as

$$\begin{aligned}
& \frac{\partial}{\partial x_1^0} (x_\tau^1 - x_\tau^0) \left[ \alpha_3 \frac{\partial}{\partial x_1^a} (x_\tau^{-1} - x_\tau^0) + \alpha_3 \frac{\partial}{\partial x_1^a} (x_\tau^1 - x_\tau^0) \right] \\
& = \frac{\partial}{\partial x_1^0} (x_\tau^1 - x_\tau^0) \left[ \alpha_3 \frac{\partial x_\tau^0}{\partial x_1^{a+1}} + \alpha_3 \frac{\partial x_\tau^0}{\partial x_1^{a-1}} - 2\alpha_3 \frac{\partial x_\tau^0}{\partial x_1^a} \right] \\
(A.13) \quad & \leq 0
\end{aligned}$$

The equality is due to the symmetry of the policy function across agents; Lemma 2 (ii) and (iii) imply that the terms in the parentheses are non-positive and the terms in the brackets are non-negative, respectively. Similarly, the first terms in the summand in (A.12) can be written as

$$\begin{aligned}
& \alpha_1 \frac{\partial}{\partial x_1^a} (x_\tau^0 - x_{\tau-1}^0) \frac{\partial}{\partial x_1^0} (x_\tau^0 - x_{\tau-1}^0) + \alpha_2 \frac{\partial}{\partial x_1^a} x_\tau^0 \frac{\partial}{\partial x_1^0} x_\tau^0 \\
& \leq \alpha_1 \frac{\partial}{\partial x_1^a} x_\tau^0 \frac{\partial}{\partial x_1^0} (x_\tau^0 - x_{\tau-1}^0) + \alpha_2 \frac{\partial}{\partial x_1^a} x_\tau^0 \frac{\partial}{\partial x_1^0} x_\tau^0 \\
& = \frac{\partial x_\tau^0}{\partial x_1^a} \left[ (\alpha_1 + \alpha_2) \frac{\partial}{\partial x_1^0} x_\tau^0 - \alpha_1 \frac{\partial}{\partial x_1^0} x_{\tau-1}^0 \right] \\
& \leq 0
\end{aligned}$$

which is nonpositive since for any  $\tau \geq 2$

$$\frac{\partial}{\partial x_1^0} x_\tau^0 \leq \frac{\alpha_1}{(\alpha_1 + \alpha_2)} \frac{\partial}{\partial x_1^0} x_{\tau-1}^0$$

due to Lemma 2 (iv). Thus, we established the non-positiveness of each term of the summand for any period  $\tau \geq 2$  in (A.12). Since, the latter is basically a finite weighted some of such terms with a negative sign in front, for any  $a \in \mathbb{A}$ ,  $\gamma_T^a \geq 0$ . Finally we



account for the coefficients multiplying  $\bar{\theta}$  in equation (A.9) and show that

$$\begin{aligned}
 \mu_T &= \frac{\partial}{\partial \bar{\theta}} E \left[ \sum_{\tau=2}^T \beta^{\tau-1} \left( -\alpha_1 (x_{\tau-1}^0 - x_{\tau}^0) \frac{\partial}{\partial x_1^0} (x_{\tau-1}^0 - x_{\tau}^0) + \alpha_2 (\theta_{\tau}^0 - x_{\tau}^0) \frac{\partial}{\partial x_1^0} x_{\tau}^0 \right. \right. \\
 (A.14) \quad &\quad \left. \left. -\alpha_3 (x_{\tau}^{-1} - x_{\tau}^0) \frac{\partial}{\partial x_1^0} (x_{\tau}^{-1} - x_{\tau}^0) - \alpha_3 (x_{\tau}^1 - x_{\tau}^0) \frac{\partial}{\partial x_1^0} (x_{\tau}^1 - x_{\tau}^0) \right) \right] \Big| (x^{t-1}, \theta^t) \\
 &\geq 0
 \end{aligned}$$

Expectation washes out all individual  $\theta_{\tau}^a$ 's and we have only  $\bar{\theta}$  apart from  $(x_1^a)_{a \in \mathbb{A}}$  in each period's expression in (A.14). By symmetry of the form in Lemma (2) (i) across agents, the weight on  $\bar{\theta}$  in  $x_t^0$ , is equal to the weight on  $\bar{\theta}$  in  $x_t^1$  and on  $x_t^{-1}$ . Thus,  $\frac{\partial}{\partial \bar{\theta}} E[(x_t^0 - x_t^1)] = \frac{\partial}{\partial \bar{\theta}} E[(x_t^0 - x_t^{-1})] = 0$ . This makes the second line of (A.14) equal to zero. By Lemma (2)-(i), the weight on  $\bar{\theta}$  in  $E[x_t^0]$ ,  $1 - \Pi_{s=2}^t C_{T-(t-1)}$  (residual of the sum of the effects of  $\{x_1^b\}$ ) is bigger than that in  $E[x_{t-1}^0]$ ,  $1 - \Pi_{s=2}^{t-1} C_{T-(t-1)}$ ; hence the term  $\frac{\partial}{\partial \bar{\theta}} E[(x_{t-1}^0 - x_t^0)] \leq 0$ . By Lemma (2)-(i),  $\frac{\partial}{\partial \bar{\theta}} E[x_t^0] \geq 0$ . By Lemma (2)-(iv),  $\frac{\partial}{\partial x_1^0} (x_t^0 - x_{t-1}^0) \leq 0$ . All these together imply that the expression in (A.14) is non-negative. Each  $E[x_{\tau}^b]$  in (A.9) can be written as a convex combination of  $(x_1^a)_{a \in \mathbb{A}}$ ,  $\bar{\theta}$ ,  $x_0^0$ ,  $\theta_1^0$ , with the help of Lemma 2-(i). Since at each iteration, convex combination structure is preserved, it is so at the end too. Then, the sum of coefficients in each of the differences involving those variables in the parentheses is zero. This in turn implies that the total sum of coefficients in (A.9) is zero. Thus, the alternative formulation in (A.11) is true. This concludes the proof of Lemma 3. Q.E.D.

By isolating the choice  $x_1^0$  on the left hand side, we can write the maximizer of the unconstrained problem as a convex combination of  $x_0^0$ ,  $\theta_1^0$ ,  $(x_1^a)_{a \neq 0}$  and  $\bar{\theta}$

$$(A.15) \quad x_1^0 = \Delta_T^{-1} \left( \alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \sum_{a \neq 0} \gamma_T^a x_1^a + \mu_T \bar{\theta} \right)$$

Each of these are elements of  $X$ , a convex set. Thus, the optimal choice of the unconstrained problem is in the feasible set of the constrained problem, hence it is its unique maximizer. The form in (A.15) implies that showing the existence of a symmetric equilibrium policy for the first period of a  $T$ -period economy is equivalent to finding the fixed point of an operator  $L_T : B((\mathbf{X} \times \Theta)^t, X) \rightarrow B((\mathbf{X} \times \Theta)^t, X)$  that acts on the

class of bounded measurable functions  $x_1 : (\mathbf{X} \times \Theta)^t \rightarrow X$  according to

$$(A.16) \quad (L_T x_1)(x^{t-1}, \theta^t) = \Delta_T^{-1} \left( \alpha_1 x_0^0 + \alpha_2 \theta_1^0 + \sum_{a \neq 0} \gamma_T^a x_1(R^a x^{t-1}, R^a \theta^t) + \mu_T \bar{\theta} \right)$$

Clearly  $L_T$  is a self-map. Using straightforward modifications of the arguments in the proof of **Step 1**, one obtains for  $x_1, \hat{x}_1 \in B((\mathbf{X} \times \Theta)^t, X)$  that

$$\left| (L_T x_1)(x^{t-1}, \theta^t) - (L_T \hat{x}_1)(x^{t-1}, \theta^t) \right| \leq \sum_{a \neq 0} \left( \frac{\gamma_T^a}{\Delta_T} \right) \|x_1 - \hat{x}_1\|_\infty$$

The coefficient  $\sum_{a \neq 0} \left( \frac{\gamma_T^a}{\Delta_T} \right) < 1$  since  $\alpha_i > 0$ ,  $i = 1, 2, 3$ . Thus,  $L_T$  is a contraction on the Banach space of bounded measurable functions  $(B((\mathbf{X} \times \Theta)^t, X), \|\cdot\|_\infty)$ , hence has a unique fixed point  $x_1^*$ . Once again, by the same token as in Lemma 1, perfect equilibria are necessarily Markovian thus we can focus attention on Markovian strategies. As in the proof of Lemma 1, it suffices to show that  $L_T(B_G) \subset B_G$ . To that effect, let  $x_1 \in B_G$  be such that there exists a  $g \in G$  for which after any history  $(x^{t-1}, \theta^t)$ , one has  $x_1(x^{t-1}, \theta^t) = g(x_0, \theta_1)$ ; let  $(c, d, e)$  be the coefficient sequence associated with  $g$ . Applying  $L_T$  to  $x_1$ , we get

$$(A.17) \quad \begin{aligned} (L_T x_1)(x^{t-1}, \theta^t) = & \Delta_T^{-1} \left( \underbrace{[\alpha_1 + \sum_{a \neq 0} \gamma_T^a c^{-a}] x_0^0}_{\Delta_T \hat{c}^0} + \underbrace{[\alpha_2 + \sum_{a_1 \neq 0} \gamma_T^{a_1} d^{-a_1}] \theta_1^0}_{\Delta_T \hat{d}^0} \right. \\ & \left. + \sum_{b \neq 0} \left\{ \underbrace{[\sum_{a \neq 0} \gamma_T^a c^{b-a}] x_0^b}_{\Delta_T \hat{c}^b} + \underbrace{[\sum_{a \neq 0} \gamma_T^a d^{b-a}] \theta_1^b}_{\Delta_T \hat{d}^b} \right\} + \underbrace{[\mu_T + e \sum_{a \neq 0} \gamma_T^a] \bar{\theta}}_{\Delta_T \hat{e}} \right) \end{aligned}$$

The expression above is linear in  $x_0, \theta_1$  and  $\bar{\theta}$ . So,  $L_T x_1$  is linear. By definition of the new coefficient sequence  $(\hat{c}, \hat{d}, \hat{e})$ , each element of the new sequence is nonnegative since each element of the original one was so and the new elements are positive weighted sums of the original ones. The total sum of the coefficients on the right hand side of (A.17) is  $\Delta_T^{-1}(\alpha_1 + \alpha_2 + \sum_{a \neq 0} \gamma_T^a + \mu_T) = 1$  since (A.16) (which is equivalent to (A.17)) is a convex combination of elements and of functions that are convex combinations of elements of the convex set  $X$ . This proves property (i). The proof of the properties (ii), (iii), and (iv) follows identical arguments as in Lemma 1. Thus, the unique fixed point  $x_1^*$  should lie in the set  $B_G$  with an associated equilibrium Markovian policy function  $g_T^{**}$ .

Therefore, when the symmetric continuation equilibrium policies are Markovian, i.e.,  $g : \mathbf{X} \times \Theta \times \{1, \dots, T-1\} \mapsto X$ , after any history  $(x^{t-1}, \theta^t)$ , the unique symmetric equilibrium policy in the first period,  $g_T^{**}$  is Markovian too. Since the choice of  $t$  was arbitrary, this must be true for any length history. Now, construct the policy function  $g^*$  as  $g_T^*(x_0, \theta_1) = g_T^{**}(x_0, \theta_1)$  for any initial  $(x_0, \theta_1)$ ; and  $g_{T-(t-1)}^*(x_{t-1}, \theta_t) = g_{T-(t-1)}(x_{t-1}, \theta_t)$ , for all  $t \in \{2, \dots, T\}$  and all  $(x_{t-1}, \theta_t)$ . But then, the function  $g^*$  is by construction the unique MPE of the  $T$ -period economy. This completes the induction step for any given  $T \geq 2$ . Therefore, the claim in Theorem 1 is true for any finite horizon economy.

**Step 3: Convergence and stationarity.** This step proves that the sequence of finite horizon symmetric Markovian equilibria tends to a stationary symmetric Markov Perfect equilibrium. To do that, we treat finite-horizon economies as finite truncations of the infinite-horizon economy. Let  $G^\infty := \prod_{t=1}^\infty G$  be the infinite-horizon Markovian strategy set. For a fixed discount factor  $\beta \in (0, 1)$ , let  $L_\beta := \{\beta_T \in [0, 1]^\infty \mid \beta_{T,t} = \beta^{t-1}, \text{ for } 1 \leq t \leq T, \text{ and } \beta_{T,t} = 0, \text{ for } t > T, \text{ where } T \in \{1, 2, \dots\} \cup \{\infty\}\}$  be the space of exponentially declining sequences (at the rate  $\beta$ ) that are equal to zero after the  $T$ -th element. Endow  $L_\beta$  with the sup norm.

