DEFAULT PENALTY AS A SELECTION MECHANISM AMONG MULTIPLE EQUILIBRIA

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Abstract

Closed exchange and production-and-exchange economies may have multiple equilibria, a fact that is usually ignored in macroeconomic models. Our basic argument is that default and bankruptcy laws are required to prevent strategic default, and these laws can also serve to provide the conditions for uniqueness. In this paper we report experimental evidence on the effectiveness of this approach to resolving multiplicity: society can assign default penalties on fiat money so the economy selects one of the equilibria. Our data show that the choice of default penalty takes the economy close to the chosen equilibrium. The theory and evidence together reinforce the idea that accounting, bankruptcy and possibly other aspects of social mechanisms play an important role in resolving the otherwise mathematically intractable challenges associated with multiplicity of equilibria in closed economies. Additionally we discuss the politico-economic meaning and experimental implications of default penalties that support an active bankruptcy-modified competitive equilibrium.

JEL: C73, C92, D51, E42, G21, G33

Keywords: Bankruptcy penalty, financial institutions, fiat money, multiple equilibria, experimental gaming

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1. INTRODUCTION

In abstract general equilibrium theorizing, closed exchange and production-and-exchange economies may have multiple equilibria. Macroeconomics and applications of many partial equilibrium models with less abstraction allow institutional and local considerations. They assume implicitly that the multiplicity problem will go away and they are probably right. In spite of its importance for reconciling policy decisions with theory, questions about multiplicity are often set aside or ignored in dynamic models of the macro-economy. The mathematical challenge of how to select among multiple competitive equilibria in abstract models with no enrichment of these models by adding politico-economic or other institutional features remains unresolved. There are several special known solutions such as the existence of gross substitutability among all goods, or the presence of a good whose valuation is manifested as a linear separable term in all utility functions.

We believe that institutions provide a politico-economic context that is sufficient to select a unique equilibrium. We suggest that fiat money, combined with bankruptcy and default laws, are sufficient, as a first order approximation, to select among multiple equilibria. The worth of virtual money is manifested as a linear separable term in all utility functions.[2] By utilizing the institutional context that is already known to be theoretically sufficient to supply the conditions for uniqueness, this approach cuts the Gordian knot of equilibrium multiplicity for all practical purposes.

This project is an attempt to understand the role of financial institutions, such as bankruptcy laws and accounting rules, in resolving the multiplicity problems in closed

[2] In essence it is merely an extension of Hicks' marginal utility of income to the negative orthant. Linearity is of little theoretical importance, as long as the disincentive is high enough at the point of default and continues to be high enough. The choice of a linear penalty is merely a mathematical convenience. In legal and institutional fact the default penalties are best described as an algorithm manifested in the form of legal procedure with considerable flexibility in ad hoc settlements.

economies. To this end, we point out the theoretical justification and conduct laboratory economies to explore whether, by introducing an appropriately chosen default (bankruptcy) penalty, the outcome of a closed economy can be directed to any targeted element in the set of equilibria of the unmodified economy. Fiat money and default laws are facts of life in any advanced economy. They are parts of the socio-political and legal context that imposes constraints on the functioning of the economy.

In attempting to construct a process model of a general equilibrium system as a playable game, if any form of borrowing is present, it is necessary to introduce default penalties to prevent strategic bankruptcy; but the bankruptcy conditions may also provide a way to select an equilibrium in presence of multiplicity. In the static models of general equilibrium theory there is no "nice" general condition that selects a unique equilibrium in an economy with only the usual restrictions on smooth concave utility functions.

The following five mathematically describable and highly restrictive conditions are known to be sufficient for the existence of a single competitive equilibrium in a closed exchange economy with n agents and m commodities:

1. There is a single agent ($n = 1$).
2. There is a single commodity ($m = 1$).
3. All individuals have the same utility function.
4. There is gross substitutability among all goods.
5. There exists a commodity that is in positive supply that is desired by all and whose worth enters into the utility functions of all as a linear separable term.

We explore the fifth condition; and more specifically the role of institutional constraints like default penalties as instruments of equilibrium selection. Qin and Shubik

(2008, 2011) suggest that penalty conditions are reasonable when one attempts to convert a general equilibrium structure into a playable game. They demonstrate that trading in markets with a fiat money and default penalties is a sufficient way to construct a model that selects among multiple equilibrium points. In essence they establish formally that a precise specification of the penalties can select any one of the available competitive equilibria in such a manner that there is no strategic default. If penalties other than these special values are utilized some individuals will elect to default and at the end of the game budgets will not balance.

In a dynamic model of an economy where individuals are strategically free to borrow, rules and penalties on default are a logical necessity (see Karatzas, Shubik, Sudderth and Geanakoplos, 2006). They are also an institutional fact in modern economies. In this paper we experimentally examine the possibility to engineer the outcome of a three-equilibrium exchange economy (constructed by Shapley and Shubik, 1977[3]) through the choice of financial institutions in the form of the default penalty regime. The selection of penalties or a value for a government money is equivalent to the fifth condition listed above. We find that the assignment of a proper value to a fiat money (which can be interpreted as a default penalty when net money holding is negative) yields laboratory outcomes in proximity to a predictable unique equilibrium.

Although the existence of multiple equilibria has been proved generally, the calculation of a good reasonably robust example is not easy. The choice of utility functions shows that the existence of a linear separable term in every utility function is not enough. One requires that the commodity with this property must be the same for all agents. In spite

[3] The Shapley and Shubik (1977) example has been generalized by Bergstrom, Shimomura and Yamato (2008) so that many other examples of economies with three equilibria can easily be generated.

of the special features of the example used we suggest that the results hold in full generality for any economy with multiple equilibria. The key point is that the grafting onto the economy of the linear term for a virtual commodity is a politico-economic act encompassing the economy. A society with creditors in political control can be expected to evolve different default and bankruptcy laws than one in which debtors are in control.

In contrast with the important goals of macroeconomic applications approached from the "top down", in this paper we are concerned with a "bottoms up" approach in utilizing a microeconomic approach to studying market economies. Building rigorous foundations of macroeconomics calls for the static general equilibrium models to be integrated with general process models. Strategic market games help us achieve this goal because setting them up forces us to specify complete and consistent process models. By their very nature they are amenable to examination by both mathematical analysis and experimental gaming.

Selection of a single equilibrium from a set of equilibria raises another question: is there any societal reason to favor a specific equilibrium over the others? Creditors and debtors have been noted above; however another reasonable condition for picking a penalty is to select the equilibrium which minimizes the need for cash. Any process model that is a playable game must specify how trade takes place and thus provides the conditions to be able to calculate the cash flows. By selecting the equilibrium that requires the least amount of money relative to overall wealth, society would economize on the use of "trust pills" as individual trust is a prerequisite for acceptability of fiat money. As a New England saying puts it: "In God we trust, all others pay cash."

The paper is organized as follows: Section 2 introduces the Shapley and Shubik (1977) economy with multiple equilibria, describes its modification by Qin and Shubik

(2008) through the introduction of a third commodity as a money into a playable game and presents testable hypotheses. Section 3 presents the experimental set up with the results in Section 4. Section 5 presents additional tests to serve as robustness checks on Section 4 results, followed by conclusions in Section 6.

2. AN ECONOMY WITH MULTIPLE EQUILIBRIA

Consider the outcomes of an economy with two commodities and two types of traders modeled as a strategic market game with three competitive equilibrium points, one of which is unstable under Walrasian dynamics. The model considered is illustrated graphically in Figure 1. It displays an exchange economy with three competitive equilibria in an Edgeworth Box. The initial endowment with goods A and B (x, y) of each trader of Type 1 is (40, 0) and the initial endowment of each trader of Type 2 is (0, 50). The utility functions of the individuals are, respectively,

$$U_1(x, y) = x + 100(1 - e^{-y/10}), \text{ and}$$

$$U_2(x, y) = y + 110(1 - e^{-x/10}). \qquad (1)$$

The initial endowment point is the upper left of the box with coordinates (40, 0) and (0, 50) in Figure 1. The dotted lines represent the individually rational indifference curves going through the initial endowment point. The Pareto optimal set of outcomes is given by $C_1D_1X\ C_2D_2$. The two curves that intersect three times on the Pareto Set are the response curves for each trader, calculated by varying price and asking each trader how much she would be willing to trade at each price. Supply equals demand only at the points of intersection of the two curves as is indicated by the three equilibria, CE_1, CE_2 and CE_3.

(Insert Figure 1 about here)

With only one trader on each side it can be regarded as a model of barter. With *n* traders on each side it provides the simplest model of an economy where a market price can be formed by aggregating many bids and offers. Here the same figure can be regarded as representing type-symmetric trade outcomes in a market with *n* players on either side.[4]

Huber, Shubik, Sunder (2009, Cowles Foundation Working Paper 1730) conducted an experimental examination of this economy in their first treatment and reported that (1) the selection of numeraire made no difference. (2) There was no convergence to any of the three CEs. (3) All runs all approached or neared a point on the Pareto surface.[5] The data were closest to the central CE and the jointly maximum outcome (assuming interpersonal comparisons) was even closer. Those data, analysis and method serve as a useful benchmark for the experiment with the economy modified through introduction of money in the current paper[6].

2.1 MODIFICATION OF THE ECONOMY INTO A PLAYABLE GAME

When a linearly separable money *M* is introduced to economy (1) in addition to goods A and B, and the utility functions are modified by adding a monetary good *z* with constant marginal utility normalized to one, we get:[7]

$$U_1(x, y, z) = \mu_1 z + x + 100(1 - e^{-y/10}) \text{ , and}$$

[4] Type symmetry means that all traders of the same type take the same action. Thus, instead of needing a diagram in 2n dimensions the 2-dimensional diagram given in Figure 1 is sufficient.

[5] There appeared to be three competing basins of attraction with no a priori tendency towards any one in particular; perhaps early moves influence the final outcomes.

[6] As the primary purpose here is to explore the power of introducing a bankruptcy penalty in selection of a type-symmetric non-cooperative equilibrium, we make several simplifying assumptions. One of them is that ten persons in a strategic market game are enough to yield outcomes for which we can use the CEs as reasonable surrogates. This would be reinforced if the average behavior of the players were myopic, more or less conditioning on the signal of the last price rather than on their oligopolistic power. In our discussion from here on we refer to the economy's CEs rather than NCEs.

[7] In order to make a meaningful comparison between equilibria, we need to normalize the economies so that the total value of all goods in the economy is the same under all equilibrium prices.

$$U_2(x,y,z) = \mu_2 z + y + 110(1 - e^{-x/10}), \qquad (2)$$

where the μ's are parameters and the initial endowments now include an amount of money. This amount equals or exceeds the transactions amount needed at any one of the CEs. This change, i.e., the introduction of money with a positive value (default penalty when net money holdings are negative) leads to a new unique equilibrium. The location of this equilibrium depends on the values of the two μ's.

