On the Likelihood of Cyclic Comparisons

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Abstract
We investigate the procedure of "random sampling" where the alternatives are random variables. When comparing any two alternatives, the decision maker samples each of the alternatives once and ranks them according to the comparison between the two realizations. Our main result is that when applied to three alternatives, the procedure yields a cycle with a probability bounded above by $\frac{8}{27}$. Bounds are also obtained for other related procedures.

Keywords: Transitivity, preference formation, the paradox of nontransitive dice

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1. **Introduction**

An experimenter would like to prove that people hold transitive preferences. He asks a psychologist, who thinks otherwise, to suggest 10 triples of lotteries that in his view are likely to lead to cycles. He requires that no two lotteries in the same triple have a common outcome and for simplicity he also requires that each lottery has three outcomes at most. The psychologist provides the experimenter with ten triple of lotteries $\{A_i, B_i, C_i\}_{i=1,\ldots,10}$. Each of the subjects is asked to make the thirty binary choices, three for each triple. A person is said to reveal a cycle in triple i if his choices from $\{A_i, B_i\}$, $\{B_i, C_i\}$, $\{A_i, C_i\}$ are either A_i, B_i, and C_i, or B_i, C_i and A_i. For each subject, the experimenter counts the number of cycles (out of a possible ten), and reports the following results:

# of cycles	0	1	2	3	4	5	6	7	8	9	10
% of subjects	73	23	3	1	0	0	0	0	0	0	0

The experimenter claims that the data can be nicely explained by a theory according to which the decision maker activates a transitive preference relation and there is a 3% chance that he makes a mistake when making a choice. We show that results which can be explained in this way are also consistent with simple procedures that are not based on the existence of well-defined preferences.

The paper focuses on some variations of a nondeterministic procedure of preference formation which we call *Random Sampling procedure*: When comparing two lotteries, the decision maker samples once from each lottery and ranks them according to their realizations. New samples are used for each of the three comparisons. This procedure is related to the S-1 procedure proposed in Osborne and Rubinstein (1998) but is different from Block and Marschak (1960)'s *Random Ordering procedure*: The decision maker has in mind a set of orderings and when comparing any two alternatives, he randomly samples one of the orderings and ranks the two alternatives according to that ordering. For a recent discussion of how the random ordering procedure can explain data which exhibits intransitivity, see Regenwetter, Jason, and Davis-Stober (2011).

Our main result offers a bound on the probability that the random sampling yields a cycle and compares this bound with that of the random ordering procedure. We then discuss the case where at each stage the decision maker recalls the samples of the previous stage. In such a scenario the order of the comparisons might matter. We discuss the bounds on the probability of a cycle assuming that the order of the three comparisons is determined by an agent who wishes to reduce the probability of a cycle, either because he wants to prove that people are rational or because he wants the agent to make a choice and a cycle makes the choice more difficult. We conclude with a brief discussion of the possible interpretations of the results.

2. Random Sampling

The main procedure we discuss in this paper is *random sampling*: To compare two random variables the decision maker draws a fresh sample from each and ranks them according to the sampled values.

Throughout the paper, all triples of random variables have finite and disjoint supports. Denote by $s(A)$ the supprt of the lottery A and by $\Pr(A > B)$ the probability that the realization of A is higher than the realization of B. By the disjoint supports assumption, $\Pr(A > B) + \Pr(B > A) = 1$. Let $\Pi(A,B,C)$ be the probability of a cycle being created by the decision maker's procedure. Applied to the random sampling procedure we have: $\Pi(A,B,C) = \Pr(A > B)\Pr(B > C)\Pr(C > A) + \Pr(A > C)\Pr(C > B)\Pr(B > A)$.