**LEMMA 4 (Compactness)**  *$L_\beta$  and  $G$  endowed with the supnorm are compact metric spaces.*

**PROOF:** Let  $(\beta_{T_n})_n$  be a sequence lying in  $L_\beta$  that converges to  $x = (x_t) \in [0, 1]^\infty$ . This implies that  $\beta_{T_n,t} \rightarrow x_t$ , for all  $t \geq 1$ , which in turn means that  $x_t \in \{0, \beta^t\}$  by the construction of  $L_\beta$ . Moreover, if  $x_t = 0$  for some  $t$ ,  $x_{t+\tau} = 0$  for all  $\tau \geq 1$  since the terms  $\beta_{T_n}$  are geometric (finite or infinite) sequences. There are two possibilities: either  $x = (1, \beta, \dots, \beta^T, 0, 0, \dots)$  or  $x = \beta^t$  for all  $t \geq 1$ . Both lie in  $L_\beta$  which means that the limit of any convergent sequence in  $L_\beta$  lies in  $L_\beta$ . This establishes that  $L_\beta$  is closed. Given any  $\epsilon > 0$ , choose  $N \geq 1$ , a natural number, s.t.  $\beta^N < \epsilon$ . It is easy to see that any element in  $L_\beta$  lies in the  $\epsilon$ -neighborhood (with respect to the sup metric) of one of the elements in the finite set  $\{\beta_1, \beta_2, \dots, \beta_N\} \subset L_\beta$ . This establishes that  $L_\beta$  is totally bounded. Therefore,  $L_\beta$  is compact. We next show that  $G$  endowed with the sup norm is compact.

Let  $H := \{x = (x^a)_{a \in \mathbb{A}} \mid x^a \leq (\frac{1}{2a}), \text{ for all } a \in \mathbb{A}\}$ . Defined by inequality constraints, this set is closed under the sup norm. We will show that it is also totally bounded. For

a given  $\epsilon > 0$ , one can find an  $N \geq 1$  s.t.  $\frac{1}{2N} < \epsilon$ . Pick a sequence  $\bar{x} \in H$ . For any  $a \in \mathbb{A}$  s.t.  $|a| \geq N$ ,  $[0, (2N)^{-1}] \subset B_\infty(x^a, \epsilon)$ , the  $\epsilon$ -ball around  $x^a$  with respect to the sup norm. For  $|a| \leq N$ , let  $Y(a) := \{0, \epsilon, 2\epsilon, \dots, k_a\epsilon, (2a)^{-1}\}$ , where  $k_a$  is the greatest integer s.t.  $k_a\epsilon \leq (2a)^{-1}$ . The set

$$\left\{ x \in H \mid x^a = \bar{x}^a, \text{ for } |a| \geq N, \text{ and } (x^{-(N-1)}, \dots, x^0, \dots, x^{N-1}) \in \prod_{|a| \leq N} Y(a), \text{ for } |a| \leq N \right\}$$

is a finite set of elements of  $H$ . Moreover, it is dense in  $H$  by construction. This establishes that  $H$  is totally bounded. Thus,  $H$  is compact under the sup norm.

Each  $g \in G$  is associated with coefficients  $((c^a, d^a)_a, e)$ . Clearly, for any sequence of policies in  $G$ ,  $g_n \rightarrow g$  in sup norm if and only if the associated coefficients  $((c_n^a, d_n^a)_a, e_n) \rightarrow ((c^a, d^a)_a, e)$  in sup norm. We know from (A.3) that  $c$  satisfies properties (i), (ii) and (iii). Thus, for any  $a \in \mathbb{A}$ ,  $c^0 > c^1 > \dots > c^{|a|}$ ,  $c^a = c^{-a}$  and  $\sum_{|b| \leq |a|} c^b < 1$ . Combining all these, we have  $2|a|c^a < \sum_{|b| \leq |a|} c^b < 1$  which in turn implies that  $c^a < \frac{1}{2|a|}$ , for all  $a \in \mathbb{A}$ . Same bounds hold for the  $d$  sequence. But then, the space of associated coefficient sequences, call it  $L_G$ , can be seen as a closed subset of  $H$ , a compact metric. Consequently,  $L_G$  is compact, thus sequentially compact. Pick a sequence  $(g_n) \in G$  and let  $(c_n, d_n, e_n)$  be the associated coefficient sequence lying in  $L_G$ . Since  $L_G$  is sequentially compact, there exists a subsequence  $(c_{m_n}, d_{m_n}, e_{m_n}) \rightarrow (c, d, e) \in L_G$ . The latter, being an admissible coefficient sequence, is associated with the policy  $g(x, \theta) := \sum_a c^a x^a + \sum_a d^a \theta^a + e\bar{\theta}$ . Thus, the respective policy subsequence  $g_{m_n} \rightarrow g \in G$ . This establishes that  $G$  is sequentially compact hence compact. This concludes the proof of Lemma 4. *Q.E.D.*

Given  $g \in G^\infty$ , let  $x^a(g)$  be agent  $a$ 's strategy induced by  $g$ , i.e.,  $x^a(g)(x^{t-1}, \theta^t) = g_t(R^a x_{t-1}, R^a \theta_t)$ , for all  $a \in \mathbb{A}$  and all  $(x^{t-1}, \theta^t)$ . Define the objective function  $U$  for agent 0 in the class of truncated economies as  $U : G^\infty \times L_\beta \times G^\infty$  as

$$U(g^0 ; \beta_T, g) := E \left[ \sum_{t=1}^{\infty} \beta_{T,t} u(x_{t-1}^0(g^0), x_t^0(g^0), \{x_t^b(g)\}_{b \in N(0)}, \theta_t^0) \mid (x_0, \theta_1) \right]$$

where  $u$  represents the conformity preferences and  $N(0) = \{-1, 1\}$  as in Assumption 1. Let the feasibility correspondence  $\Gamma : L_\beta \times G^\infty \rightarrow G^\infty$  be defined for  $T < \infty$  as  $\Gamma(\beta_T, g) = \{g^0 \in G^\infty \mid g_t^0(x, \theta) = \bar{\theta}, \forall t > T, \forall (x, \theta) \in \mathbf{X} \times \Theta\}$ , and for  $T = \infty$  as  $\Gamma(\beta_\infty, g) = G^\infty$ . It

is easy to see, thanks to the monotonicity of  $\Gamma$  in  $T$  (through  $\beta_T$ ) and the compactness of  $G$  that  $\Gamma$  is a compact-valued and continuous correspondence. Moreover, as the next Lemma shows, the parameterized objective function  $U$  is continuous in  $g^0$ , the choice variable.

**LEMMA 5 (Continuity)** *For any given  $(\beta_T, g) \in L_\beta \times G^\infty$ ,  $U(\cdot; \beta_T, g)$  is continuous on  $\Gamma(\beta_T, g)$  with respect to the product topology.*

**PROOF:** Since  $G$  endowed with the sup norm is a compact metric space due to Lemma 4, the metric  $d(g, g') := \sum_{t=1}^{\infty} 2^{-t} \|g_t - g'_t\|_\infty$  induces the product topology on  $G^\infty$  (see e.g., Aliprantis and Border (2006, p. 90)), where  $\|\cdot\|_\infty$  is the supnorm as before. Let  $(\beta_T, g) \in L_\beta \times G^\infty$  and  $\epsilon > 0$  be given. Set  $\epsilon' := (\frac{1-\beta}{1-\beta^{T+1}})\epsilon$ . The period utility  $u$  is uniformly continuous since  $X$  is compact. Thus, one can choose a  $\delta' > 0$  such that for any  $t$ ,  $|x_t^0 - y_t^0| < \delta'$  implies

$$|u(x_{t-1}^0, x_t^0, \{x_t^b(g)\}_{b \in N(0)}, \theta_t^0) - u(y_{t-1}^0, y_t^0, \{x_t^b(g)\}_{b \in N(0)}, \theta_t^0)| < \epsilon'.$$

Set  $\delta = 2^{-T}\delta'$ . Pick  $g^0, g'^0 \in \Gamma(\beta^T, g)$  such that  $d(g^0, g'^0) < \delta$ . This implies that for all  $t \leq T$ ,  $\|g_t^0 - g_t'^0\|_\infty < 2^T\delta = \delta'$  hence  $|x_t^0(g^0) - x_t^0(g'^0)| < \delta$ . Uniform continuity of  $u$  then implies that the period utility levels are uniformly bounded above by  $\epsilon'$  for all periods  $t \leq T$ . The claim therefore follows from

$$|U(g^0; \beta_T, g) - U(g'^0; \beta_T, g)| < \frac{1 - \beta^{T+1}}{1 - \beta} \epsilon' = \epsilon$$

*Q.E.D.*

For every  $T$ -period symmetric Markovian equilibrium policy sequence  $g^{*T}$ , define  $g^{**T} \in G^\infty$  as

$$\forall t, \forall (x, \theta) \in \mathbf{X} \times \Theta, g_t^{**T}(x, \theta) := \begin{cases} g_{T-(t-1)}^{*T}(x, \theta), & \text{if } t \leq T \\ \bar{\theta}, & \text{if } t > T \end{cases}$$

$G^\infty$  endowed with the product topology is compact since each  $G$  endowed with the supnorm is compact from Lemma 4. Since product topology is metrizable, say with metric  $d$ ,<sup>3</sup>  $(G^\infty, d)$  is a compact metric space hence the sequence  $(g^{**T})_T$  has a convergent

---

<sup>3</sup>See Footnote 4 for an example of metrization of product topology.

subsequence  $(g^{**T_n})_{T_n}$  in  $G^\infty$  that converges say to  $g^* \in G^\infty$ . Let  $M : L_\beta \times G^\infty \rightarrow G^\infty$  be the correspondence of maximizers of  $U$  given the value of the parameters. Also, let  $\mathcal{E} : L_\beta \rightarrow G^\infty$  be the symmetric equilibrium correspondence for the sequence of finite economies. Since  $g^{*T_n}$  is a symmetric Markovian equilibrium for any  $T_n$ , for all  $g^{T_n} \in G^\infty$  we have

$$\begin{aligned}
U(g_{T_n}^* ; \beta_{T_n}, g_{T_n}^*) &= E \left[ \sum_{t=1}^{\infty} \beta_{T_n,t} u \left( x_{t-1}^0(g^{*T_n}), x_t^0(g^{*T_n}), \{x_t^b(g^{*T_n})\}_{b \in N(0)}, \theta_t^0 \right) \mid (x_0, \theta_1) \right] \\
&= E \left[ \sum_{t=1}^{T_n+1} \beta^{t-1} u \left( x_{t-1}^0(g^{*T_n}), x_t^0(g^{*T_n}), \{x_t^b(g^{*T_n})\}_{b \in N(0)}, \theta_t^0 \right) \mid (x_0, \theta_1) \right] \\
&\geq E \left[ \sum_{t=1}^{T_n+1} \beta^{t-1} u \left( x_{t-1}^0(g^{T_n}), x_t^0(g^{T_n}), \{x_t^b(g^{*T_n})\}_{b \in N(0)}, \theta_t^0 \right) \mid (x_0, \theta_1) \right] \\
&= E \left[ \sum_{t=1}^{\infty} \beta_{T_n,t} u \left( x_{t-1}^0(g^{T_n}), x_t^0(g^{T_n}), \{x_t^b(g^{*T_n})\}_{b \in N(0)}, \theta_t^0 \right) \mid (x_0, \theta_1) \right] \\
&= U(g_{T_n} ; \beta_{T_n}, g_{T_n}^*)
\end{aligned}$$

Thus,  $g_{T_n}^* \in M(\beta_{T_n}, g_{T_n}^*)$  for all  $T_n$ . Since  $U$  is continuous in the choice dimension due to Lemma 5 and that the feasibility correspondence  $\Gamma$  is continuous, by the Maximum Theorem (see Berge (1963), p. 115), the correspondence of maximizers,  $M$ , is upper hemi-continuous. This implies that if  $(\beta_{T_n}, g_{T_n}^*) \rightarrow (\beta_\infty, g^*)$ , then  $g^* \in M(\beta_\infty, g^*)$  hence  $g^*$  is a symmetric MPE of the infinite-horizon economy. This implies immediately that the equilibrium correspondence  $\mathcal{E}$  is upper hemi-continuous too.