Parameters μ_i define the expected value of money at the end of the game fixed by the experimenter. In the theory they can be interpreted as expectations, or (when borrowing is permitted and thus negative holdings are possible) they can be interpreted as default parameters set by a society. In either instance, if these parameters are under the control of the society (or experimenter in the game) and there is a setting associated with each of the three equilibria, the parameters may be selected in a way that fixes the value of money associated with one of the equilibria.

For interpretation of the parameters μ_1 and μ_2 note that in an exchange economy with a fiat money there are two quite different forces that support the valuation of the fiat. The first is expectations of the future worth of money in exchange; this is essentially dynamic (see Bak, Norrellyke and Shubik, 1999). The second involves the magnitude of the penalties imposed by a society on individuals who default on their debts. In equilibrium in a society that uses a fiat, money must have the marginal utility of a unit of income equal at least to the marginal disutility of ending with a unit of debt.

Although we have introduced a linear separable money, when the bankruptcy penalties are selected so they coincide with the Lagrangians of one of the CEs, there are two ways we can model the game, with some form of outside or physical money present that does

not net to zero; or with each individual granted a credit line based on (his correctly) forecasted income. If we use the credit line at equilibrium after trade all return their credit lines and final credit holdings net to zero. In this instance the selection of the penalty selects the equilibrium without the use of a quasi-linear money. This structure implies the existence of an agency with omniscient forecasting skills. An alternative approach is to consider an economy with a small amount of quasi-linear money that provides enough liquidity to absorb a reasonable amount of error or heterogeneous behavior without causing bankruptcy. In essence a quasi-side payment game has been created that provides this flexibility to the dynamics.

When net trade in equilibrium is zero the game can be regarded as an NSP game and there is no need for money. When the penalties are different from any of the CEs of the model, the economy in essence selects a CE in the three dimensional space that involves a net transfer of money. In this case, the distortion of the price system will favor the individuals with negative cash flow. Thus the presence of the quasi-linear money absorbs error both in individual behavior and in setting penalties

We may rewrite the utility functions (2) in the form:

$$U_1(x,y,z) = z + \frac{1}{\mu_1}(x + 100(1 - e^{-y/10}))\text{, and}$$

$$U_2(x,y,z) = z + \frac{1}{\mu_2}(y + 110(1 - e^{-x/10}))\,. \qquad (3)$$

If individuals were permitted to borrow, and the marginal disutility of debt were less than the marginal utility of income, it would pay individuals to borrow more and to default. In our experiment we did not allow borrowing. When outside money is present, individual

spending in excess of their income is the equivalent to default in an economy with no outside money. We therefore have the subjects earn points for their net money holdings at the end.

In this economy subjects trade goods A and B for money in separate markets. The trader strategy has two dimensions, with type 1 offering a quantity of good A for sale and bidding a quantity of fiat money to buy good B (and vice-versa for traders of type 2). The introduction of fiat money with the parameters μ_i is enough to guarantee a unique competitive equilibrium point for non-zero amounts of money (see Qin and Shubik 2008).

We fix μ_1 =1 and vary μ_2 in three treatments to target the three equilibrium points.[8] Specifically, in treatments COa, COb, and COc,[9] we set μ_2 = 0.28, 0.75, and 5.07, respectively,[10] since these values correspond to the respective marginal value of income[11] at the three competitive equilibria of the economy, and examine the effect of varying μ_2 on outcomes of the economy.[12]

We conduct and present games in which after each move resources were reinitialized, as well as games in which they were carried over from one period to the next. Theoretically, if resources are not reinitialized, the agents could end up trading to anywhere on the contract curve. However, as will be seen in the results later, even without reinitialization, the three

[8] With the same bankruptcy laws applicable to all, how could the penalties differ across traders? As demonstrated by the bankruptcies of General Motors and Chrysler in Spring 2009, default penalties are tailored by the legal process, and yield very different opportunity costs for different agents.

[9] Treatment label CO stands for commodity and money balances being carried over from end of one period to the beginning of the next, as contrasted with label RI for treatments in which the commodity and money endowments of subjects were reinitialized at the beginning of each period (to be described below).

[10] The same μ_2's are used in treatments RIa, RIb, and RIc, respectively, with RI standing for reinitialized.

[11] A default penalty needs to be at least this strong to discourage default.

[12] Multiple equilibria are rare in general as was shown by Debreu but are highly important in preventing any strong welfare interpretation of competitive market and also are a stumbling block in the development of dynamics. For this example Kumar and Shubik (2003) performed a sensitivity analysis to show precisely the somewhat narrow range of changes in the distribution of endowments of the two player types that would preserve the property of multiple equilibria. In other words a slight redistribution of resources given to a trader (more of one commodity and less of the other) will produce an economy with one equilibrium unless the redistribution is within an appropriately narrow range,

initial CEs serve as predictable basins of attraction and the outcomes of the experimental economies are clustered in a narrow band around the CE targeted by the choice of μ_2.

The value of learning is more limited in the CO treatments, as holdings of goods are not reinitialized. Even in a static theory this makes a difference. If we reinitialize the holdings, traders have the opportunity to learn costlessly. Reinitialization clearly makes learning easier, as different strategies can be tried and individual decisions can be improved. In the other instance, i.e., when holdings of goods are carried over, a subject does not have the opportunity to recover from poor decisions made in the past. In particular, as the competitive equilibrium moves with each change in endowment point the no-reinitialization process is stacked against going to the initial CE. To account for this, and to observe whether learning takes place, we also conduct three sub-treatments RIa, RIb, and RIc (with "RI" standing for re-initialization.[13]

Conjecture 1: *In Treatments COa, COb, and COc, as well as in RIa, RIb, and RIc, the economy converges, and can be made to converge, to any of the three equilibria guided by the selection of parameters* μ.[14]

As the money is a linear term, equilibrium can be reached with any net money holdings, depending on how prices evolve. We conjecture that subjects with relatively high marginal utility for procuring more goods will be ready to incur negative net money holdings (i.e., spend more than they earn). The theoretical possibility of zero net money holdings

[13] Ghosal and Morelli (2004) is a related paper concentrating on the theory of dynamics of perfect equilibria in strategic market games. Examination of this model in light of their work would be a natural extension of the present paper, which we leave for future research.

[14] If the μ's (μ_1, μ_2) are not selected to coincide with the Lagrangians the books are balanced by a transfer of money as is shown in the robustness check, presented in Section 5.

should occur for penalties set appropriately for any one of the three CEs. Thus for them the null hypothesis is.

Conjecture 2: *In Treatments COa, COb, as well as in COc and RIa, RIb, and RIc, net money holdings will be equal to the equilibrium level of zero.*

Ordinary individuals rarely make conscious economic decisions at a global level. Therefore, for understanding and analyzing an economy populated by agents whose behavior is mostly local, it is possible that the multiple equilibria obtained from global optimization in a formal mathematical model may be misleading or irrelevant. Moreover virtually all experimental gaming has been conducted with open or partial equilibrium systems and we cannot assume that those results necessarily generalize to closed systems. On the other hand, global optima may form domains of attraction even in environments dominated by local behavior (e.g., Gode and Sunder, 1993). Whether this is the case remains an empirical question on which the present exploration can be expected to shed some light.

2.2 A Caveat

Selection of the three levels of linear penalties associated with each of the underlying CEs has the strong property that all budget constraints are met and the net transfer of money is zero. When the penalties chosen are other than the values that select one of the three equilibrium points there is active bankruptcy and a need for the transfer of money to balance the books. For experimental simplicity, we have utilized a linear separable money that may be regarded as producing a unique equilibrium in a three- rather than a two-commodity world.

By using the linear penalty and settlement in our experiments we do not address the complexities of the income effect of bankruptcy laws. Doing so would call for a

consideration of the highly important aspects of collateral and secured lending. Hellwig (1982) was the first to consider collateral. Geanakoplos and Zame (2007) have developed an understanding of the role of collateral in providing settlements based more endogenously on the ownership of real assets than on the pure political enforcement of a bankruptcy law.

3. EXPERIMENTAL SETUP

Subjects were given endowments of goods and money (A, B, M endowments of 40, 0, 100 for the five subjects of one type and 0, 50, 100 for the five subjects of the second type). The first type of traders who were endowed with good A were asked to state the number of units of A they wished to sell (out of their endowment or the balance) and the number of units of money they wished to tender to buy good B. Similarly, the second type of traders who were endowed with good B were asked to state the number of units of B they wished to sell (out of their endowment or the balance) and the number of units of money they wished to tender to buy good A. Negative holdings in goods or money were not possible. Computer added the total amount of money bid for good A by the five subjects of the second type and divided it by the total number of units of good A offered for sale by the five subjects of the first type to determine the price of good A, and implemented the appropriate transfers of good A and money among the subjects. Similarly, computer also added the total amount of money bid for good B by the five subjects of the first type and divided it by the total number of units of good B offered for sale by the five subjects of the second type to determine the price of good B, and implemented the appropriate transfers of good B and money among the subjects.

Subjects' earnings functions were common knowledge and were provided to them algebraically as well as numerically in a 50x50 payoff table (see Appendix A for condensed versions).

First type of traders: Points earned = $A + 100 * (1-e^{(-B/10)}) + \text{NET MONEY}$,

Second type of traders: Points earned = $(1/\mu_2) * ((B + 110 * (1-e^{(-A/10)})) + \text{NET MONEY}$,

where $\mu_2 = 0.28$ in sub-treatments a, 0.75 in b, and 5.07 in c.

In the CO-treatments holdings of goods and money are carried over from one period to the next. The final payout is determined by the final holdings of goods and the net change in money holdings.

To allow less constrained learning and observe possible learning effects, holdings of goods are reinitialized after each period in three RI-treatments (μ_2 is varied again with values 0.28, 0.75, and 5.07). In RI_a, Rb_b and RI_c, subjects start each of the 15 periods with 40, 0, 100 or 0, 50, 100 of goods A, B, money, and they have only one transaction to reach their desired holdings of the goods and money. Points earned by each subject are added up over periods and converted into money at a predetermined rate. Average payment was 20 dollar for each subject in each of the approximately 60-minute sessions.

We conducted six independent runs for each of COa, COb, and COc, and four independent runs for each of RIa, RIb, and RIc, each with a different cohort of 10 students. We thus have 30 runs with a total of 300 students. Nine runs were conducted at Yale University, and 21 at the University of Innsbruck, Austria. All students were BA or MA students in Management or Economics. All sessions were carried out using a program written in z-Tree (Fischbacher, 2007).

4. RESULTS

Figure 2 presents the development of end-of-period holdings of goods A and B for COa (left panel), COb (center), and COc (right). In the top row of panels average holdings of the traders in each of the six runs of each sub-treatment are displayed. Holdings at the end of each period are marked with a diamond and periods of a single run are connected with a black line. In the lower row of panels, a single run (always the second of the six) of each sub-treatment is displayed. In addition to the average individual holdings of the groups of traders, the holdings of the individual traders are also displayed by small circles to convey the dispersion of holdings.