Claim 1: **The maximal probability that the procedure of random sampling yields a cycle is $\frac{8}{27}$.**

Proof: Consider the three random variables presented in the following table:

value	A	B	C
4	$\frac{1}{3}$		
3		$\frac{2}{3}$	
2			1
1	$\frac{2}{3}$		
0		$\frac{1}{3}$	

In this case, $\Pr(A > B) = \frac{5}{9}$, $\Pr(B > C) = \frac{2}{3}$, $\Pr(C > A) = \frac{2}{3}$ and the probability of a cycle $\Pi(A,B,C) = \frac{20}{81} + \frac{4}{81} = \frac{8}{27}$.

In order to prove that this is the upper bound, let $x_1 > x_2 >\ldots> x_n$ be the values in the supports of the three random variables A,B and C. Denote by $X_i \in \{A,B,C\}$ the random variable that contains x_i in its support. Let $\pi_i = \Pr(X_i = x_i) > 0$.

First, we assume without loss of generality that for all i, $X_i \neq X_{i+1}$; otherwise, if $X_i = X_{i+1} = A$, let A' be the random variable which differs from A by $\Pr(A' = x_i) = \pi_i + \pi_{i+1}$ and $\Pr(A' = x_{i+1}) = 0$. Then, $\Pi(A',B,C) = \Pi(A,B,C)$.

Next, assume that for some i, $X_i = X_{i+2} \neq X_{i+1}$ (without loss of generality $X_i = A$ and $X_{i+1} = B$). Then we can (weakly) increase the probability of a cycle by replacing A with A_ε, a random variable which differs from A by either moving a probability mass $\varepsilon > 0$ from x_i to x_{i+2} or from x_{i+2} to x_i. Clearly, $\Pr(C > A_\varepsilon) = \Pr(C > A)$ and $\Pr(A_\varepsilon > B)$ is linear in ε.

Since

$$\Pi(A_\varepsilon, B, C) = \Pr(A_\varepsilon > B)[\Pr(B > C)\Pr(C > A)] + (1 - \Pr(A_\varepsilon > B))\Pr(A > C)\Pr(C > B)$$

shifting probability mass from x_{i+2} to x_i or the other way around (according to the sign of $\Pr(B > C)\Pr(C > A) - \Pr(A > C)\Pr(C > B)$) will (weakly) increase the probability of a cycle.

Thus, without loss of generality we can assume that the sequence $\{X_i\}$ is of the form $A, B, C, \ldots, A, B, C, \ldots$ ending with $X_{n-2} = A$, $X_{n-1} = B$ and $X_n = C$.

Next we show that if the three random variables (A, B, C) maximize Π and if $n > 6$, then there is a triple of random variables that maximizes Π with less than n values in their joint supports. First note that:

$$\Pi(A, B, C) = \Pr(C > A)[\Pr(B > C)\Pr(A > B) - \Pr(B > A)\Pr(C > B)] + \Pr(B > A)\Pr(C > B)$$

Changing C does not affect $\Pr(B > A)$. Consider the set of all C' with a support that is a subset of C such that $\Pr(B > C') = \Pr(B > C)$. For all such C', denote by γ_i the probability that C' yields the outcome x_i. This is the set of all vectors $(\gamma_i)_{x_i \in s(C)}$ such that $\gamma_i \geq 0$ for all i and the following two linear equations hold:

$$\sum_{x_i \in s(C)} \gamma_i = 1$$

$$\sum_{x_i \in s(C)} \gamma_i \times \sum_{j<i \text{ and } x_j \in s(B)} \pi_j = \Pr(B > C)$$

Since $n > 6$ and $X_n = C$, there are at least $m \geq 3$ points in the support of C. The set C' is therefore non empty and is given by the intersection of R^m_{++} and the above two $m-1$ dimensional hyperplanes. The two hyperplanes intersect at C, thus the set is the intersection of R^m_{++} and a linear space of dimension $m - 2 > 0$.