Uniqueness of finite-horizon symmetric MPEs imply that  $\mathcal{E}$  is single-valued hence continuous for  $T < \infty$ . Define  $\mathcal{F}(\beta_T) := \mathcal{E}(\beta_T)$ , for  $T < \infty$  and let  $\mathcal{F}(\beta_\infty) = g^*$ . This way,  $\mathcal{F}$  is continuous on the space  $L_\beta$ , which is compact under the supnorm by Lemma 4. Consequently,  $\mathcal{F}$  is uniformly continuous. This means, for a given  $\epsilon > 0$ , we can pick  $\delta > 0$  small enough so that  $\|\beta_T - \beta_{T'}\|_\infty < \delta$  implies  $d(\mathcal{F}(\beta_T), \mathcal{F}(\beta_{T'})) < \frac{\epsilon}{2}$ . We know from the previous approximation that for  $\beta_T \rightarrow \beta_\infty$  there is a subsequence  $g^{*T_n} \rightarrow g^*$ . Since  $(\beta_T)_T$  is convergent, it is Cauchy. So, choose  $T(\delta)$  large enough such that  $\forall T, T' \geq T(\delta)$ ,  $\|\beta_T - \beta_{T'}\| < \delta$  and  $\forall T_n \geq T(\delta)$ ,  $\|g^{*T_n} - g^*\|_\infty < \frac{\epsilon}{2}$ . Pick, then, any element  $T_n$  of the subsequence and any other element,  $T'$  such that  $T_n, T' \geq T(\delta)$ . We have

$$\begin{aligned}
d(g^{*T'}, g^*) &= d(\mathcal{F}(\beta_{T'}), \mathcal{F}(\beta_\infty)) \\
&\leq d(\mathcal{F}(\beta_{T'}), \mathcal{F}(\beta_{T_n})) + d(\mathcal{F}(\beta_{T_n}), \mathcal{F}(\beta_\infty)) \\
&< \frac{\epsilon}{2} + d(g^{*T_n}, g^*) \\
&< \epsilon
\end{aligned}$$

The first inequality is the triangle inequality; the second is due to the uniform continuity of  $\mathcal{F}$  and the third is by the fact that  $g^{*T_n} \rightarrow g^*$  uniformly. This proves that the whole sequence  $g^{*T} \rightarrow g^*$  uniformly. The implication of this latter is that, as the finite-horizon economies approach the infinite-horizon economy, every two consecutive period, we make choices approximately with respect to the same MPE policy, hence  $g^*$  is stationary. This concludes **Step 3** which in turn establishes the proof of the statement of Theorem 1. ■

## APPENDIX B: PROOF OF THE RECURSIVE COMPUTATION THEOREM

Here we prove Theorem 3. Consider a finite-horizon  $T$ -period economy with conformity preferences ( $\alpha_i > 0, i = 1, 2, 3$ ) and complete information. For part (i), we simply assume that  $T = 1$  and show that one can fit an exponentially declining sequence into equation A.2. Since that equation has a unique solution as argued in the existence proof, that solution must have exponentially declining coefficients. Matching the coefficients of the policy function using equation A.2, one gets for  $a \neq 0$

$$d_1^{a+1} = \left(\frac{\alpha_3}{\Delta_1}\right) d_1^{a+2} + \left(\frac{\alpha_3}{\Delta_1}\right) d_1^a$$

Dividing both sides by  $d_1^a$  and multiplying them by  $\left(\frac{\Delta_1}{\alpha_3}\right)$ , one gets

$$\left(\frac{\Delta_1}{\alpha_3}\right) \underbrace{\left(\frac{d_1^{a+1}}{d_1^a}\right)}_{r_1} = \underbrace{\left(\frac{d_1^{a+2}}{d_1^{a+1}}\right)}_{r_1} \underbrace{\left(\frac{d_1^{a+1}}{d_1^a}\right)}_{r_1} + 1$$

which induces a quadratic equation

$$r_1^2 - \left(\frac{\Delta_1}{\alpha_3}\right) r_1 + 1 = 0$$

whose determinant  $\left(\frac{\Delta_1}{\alpha_3}\right)^2 - 4 > 0$  since  $\Delta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 > 2\alpha_3$  (remember that  $\alpha_i > 0$  for  $i = 1, 2, 3$ ). The equation has two positive roots, one bigger and one smaller than 1. The bigger root cannot work since it is explosive as  $|a| \rightarrow \infty$ . We pick the smaller root

$$(B.1) \quad 0 < r_1 = \left(\frac{\Delta_1}{2\alpha_3}\right) - \sqrt{\left(\frac{\Delta_1}{2\alpha_3}\right)^2 - 1} < 1$$

which is decreasing in  $\left(\frac{\Delta_1}{2\alpha_3}\right)$  spanning the interval  $(0, 1)$  for different values of the former in the interval  $(1, \infty)$ . Finally, the sum of coefficients can be written

$$(B.2) \quad \sum_{a \in \mathbb{A}} d_1^a = \sum_{a \in \mathbb{A}} d_1^0 r_1^{|a|} = d_1^0 + 2 d_1^0 \frac{r_1}{1 - r_1} = \frac{\alpha_2}{\alpha_1 + \alpha_2}$$

The first equality is due to the exponentiality of the sequence; the third uses the same argument as in (A.8) with  $\mu_1 = 0$ , for the coefficient sequence  $(d_1^a)_{a \in \mathbb{A}}$ . Solving for  $d_1^0$  from above, we obtain

$$d_1^0 = \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right) \left(\frac{1 - r_1}{1 + r_1}\right)$$

and finally thanks to exponentiality

$$d_1^a = r_1^{|a|} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right) \left(\frac{1 - r_1}{1 + r_1}\right), \quad \text{for } a \in \mathbb{A}$$

The argument for the sequence  $(c_1^a)_{a \in \mathbb{A}}$  is identical with one change: The sum of coefficients  $\sum_a c_1^a = \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)$ . This proves part (i) of the theorem.

For part (ii), observe that the parameters of the maps  $L_s$ , namely  $\Delta_s, (\gamma_s^a), \mu_s$  are functions only of the continuation policy coefficients  $(c_\tau^*, d_\tau^*, e_\tau^*)_{\tau=1}^{s-1}$  as defined in (A.10), (A.12), and (A.14), simply because these are “forward-looking” expressions. We saw in the induction step (**Step 2**) of the existence proof that  $L_s$  defined in this fashion becomes a contraction and has a unique fixed point, which is the coefficient sequence of the first-period policy of an  $s$ -period continuation. This establishes part (ii).

For part (iii), observe that each  $g \in G$  is associated with coefficients  $((c^a, d^a)_a, e)$ . Clearly, for any sequence of policies in  $G$ ,  $g_n \rightarrow g$  in sup norm if and only if the associated coefficients  $((c_n^a, d_n^a)_a, e_n) \rightarrow ((c^a, d^a)_a, e)$  in sup norm. In **Step 3** of the existence proof, we establish the convergence of the finite-horizon equilibrium policies to the stationary



infinite-horizon MPE policy as the horizon expands. But this implies that the associated unique coefficient sequence also should converge, then, to the coefficient sequence of the infinite-horizon stationary MPE policy. This establishes part (iii) of Theorem 3. ■

## APPENDIX C: PROOF OF ERGODICITY

This section proves Theorem 4. Suppose that the process  $((\theta_t^a)_{t=-\infty}^\infty)_{a \in \mathbb{A}}$  is i.i.d. with respect to  $a$  and  $t$  according to  $\nu$ . Let  $\pi$  be the initial measure on the configuration space  $\mathbf{X}$  which is the distribution of

$$(C.1) \quad x_0 = \left( \frac{e\bar{\theta}}{1-C} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} (d^{b_s} \theta_{1-s}^{a+b_1+\cdots+b_s}) \right)_{a \in \mathbb{A}}$$

Given that  $(x_t \in \mathbf{X})_{t=0}^\infty$  is an equilibrium process generated by the stationary MPE  $g^*$  in Theorem 1, given  $x_0$ , one obtains on the equilibrium path

$$\begin{aligned} x_1^a &= \sum_{b_1 \in \mathbb{A}} c^{b_1} x_0^{a+b_1} + \sum_{b_1 \in \mathbb{A}} d^{b_1} \theta_1^{a+b_1} + e\bar{\theta} \\ &= \sum_{b_1 \in \mathbb{A}} c^{b_1} \left( \frac{e\bar{\theta}}{1-C} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} (d^{b_s} \theta_{1-s}^{a+b_1+\cdots+b_s}) \right) \\ &\quad + \sum_{b_1 \in \mathbb{A}} d^{b_1} \theta_1^{a+b_1} + e\bar{\theta} \\ &= \frac{e\bar{\theta}}{1-C} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} (d^{b_s} \theta_{2-s}^{a+b_1+\cdots+b_s}) \end{aligned}$$

which has the same form as in (C.1). Since the process  $((\theta_t^a)_{t=-\infty}^\infty)_{a \in \mathbb{A}}$  is i.i.d.,  $x_0^a$  and  $x_1^a$  are distributed identically when the initial measure is  $\pi$ . Since the choice of  $a$  was arbitrary,  $\pi$  is a stationary distribution of the Markov process  $(x_t)_{t=0}^\infty$ . Moreover, from Lemma 2 for a stationary policy function, on any path  $(\theta_1, \theta_2, \dots)$  of the stochastic process

$$\begin{aligned} x_t^a &= \sum_{b_1} \cdots \sum_{b_t} c^{b_1} \cdots c^{b_t} x_0^{a+b_1+\cdots+b_t} \\ &\quad + \sum_{s=1}^t \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left( d^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e\bar{\theta} \right) \end{aligned}$$

$$\begin{aligned}
&= C^t \sum_{b_1} \cdots \sum_{b_t} \left( \frac{c^{b_1} \cdots c^{b_t}}{C^t} \right) x_0^{a+b_1+\cdots+b_t} \\
\text{(C.2)} \quad &+ \sum_{s=1}^t \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left( d^{b_s} \theta_{t+1-s}^{a+b_1+\cdots+b_s} + e \bar{\theta} \right)
\end{aligned}$$

Thus, independent of the initial conditions,  $x_t^a$  converges pointwise to  $x^a \in X$  where

$$\begin{aligned}
\text{(C.3)} \quad x^a := \lim_{t \rightarrow \infty} x_t^a &= \lim_{t \rightarrow \infty} \left[ C^t \sum_{b_1} \cdots \sum_{b_t} \left( \frac{c^{b_1} \cdots c^{b_t}}{C^t} \right) x_0^{a+b_1+\cdots+b_t} \right. \\
&\quad \left. + \sum_{s=1}^t \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left( d^{b_s} \theta_{t+1-s}^{a+b_1+\cdots+b_s} + e \bar{\theta} \right) \right]
\end{aligned}$$

The first term of the previous expression  $C^t \rightarrow 0$  since  $C < 1$  due to  $\alpha_i > 0$ , for all  $i$ . The first term in the parentheses in the summand is a convex combination of uniformly bounded terms. Hence, the first part of the above expression goes to 0 as  $t \rightarrow \infty$ . Moreover, since the equilibrium is symmetric, the convergence is uniform across agents:  $x_t \rightarrow x = (x^a)$  uniformly. Since the exogenous shock process is i.i.d, the law for the part after the plus sign in (C.3) is identical to the law for

$$\sum_{s=1}^t \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left( d^{b_s} \theta_{1-s}^{a+b_1+\cdots+b_s} + e \bar{\theta} \right)$$

which is the ‘ $t$ -translated-into-the-past’ version of the former. Thus, for any given initial value  $x_0$ , and a path  $(\dots, \theta_{-1}, \theta_0)$ , the pointwise limit of  $x_t^a$  can be written as

$$\text{(C.4)} \quad x^a = \frac{e \bar{\theta}}{1 - C} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left( d^{b_s} \theta_{1-s}^{a+b_1+\cdots+b_s} \right)$$