(Insert Figure 2 about here)

Defining the salvage value and default penalty of money leads the economy towards a unique equilibrium in each sub-treatment. This unique equilibrium is shown as a black diamond for sub-treatments a, a triangle for b, and a square for c, while we still display the former equilibria in unfulfilled white markers for the sake of easier comparison across sub-treatments. The paths in the three sub-treatments are distinct from each other, and each path approaches the vicinity of its respective equilibrium. To test whether manipulation of salvage values/default penalties for money (parameter μ_2) can select different equilibria as claimed in Conjecture 1, we supplement the graphic presentation in Figure 2 with 2-sided Mann-Whitney U-tests comparing average final holdings of good A for COa with those of COb, COa vs. COc, and COb vs. COc. This is then repeated for good B. All six statistical tests yield p-values smaller than 0.01 (N=6), confirming that the choice of different μ_2 generated significantly different final holdings. To test whether the targeted equilibria were approached we conduct 2-sided t-test (N=6) to compare average end-holdings of goods A and B to the

holdings in the respective equilibrium. Here none of the six tests delivers a significant result, i.e., holdings of A are indistinguishable from the respective equilibria in all three tests, as are those of B from their respective equilibria. Thus, all test results are in line with the theoretical predictions of the model. We cannot reject Conjecture 1 on the choice of default penalty leading the economy to the targeted equilibrium.

The three sub-treatments differ with respect to the trading volume required to reach the respective equilibria. In COa ($\mu_2 = 0.28$), with holdings of goods relatively more valuable for traders initially endowed with good B, those endowed with A should sell most (36.78 out of 40) of their holdings of A, while those endowed with B should hold on to most (39.77 out of 50) of their goods to reach equilibrium. The development of cumulative market trading volume over periods is displayed in Figure 3 (market volumes should be five times the per capita trades given above). In each market the trading volume is high in early periods and falls off rapidly until trading stops between periods 8 and 15 (when the volume drops below the threshold of 0.2 units of either good). We also see that the predicted market volumes (horizontal lines with diamond markers) provide support for the observed market volumes in the six runs in each of the three panels of Figure 3.[15]

(Insert Figure 3 about here)

Figure 4 shows that efficiency increases markedly over time, as subject's holdings of goods A and B approach the respective equilibria. In all but one of the runs trading stops in the period with the highest overall efficiency or one period later, which is in line with

[15] All six 2-sided Mann-Whitney U-tests comparing average trading volume between the three sub-treatments deliver p-values below 0.01 (N=6), while all 2-sided t-tests comparing average trading volume to its respective equilibrium prediction deliver insignificant results (2 results are significant on the 10-percent level, but none on the 5- or 1-percent level).

rational expectations. The final efficiency levels reached are between 92.0 and 99.6 percent with an average of 97.8 percent.

(Insert Figure 4 about here)

Theory predicts that holdings of goods A and B should approach the respective equilibrium levels, while money holdings should remain unchanged at the endowment level. This prediction for goods is generally supported by the data, but is not supported for money. In COa, where goods are more valuable to traders initially endowed with good B, who thus bought, while those endowed with goods A mostly sold, money accumulated with those initially endowed with good A. The reverse holds in COc, where money mostly ends up with traders initially endowed with good B. To provide a quantitative measure Figure 5 shows the development of average absolute deviations of holdings of goods A, B, and money from the respective equilibrium predictions over periods. To make the numbers comparable they are given in percent of initial holdings (40, 50, and 100, respectively, for goods A, B, and money). Especially in COa and COc holdings of goods move closer to equilibrium predictions over time, while holdings of money move away, i.e., money moves from one trader type to the other. Two-sided t-tests reveal that in COa and COc dispersion of money holdings is significantly larger than dispersion in holdings of goods A and B.[16] Thus, heterogeneity in the end is mostly for money holdings, while holdings of goods A and B are more homogeneous and much closer to the equilibrium predictions.

(Insert Figure 5 about here)

Sub-treatments RIa, RIb, RIc (holdings of goods and money reinitialized)

[16] p-values below 0.05, N=6 for all six tests in COa and COc. No significant difference in COb with p-values between 0.143 and 0.229 for the four tests. There is no significant difference between the average absolute deviations of holdings of goods A and B with all p-values above 0.2.

To allow for (costless) learning and to better observe possible learning effects we implement three RI sub-treatments, where holdings of goods are re-initialized after each period.

Figure 6 displays period-by-period average holdings of goods A and B in RIa, RIb, and RIc (with the final period shown by an enlarged marker). The lines connecting the markets allow us to follow the outcome of trading, i.e., the average end-of-period holdings over the sequence of 15 periods. The top panels show the first and second independent runs for each of the three sub-treatments, while the bottom panels present the third and fourth. The paths in the three sub-treatments are quite distinct from one another in each of the panels, and each run approaches its respective equilibrium.[17] The RIb- and RIc-equilibria are essentially reached in the second period, while in Ra it took a few periods longer to approach the equilibrium. To explore whether average final holdings differed across sub-treatments we conduct Mann-Whitney U-Tests with one observation per run (average holdings of goods A, B in the last period), separately for goods A and B, comparing the four runs of RIa with the four of RIb, RIa to RIc, and RIb to RIc. All six tests delivered p-values of 0.029 on a two-sided test, thus confirming that different μ_2's suffice to produce different outcomes.[18]

The RI treatments demonstrate that the selection of the default penalty is suitable to select among multiple equilibria, thus corroborating the result from CO treatments that Conjecture 1 is not rejected.

[17] Note that each marker in Figure 6 shows the holding achieved in a period starting every period with the endowment point in the northwest corner. We have joined the markers with a line to indicate the sequence of periods in order to point out that the outcomes got generally closer to the respective equilibrium holdings in the later periods of the runs. In contrast, in the CO-treatments (without reinitialization), the northwest corner was the endowment point only at the beginning of period 1, and the change in holdings in all subsequence periods was incremental relative to the end of the preceding period.

[18] We also conducted six two-sided t-tests (N=4) to test whether final holdings of goods A and B were different from the respective equilibrium predictions. There is no significant difference in five of the six tests; only holdings of good A in RIa are significantly different from the equilibrium.

(Insert Figure 6 about here)

In all three RI sub-treatments the largest increase in efficiency occurred during the first period, moving from autarky to the market economy. As can be seen in Figure 7 this is followed by smaller increases in efficiency of repetitions over subsequent periods (period 0 is efficiency associated with autarky and 100 percent is the efficiency of the respective competitive equilibria). In most runs efficiency levels of more than 90% are reached in the first period, i.e., when moving from autarky to a market economy.

(Insert Figure 7 about here)

The upper row of panels in Figure 8 replicates for the RI-treatments what Figure 5 presented for CO: the development of the average absolute deviation of holdings of goods A, B, and money from the respective equilibrium predictions over time. Again all numbers are given in percent of initial holdings. The results are comparable to Figure 5: in RIa and RIc money holdings move away from the equilibrium prediction of zero, while holdings of goods A and B quickly drop toward equilibrium holdings. Again, heterogeneity in final holdings is greater for money than for goods. The lower row of panels presents the averages of the four runs given in the upper panels.

(Insert Figure 8 about here)

Net Money Holdings in CO and RI Treatments

In all our sub-treatments the respective equilibria can be reached with net money holdings of all traders at zero or at any other desired level, as money holdings are a result of prices which are set endogenously by traders' bids and offers. Net money holdings of zero are the equilibrium prediction and they are achieved when the ratio between the prices of the

two goods is equal to the respective values of μ_2.[19] However, this would lead to a very uneven distribution of final points earned, e.g., in COc and RIc with μ_2 = 5.07 subjects starting with good B would have to buy 7.74 units of good A at a price five times higher than the price they get for each of the 39.26 units of B they sell in equilibrium. They would end up with relatively small holdings of the goods and thus earn only 10 percent of the points that A-holders earn.

Figure 9 presents the development of average net money holdings of traders initially endowed with good A over time in all six sub-treatments (the runs of COa and RIa are in the left panel, COb and RIb are in the center and COc and RIc are on the right) contrasted with the CE holdings of zero.[20] We see that the CE-proposition does not serve as a good benchmark in most of the runs, as net money holdings are rarely close to zero and mostly move away from zero over time. Only in COb and RIb do the net money holdings remain relatively small. Net money holdings of subjects in the last period are significantly different from zero in all 20 runs of COa, RIa, COc, and RIc ($p < 0.01$, Mann-Whitney U-Tests, N=10 for each test). For COb and RIb the final money holdings in five of the ten runs are significantly different from zero ($p < 0.05$). Thus Conjecture 2 is rejected.

(Insert Figure 9 about here)

Conjecture 2, while consistent with theory, leaves out both the imperfections of learning and error to be expected in even as simple an environment as this. To explore if the subjects took into account the impact of their offered volume on prices, we calculated the correlation coefficients between the changes in prices from period t-1 to period t and the change in total volume offered from period t to period t+1. This is done separately for good

[19] Recall that $\mu_1 = 1$ in all treatments.

[20] Average net holdings of traders initially endowed with good B are simply the net holdings of A multiplied with (-1).

A and good B in each of the 12 runs of RIa, RIb, and RIc. Learning subjects should offer fewer goods when prices are comparatively low and more goods when prices are high, i.e., we should observe positive correlation coefficients. We find that in 21 of 24 cases the correlation is positive, with an average correlation of 0.30 for good A and 0.34 for good B. This suggests that subjects reacted to changes in relative price levels, i.e., they offered fewer goods after prices dropped, and more goods when prices rose in the preceding period.

Furthermore, the perceived extreme asymmetry especially of the CEs in RIa and RIc is such that we might expect a deviation from the balanced budget condition. We think an aversion to results with a very uneven earnings distribution, which might be considered unfair, play a role here (see e.g., Ernst Fehr on this topic). Further experimentation is called for to resolve why budgets do not balance, as predicted by equilibrium.

5. ROBUSTNESS CHECK

We have shown that the economy can be guided to any of the desired equilibria of the original economy by proper selection of default penalty. But what happens if the parameters selected do not coincide with the marginal values of income at any of the three original equilibria? An equilibrium will exist for any parameter value above zero but it will not be one of the three CEs of the original economy. Any selection of penalty which is different from the CE penalties will lead to a unique equilibrium with a net transfer of money from one class of agents to the other.[21] We examine this possibility as a robustness check by setting $\mu_1 = \mu_2 = 1$ in order to consider a case where the solution should be a unique equilibrium with allocations different from all three equilibria in the original model. When

[21] Thus, for a complete representation we would need a three-dimensional diagram.

both μ's are set to 1 the unique equilibrium coincides with the joint maximum, i.e., the point where the sum of the earnings of the two trader types is maximized.[22]

One might ask why bother with penalty levels other than those that support one of the three original equilibria. One reason is to stress our concern with the role of rules and institutions in the economy. We believe that the government can have only an approximate and general knowledge of the preferences and assets in the economy which is normally insufficiently accurate to guess a penalty level that would support one of multiple equilibria. If it guesses incorrectly the number of bankruptcies would signal that it needs to adjust the penalties. This could be tested experimentally by having the government as a player trying to select an appropriate penalty but having some uncertainty concerning endowments and preferences.[23] The present experiment does not include such a test, and is confined to verifying if the predictability of the outcomes is robust to the choice of penalties that deviate from the original equilibrium levels.