Replacing C with C' will increase the probability of a cycle if $\Pr(B > C)\Pr(A > B) - \Pr(B > A)\Pr(C > B)$ and $\Pr(C' > A) - \Pr(C > A)$ have the same sign. The expression

$$\Pr(C', A) = \sum_{x_i \in s(C)} \gamma_i \times \sum_{j>i \text{ and } x_j \in s(A)} \pi_j$$

is a linear function in $(\gamma_i)_{x_i \in s(C)}$. Therefore, we can (weakly) increase $\Pi(C', A)$ by moving in some direction until we reach the boundary where $\gamma_i = 0$ for some x_i in the support of C.

We can therefore narrow our attention to the sequence of variables $(X_i)_{i=1\ldots6}$ which is of the form A, B, C, A, B, C. Denote by α, b, γ the probabilities that the variables A , B, and C

obtain the highest prize in their supports. Then,

$$\Pi(A,B,C) = (1-b+\alpha b)(1-\gamma+b\gamma)(\gamma-\alpha\gamma)+(b-\alpha b)(\gamma-b\gamma)(1-\gamma+\alpha\gamma) =$$
$$= \gamma^2(1-\alpha)(b-1)+\gamma(1-\alpha)(\alpha b-b^2+1)$$

Assuming that both $1 > \alpha$ and $\beta > 0$, the last expression is strictly increasing in γ within the interval $[0,1]$. Thus, it attains its maximum at $\gamma = 1$. We conclude that in the optimum, one of the three variables must be degenerate and without loss of generality the sequence $(X_i)_{i=1...5} = (B,C,A,B,C)$. Then,

$$\Pi = \gamma^2(\beta-1)+\gamma(-\beta^2+1) = \gamma^2\beta-\gamma^2-\gamma\beta^2+\gamma$$

This expression has a unique maximum point at $\beta = \frac{1}{3}$ and $\gamma = \frac{2}{3}$ and a maximization value of $\Pi = \frac{8}{27}$. ■

In Claim 1 we obtained the upper bound on the probability that the procedure of random realizations yields one of the two possible cycles $A \succ C \succ B \succ A$ or $A \succ B \succ C \succ A$. Claim 2 identifies the highest probability that the procedure yields a particular cycle.

Claim 2: **The maximal probability that the procedure of random sampling yields a particular cycle is** $\frac{1}{4}$.

Proof: Let A to be the random variable that receives the values 4 or 1 with equal probabilities. Let $B \equiv 3$ and let $C \equiv 2$. Then, $\Pr(A > B)\Pr(B > C)\Pr(C > A) = \frac{1}{4}$.

Now, let A, B and C be random variables with $n > 6$ values in their joint support. We show that there is another triple of random variables that yields the cycle $A \succ B \succ C \succ A$ with at least as high a probability and with less than n values in their joint support.

As in the proof of Claim 1 we can easily reduce n if $X_i = X_{i+1}$. If $X_i = A$, then $X_{i+1} = B$ (and similarly if $X_i = B$, then $X_{i+1} = C$ and if $X_i = C$, then $X_{i+1} = A$) since if $X_{i+1} = C$ we can increase $\Pr(C > A)$ without affecting $\Pr(A > B)$ and $\Pr(B > C)$ by shifting the probability mass in A from x_i to x_{i+1} and shifting the probability mass in C from x_{i+1} to x_i.

Without loss of generality, let $X_n = C$. As in the proof of Claim 1, we can modify C to C' such that $\Pr(B > C') = \Pr(B > C)$ and increase (at least weakly) $\Pr(C > A)$ until we reach the boundary where probability 0 is assigned to one of the outcomes in the support of C. Thus, we can assume that there is a triple which maximizes the probability of the cycle with $n \leq 6$ points in their joint support. Let α, β, and γ be the the probabilities that the variables