Let  $\mathbb{P}_{\infty}(\cdot) := \prod_{t=0}^{\infty} \mathbb{P}(\cdot)$  and  $\theta := (\dots, \theta_{-1}, \theta_0)$ . Pick any  $f \in C(\mathbf{X}, \mathbb{R})$ , the set of bounded, continuous, and measurable, real-valued functions from  $\mathbf{X}$  into  $\mathbb{R}$ . Let  $\pi_0$  be an arbitrary initial distribution for  $x_0$ . We have

$$\begin{aligned}
\lim_{t \rightarrow \infty} \int f(x_t) \pi_t(dx_t) &= \lim_{t \rightarrow \infty} \int f \left( \left( C^t \sum_{b_1} \cdots \sum_{b_t} \left( \frac{c^{b_1} \cdots c^{b_t}}{C^t} \right) x_0^{a+b_1+\cdots+b_t} \right. \right. \\
&\quad \left. \left. + \sum_{s=1}^t \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left( d^{b_s} \theta_{1-s}^{a+b_1+\cdots+b_s} + e \bar{\theta} \right) \right)_{a \in \mathbb{A}} \right) \mathbb{P}_\infty(d\theta) \pi_0(dx_0) \\
&= \int f \left( \left( \frac{e \bar{\theta}}{1-C} + \sum_{s=1}^\infty \sum_{b_1} \cdots \sum_{b_s} c^{b_1} \cdots c^{b_{s-1}} \left( d^{b_s} \theta_{1-s}^{a+b_1+\cdots+b_s} \right) \right)_{a \in \mathbb{A}} \right) \mathbb{P}_\infty(d\theta) \pi_0(dx_0) \\
&= \int f(x) \pi(dx)
\end{aligned}$$

The first equality is from (C.2); the second is due to Lebesgue Dominated Convergence theorem (see e.g. Aliprantis and Border (2006), p. 415); third is due to the continuity of  $f$  and the pointwise limit of  $x_t$  in (C.4). Thus, for any  $f \in C(\mathbf{X}, \mathbb{R})$ ,  $\lim_{t \rightarrow \infty} \int f d\pi_t = \int f d\pi$ , meaning that the sequence of equilibrium distributions  $\pi_t$  generated by the exogenous law  $\mathbb{P}$  and the stationary MPE policy  $g^*$  converges weakly to the invariant distribution  $\pi$ . The choice of  $\pi_0$  was arbitrary. Hence, for any initial distribution, the induced equilibrium process converges weakly to the same invariant distribution  $\pi$ . Therefore,  $\pi$  is the unique invariant distribution of the equilibrium process. Here is why: Suppose that  $\hat{\pi}$  is another invariant distribution. This implies that the induced process starting with  $\pi_0 = \hat{\pi}$  should satisfy  $\pi_t = \hat{\pi}$ , for all  $t = 1, 2, \dots$ . From the above convergence argument  $\pi_t \rightarrow \pi$  weakly. Hence  $\hat{\pi} = \pi$ .

Finally, to show ergodicity, pick an  $f \in B(\mathbf{X}, \mathbb{R})$ , the set of bounded, measurable, real-valued functions from  $\mathbf{X}$  into  $\mathbb{R}$ . The process starting with  $\pi$  is stationary, hence  $\pi_t = \pi$  for all  $t = 0, 1, \dots$ . Since the process  $x_t$  is stationary, so is the process  $(f(x_t))$ . We can then use Birkhoff's Ergodic Theorem (see e.g. Aliprantis and Border (2006), p. 659) on the process  $(f(x_t))$  to obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T f(x_t) = \int f(x_t) \pi(dx_t)$$

almost surely. Since the choice of  $f$  was arbitrary, the last expression holds for all  $f \in B(\mathbf{X}, \mathbb{R})$ . Thus the equilibrium process  $(x_t \in \mathbf{X})_{t=0}^\infty$  starting from initial distribution  $\pi$  is ergodic. This concludes the proof of Theorem 4.  $\blacksquare$

## APPENDIX D: TAIL CONVERGENCE MONOTONICITY

We prove here Proposition 1. The proof is by induction on  $T$ . For  $T = 1$ , we know from Theorem 3 (i) that the policy coefficient sequence  $(d_1^a)_{a \in \mathbb{A}}$  is exponentially declining on both sides of the origin, at the rate  $r_1$ . From the form of the policy function in Theorem 1, the conditional covariance between agents 0 and  $a + 1$ , with  $a \geq 0$  w.l.o.g., given  $x_0$  is

$$\begin{aligned}
 \text{Cov} \left( x_1^0, x_1^{a+1} \mid x_0 \right) &= \text{Cov} \left( \sum_{b_1 \in \mathbb{A}} d_1^{b_1} \theta_1^{b_1}, \sum_{b_2 \in \mathbb{A}} d_1^{b_2} \theta_1^{a+1+b_2} \right) \\
 &= \sum_{b_1 \in \mathbb{A}} d_1^{b_1} \text{Cov} \left( \theta_1^{b_1}, \sum_{b_2 \in \mathbb{A}} d_1^{b_2} \theta_1^{a+1+b_2} \right) \\
 \text{(D.1)} \quad &= \text{Var}(\theta) \sum_{b_1 \in \mathbb{A}} d_1^{b_1} d_1^{b_1-(a+1)}
 \end{aligned}$$

We will focus on the summation term in the last expression in (D.1). Write it as

$$\begin{aligned}
 \sum_{b_1 \in \mathbb{A}} d_1^{b_1} d_1^{b_1-(a+1)} &= \sum_{b_1 < 0} d_1^{b_1} d_1^{b_1-(a+1)} + d_1^0 d_1^{-a-1} + \sum_{b_1 > 0} d_1^{b_1} d_1^{b_1-(a+1)} \\
 &= \sum_{b_1 < 0} d_1^{b_1} d_1^{b_1-(a+1)} + d_1^0 d_1^{-a-1} + \sum_{b_1 \geq 0} d_1^{b_1+1} d_1^{b_1-(a)} \\
 &= \sum_{b_1 < 0} d_1^{b_1} (r_1 d_1^{b_1-a}) + (d_1^0)^2 (r_1)^{a+1} + \sum_{b_1 \geq 0} (r_1 d_1^{b_1}) d_1^{b_1-a} \\
 &= r_1 \sum_{b_1 \in \mathbb{A}} d_1^{b_1} d_1^{b_1-a} + (d_1^0)^2 (r_1)^{a+1} \\
 &= r_1 \text{Var}(\theta)^{-1} \text{Cov} \left( x_1^0, x_1^a \mid x_0 \right) + (d_1^0)^2 r_1^{a+1}
 \end{aligned}$$

The first equality is a partitioning, the second a simple change of variable, and the third is due to the symmetry and the exponentiality of the  $d_1^a$  sequence. Substituting the final expression back in (D.1) yields, for all  $a \geq 0$

$$\text{(D.2)} \quad \text{Cov} \left( x_1^0, x_1^{a+1} \mid x_0 \right) = r_1 \text{Cov} \left( x_1^0, x_1^a \mid x_0 \right) + r_1^{a+1} \text{Var}(\theta) (d_1^0)^2$$

which implies that the rate of decay of the covariances is greater than  $r_1$ , for any  $a \geq 0$ . Since the second term on the right hand side of (D.2) decays at the rate  $r_1$ , this implies

that the ratio

$$(D.3) \quad \frac{r_1^{a+1} \text{Var}(\theta) (d_1^0)^2}{\text{Cov}\left(x_1^0, x_1^a \mid x_0\right)}$$

decreases monotonically, and being non-negative, it converges. Actually it converges to zero. Here is why: Since the ratio is less than 1, suppose that it converges to  $k \in (0, 1)$ . This means from (D.2) that the limit rate of decay of the covariances is  $r_1 + k$ , greater than the rate for the term in the numerator in (D.3). Thus, the ratio in (D.3) should converge to zero at the limit, a contradiction to  $k \in (0, 1)$ . So, the limit of (D.3) is zero, which in turn implies from (D.2), after dividing both sides by  $\text{Cov}\left(x_1^0, x_1^a \mid x_0\right)$ , that

$$(D.4) \quad \lim_{a \rightarrow \infty} \frac{\text{Cov}\left(x_1^0, x_1^{a+1} \mid x_0\right)}{\text{Cov}\left(x_1^0, x_1^a \mid x_0\right)} = r_1$$

The argument is symmetric for  $a \leq 0$ ; hence the sequence  $\left\{\text{Cov}\left(x_1^0, x_1^a \mid x_0\right)\right\}_{a \in \mathbb{A}}$  declines exponentially on both tails at the same rate  $r_1$  and the statement is true for  $T = 1$ .

Now assume that the statement in Proposition 1 is true for economies up to  $T - 1$  period. We will show that it should also hold for  $T$ -period economies. We will base the main induction arguments on the following Lemma.

**LEMMA 6** *The sequence  $\{\gamma_T^b\}_{b \in \mathbb{A}}$  in Lemma 3 and the equilibrium coefficient sequence  $(c_T, d_T)$  for the first-period policy of a  $T$ -period economy have the following properties: The rate at which they decline at the tail satisfies, for  $T \geq 2$*

$$\lim_{a \rightarrow \infty} \left( \frac{\gamma_T^{a+1}}{\gamma_T^a} \right) = \lim_{a \rightarrow -\infty} \left( \frac{\gamma_T^{a-1}}{\gamma_T^a} \right) = r_{T-1},$$

and

$$\lim_{a \rightarrow \infty} \left( \frac{d_T^{a+1}}{d_T^a} \right) = \lim_{a \rightarrow -\infty} \left( \frac{d_T^{a-1}}{d_T^a} \right) = r_T > r_{T-1}.$$

**PROOF:** Let  $u(t) := u\left(x_{t-1}^0, x_t^0, \{x_t^b\}_{b \in \{-1, 1\}}, \theta_t^0\right)$  where  $u$  represents the conformity preferences in Assumption 1. Let  $u_0(t) := \frac{\partial}{\partial x_1^0} u(t)$ . From equation (A.12),  $\gamma_T^a$  can be

written as

$$\begin{aligned}\gamma_T^a &:= \alpha_3 I_{\{a \in \{-1, 1\}\}} \\ &+ \sum_{\tau=2}^T \beta^{\tau-1} \left[ \left( \frac{\partial x_{\tau-1}^0}{\partial x_1^a} \right) \frac{\partial}{\partial x_{\tau-1}^0} u_0(\tau) + \left( \frac{\partial x_{\tau}^0}{\partial x_1^a} \right) \frac{\partial}{\partial x_{\tau}^0} u_0(\tau) \right. \\ &\quad \left. + \left( \frac{\partial x_{\tau}^{-1}}{\partial x_1^a} \right) \frac{\partial}{\partial x_{\tau}^{-1}} u_0(\tau) + \left( \frac{\partial x_{\tau}^1}{\partial x_1^a} \right) \frac{\partial}{\partial x_{\tau}^1} u_0(\tau) \right]\end{aligned}$$

We will present the argument for the second term inside the summand and the method of proof will apply to the remaining terms straightforwardly. Assume w.l.o.g. that  $a \geq 0$ .

$$\begin{aligned}\left( \frac{\partial x_{\tau}^0}{\partial x_1^a} \right) \frac{\partial}{\partial x_{\tau}^0} u_0(\tau) &= \sum_{s \in \mathbb{A}} \left( \frac{\partial x_2^s}{\partial x_1^a} \right) \left( \frac{\partial x_{\tau}^0}{\partial x_2^s} \right) \frac{\partial}{\partial x_{\tau}^0} u_0(\tau) \\ &= \sum_{s \in \mathbb{A}} c_2^{a-s} \left( \frac{\partial x_{\tau}^0}{\partial x_2^s} \right) \frac{\partial}{\partial x_{\tau}^0} u_0(\tau)\end{aligned}$$

and the corresponding term for  $\gamma_T^{a+1}$  is

$$\sum_{s \in \mathbb{A}} c_2^{a+1-s} \left( \frac{\partial x_{\tau}^0}{\partial x_2^s} \right) \frac{\partial}{\partial x_{\tau}^0} u_0(\tau)$$

as  $a \rightarrow \infty$  the tail convergence rate for the continuation kicks in and

$$\begin{aligned}\lim_{a \rightarrow \infty} \sum_{s \in \mathbb{A}} c_2^{a+1-s} \left( \frac{\partial x_{\tau}^0}{\partial x_2^s} \right) \frac{\partial}{\partial x_{\tau}^0} u(\tau) &= \lim_{a \rightarrow \infty} \sum_{s \in \mathbb{A}} r_{T-1} c_2^{a-s} \left( \frac{\partial x_{\tau}^0}{\partial x_2^s} \right) \frac{\partial}{\partial x_{\tau}^0} u(\tau) \\ &= r_{T-1} \lim_{a \rightarrow \infty} \sum_{s \in \mathbb{A}} c_2^{a-s} \left( \frac{\partial x_{\tau}^0}{\partial x_2^s} \right) \frac{\partial}{\partial x_{\tau}^0} u(\tau)\end{aligned}$$

thus

$$\lim_{a \rightarrow \infty} \left( \frac{\partial x_{\tau}^0}{\partial x_1^{a+1}} \right) \frac{\partial}{\partial x_{\tau}^0} u(\tau) = r_{T-1} \lim_{a \rightarrow \infty} \left( \frac{\partial x_{\tau}^0}{\partial x_1^a} \right) \frac{\partial}{\partial x_{\tau}^0} u(\tau)$$

Hence at the tail, the second term of the sum inside the brackets of [D.5](#) decays at the rate  $r_{T-1}$ . The same one-step transition argument applies to each terms of the sum, in equation [\(D.5\)](#). Moreover, since [\(D.5\)](#) is a discounted sum, the entire expression is summable and it implies that

$$\lim_{a \rightarrow \infty} \left( \frac{\gamma_T^{a+1}}{\gamma_T^a} \right) = r_{T-1}$$

inheriting the rate of tail convergence of the continuation economy, as is argued in the Lemma. The method of proof for  $a \leq 0$  is identical thanks to the symmetry of the environment.