Conjecture 3: *In the robustness check the unique equilibrium defined by the chosen default penalties μ_1 and μ_2 is approached.*

To ensure comparability with the main experiment, we conduct one sub-treatment where holdings of goods are carried over (CO-R) and one in which the endowments are reinitialized after each period (RI-R). Two runs with one cohort of 10 subjects are conducted for CO-R and one run with a different cohort of students for RI-R.

[22] For any value of μ the equilibrium is also a joint maximum. The important game theoretic distinction between the treatment where all μ's are set to 1, and the three other cases involving the CEs, is that the latter illustrate equilibria in no-side-payment games which means that the books balance and there is no net transfer of money. In the other instance the books do not balance and there is a net transfer of money with bankruptcy possible.

[23] When there is no exogenous uncertainty active bankruptcy is caused by inappropriate penalties or human error. In an economy with exogenous uncertainty an optimal bankruptcy law can only be reflected by taking into account society's attitude towards risk. It is a form of public good; and even without human error it will involve active bankruptcy.

5.1 Results

Figure 10 presents the development of holdings of goods over time in the two runs of CO-R. The two runs are quite similar to each other, and end in the vicinity of the unique equilibrium (joint maximum) marked by a dark triangle. Final holdings of goods are not significantly different from the holdings in the joint maximum in both runs (Mann-Whitney U-test, $p > 0.1$ in both runs, N=10), and Conjecture 3 is not rejected. Two panels of Figure 11 show the development of cumulative trading volumes and efficiency of time. With convex paths for both, these figures show no remarkable departures from the results of the main experiment presented in Section 4.

(Insert Figure 10 and 11 about here)

This is corroborated by the run RI-R (with re-initialization), presented in Figure 12. In the left panel we see that the average final holdings of traders of the first type are in the vicinity of the unique equilibrium. The right panel shows that efficiency increases over time (it is high in the first period, lower in the next two, and increases gradually but steadily from period 3 to the end).

(Insert Figure 12 about here)

Equilibrium prediction is that the final net money holdings will be -2.3 for traders endowed with 40/0 and +2.3 for traders endowed with 0/50 of goods A/B. In the two runs of CO-R average final money holdings are -3.9 and +3.9 for traders endowed with 40/0 and 0/50, respectively (in RI-R final money holdings are -1.4 and +1.4, respectively). The algebraic sign of the net money holds is the same as the equilibrium predictions, although the actual amounts deviate significantly (just as they do in the main experiment in Section 4).

6. CONCLUSIONS

Societies, and the politicians acting on their behalf, continually face choices among a multiplicity of possible outcomes. Long term legislative actions often involve trading off utilities across individuals or groups. In this paper we explore empirically the practical feasibility of such choices. We demonstrate that by introducing an appropriately chosen default (bankruptcy) penalty,[24] the outcome of a closed economy can be directed to any targeted element in the set of its equilibria. Two goods were traded for money in laboratory markets. Our experiment showed that default (bankruptcy) penalty of a fiat money can be chosen to achieve any of the given equilibria of the economy, or more generally, any desired point on the contract curve (see the robustness check). Our central experimental task was to examine the suggestion from theory that the institutional arrangements in a society provide the means to resolve the possibility of multiple equilibria in an economy. Our sub-treatments where three different default penalties were chosen show that the laboratory economies approached the proximity of the respective equilibria. We also observed that final holdings of money were more heterogeneous than of goods.

The results provide some empirical support for the attitudes of macroeconomists who do not regard the non-uniqueness of competitive equilibria as a problem of practical significance for their work. Societies may implicitly solve the uniqueness problem in the guidance of a competitive economy by selecting default penalties that link the value of money directly to the preferences of individuals. The need for societies to add the

[24] In actuality the government selection is made under lack of common knowledge, hence at best it is a crude guess. In fact in any modern society there is some percentage of the members of the economy who wind up in bankruptcy.

institutional details and extra parameters is forced by the requirement to specify how to handle all outcomes from a dynamic process.

The robustness check demonstrated that by proper selection of a default penalty any desired outcome on the contract set can be targeted and approximately attained. However, we stress that although a societal selection of the extra parameters is sufficient to obtain a unique equilibrium, unless the parameters coincide with the values of the Lagrangian variables at an equilibrium of a static exchange economy, the static equilibrium solution to the new game will not coincide with any of the original equilibria.

Both, formal economic analysis and experimental gaming, in their own albeit different ways, call for stringent simplification. Our introduction of both a bankruptcy penalty and linearly separable money is a radical simplification. In an error free world, as a means for selecting among competitive equilibria, only the penalty is required as the "money" nets to zero. In actual economies, money coexists, with many assets which permit much secured lending that helps ameliorate the damage from defaults and provides a more economical solution to the distribution of risk and the redistribution of assets. In our models the ideal money we introduce is a crude metaphor for the more complex arrangements in an asset rich economy

Summarizing, we showed how a pair of socially engineered parameters could serve to select any of the equilibria, but this requires a "fine tuning" of the equilibrium values and detailed knowledge of the preferences and parameters of the economy. In a society with dispersed knowledge and perennial political and bureaucratic battles, neither such knowledge nor the fine tuning seems feasible or likely. Fortunately, even poorly tuned parameters

resolve the multiplicity problem, and societies may resort to successive adjustments of their values over extended periods of time to discover acceptable levels.

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Figure 1: An Exchange Economy with Two Goods and Three Competitive Equilibria

(Reproduced from Shapley and Shubik 1977, Fig. 1, p. 874)

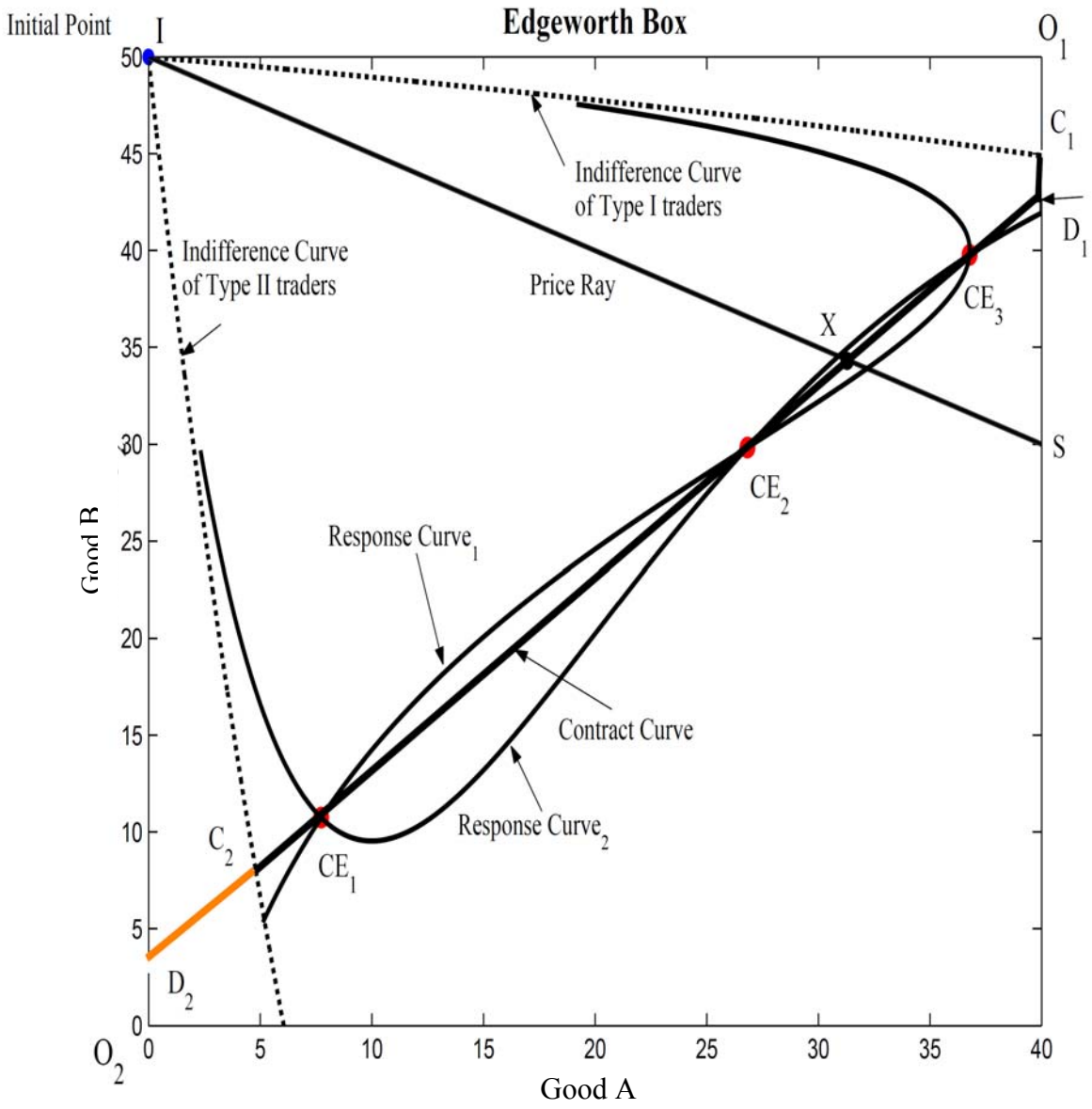


Figure 2: Paths of Holdings of Goods A and B over Periods in treatment CO
(with goods and money balances carried over period-to-period).
Top Panels: Trader-type averages in six independent runs for each default penalty.
Bottom Panels: Trader-type average and individual holding details of run 2 for each default penalty.