$X_1 = A$, $X_2 = B$, and $X_3 = C$ obtain the highest prize in their supports. Using the inequality of geometrical and arithmetic averages we obtain that $\Pr(A > B)\Pr(B > C)\Pr(C > A) =$

$$(1-\beta+\alpha\beta)(1-\gamma+\beta\gamma)(\gamma(1-\alpha)) = [(1-\beta+\alpha\beta)\gamma][(1-\gamma+\beta\gamma)(1-\alpha)] \leq$$
$$\left(\frac{(1-\beta+\alpha\beta)\gamma+(1-\gamma+\beta\gamma)(1-\alpha)}{2}\right)^2 = \frac{(1-\alpha+\alpha\gamma)^2}{4} \leq \frac{1}{4}$$

■

Comments: (a) The problem we dealt with in this section is related to the so-called "paradox of nontransitive dice" (see Gardner (1970) who credits it to the statistician Bradley Efrom). This "paradox" involves three independent random variables: A, B and C, where $Pr(A > B)$, $Pr(B > C)$ and $Pr(C > A)$ all exceed 0.5. Savage (1994) further proved that $max_{A,B,C} \min\{\Pr(A > B), \Pr(B > C), \Pr(C > A)\} = (\sqrt{5} - 1)/2$.

(b) It follows from Claims 1 and 2 that for every three distributions F,G and H with a bounded domain and which do not have an atom in the same point:
$\int FdG \int GdH \int HdF + \int FdH \int HdG \int GdF \leq \frac{8}{27}$ and $\int FdG \int GdH \int HdF \leq \frac{1}{4}$.

(c) When a decision maker applies the ordering sample procedure to a set of size n, the maximum probability that his ranking is acyclic goes to zero as the number of alternatives increases to infinity. To see it consider n random variables which are uniform on the interval $[0, 1]$ (and obviously could be approximated by random variables with finite and disjoint supports). For any two of these random variables, the probability that the realization of one is higher than of the other is $\frac{1}{2}$. By Moon and Moser (1962), the probability that the realized tournament is irreducible (i.e., there are no two non-empty disjoint sets such that every node in one set "beats" every node in the other) goes to 1 as $n \to \infty$. By Moon (1966), a tournament with n nodes has a cycle of length n (and therefore is not acycilcal) if and only if it is irreducible. Thus, the probability that the decision maker's comparisons of n uniform random variables yields a cycle of size n goes to 1 as $n \to \infty$.

3. The Random Ordering Procedure

In the random ordering procedure (Block and Marschak (1960)) the decision maker is characterized by π, a probability measure over the six orderings of the three alternatives A, B, and C. When comparing any pair of alternatives, the decision maker draws an ordering that will determine his ranking of these alternatives. Thus, he might apply different orderings in ranking two different pairs of alternatives. In this section we show that the bounds we obtained in the previous section are lower than the bounds on the probability of a cycle in the random ordering procedure.

Claim 3: **The maximal probability that the random ordering procedure yields a cycle is** $\frac{1}{3}$.

Proof: Consider π to be a probability measure on the orderings that assigns equal probabilities to the three orderings $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$ and $C \succ_3 A \succ_3 B$. Then, $\Pr(A \succ B) = \Pr(B \succ C) = \Pr(C \succ A) = \frac{2}{3}$ and the probability of a cycle is $\frac{8}{27} + \frac{1}{27} = \frac{1}{3}$.

To see that $\frac{1}{3}$ is indeed the bound, note that by the inequality of arithmetic and geometric means:

$$\Pi(A,B,C) = \Pr(A \succ B)\Pr(B \succ C)\Pr(C \succ A) + \Pr(A \succ C)\Pr(C \succ B)\Pr(B \succ A) \leq$$
$$[\Pr(A \succ B) + \Pr(B \succ C) + \Pr(C \succ A)]^3/27 + [\Pr(A \succ C) + \Pr(C \succ B) + \Pr(B \succ A)]^3/27$$

Since every ordering must satisfy at least one and at most two of $A \succ B$, $B \succ C$ and $C \succ A$, we obtain: $1 \leq [\Pr(A \succ B) + \Pr(B \succ C) + \Pr(C \succ A)] \leq 2$. The function $x^3 + (3-x)^3$ is convex in the interval $[1,2]$ and obtains its maximum at $x = 1$ and $x = 2$. Thus $\Pi(A,B,C) \leq \frac{1}{27} + \frac{8}{27} = \frac{1}{3}$. ■

Claim 4: **The maximal probability that the procedure of random ordering yields a particular cycle is** $\frac{8}{27}$.