For the second part of Lemma 6, let  $D(r_{T-1})$  be the space of sequences that satisfies the properties in (A.3) and that converges at the tail at a rate  $r_T \geq r_{T-1}$ . This is a closed subset of the space of sequences that satisfy only the properties in (A.3), hence a complete metric space itself. Consequently, the unique coefficient sequence  $d_T$  that is the fixed point of the map in (A.15) should lie in  $D(r_{T-1})$ . Let  $D'(r_{T-1}) \subset D(r_{T-1})$  be the space of sequences in  $D(r_{T-1})$  whose convergence at the tail is *strictly greater* than  $r_{T-1}$ . We will show below that the map in (A.15) maps elements of  $D(r_{T-1})$  into the set  $D'(r_{T-1})$ , which will imply that the unique solution of the map (A.15) converges at a rate  $r_T > r_{T-1}$  at the tail.

Pick agent  $2a + 1$  and assume w.l.o.g. that  $a \geq 0$ . Let  $d_T \in D'(r_{T-1})$ . From (A.15) by matching coefficients

$$\begin{aligned}
 \hat{d}_T^{2a+1} &= \Delta_T^{-1} \left[ \sum_{b \neq 0} \gamma_T^b d_T^{2a+1-b} \right] \\
 &= \Delta_T^{-1} \left[ \sum_{b > a} \gamma_T^b d_T^{2a+1-b} + \gamma_T^a d_T^{a+1} + \sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a+1-b} \right] \\
 &= \Delta_T^{-1} \left[ \sum_{b \geq a} \gamma_T^{b+1} d_T^{2a-b} + \gamma_T^a d_T^{a+1} + \sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a+1-b} \right] \\
 \text{(D.5)} \quad &= \Delta_T^{-1} \left[ \sum_{b \geq a} \gamma_T^b d_T^{2a-b} \left( \frac{\gamma_T^{b+1}}{\gamma_T^b} \right) + \gamma_T^a d_T^{a+1} + \sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a-b} \left( \frac{d_T^{2a+1-b}}{d_T^{2a-b}} \right) \right]
 \end{aligned}$$

The second equality is a partitioning of the sum taking agent  $a$  as the ‘middle’; the first sum after the third equality is a simple shift and change of the dummy variable  $b$ ; the first term after the first equality sign is by multiplying and dividing each term in the summand by  $\gamma_T^b$ ; finally the last term after the fourth equality sign is by multiplying and dividing each term in the summand by  $d_T^{2a-b}$ . Since all elements involved are non-zero, the algebraic manipulation above is feasible. We can add to and subtract from equation

(D.5) the term  $\Delta_T^{-1} \sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a-b} r_{T-1}$  and rearrange the order of the terms to obtain

$$(D.6) \quad \hat{d}_T^{2a+1} = \Delta_T^{-1} \left[ \underbrace{\sum_{b \geq a} \gamma_T^b d_T^{2a-b} \left( \frac{\gamma_T^{b+1}}{\gamma_T^b} \right) + \sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a-b} r_{T-1}}_A + \underbrace{\gamma_T^a d_T^{a+1}}_B \right. \\ \left. + \underbrace{\sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a-b} \left\{ \left( \frac{d_T^{2a+1-b}}{d_T^{2a-b}} \right) - r_{T-1} \right\}}_C \right]$$

The analogous expression for  $\hat{d}_T^{2a}$  is given, after a similar partitioning with agent  $a$  as the middle agent, by

$$(D.7) \quad \hat{d}_T^{2a} = \Delta_T^{-1} \left[ \sum_{b \neq 0} \gamma_T^b d_T^{2a-b} \right] \\ = \Delta_T^{-1} \left[ \underbrace{\sum_{b \geq a} \gamma_T^b d_T^{2a-1-b} \left( \frac{\gamma_T^{b+1}}{\gamma_T^b} \right) + \sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a-1-b} r_{T-1}}_{A'} + \underbrace{\gamma_T^a d_T^a}_{B'} \right. \\ \left. + \underbrace{\sum_{b < a, b \neq 0} \gamma_T^b d_T^{2a-1-b} \left\{ \left( \frac{d_T^{2a-b}}{d_T^{2a-1-b}} \right) - r_{T-1} \right\}}_{C'} \right]$$

We showed above in the first part of the proof of Lemma 6 that as  $b \rightarrow \infty$ , the ratio  $(\gamma_T^{b+1}/\gamma_T^b) \rightarrow r_{T-1}$ . Consequently, the following limits hold

$$(D.8) \quad \lim_{a \rightarrow \infty} \Delta_T^{-1} A = r_{T-1} \lim_{a \rightarrow \infty} \hat{d}_T^{2a}$$

$$(D.9) \quad \lim_{a \rightarrow \infty} \Delta_T^{-1} A' = r_{T-1} \lim_{a \rightarrow \infty} \hat{d}_T^{2a-1}$$

$$(D.10) \quad \lim_{a \rightarrow \infty} \Delta_T^{-1} C = (r_T - r_{T-1}) \lim_{a \rightarrow \infty} \hat{d}_T^{2a}$$

$$(D.11) \quad \lim_{a \rightarrow \infty} \Delta_T^{-1} C' = (r_T - r_{T-1}) \lim_{a \rightarrow \infty} \hat{d}_T^{2a-1}$$



The expressions in (D.8) and (D.10) put together imply that as  $a$  gets large the ratio

$$(D.12) \quad \left( \frac{\hat{d}_T^{2a+1}}{\hat{d}_T^{2a}} \right) \approx r_T + \frac{\gamma_T^a d_T^{a+1}}{\hat{d}_T^{2a}} \geq r_T$$

and the expressions in (D.9) and (D.11) put together imply that

$$(D.13) \quad \left( \frac{\hat{d}_T^{2a}}{\hat{d}_T^{2a-1}} \right) \approx r_T + \frac{\gamma_T^a d_T^a}{\hat{d}_T^{2a-1}} \geq r_T$$

The last two expressions imply that the ratios  $\frac{\gamma_T^a d_T^{a+1}}{\hat{d}_T^{2a}}$  and  $\frac{\gamma_T^a d_T^a}{\hat{d}_T^{2a-1}}$  converge. This is because as  $a$  gets arbitrarily large, the numerator converges at the rate  $r_{T-1} r_T$  and the denominator at a rate greater than equal to  $r_T^2$ . Since both ratios are strictly less than one, one of the following two possibilities must hold: either (i) they converge to a positive constant less than one (the case where  $r_T = r_{T-1}$ ) or (ii) they converge to zero (the case where  $r_T > r_{T-1}$ ). The first case is not possible. Suppose it is. Then,  $r_T = r_{T-1}$ . This implies from (D.6) along with (D.8) and (D.10) that

$$(D.14) \quad r_T = \lim_{a \rightarrow \infty} \left( \frac{\hat{d}_T^{2a+1}}{\hat{d}_T^{2a}} \right) = r_T + \lim_{a \rightarrow \infty} \left( \frac{\gamma_T^a d_T^{a+1}}{\hat{d}_T^{2a}} \right) > r_T$$

a contradiction. Therefore the second case (ii) must be true. The argument for  $a \leq 0$  is symmetric. This means that any sequence  $d_T \in D(r_{T-1})$  is mapped to a sequence  $\hat{d}_T \in D'(r_{T-1})$ , meaning that it converges at the rate  $r_T > r_{T-1}$  at the tail. So our claim in the beginning is true and the unique sequence  $d_T$  that satisfies properties in (A.3) and the equation (A.15) converges at the tail at a rate  $r_T > r_{T-1}$ . This concludes the proof of Lemma 6. Q.E.D.

For the rest of the proof of Proposition 1, assume that the statement in Proposition 1 is true for economies up to  $T - 1$  period. We will show that it should also hold for  $T$ -period economies. Consider first the covariance between agents 0 and  $2a + 1$ , with  $a \geq 0$  w.l.o.g.

$$\begin{aligned} Cov \left( x_1^0, x_1^{2a+1} \middle| x_0 \right) &= Var(\theta) \sum_{b \in \mathbb{A}} d_1^b d_1^{b-(2a+1)} \\ &= Var(\theta) \left[ \sum_{b \leq a} d_T^b d_T^{b-(2a+1)} + d_T^{a+1} d_T^{-a} + \sum_{b \geq a+2} d_T^b d_T^{b-(2a+1)} \right] \end{aligned}$$

$$\begin{aligned}
&= \text{Var}(\theta) \left[ \sum_{b \leq a} d_T^b d_T^{b-(2a+1)} + d_T^{a+1} d_T^{-a} + \sum_{b \geq a+1} d_T^{b+1} d_T^{b+1-(2a+1)} \right] \\
&= \text{Var}(\theta) \left[ \sum_{b \leq a} d_T^b d_T^{b-(2a)} \left( \frac{d_T^{b-(2a+1)}}{d_T^{b-(2a)}} \right) + \sum_{b \geq a+1} d_T^b d_T^{b-(2a)} \left( \frac{d_T^{b+1}}{d_T^b} \right) + d_T^{a+1} d_T^{-a} \right]
\end{aligned}$$

where the algebraic manipulation is the same as in the proof of Lemma 6. The analogous expression for agent  $2a$ , taking agent  $a$  as the agent in the middle, is

$$\text{Cov} \left( x_1^0, x_1^{2a} \mid x_0 \right) = \text{Var}(\theta) \left[ \sum_{b \leq a} d_T^b d_T^{b-(2a-1)} \left( \frac{d_T^{b-2a}}{d_T^{b-(2a-1)}} \right) + \sum_{b \geq a+1} d_T^b d_T^{b-(2a-1)} \left( \frac{d_T^{b+1}}{d_T^b} \right) + d_T^a d_T^{-a} \right]$$

We know from Lemma 6 that as  $a \rightarrow \infty$ , the ratio  $(d_T^{a+1}/d_T^a) \rightarrow r_T > r_{T-1}$ . This implies, from the expressions above for the covariance terms, that for large  $a$ ,

$$\text{(D.15)} \quad \frac{\text{Cov} \left( x_1^0, x_1^{2a+1} \mid x_0 \right)}{\text{Cov} \left( x_1^0, x_1^{2a} \mid x_0 \right)} \approx r_T + \frac{d_T^{a+1} d_T^{-a}}{\text{Cov} \left( x_1^0, x_1^{2a} \mid x_0 \right)} \geq r_T > r_{T-1}$$

and

$$\text{(D.16)} \quad \frac{\text{Cov} \left( x_1^0, x_1^{2a} \mid x_0 \right)}{\text{Cov} \left( x_1^0, x_1^{2a-1} \mid x_0 \right)} \approx r_T + \frac{d_T^a d_T^{-a}}{\text{Cov} \left( x_1^0, x_1^{2a-1} \mid x_0 \right)} \geq r_T > r_{T-1}$$

and straightforward modifications of the argument used in the proof of Lemma 6 implies that the ratios  $d_T^{a+1} d_T^{-a} \text{Cov} \left( x_1^0, x_1^{2a} \mid x_0 \right)^{-1}$  and  $d_T^a d_T^{-a} \text{Cov} \left( x_1^0, x_1^{2a-1} \mid x_0 \right)^{-1}$  both converge to zero and one obtains

$$\text{(D.17)} \quad \lim_{a \rightarrow \infty} \frac{\text{Cov} \left( x_1^0, x_1^{2a+1} \mid x_0 \right)}{\text{Cov} \left( x_1^0, x_1^{2a} \mid x_0 \right)} = r_T > r_{T-1}$$

$$\text{(D.18)} \quad \lim_{a \rightarrow \infty} \frac{\text{Cov} \left( x_1^0, x_1^{2a} \mid x_0 \right)}{\text{Cov} \left( x_1^0, x_1^{2a-1} \mid x_0 \right)} = r_T > r_{T-1}$$

thus the statement of Proposition 1 is true for any finite  $T$ -period economy. Clearly,

$r_T \leq 1$  for any  $T \geq 1$  since the non-negative  $d$  sequences sum up to less than 1. Hence, what we have is a monotone increasing sequence bounded from above by 1. Hence, the limit  $r_\infty = \lim_{T \rightarrow \infty} r_T$  exists and is less than or equal to 1. Moreover, we know from Theorem 3 that the sequence of finite-horizon MPE coefficients converges to that of the infinite-horizon MPE coefficient sequence  $d$ , thus  $r_\infty$  is the tail convergence rate of the infinite-horizon MPE coefficient sequence  $d$ . Therefore  $r_\infty < 1$  since otherwise that would contradict the summability of the sequence  $d$ . This establishes the proof of Proposition 1. ■

## APPENDIX E: PROOF OF INEFFICIENCY

We give here the proof of Theorem 5 for economies with complete information. The extension of the line of proof to the incomplete information economies is straightforward.