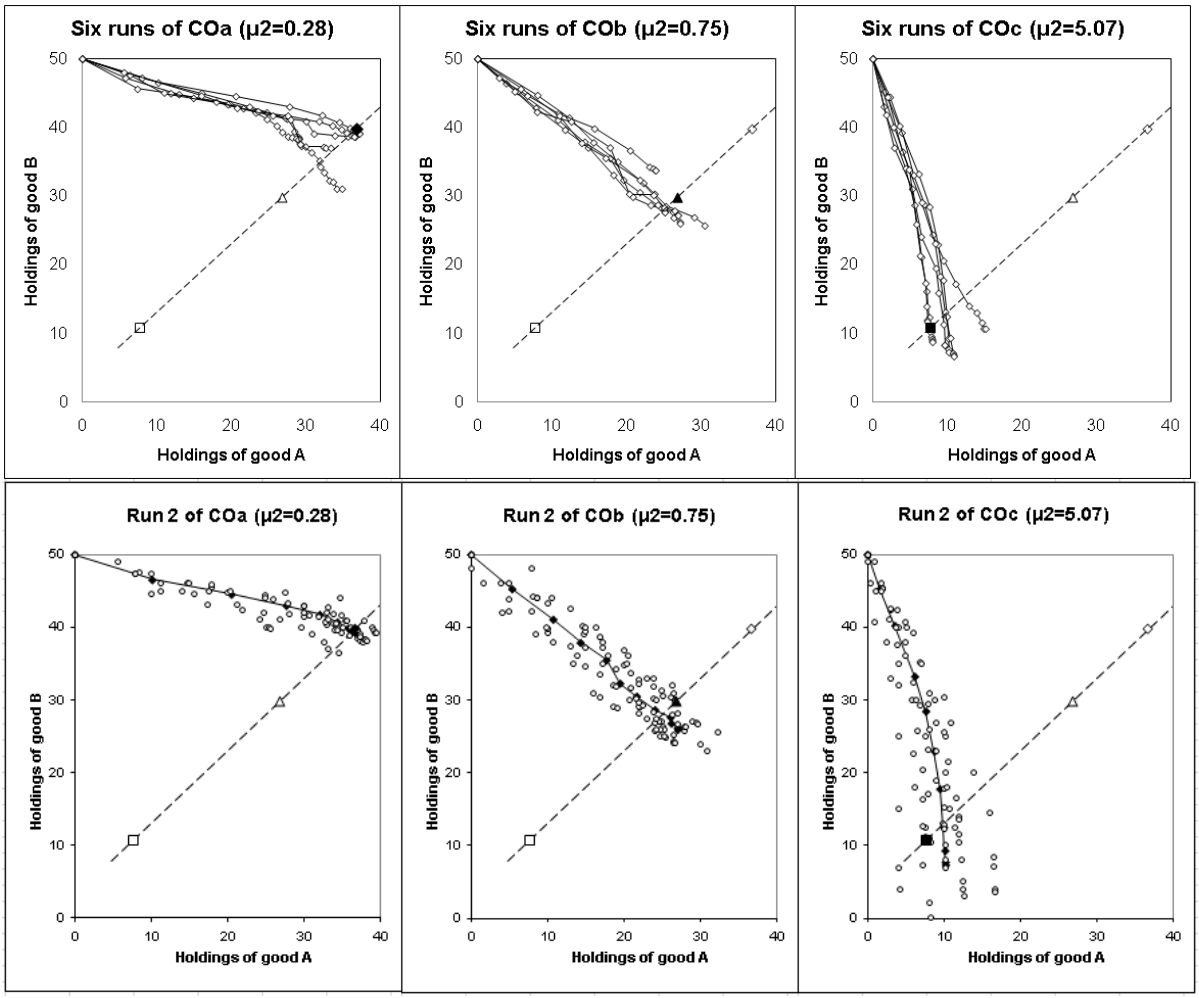


Figure 3: Time Series of Cumulative Trading Volume for Goods A and B (in six independent runs for each default penalty with money and goods carried over period-to-period; Period 0=autarky)

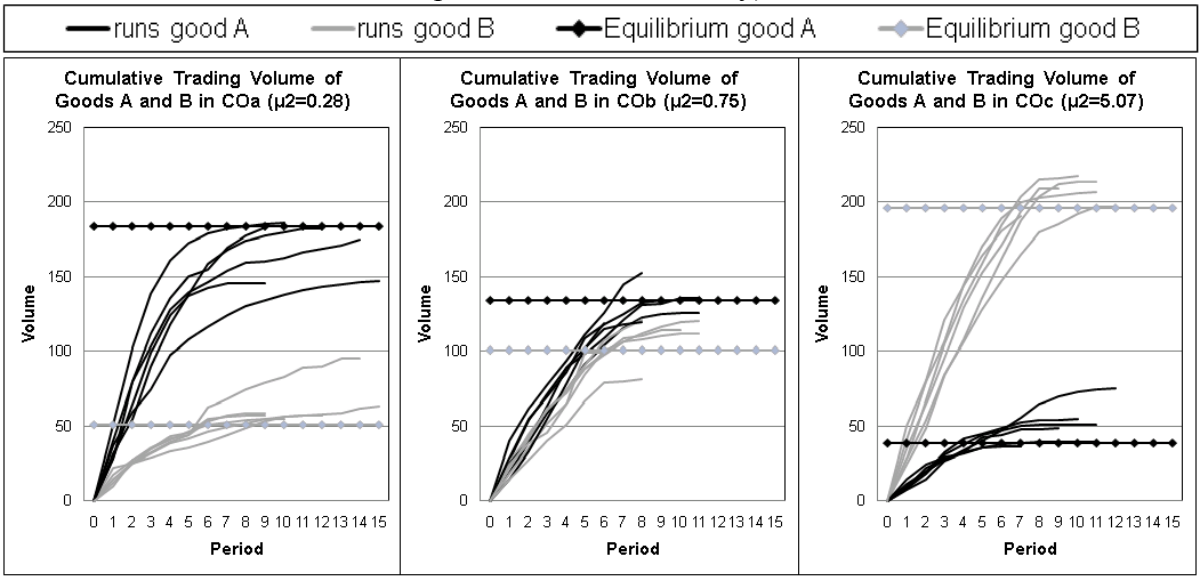


Figure 4: Development of Efficiency (in six independent runs for each default penalty with money and good carried over; Period 0 = autarky)

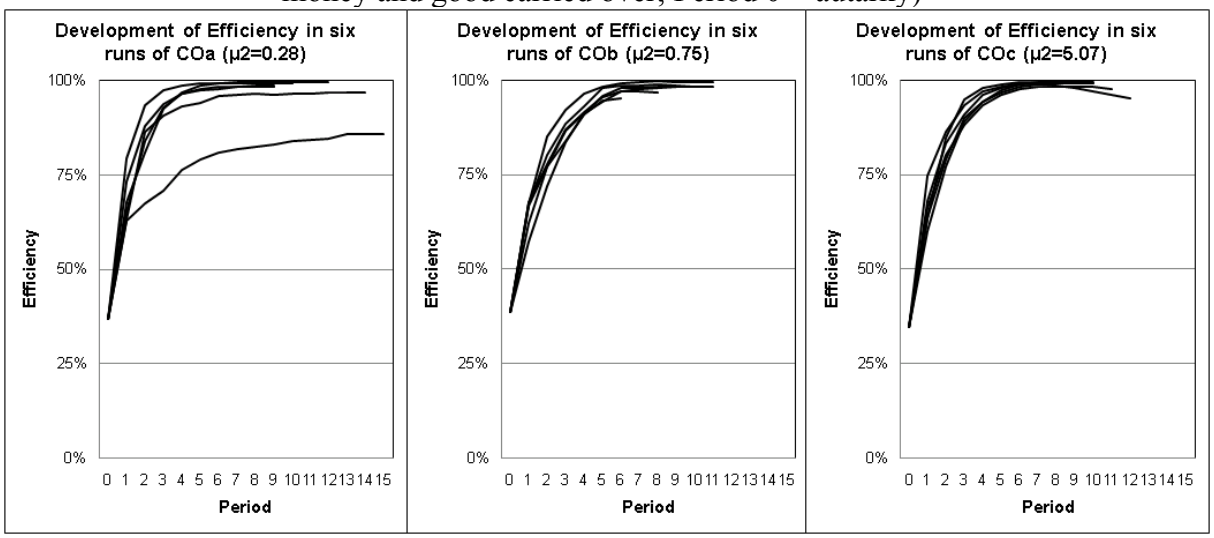


Figure 5: Development of Average Absolute Deviation of Holdings of Goods A, B and Money Holdings from the Respective Equilibrium Predictions in Percent of Initial Holdings in Treatment CO (six runs for each default penalty, Period 0 = autarky)

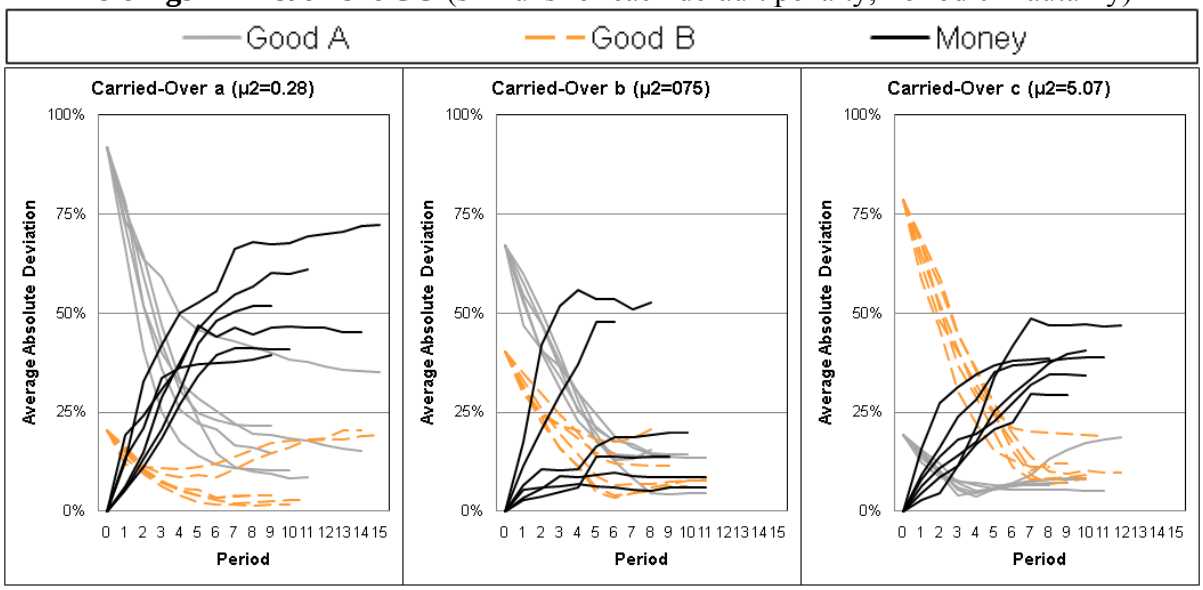


Figure 6: Average Holdings of Goods A and B in Treatment RI (holdings reinitialized)