Proof : The above example attains the bound. To prove that the bound is $\frac{8}{27}$, note that $\Pr(A \succ B)\Pr(B \succ C)\Pr(C \succ A) \leq [\Pr(A \succ B) + \Pr(B \succ C) + \Pr(C \succ A)]^3/27$. The function x^3 in the interval $[1,2]$ attains the maximum at 2 and thus the inequality follows. ■

Note that the above example is the only one in which the probability of a cycle is $\frac{8}{27}$. To see this, count the six orderings: $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$, $C \succ_3 A \succ_3 B$, $A \succ_4 C \succ_4 B$, and $B \succ_5 A \succ_5 C$, $C \succ_6 B \succ_6 A$. Denote by π_i the probability of $\succ_i$. Then, $\Pr(A \succ B)\Pr(B \succ C)\Pr(C \succ A) = (\pi_1 + \pi_3 + \pi_4)(\pi_1 + \pi_2 + \pi_5)(\pi_2 + \pi_3 + \pi_6)$. The maximum is attained only when $\pi_4 = \pi_5 = \pi_6 = 0$ and $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$.

4. The Random Sampling Procedure with Partial Recall

In the procedure discussed in Section 2 each comparison is done independently of the other two comparisons. A decision maker who compares first A and B and moves to compare B and C does not recall the previous value of B. Thus the existence of a cycle did not depend on the order by which the comparisons were done. In contrast, in this section we assume that the decision maker carries out the comparisons sequentially in three stages and at each stage he remembers the realizations of the previous stage, but not those of two stages earlier. In other words, he applies the following procedure:

Random Sampling procedure with partial recall:

When applied to the sequence of three lotteries (A, B, C):

(i) Compare A and B by sampling each once.

(ii) Compare B and C by sampling C once and compare the outcome with that of the previous-stage sampling of B.

(iii) Compare C and A by sampling A again and compare the outcome with that of the previous-stage sampling of C.

The probability that the procedure yields a cycle is:
$\Pi(A, B, C) = \Pr(A_1 > B > C > A_2) + \Pr(A_2 > C > B > A_1)$ where A_1 and A_2 are copies of A, i.e., they are i.i.d and distributed like A. Note that $\Pi(A, B, C)$ might differ from $\Pi(B, A, C)$ but that $\Pi(A, B, C) = \Pi(A, C, B)$.

Claim 5: **The maximal probability that the random sampling procedure with partial recall yields a cycle is $\frac{1}{4}$.**

Proof: Even though we use here a different procedure, the probability of a cycle is $\frac{1}{4}$ for the same triple of variables used at the beginning of the proof of Claim 2. To see that $\frac{1}{4}$ is the bound, denote by Π_b the probability of a cycle given that the value of B is b:

$$\Pi_b = \Pr(A_1 > b > C > A_2) + \Pr(A_2 > C > b > A_1) \leq$$
$$\Pr(A_1 > b > C)\Pr(b > A_2) + \Pr(A_2 > b)\Pr(C > b > A_1) =$$
$$\Pr(A > b)\Pr(b > C)\Pr(b > A) + \Pr(A > b)\Pr(C > b)\Pr(b > A) =$$
$$\Pr(b > A)\Pr(A > b)[\Pr(b > C) + \Pr(C > b)] \leq \frac{1}{4}$$

Since $\Pi_b \leq \frac{1}{4}$ for every possible realization of B, $\Pi(A, B, C) \leq \frac{1}{4}$ as well. ■