**Finite-Horizon:** Take any finite horizon economy ( $T < \infty$ ). We will use continuity arguments so endow the underlying space  $\mathbf{X} \times \Theta$  with the product topology. Product topology is metrizable, say by metric  $d^4$ . In the final period of this finite horizon economy, with absolutely continuous distribution  $\pi_{T-1}$  on the space of choice profiles  $x_{T-1}$ <sup>5</sup> with a positive density, the planner maximizes ex-ante (before the realization of  $\theta_T$ ) the expected utility of a given agent, say of agent  $0 \in \mathbb{A}$ , by choosing a symmetric policy function  $h \in CB(\mathbf{X} \times \Theta, X)$ , the space of bounded, continuous, and  $X$ -valued measurable functions.<sup>6</sup>

The space  $\mathbf{X} \times \Theta$  is compact with respect to the product topology since  $X$  and  $\Theta$  are compact. Since the utility function is continuous and strictly concave in all arguments, the maximizer exists and it is unique. The necessary condition for optimality is summarized in the following lemma.

---

<sup>4</sup>Let  $|\cdot|$  be the usual Euclidean norm. For any  $(x, \theta), (x', \theta') \in \mathbf{X} \times \Theta$ , let

$$d((x, \theta), (x', \theta')) := \sum_{a \in \mathbb{A}} 2^{-a} (|x_a - x'_a| + |\theta_a - \theta'_a|)$$

Since  $X = \Theta = [\underline{x}, \bar{x}]$  is a compact interval, this is a well-defined metric that metrizes the product topology on  $\mathbf{X} \times \Theta$ . See also Aliprantis and Border (2006), p. 90.

<sup>5</sup>Starting with an initial  $\pi_0$  which is absolutely continuous, the MPE policy function and the absolutely continuous preference shocks induce a sequence  $(\pi_t)$  of absolutely continuous distributions on  $t$ -period equilibrium choice profiles.

<sup>6</sup>Since the planner's choice rule is symmetric, the choice of agent 0 rather than another agent is inconsequential.

LEMMA 7 For any  $(x_{T-1}, \theta_T) \in \mathbf{X} \times \Theta$ ,

$$\begin{aligned} 0 &= \alpha_1 (x_{T-1}^0 - h(x_{T-1}, \theta_T)) + \alpha_2 (\theta_T^0 - h(x_{T-1}, \theta_T)) \\ &+ \alpha_3 (h(R^{-1} x_{T-1}, R^{-1} \theta_T) - h(x_{T-1}, \theta_T)) + \alpha_3 (h(R x_{T-1}, R \theta_T) - h(x_{T-1}, \theta_T)) \\ &- \alpha_3 (h(x_{T-1}, \theta_T) - h(R x_{T-1}, R \theta_T)) - \alpha_3 (h(x_{T-1}, \theta_T) - h(R^{-1} x_{T-1}, R^{-1} \theta_T)) \end{aligned}$$

PROOF: The proof uses an extension of the usual calculus of variation techniques to our symmetric strategic environment. We prove it for the class of bounded, continuous, and measurable, real-valued functions on  $\mathbf{X} \times \Theta$ . Then, we use the restriction of the result to a subset of it, the space of bounded, continuous, and measurable,  $X$ -valued functions. Suppose that the function  $h$  provides the maximum for the planner's problem. For any other admissible function  $h'$ , define  $k = h' - h$ . Consider now the expected utility from a one-parameter deviation from the optimal policy  $h$ , i.e.,

$$\begin{aligned} J(a) &:= \int u(x_{T-1}^0, (h + ak)(x_{T-1}, \theta_T), (h + ak)(R^{-1} x_{T-1}, R^{-1} \theta_T), \\ &\quad (h + ak)(R x_{T-1}, R \theta_T), \theta_T^0) \mathbb{P}(d\theta_T) \pi_{T-1}(dx_{T-1}) \end{aligned}$$

where  $a$  is an arbitrary real number and  $u$  represents the conformity preferences in Assumption 1.. Since  $h$  maximizes the planner's problem, the function  $J$  must assume its maximum at  $a = 0$ . Leibnitz's rule for differentiation under an integral along with the chain rule for differentiation gives us

$$J'(a) := \int (u_2 k + u_3 k \circ R^{-1} + u_4 k \circ R) d\mathbb{P} d\pi_{T-1}$$

where  $u_i$  is the partial derivative of  $u$  with respect to the  $i$ -th argument. For  $J$  to assume its maximum at  $a = 0$ , it must satisfy

$$\begin{aligned} J'(0) &:= \int \left[ u_2 (x_{T-1}^0, h(x_{T-1}, \theta_T), h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R x_{T-1}, R \theta_T), \theta_T^0) k(x_{T-1}, \theta_T) \right. \\ &+ u_3 (x_{T-1}^0, h(x_{T-1}, \theta_T), h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R x_{T-1}, R \theta_T), \theta_T^0) k(R^{-1} x_{T-1}, R^{-1} \theta_T) \\ &+ u_4 (x_{T-1}^0, h(x_{T-1}, \theta_T), h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R x_{T-1}, R \theta_T), \theta_T^0) k(R x_{T-1}, R \theta_T) \left. \right] \\ &\times \mathbb{P}(d\theta_T) \pi_{T-1}(dx_{T-1}) = 0 \end{aligned}$$

for any arbitrary admissible deviation  $k$ . Suppose that the statement of the lemma is

not true. This implies that there is an element  $(\bar{x}, \bar{\theta}) \in \mathbf{X} \times \Theta$  such that

$$\begin{aligned}
 0 &\neq u_2(\bar{x}^0, h(\bar{x}, \bar{\theta}), h(R^{-1}\bar{x}, R^{-1}\bar{\theta}), h(R\bar{x}, R\bar{\theta}), \bar{\theta}^0) \\
 &+ u_3(\bar{x}^1, h(R\bar{x}, R\bar{\theta}), h(\bar{x}, \bar{\theta}), h(R^2\bar{x}, R^2\bar{\theta}), \bar{\theta}^1) \\
 \text{(E.1)} \quad &+ u_4(\bar{x}^{-1}, h(R^{-1}\bar{x}, R^{-1}\bar{\theta}), h(R^{-2}\bar{x}, R^{-2}\bar{\theta}), h(\bar{x}, \bar{\theta}), \bar{\theta}^{-1})
 \end{aligned}$$

Assume w.l.o.g. that the above expression takes a positive value (the proof for the case with a negative value is identical). Since the utility function, its partials, and the deviation functions are all continuous with respect to the product topology, and that the measures  $\pi$  and  $\mathbb{P}$  have positive densities, there exists a  $(\pi \times \mathbb{P})$ -positive measure neighborhood  $A \subset \mathbf{X} \times \Theta$  around  $(\bar{x}, \bar{\theta})$  such that the above expression stays positive for all  $(x_{T-1}, \theta_T) \in A$ .<sup>7</sup> Assume that  $a_1 = (\bar{x}, \bar{\theta})$ ,  $a_2 = (R\bar{x}, R\bar{\theta})$ , and  $a_3 = (R^{-1}\bar{x}, R^{-1}\bar{\theta})$  are distinct points. Otherwise, since the underlying space  $X$  is a real interval and the maps  $R$  and  $R^{-1}$  are right and left shift maps, one can always pick a point in  $A$  that has that property.

Now choose  $\epsilon > 0$  small enough so that the  $\epsilon$ -balls  $B_\epsilon(a_1)$ ,  $B_\epsilon(a_2)$ , and  $B_\epsilon(a_3)$  are disjoint.  $R$  and  $R^{-1}$  being both continuous are homeomorphisms. So, one can find  $\epsilon > \delta_1 > 0$  and  $\epsilon > \delta_2 > 0$  such that  $R(B_{\delta_1}(a_1)) \subset B_\epsilon(a_2)$  and  $R^{-1}(B_{\delta_2}(a_1)) \subset B_\epsilon(a_3)$ . Let  $\delta = \min\{\delta_1, \delta_2\}$  and  $A_1 := B_\delta(a_1)$ . We next define a particular deviation  $k$ . Let the function  $k$  be defined as

$$\text{(E.2)} \quad k(x, \theta) = k(Rx, R\theta) = k(R^{-1}x, R^{-1}\theta) = \begin{cases} \gamma [\delta - d((x, \theta), a_1)], & \text{if } (x, \theta) \in A_1 \\ 0, & \text{otherwise.} \end{cases}$$

where  $\gamma > 0$  is a scalable constant. This is possible because  $A_1$ ,  $R(A_1)$  and  $R^{-1}(A_1)$  are disjoint sets. Constructed this way,  $k$  is a bounded, continuous, and measurable function<sup>8</sup>. Substitute  $k$  into equation (E.1). By construction, the only set on which  $k$  is positive is the set  $A_1$  which is itself a subset of  $A$ , the set of elements of  $\mathbf{X} \times \Theta$  for which the expression (E.1) is positive. Hence, evaluated with the constructed deviation function  $k$ ,  $J'(0) > 0$ , a contradiction to the fact that the policy function  $h$  was optimal.

---

<sup>7</sup>Endowed with the product topology, the space  $\mathbf{X} \times \Theta$  is metrizable by the metric  $d$ . See footnote 4. Product topology and the associated metric allows us to choose positive measure proper subsets of  $X$  for choices of near-by agents and the whole sets  $X$  and  $\Theta$  for far-away agents, staying at the same time in the close vicinity of the point  $(\bar{x}, \bar{\theta})$ .

<sup>8</sup>We endow the range space, the real line, with the Borel  $\sigma$ -field hence any continuous function into the real line is automatically measurable.