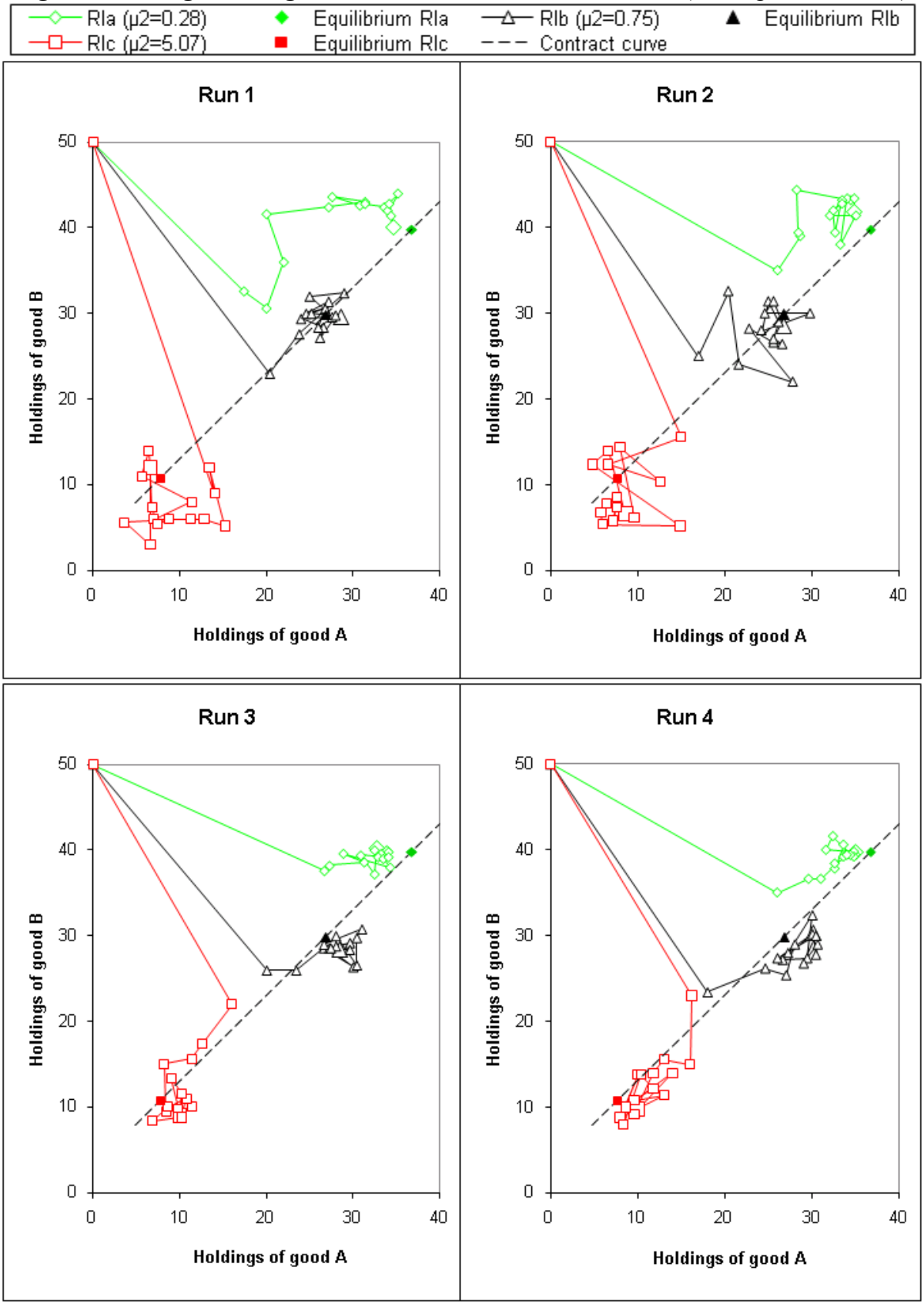


Figure 7: Development of Efficiency in Treatment RI (reinitialized, Period 0 = autarky)

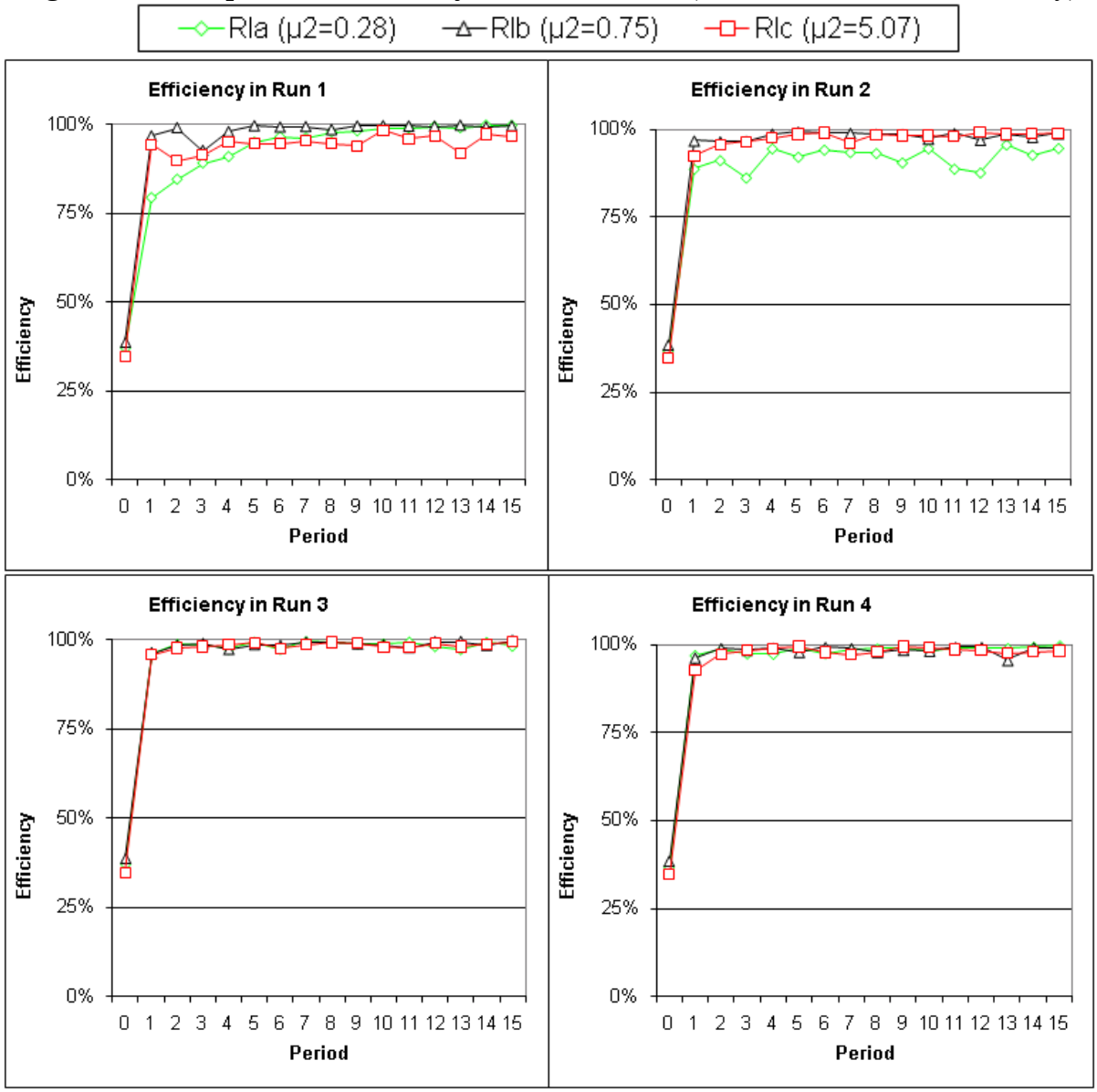
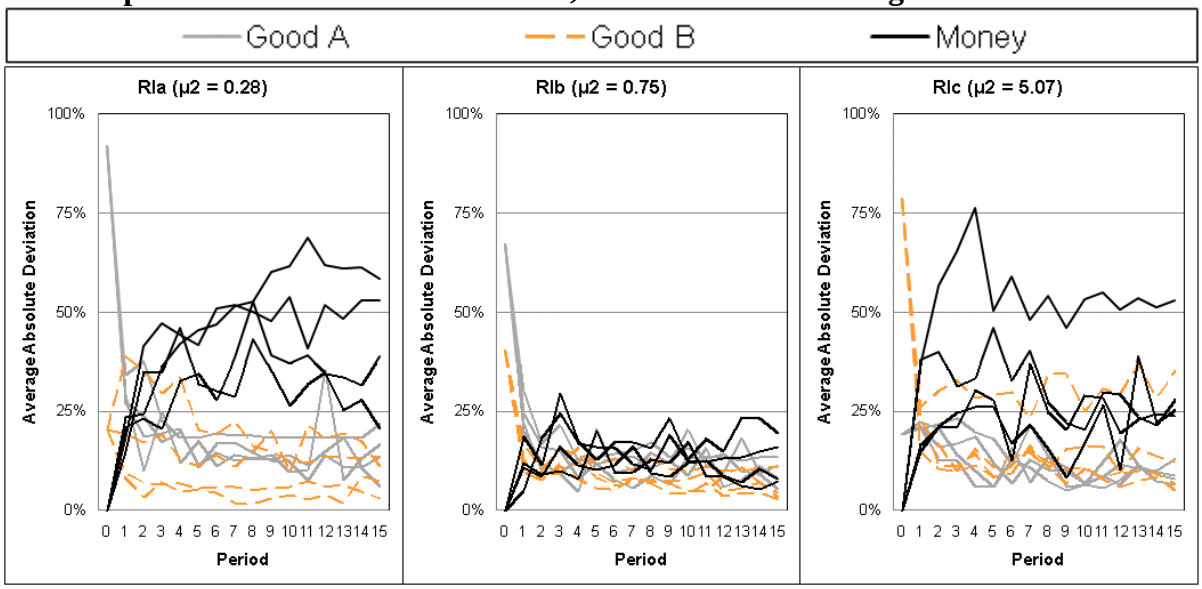


Figure 8: Development of Average Absolute Deviation of Holdings of Goods A, B and Money Holdings from the Respective Equilibrium Predictions in Percent of Initial Holdings in Treatment RI (four runs for each default penalty, Period 0 = autarky)
Top Panels: data of individual runs; Bottom Panels: Averages across 4 runs



For comparison: the respective averages

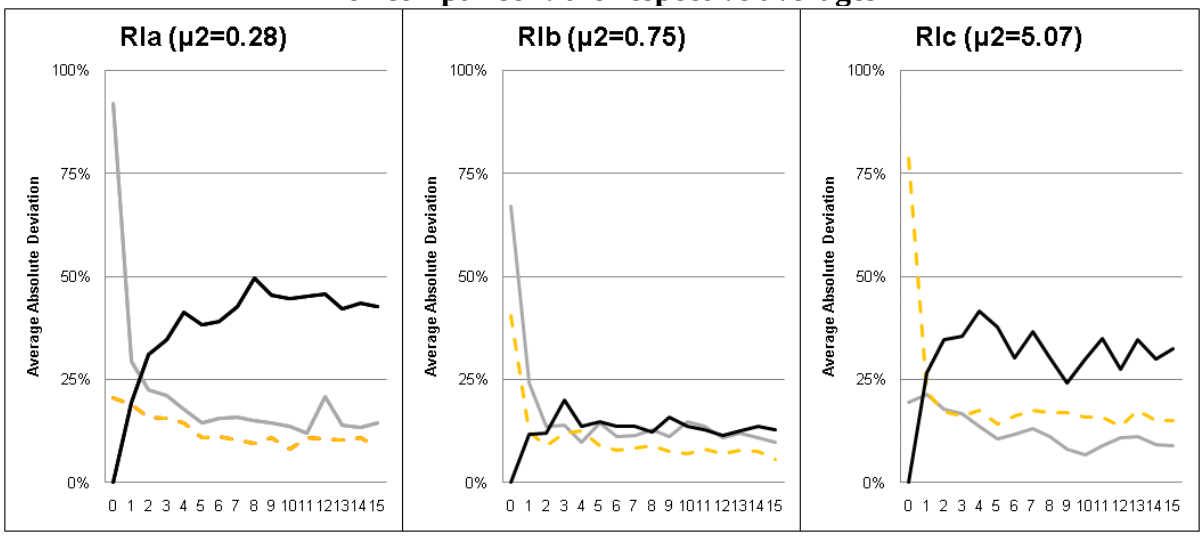


Figure 9: Development of Average Net Money Holdings for Subjects endowed with Good A in Treatments CO (holdings carried over) **and RI** (holdings re-initialized)