Imagine now that the order in which the alternatives are presented to the decision maker is determined by a "master of ceremonies" (MC) who wants the decision maker having a clear ordering of the alternatives. Let $V(A,B,C) = \min\{\Pi(A,B,C),\Pi(B,C,A),\Pi(C,A,B)\}$ be the probability of a cycle given that the MC chooses the order of the comparisons of the three variables A, B and C in order to minimize the probability of the cycle. In the example used in the above proof $\Pi(A,B,C) = \frac{1}{4}$ but $\Pi(B,C,A) = 0$ and thus $V(A,B,C) = 0$. On the other hand, if A,B,C are uniformly distributed over $[0,1]$ then $V(A,B,C) = \Pi(A,B,C) = \frac{1}{12}$ (each ordering of four identical random variables has the same probability of $\frac{1}{24}$ and therefore $\Pr(A_1 > B > C > A_2) + \Pr(A_2 > C > B > A_1) = \frac{1}{12}$). We succeeded to find the bound on V for only a limited family of random variables.

Claim 6: **The maximal $V(A,B,C)$ for three binary random variables is $\frac{1}{16}$.**

Proof: First note that for the following three variables $V(A,B,C) = \frac{1}{16}$.

value	A	B	C
5	$\frac{1}{2}$		
4		$\frac{1}{2}$	
3			$\frac{1}{2}$
2	$\frac{1}{2}$		
1		$\frac{1}{2}$	
0			$\frac{1}{2}$

If the three variables are such that between the two values of one of the lotteries, say A, there are no values of another lottery, say C, then $\Pi(A,B,C) = 0$. Thus, we need to consider only the case in which the values of the three lotteries can be ordered as A,B,C,A,B,C. Denote by α,β,γ the probabilities of the highest value of each of the three lotteries A,B,C respectively. Then, $\Pi(A,B,C) = \alpha\beta\gamma(1-\alpha)$, $\Pi(B,A,C) = \beta\gamma(1-\alpha)(1-\beta)$ and $\Pi(C,A,B) = \gamma(1-\alpha)(1-\beta)(1-\gamma)$.

Note that by the continuity of Π, at a maximum point of $V(A,B,C)$ it must be that two of the terms $\Pi(A,B,C)$, $\Pi(B,C,A)$, $\Pi(C,A,B)$ are equal and are weakly less than the third. If $\Pi(B,A,C)$ is minimal then $\Pi(B,C,A) = \beta\gamma(1-\alpha)(1-\beta) = \min\{\alpha\beta\gamma(1-\alpha),\ \gamma(1-\alpha)(1-\beta)(1-\gamma)\}$. It follows that $1-\beta \leq \alpha$ and $\beta \leq 1-\gamma$ and thus, $\Pi(B,C,A) \leq \beta(1-\beta)(1-\alpha)\alpha \leq \frac{1}{16}$. If $\Pi(B,C,A)$ is not minimal then at the maximum point of V, $\beta\gamma(1-\alpha)(1-\beta) > \alpha\beta\gamma(1-\alpha) = \gamma(1-\alpha)(1-\beta)(1-\gamma)$, hence $1-\alpha > \beta$ and $\beta > 1-\gamma$.

The maximum with respect to β of the function $\alpha\beta\gamma(1-\alpha)$ (which is linear in β) given the linear constraints $\alpha\beta = (1-\beta)(1-\gamma)$ and $(1-\alpha) \geq \beta \geq (1-\gamma)$ must be obtained where either $\beta = 1-\alpha$ or $\beta = 1-\gamma$. In the former case $\alpha\beta\gamma(1-\alpha) = (1-\beta)(1-\gamma)\gamma\beta \leq \frac{1}{16}$ while in the latter $\alpha\beta\gamma(1-\alpha) = \alpha(1-\gamma)\gamma(1-\alpha) \leq \frac{1}{16}$. ■