Therefore the statement of the lemma must be true. This concludes the proof. *Q.E.D.*

This implies that

$$(E.3) \quad h(x_{T-1}, \theta_T) = (\alpha_1 + \alpha_2 + 4\alpha_3)^{-1} \left( \alpha_1 x_{T-1}^0 + \alpha_2 \theta_{T-1}^0 \right. \\ \left. + 2\alpha_3 h(R^{-1} x_{T-1}, R^{-1} \theta_T) + 2\alpha_3 h(R x_{T-1}, R \theta_T) \right)$$

As in the proof of existence, the operator induced by (E.3) is a contraction on the Banach space of bounded, continuous, measurable functions with the supnorm, whose unique fixed is in  $G$ , defined in (A.3). Therefore, one can fit the following solution

$$h(x_{T-1}, \theta_T) = \sum_a c_P^a x_{T-1}^a + \sum_a d_P^a \theta_T^a$$

substituting, we get

$$\sum_a c_P^a x_{T-1}^a + \sum_a d_P^a \theta_T^a = (\alpha_1 + \alpha_2 + 4\alpha_3)^{-1} \left[ \alpha_1 x_{T-1}^0 + \alpha_2 \theta_T^0 \right. \\ \left. + 2\alpha_3 \left( \sum_a c_P^a x_{T-1}^{a-1} + \sum_a d_P^a \theta_T^{a-1} \right) + 2\alpha_3 \left( \sum_a c_P^a x_{T-1}^{a+1} + \sum_a d_P^a \theta_T^{a+1} \right) \right]$$

By matching coefficients, we get for all  $a \in \mathbb{A}$

$$c_P^a = (\alpha_1 + \alpha_2 + 4\alpha_3)^{-1} \left[ 2\alpha_3 c_P^{a-1} + 2\alpha_3 c_P^{a+1} + \alpha_1 \mathbf{1}_{\{a=0\}} \right] \\ d_P^a = (\alpha_1 + \alpha_2 + 4\alpha_3)^{-1} \left[ 2\alpha_3 d_P^{a-1} + 2\alpha_3 d_P^{a+1} + \alpha_2 \mathbf{1}_{\{a=0\}} \right]$$

The same method as in the proof of Theorem 3 yields for any  $a \in \mathbb{A}$ ,

$$(E.4) \quad c_P^a = r_P^{|a|} \left( \frac{\alpha_1}{\alpha_1 + \alpha_2} \right) \left( \frac{1 - r_P}{1 + r_P} \right) \quad \text{and} \quad d_P^a = r_P^{|a|} \left( \frac{\alpha_2}{\alpha_1 + \alpha_2} \right) \left( \frac{1 - r_P}{1 + r_P} \right)$$

$$(E.5) \quad r_P = \left( \frac{\Delta_P}{2\alpha_3} \right) - \sqrt{\left( \frac{\Delta_P}{2\alpha_3} \right)^2 - 1} \quad \text{with} \quad \Delta_P = \alpha_1 + \alpha_2 + 4\alpha_3.$$

We next compare the equilibrium policy sequence in Theorem 3 with the planner's optimal choice coefficient sequence. Notice that

$$\left( \frac{\Delta_P}{2\alpha_3} \right) = \frac{\alpha_1 + \alpha_2 + 4\alpha_3}{2\alpha_3} = 2 + \frac{\alpha_1 + \alpha_2}{2\alpha_3} < 2 + \frac{\alpha_1 + \alpha_2}{\alpha_3} = \left( \frac{\Delta_1}{\alpha_3} \right)$$

which implies that  $r_P > r_1$  since  $r_P$  is decreasing in  $\Delta_P$  by (E.5). Thus, the planner's optimal policy coefficient sequence converges to zero slower than the equilibrium policy

coefficient sequence. Moreover, the equilibrium policy cannot satisfy the FOC of the planner's problem. Therefore, the equilibrium is inefficient for finite-horizon economies.

**Infinite-Horizon:** The argument here is very similar to the one in the finite horizon case. We know from Theorem 1 that the equilibrium has the following structure

$$g(x_{T-1}, \theta_T) = \sum_a c^a x_{T-1}^a + \sum_a d^a \theta_T^a + e \bar{\theta}$$

We argue that this solution cannot satisfy the planner's problem's optimality condition. For a given function  $h \in G$  (see (A.3)), define  $H : \mathbf{X} \times \Theta \rightarrow \mathbf{X}$  as

$$(E.6) \quad H(x_{T-1}, \theta_T) := \left( h(R^a x_{T-1}, R^a \theta_T) \right)_{a \in \mathbb{A}}$$

Let  $V^h$  be the continuation value of using the function  $h$  in the future, defined recursively as

$$(E.7) \quad \begin{aligned} V^h(x_{T-1}, \theta_T) &= u(x_{T-1}^0, h(x_{T-1}, \theta_T), h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R x_{T-1}, R \theta_T), \theta_T^0) \\ &+ \beta \int V^h(H(x_{T-1}, \theta_T), \theta_{T+1}) \mathbb{P}(d\theta_{T+1}) \end{aligned}$$

where  $u$  is as in Assumption 1. Since the policy  $h \in G$  is linear and the utility function is continuously differentiable and strictly concave with respect to all arguments, elementary dynamic programming techniques (see e.g. Stokey and Lucas (1989)) guarantee that the value function  $V^h$  exists, it is bounded, continuous, strictly concave and continuously differentiable. Denote by  $V_a^h$  the partial derivative of  $V^h$  with respect to agent  $a$ 's initial choice. Given an initial absolutely continuous distribution  $\pi_{T-1}$  on the space of previous period's choice profiles with positive density, the planner maximizes agent 0's expected discounted utility. So, the planner's problem is

$$\begin{aligned} \max_{\{h \in G\}} & \int \left[ u(x_{T-1}^0, h(x_{T-1}, \theta_T), h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R x_{T-1}, R \theta_T), \theta_T^0) \right. \\ & \left. + \beta \int V^h(H(x_{T-1}, \theta_T), \theta_{T+1}) \mathbb{P}(d\theta_{T+1}) \right] \mathbb{P}(d\theta_T) \mathbb{P}(d\theta_{T+1}) \pi_{T-1}(dx_{T-1}) \end{aligned}$$

Once again, the solution exists and it is unique thanks to the compactness (with respect to the product topology) of the underlying space  $\mathbf{X} \times \Theta$  and the continuity and strict concavity of the utility and value functions. A straightforward modification of the first

order condition argument in the finite case yields, for any  $(x_{T-1}, \theta_T) \in \mathbf{X} \times \Theta$

$$\begin{aligned} 0 &= u_2(x_{T-1}^0, h(x_{T-1}, \theta_T), h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R x_{T-1}, R \theta_T), \theta_T^0) \\ &+ u_3(x_{T-1}^1, h(R x_{T-1}, R \theta_T), h(x_{T-1}, \theta_T), h(R^2 x_{T-1}, R^2 \theta_T), \theta_T^1) \\ &+ u_4(x_{T-1}^{-1}, h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R^{-2} x_{T-1}, R^{-2} \theta_T), h(x_{T-1}, \theta_T), \theta_T^{-1}) \\ &+ \beta \sum_a \int V_a^h (H(R^{-a} x_{T-1}, R^{-a} \theta_T), \theta_{T+1}) \mathbb{P}(d\theta_{T+1}) \end{aligned}$$

But the FOC for equilibrium is

$$\begin{aligned} 0 &= \int u_2(x_{T-1}^0, h(x_{T-1}, \theta_T), h(R^{-1} x_{T-1}, R^{-1} \theta_T), h(R x_{T-1}, R \theta_T), \theta_T^0) \\ &+ \beta \int V_a^h (H(x_{T-1}, \theta_T), \theta_{T+1}) \mathbb{P}(d\theta_{T+1}) \end{aligned}$$

For the equilibrium policy to be efficient, it needs to satisfy both FOCs for any  $(x_{T-1}, \theta_T) \in \mathbf{X} \times \Theta$ . For the quadratic specification, this entails

$$\begin{aligned} 0 &= 2\alpha_3 (h(x_{T-1}, \theta_T) - h(R x_{T-1}, R \theta_T)) + 2\alpha_3 (h(x_{T-1}, \theta_T) - h(R^{-1} x_{T-1}, R^{-1} \theta_T)) \\ &+ \beta \sum_{a \neq 0} \int V_a^h (H(R^{-a} x_{T-1}, R^{-a} \theta_T), \theta_{T+1}) \mathbb{P}(d\theta_{T+1}) \end{aligned}$$

Substituting the equilibrium policy function  $g$  for  $h$  and recollecting terms

$$\begin{aligned} &2\alpha_3 \left[ \sum_a c^a (2x_{T-1}^a - x_{T-1}^{a-1} - x_{T-1}^{a+1}) + \sum_a d^a (2\theta_{T-1}^a - \theta_{T-1}^{a-1} - \theta_{T-1}^{a+1}) \right] \\ \text{(E.8)} \quad &+ \beta \sum_{a \neq 0} \int V_a^h (H(R^{-a} x_{T-1}, R^{-a} \theta_T), \theta_{T+1}) \mathbb{P}(d\theta_{T+1}) = 0 \end{aligned}$$

The next lemma says that there exists a positive measure subset of the underlying space on which the expression in (E.8) is non-zero.

**LEMMA 8** *Let  $(\hat{x}, \hat{\theta}) \in \mathbf{X} \times \Theta$  be the point where  $\hat{x}_a = \bar{x}$  and  $\hat{\theta}_a = \bar{x}$ , for all  $a \in \mathbb{A}$ <sup>9</sup>. The expression in (E.8) is negative on a positive measure subset  $E \subset \mathbf{X} \times \Theta$ , that includes  $(\hat{x}, \hat{\theta})$ .*

---

<sup>9</sup>Recall from Assumption 1 that  $\bar{x}$  is the upper boundary of the feasible action and type sets  $X$  and  $\Theta$ .



PROOF: Using (E.7) iteratively, one can write for any  $(x_T, \theta_{T+1}) \in \mathbf{X} \times \Theta$

$$\begin{aligned}
 V_a^h(x_T, \theta_{T+1}) &= \int \sum_{t=T+1}^{T+N} \beta^{t-T-1} \left[ 2\alpha_1 (x_t^0 - x_{t-1}^0) \frac{\partial}{\partial x_T^a} (x_{t-1}^0 - x_t^0) + 2\alpha_2 (\theta_t^0 - x_t^0) \frac{\partial}{\partial x_T^a} x_t^0 \right. \\
 &\quad + 2\alpha_3 (x_t^{-1} - x_t^0) \frac{\partial}{\partial x_T^a} (x_t^{-1} - x_t^0) + 2\alpha_3 (x_t^1 - x_t^0) \frac{\partial}{\partial x_T^a} (x_t^1 - x_t^0) \\
 &\quad \left. + \beta^{N+1} V_a^h(x_{t+N}, \theta_{t+N+1}) \right] \prod_{i=1}^N \mathbb{P}(d\theta_{T+1+i})
 \end{aligned}
 \tag{E.9}$$

where  $x_t$  is written as, using iterations of the policy function  $g$  and Lemma 2 (i) with  $x_T$  instead of  $x_1$

$$\begin{aligned}
 x_t^a &= \sum_{b_1 \in \mathbb{A}} \dots \sum_{b_{t-T} \in \mathbb{A}} c^{b_1} \dots c^{b_{t-T}} x_T^{a+b_1+\dots+b_{t-T}} \\
 &+ \sum_{s=1}^{t-T} \sum_{b_1 \in \mathbb{A}} \dots \sum_{b_{s-1} \in \mathbb{A}} c^{b_1} \dots c^{b_{s-1}} \left( \sum_{b_s \in \mathbb{A}} d^{b_s} \theta_{t-(s-1)}^{a+b_1+\dots+b_s} + e \bar{\theta} \right)
 \end{aligned}
 \tag{E.10}$$

At the point  $(\hat{x}, \hat{\theta})$ ,  $x_T^a = \bar{x}$  for all  $a \in \mathbb{A}$ . So, the first part after the equality sign in (E.10) is the same for all agents. Since the preference shocks are i.i.d., the second part will be the same for all agents in expectations, which eliminates the terms in the second line after the equality sign in (E.9). Thanks to Lemma 2 (i),  $\frac{\partial}{\partial x_T^a} x_t^0 > 0$  for any  $a \in \mathbb{A}$ , and for all  $t = T+1, \dots, T+N$ . But then, the second term in (E.9) after the first bracket is negative in expectations. This is because using (E.10)  $E[x_t^0 | (x_T, \theta_{T+1})] = C^{t-T} \bar{x} + (1 - C^{t-T}) \bar{\theta} > \bar{\theta}$ , where  $C = \sum_a c^a$ . The first term after the bracket sign too is negative in expectations. Here is why: The term

$$\begin{aligned}
 E[(x_t^0 - x_{t-1}^0) | (x_T, \theta_{T+1})] &= C^{t-T} \bar{x} + (1 - C^{t-T}) \bar{\theta} - C^{t-1-T} \bar{x} - (1 - C^{t-1-T}) \bar{\theta} \\
 &= C^{t-1-T} (1 - C) (\bar{\theta} - \bar{x}) < 0
 \end{aligned}$$

for any  $t = T+1, \dots, T+N$ . So, one can write

$$\begin{aligned}
 E[2\alpha_1 (x_t^0 - x_{t-1}^0) \frac{\partial}{\partial x_T^a} (x_{t-1}^0 - x_t^0) | (x_T, \theta_{T+1})] &< E[2\alpha_1 (x_t^0 - x_{t-1}^0) \frac{\partial}{\partial x_T^a} x_{t-1}^0 | (x_T, \theta_{T+1})] \\
 &< 0
 \end{aligned}$$

which shows that the summand in (E.9) is negative in expectations in every period. In turn, the whole sum, then, until the last line of (E.9), is negative in expectations for any arbitrary  $N$ . The choice of  $a$  was arbitrary and that  $V_a^h$  is continuous on  $\mathbf{X} \times \Theta$

for any  $a \in \mathbb{A}$ . The latter is compact with respect to the product topology. Hence,  $V_a^h$  is bounded. So, one can choose an  $N$  large enough to make the  $\beta^{N+1} V_a^h(x_{t+N}, \theta_{t+N+1})$  term arbitrarily small. This implies that the whole expression in (E.9) is negative, which in turn means that  $V_a^h(\hat{x}, \hat{\theta}) < 0$  for any  $a \in \mathbb{A}$ .