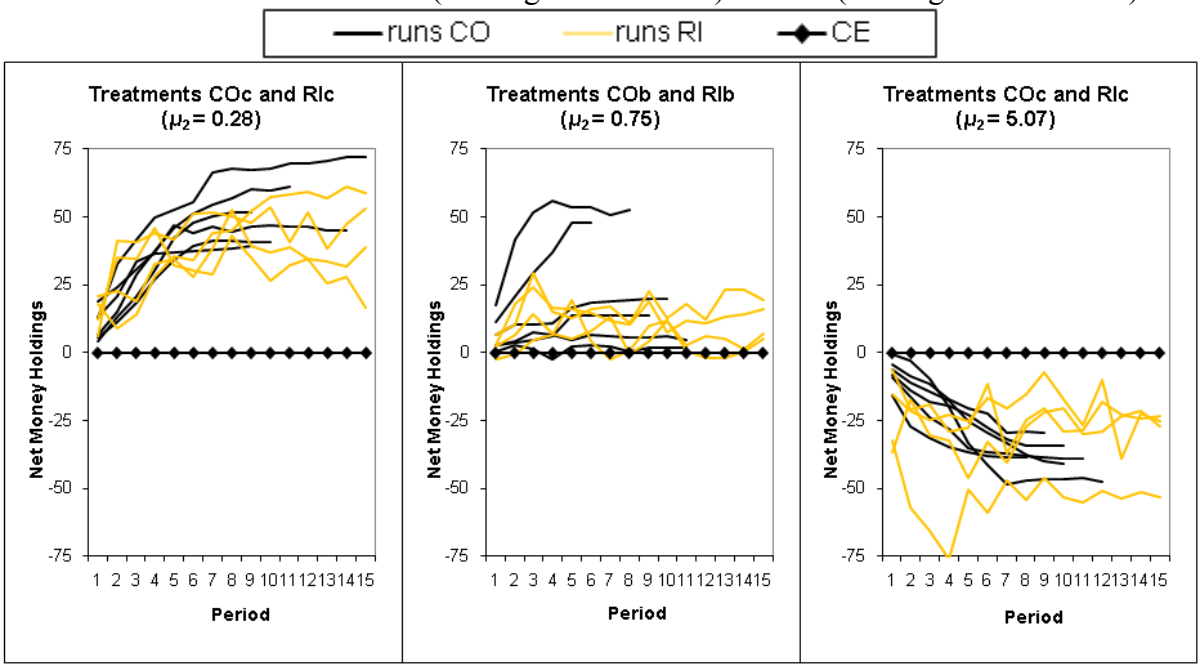


Figure 10: Holdings of Goods A and B in the two Runs of Treatment CO-R
(Robustness check with holdings of goods carried over from one period to the next)

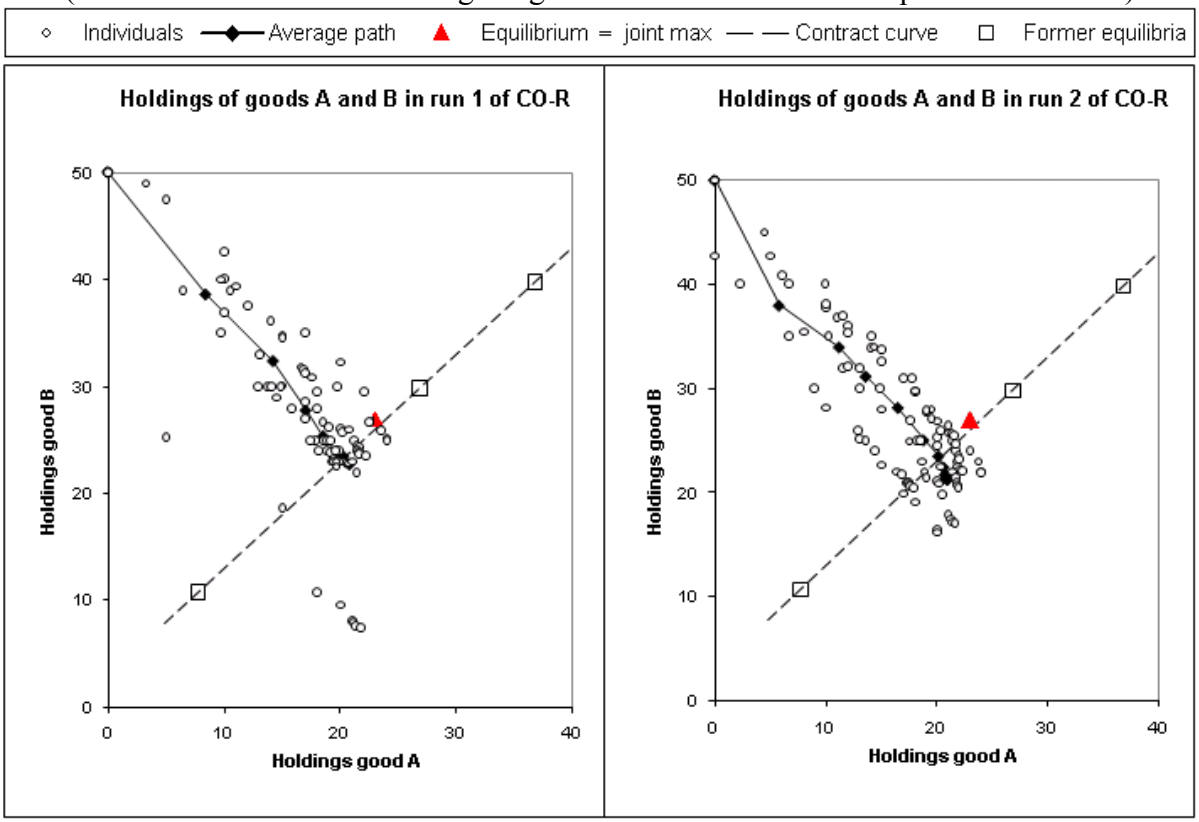


Figure 11: Time Series of Cumulative Trading Volume (left Panel) and Efficiency (right Panel) per Period in Treatment CO-R. Run 2 is shaded grey.

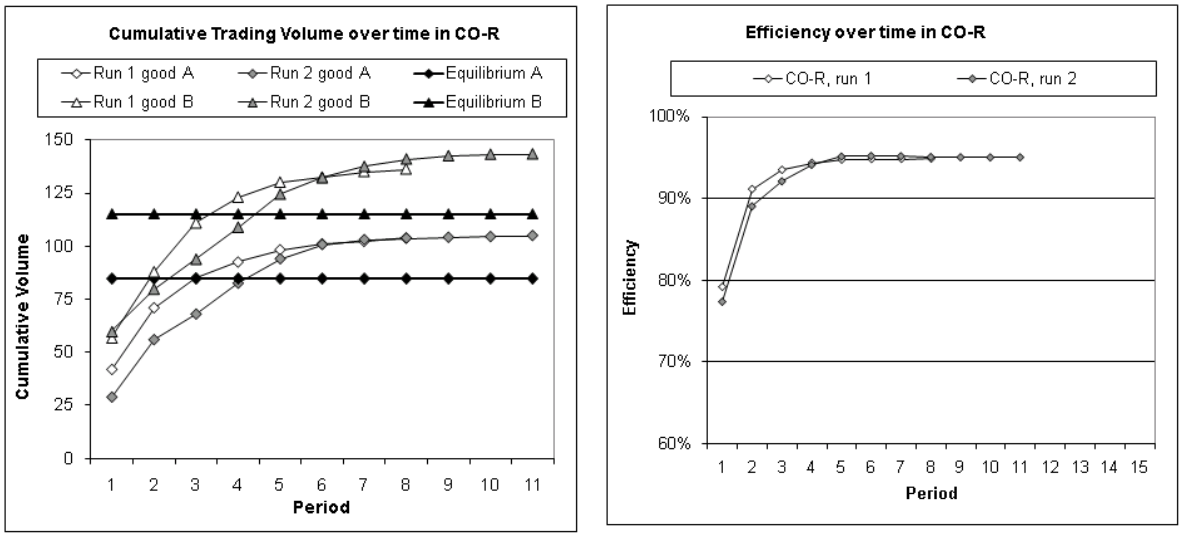
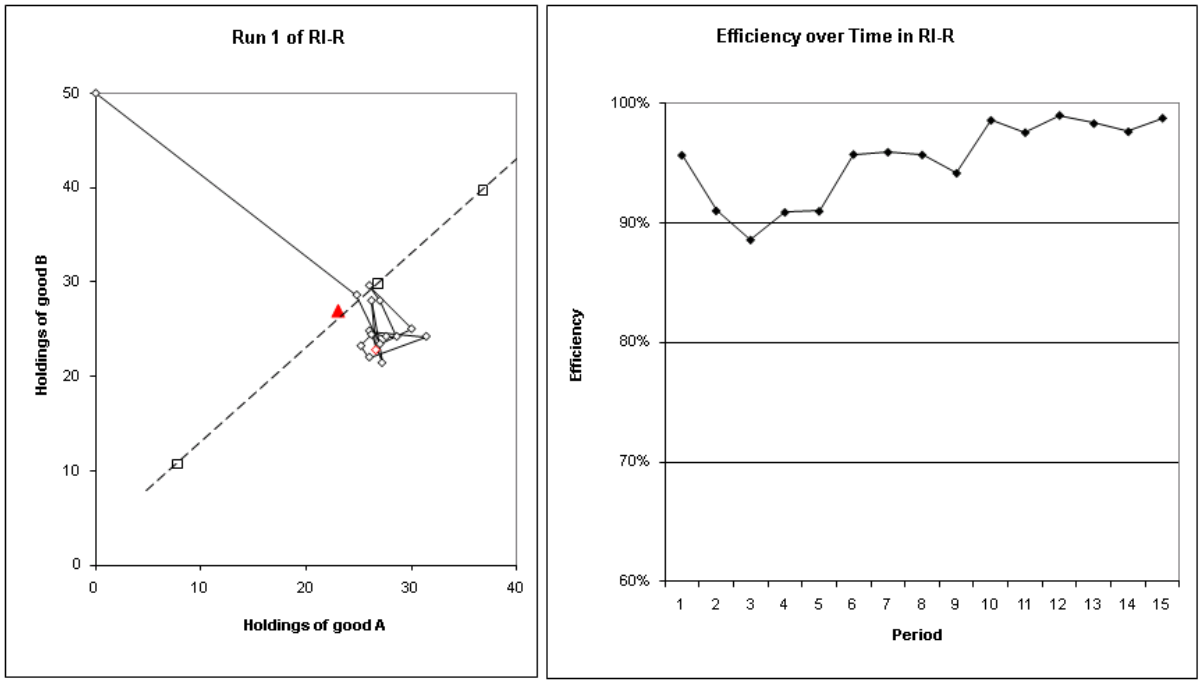


Figure 12: Holdings of goods A and B (left Panel) and Efficiency (right Panel) per Period in Treatment RI-R (holdings re-initialized)



Appendix: Instructions for Treatments RI and RI-R (only μ_2 varied)

General

This is an experiment in market decision making. If you follow these instructions carefully and make good decisions, you will earn more money, which will be paid to you at the end of the session.