When the support of each of the random variables has at most three points, numerical methods prove that the maximum of $V(A,B,C)$ is roughly 0.0910 and is attained near the triple of random variables:

value	A	B	C
6	0.19		
5		0.37	
4			0.63
3		0.63	
2	0.62		
1			0.37
0	0.19		

The probability of a cycle can be reduced even further if the MC can choose the first couple of alternatives and only after he observes their realizations he determines which of the two alternatives will be compared with the third one at the second stage. Using numerical methods we conclude that for any triple of lotteries with no more than three outcomes the MC can present the comparisons such that the probability of a cycle is not greater than $\frac{1}{32}$. Moreover, if each lottery has at most two outcomes cycles can be eliminated:

Claim 7: **Let** A, B, C **be three binary random variables**. **If the decision maker follows the Random Sampling procedure with partial recall then the MC who observes the realizations can arrange the order of comparisons so that no cycles emerge**.

Proof: Suppose that between the outcomes of one lottery, say A, there are no outcomes of another lottery, say B. Then the MC will ask the decision maker to compare A and B and then B and C. Assume $B \succ A$. If $B \succ C$ then there is no cycle. If $C \succ B$ then the fresh realization of C is higher than both values of A and at the third stage $C \succ A$. The case that

$A \succ B$ is similar.

Suppose that the outcomes are ordered $a_1 > b_1 > c_1 > a_2 > b_2 > c_2$. The MC's instructions could be the following:

Start by comparing A and B. Then,

1. If the realization of A is a_1 continue with comparing A and C. Whatever is the realization of C, $A \succ C$ and hence no cycle.

2. If the realizations are a_2 and b_1 ($B \succ A$) then continue by comparing B and C. Whatever is the realization of C, $B \succ C$, hence no cycle.

3. If the realizations are a_2 and b_2 then $A \succ B$. Proceed to compare B and C. If the realization is c_1 then $C \succ B$, hence no cycle. If the realization is c_2 then $B \succ C$ and when A and C are compared (using c_2) then $A \succ C$ and there is no cycle. ■

5. Conclusion

The results of this paper are relevant for two issues:

The first is related to experimental choice theory. Researchers in the field should be aware of the fact that procedures of choice which are not based on transitive preferences might still yield transitivity among three alternatives with fairly high probability. Low frequency of cycles in responses of subjects to comparisons of three alternatives could be not only an outcome of some error rate. It might be consistent with the subjects using procedure of random choice and some manipulation by researchers who wish to show that people are rational. For example, the hypothetical data presented in the introduction could be explained by the assumption that the decision maker makes a mistake in each comparison with probability of 3%. But, it can be also explained by the decision maker following a random sampling procedure with partial recall and the experimenter manipulating the order by which he requires the subjects to make the comparisons.

The second issue involves the money pump argument which is brought in the literature as a normative support for the transitivity assumption (Yaari (1998)). According to this argument intransitivity of preferences exposes a decision maker to manipulation. If a decision maker has in mind a cycle $A \succ C \succ B \succ A$ where A, B and C are three objects, then a manipulator could promise the decision maker an object A and then seduce him into exchanging it for B and so on in an unending chain of exchanges of A, B and C for B, C and A, respectively. Each time, the manipulator receives some fee from the decision maker, thus eventually bankrupting him (especially if the cycle is embedded into an automatic trading program). The manipulator can execute the money pump without actually possessing any of the three objects. If, however, the individual’s preference are random, then eventually the manipulator may need to fulfill the promise and actually buy the object. The profitability of the money pump will depend on the size of the fee the manipulator can charge and the probability that the decision maker’s sampling method will yield a cycle. By claims 2 and 5, decision makers who apply the procedures of random sampling and random ordering will complete a particular cycle with probability not larger than $\frac{1}{4}$. This low probability may mean that the above money pump argument will not necessarily hold since the manipulator will need soon to supply the object he promised before collecting sufficient fees from the decision maker to cover its cost.

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