At the point  $(\hat{x}, \hat{\theta})$ , the first line of (E.8) is zero and the second line is negative, as we just showed, which makes the whole expression in (E.8) negative. Since the first line in (E.8) is continuous and so are  $V_a$  for any  $a \in \mathbb{A}$ , the whole expression in (E.8) is continuous. Hence, as in the proof of Lemma 7, there exists a  $(\pi \times \mathbb{P})$ -positive measure neighborhood  $E \subset \mathbf{X} \times \Theta$  around  $(\bar{x}, \bar{\theta})$  such that the above expression stays negative for all  $(x_{T-1}, \theta_T) \in E$ . This concludes the proof. Q.E.D.

The statement of Lemma 8 leads to a contradiction since it means that the planner's optimal rule and the equilibrium policy function  $g$  does not agree on  $E$ . Therefore,  $g$  is inefficient. This concludes the proof. ■

## APPENDIX F: IDENTIFICATION

### F.1. *Rationality vs. Myopia*

We prove here Proposition 2. We showed in the proof of Proposition 1 (Tail Convergence Monotonicity) that, for  $T = 1$ , the ratio  $\left(\frac{\rho_{a+1,T}}{\rho_{a,T}}\right)$  is necessarily monotonically decreasing in  $a$  for any underlying preference parameter vector  $\alpha$ , converging eventually, at the tail, to the rate  $r_1$  given in Theorem 3 (i). Moreover, as we showed in Section 4.2, the cross-sectional covariances at the stationary distribution can be written recursively given the weights of the policy function. For the myopic policy function, they take the form

$$(F.1) \quad Cov(x^0, x^a) = \sum_{a_1 \in \mathbb{A}} \sum_{b_1 \in \mathbb{A}} c_1^{a_1} c_1^{b_1} Cov(x^{a_1}, x^{a+b_1}) + Var(\theta) \sum_{a_1 \in \mathbb{A}} d_1^{a_1} d_1^{a_1-a},$$

Since the  $c_1$  and  $d_1$  sequences are exponential at the rate  $r_1$  from Theorem 3 (i), by straightforward modifications of the arguments in the first part of the proof of Proposition 1, the ratio of consecutive covariances for the myopic,  $\left(\frac{Cov(x^0, x^{a+1})}{Cov(x^0, x^a)}\right)$  converges monotonically as  $a$  gets large.

However, the above ratio for the stationary policy function is non-monotonic for a set of parameter values. See the table below where we plot the above ratio for any parameter

vector  $\bar{\alpha}$  where  $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3$ .

$$r_a := \left( \frac{\rho_{a+1}}{\rho_a} \right)$$

$r_1$	$\dots$	$r_5$	$\dots$	$r_9$	$\dots$	$r_\infty$
.4757	$\dots$	.3543	$\dots$	.3571	$\dots$	.3680

The values dip first then come back up which proves non-monotonicity of  $\left( \frac{\rho_{a+1}}{\rho_a} \right)$  in  $a$  for the chosen parameter vector  $\bar{\alpha}$ . Moreover, the map in (3.2) that generates the policy weights as fixed points is continuous in the parameters  $(\alpha_1, \alpha_2, \alpha_3)$  of the utility function. Thus, there is an open-neighborhood around  $\bar{\alpha}$  such that for each element  $\hat{\alpha}$  of that neighborhood, the same non-monotonicity property obtains. This concludes the proof.  $\blacksquare$

We conjecture that this non-monotonicity property should hold for the entire admissible parameter space. As an illustration we present in Figure 1, the ratio  $\left( \frac{\rho_{a+1,T}}{\rho_{a,T}} \right)$  ( $y$ -axis) as a function of  $a$  ( $x$ -axis), at the stationary distribution, for different levels of strength of interaction proxied by the ratio  $\left( \frac{2\alpha_3}{\Delta_1} \right)$ .<sup>10</sup> Clearly, for a large set of parameters, non-monotonicity obtains at the stationary distribution. The limit auto-correlation function for the myopic model, on the contrary, inherits the behavior of its one-step transition counterpart: it converges at a monotonically decreasing rate.

## F.2. Lack of Identification in Infinite Horizon Economies

We prove here Proposition 4. In the case of complete information, the policy function is:

$$x_t^a = \sum_{b \in \mathbb{A}} c^b(\alpha) x_{t-1}^{a+b} + \sum_{b \in \mathbb{A}} d^b(\alpha) \theta_t^{a+b} + e \bar{\theta}$$

---

<sup>10</sup>More precisely, we set  $\left( \frac{\alpha_1}{\alpha_2} \right) = 1$ ,  $\left( \frac{2\alpha_3}{\Delta_1} \right) \in \{0.1, 0.2, 0.75, 0.9\}$  and  $\beta = .95$ . Note that the results are independent of particular values of  $\alpha_i$  as long as the ratio  $\left( \frac{2\alpha_3}{\Delta_1} \right)$  is the same.

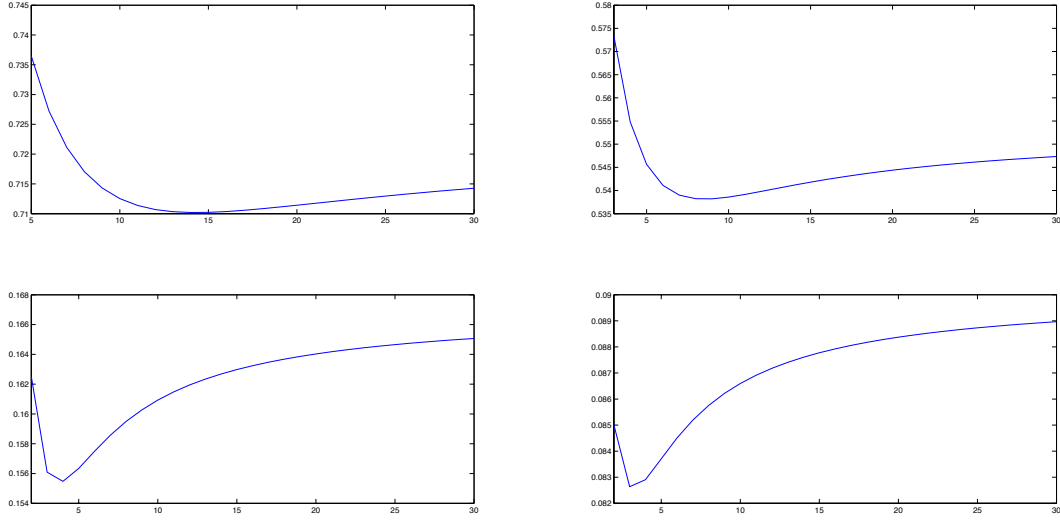


FIGURE 1.— The ratio  $r_a$  as a function of  $a$  for different values of  $\left(\frac{2\alpha_3}{\Delta_1}\right)$  at the stationary distribution. Namely, all parameter vectors  $\alpha$  such that  $\left(\frac{2\alpha_3}{\Delta_1}\right) = 0.9, 0.75, 0.2, 0.1$ , respectively for the *Top-Left*, *Top-Right*, *Bottom-Left*, and *Bottom-Right* panels.

As we saw in Lemma 2, one can obtain by iteration the reduced form<sup>11</sup>

$$x_t^a = \sum_{b_1} \cdots \sum_{b_t} c(\alpha)^{b_1} \cdots c(\alpha)^{b_t} x_0^{a+b_1+\cdots+b_t} \\ + \sum_{s=1}^t \sum_{b_1} \cdots \sum_{b_{s-1}} c(\alpha)^{b_1} \cdots c(\alpha)^{b_{s-1}} \left( \sum_{b_s} d(\alpha)^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e(\alpha) \bar{\theta} \right)$$

Consider now the alternative specification with *no interactions* between agents ( $\hat{\alpha}_3 = 0$ ) and *no habits* ( $\hat{\alpha}_1 = 0$ ), a preference shock process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  and *own type effects* with  $\hat{\alpha}_2 > 0$ . For this economy, equilibrium choice of agent  $a$  at time  $t$  is given by

$$x_t^a = \hat{\theta}_t^a$$

Defining the new preference shock process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  as

<sup>11</sup>In Lemma 2, the iteration stops once it reaches period 1. But, since a stationary MPE exists by Theorem 1, we iterate here once more on the form in Lemma 2 using the stationary policy function and write equilibrium choices as a function of the initial conditions  $x_0$ .

$$\begin{aligned}\hat{\theta}_t^a : &= \sum_{b_1} \cdots \sum_{b_t} c(\alpha)^{b_1} \cdots c(\alpha)^{b_t} x_0^{a+b_1+\cdots+b_t} \\ &+ \sum_{s=1}^t \sum_{b_1} \cdots \sum_{b_s} c(\alpha)^{b_1} \cdots c(\alpha)^{b_{s-1}} \left( d(\alpha)^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} + e(\alpha) \bar{\theta} \right)\end{aligned}$$

would imply that for an arbitrary initial distribution  $\pi_0$  for  $x_0$ , the joint probability distributions that the two specifications (with and without interactions) generate on the process  $\{x_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$ , are identical. Moreover, if one allows for infinite histories, one can define the preference shock process  $\{\hat{\theta}_t^a\}_{t \geq 1}^{a \in \mathbb{A}}$  as before

$$\hat{\theta}_t^a := \frac{e(\alpha) \bar{\theta}}{1 - C(\alpha)} + \sum_{s=1}^{\infty} \sum_{b_1} \cdots \sum_{b_s} (c(\alpha))^{b_1} \cdots (c(\alpha))^{b_{s-1}} \left( (d(\alpha))^{b_s} \theta_{t-(s-1)}^{a+b_1+\cdots+b_s} \right)$$

and obtain observational equivalence once again. Hence, we conclude that identification is not possible. This concludes the proof.  $\blacksquare$

## APPENDIX G: DETAILS ABOUT THE SIMULATIONS

We build an artificial economy that consists of a large number of agents ( $|\mathbb{A}| = 1300, 2500$ , and  $5000$ , depending on the treatment) distributed on the one-dimensional integer lattice. At both ends “buffer” agents that act randomly are added to smooth boundary effects. Depending on the treatment, we start the economy with the following initial configuration of choices: (i) the highest action for all agents; (ii) the lowest action for all agents, (iii) the action equal to the mean shock for all agents.

The core engine behind the simulations is a Matlab code, **g.m**, which computes the equilibrium policy weights recursively as outlined in Section 3.2 of the paper. The code is posted on Özgür’s webpage, <http://www.sceco.umontreal.ca/onurozgur/>, at the Université de Montréal; the code contains also detailed explanations. The correlation computations use another code, **cor.m**, also available on Özgür’s webpage.

Both codes use as input parameters values of the preference parameters  $\alpha_i$ ,  $i = 1, 2, 3$ , the discount factor  $\beta$ , the horizon for the economy  $T$ , the number of agents  $|\mathbb{A}|$ , and the longest distance between agents for which the equilibrium correlation is computed  $M$ .

For the limit distributions results, once **g.m** computes the policy weights, we let the computer draw  $(\theta_t^a)_{a=1}^{|\mathbb{A}|}$  from the interval  $[-D, D]$  according to the uniform distribution (this is for simplicity since all results in the paper are distribution-free).

## REFERENCES

- ALIPRANTIS, D., AND BORDER, K. C. (2006), *Infinite Dimensional Analysis*, third edition. Springer-Verlag, Berlin.
- BERGE, C. (1963): *Topological Spaces*. New York, Macmillan.
- STOKEY, N., LUCAS, R. E. WITH PRESCOTT, E. (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, MA.