This session consists of several periods and has 10 participants. At the beginning of each period, each of the five participants will receive 40 units of good A, and each of the other five will receive 50 units of good B. In addition each participant will receive 100 units of money at the start of each period. In each of some 10 to 20 period you will have the opportunity to offer your goods for sale and to buy the other goods.

Each participant is free to offer for sale any part or all the goods he/she owns each period. You earn points for your holdings of good and money at the end of each period. Holdings of goods and money are not carried over from period to period; you start each period with 100 units of money and either 40 units of A or 50 units of B.

During each period we conduct a market in which the price per unit of A and B will be determined. All units of A and B put up for sale will be sold at their respective price, and you can buy units of A and B at the same price. The following paragraphs describe how the price per unit of A and B will be determined.

In each period, you are asked to enter the cash you are willing to pay to buy the good you do not own (say A), and the number of units of the good you own that you are willing to sell (say B) (see the center of Screen 1). The cash you **bid to buy cannot exceed your money balance (100)**, and the units you **offer to sell cannot exceed your holdings of that good (40 of A or 50 of B)**. You receive the income from the sale of any goods to be paid in money at the end of each period.

The computer will calculate the sum of the amounts of good A offered by all participants (= Sum_A). It will also calculate the total number of units of money offered to buy the goods ($\$Sum_A$) and determine the price of A expressed in terms of money, $p_A = \$Sum_A/Sum_A$. The same is done with good B.

If you offer q_A units of A for sale, you will get an income of $q_A * P_A$. If you bid b_A units of money to purchase A, you will get b_A/P_A units of good A.

Screen 1: trading screen for a trader endowed with good B

Both goods are perishable and must be either sold or consumed in the current period. The number of units of A and B you own at the end of the period, c_A and c_B (unsold units of owned good and purchased units of the other good) will be consumed and determine the number of points you earn for the period. Traders initially endowed with A earn:

Points = $(1/\mu) * (A + 100 * (1-e^{(-B/10)})) + \text{NET MONEY}$

COMMENT: $\mu = 1$ in R-R, 0.28 in Ra, 0.75 in Rb, and 5.07 in Rc.

And traders endowed with B earn

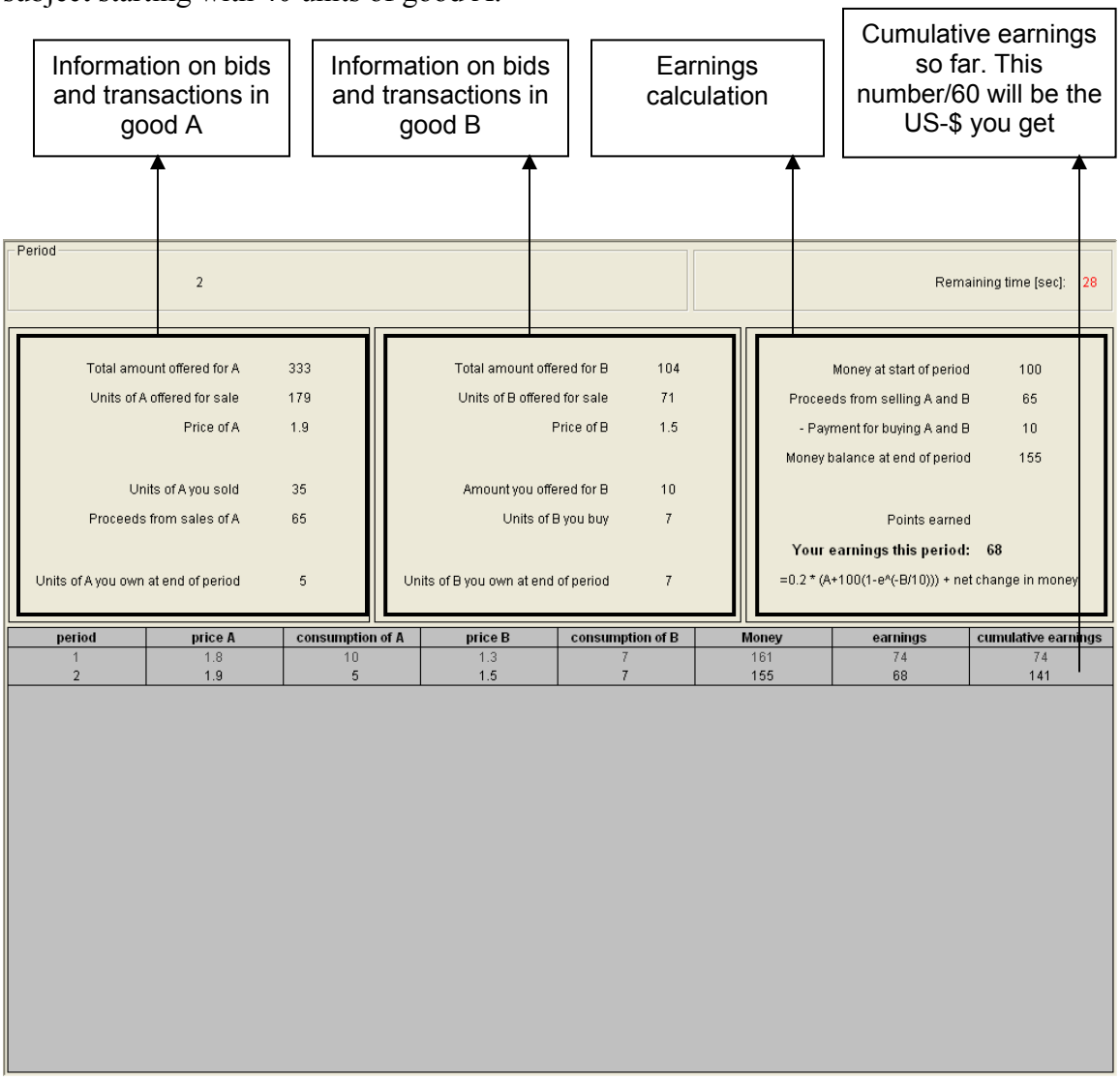
Points = $B + 110 * (1-e^{(-A/10)}) + \text{NET MONEY}$

Example: If at the end of any period you are endowed with B and have 30 units of A and 15 units of B you earn 15 + 110 (1-e^(-30/10)) = 119.5 points.*

Your cash balance holdings will help determine the points you earn. At the end of each period the starting endowment of 100 units of money will be deducted from your final

money holdings. The resulting net holdings (which may be negative) will be added to (or subtracted from) your total points earned.

Screen 2 shows an example of calculations for Period 2. There are 10 participants in the market, and half of them have 40 units of A, the other half 50 units of B. Here we see a subject starting with 40 units of good A.



The earnings of each period are added up in the last column. At the end they will be converted into real Dollars at the rate of 60 points = 1 US$ and this amount will be paid out to you.

How to calculate the points you earn (in sub-treatments COc and RIc):

The points those initially endowed with A earn each period are calculated as:

$$\text{Points} = (A + 100 * (1-e^{\wedge(-B/10)})) + \text{Net Money}$$

And the points those initially endowed with B earn each period are calculated as:

$$\text{Points} = 1/5.07 * (B + 110 * (1-e^{\wedge(-A/10)}) + \text{Net Money}$$

The following tables may be useful to understand this relationship. They show the resulting points from different combinations of goods A and B (assuming net money to be zero).

Table for those initially endowed with A:

		Units of good A you hold at the end of a period										
		0	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
Units of B you hold	**0**	0.0	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0
	5	39.3	44.3	49.3	54.3	59.3	64.3	69.3	74.3	79.3	84.3	89.3
	10	63.2	68.2	73.2	78.2	83.2	88.2	93.2	98.2	103.2	108.2	113.2
	15	77.7	82.7	87.7	92.7	97.7	102.7	107.7	112.7	117.7	122.7	127.7
	20	86.5	91.5	96.5	101.5	106.5	111.5	116.5	121.5	126.5	131.5	136.5
	25	91.8	96.8	101.8	106.8	111.8	116.8	121.8	126.8	131.8	136.8	141.8
	30	95.0	100.0	105.0	110.0	115.0	120.0	125.0	130.0	135.0	140.0	145.0
	35	97.0	102.0	107.0	112.0	117.0	122.0	127.0	132.0	137.0	142.0	147.0
	40	98.2	103.2	108.2	113.2	118.2	123.2	128.2	133.2	138.2	143.2	148.2
	45	98.9	103.9	108.9	113.9	118.9	123.9	128.9	133.9	138.9	143.9	148.9
	50	99.3	104.3	109.3	114.3	119.3	124.3	129.3	134.3	139.3	144.3	149.3

Table for those initially endowed with B:

		Units of good A you hold at the end of a period										
		0	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
Units of B you hold	**0**	0.0	8.5	13.7	16.9	18.8	19.9	20.6	21.0	21.3	21.5	21.6
	5	1.0	9.5	14.7	17.8	19.7	20.9	21.6	22.0	22.3	22.4	22.5
	10	2.0	10.5	15.7	18.8	20.7	21.9	22.6	23.0	23.3	23.4	23.5
	15	3.0	11.5	16.7	19.8	21.7	22.9	23.6	24.0	24.3	24.4	24.5
	20	3.9	12.5	17.7	20.8	22.7	23.9	24.6	25.0	25.2	25.4	25.5
	25	4.9	13.5	18.6	21.8	23.7	24.8	25.5	26.0	26.2	26.4	26.5
	30	5.9	14.5	19.6	22.8	24.7	25.8	26.5	27.0	27.2	27.4	27.5
	35	6.9	15.4	20.6	23.8	25.7	26.8	27.5	27.9	28.2	28.4	28.5
	40	7.9	16.4	21.6	24.7	26.6	27.8	28.5	28.9	29.2	29.3	29.4
	45	8.9	17.4	22.6	25.7	27.6	28.8	29.5	29.9	30.2	30.3	30.4
	50	9.9	18.4	23.6	26.7	28.6	29.8	30.5	30.9	31.2	31.3	31.4