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Essays on Microeconomics with Incomplete Information

by

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Abstract

My dissertation devotes to the understanding of people's interactions under uncertainty.. It contains four essays on Microeconomics with Incomplete Information.[1]

Chapter 1 focuses on the existence of rational bubbles in an Allen-Morris-Postlewaite (1993) setting, and finds positive and negative results for bubbles in an asset market featuring rational expectations equilibrium. An expected bubble is said to exist if it is mutual knowledge that the price of the asset is higher than the expected dividend. Similarly we call it a strong bubble if everyone knows that the price is higher than the maximum possible dividend. Substituting common knowledge for mutual knowledge, I develop the new concepts of a common expected bubble and a common strong bubble. In a simple finite horizon model with asymmetric information and short sales constraints, I show that the following results hold for any finite number of agents. First, under the implicit assumption of perfect memory, common strong bubbles never exist in any rational expectations equilibrium. Second, it is possible to have one that is both a strong bubble and a common expected bubble in a rational expectations equilibrium. Based on these results, this paper, as well as Conlon (2004) and many others, provides a partial answer to the question: What properties do rational bubbles have in a rational expectations equilibrium?

In Chapter 2, I study the relationship between information improvement and welfare outcomes in a finite-player finite-state model with incomplete information. In a context

[1] Chapter 3 is based on joint work with John Conlon (University of Mississippi).

of strategic interactions, it is possible that people may prefer to be ignorant rather than knowledgeable. Three simple examples are studied carefully in order to provide economic insight for this observation: if players were allowed to (not forced to) forget at no cost, they might have incentives to do so in equilibrium, and their expected payoff could actually be improved. In a general setting where players simultaneously choose whether to forget or not before the state of the world is realized, I show that players' actions would reveal additional information and that their preferences must be negatively correlated, for forgetfulness to be part of a possible equilibrium strategy. This finding indicates that in a world of incomplete information, people may not be made better off by obtaining more information, and they may even have incentive to be forgetful. These results will have important applications in policy design.

Many economic models of rational bubbles are not very robust to perturbations. The existence of bubbles in these models requires strong conditions to be satisfied. In Chapter 3, we first study the bubble examples in the first Chapter and show that those bubbles are robust to both *strongly symmetric perturbations* in beliefs and *very symmetric perturbations* in dividends, but not robust to general perturbations. Then we construct a new three-period two-agent robust bubble example where small variations in parameters do not eliminate the bubble equilibria. The idea is that assuming continuum of states can lead to a robust bubble equilibrium where each bad type of the seller pools with some good type of the seller. This provides a new answer to the question: How robust can rational bubbles be in a finite horizon model?

Morris, Shin and Postlewaite (1993) show an upper bound of asset prices in Rational Expectations Equilibrium. Chapter 4 is a note that strengthens their result by providing a tighter upper bound and hence offers a better answer to the question: How large can a bubble be in equilibrium?

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Chapter 1 Strong Bubbles and Common Expected Bubbles in a Finite Horizon Model

1.1 Introduction

Bubbles exist in many markets, not only those where assets have fundamental values hard to determine or observe (stocks, for instance), but also some where assets have fundamental values known to be less than their prices (fiat money, for instance). How can bubbles be explained and what must be true for the existence of bubbles? Though claiming that most bubbles are irrational is much easier than interpreting bubbles in a rational way, economists have made and are still making efforts to deal with the latter.

Among the huge literature on the existence of bubbles, one strand has developed models based on the existence of some irrational agents, often called noise traders in the literature (see, for example, De Long, Shleifer, Summers and Waldmann (1990), Abreu and Brunnermeier (2003), and Zurita (2004)). Papers in this strand interpret bubbles by the interaction between the rational and the irrational.[2]

Another strand of the literature, has tried to model bubbles under the assumption that all agents are rational.[3] In such settings, an asset bubble can be explained either by the assumption of an infinite horizon or by the infinite presence of new agents (see Tirole (1982) and Tirole (1985) for example). However, in order to interpret the existence of a

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[2] Though the rational agents have incentive to take advantage of the irrational, it is possible that noise traders may actually earn a higher expected return than rational investors do. For details, see De Long, Shleifer, et al. (1990).

[3] In fact it is assumed that the rationality of the agents is common knowledge in most papers of this strand. Under the assumption of rational expectations, these two are equivalent.

finite horizon bubble[4] in a rational expectations equilibrium with a finite number of agents, either a change of standard assumptions (for instance, symmetric information) or the introduction of specific requirements (for instance, short sales constraints) has to be made. Thus the question becomes: What is the minimum requirement for the existence of such a rational bubble?

By the well-known no-trade theorem of Milgrom and Stokey (1982), under the standard setting, if the initial allocation is efficient relative to each agent's belief, then the common knowledge of feasibility of and voluntary participation in trade will give agents no incentive to trade, no matter whether they have private information or not. If there is no trade in a finite horizon economy, there is certainly no bubble. Hence the ex ante inefficiency of the endowment allocation, or the existence of potential gains from trade, is one necessary condition for such a bubble to exist.[5]

Allen, Morris and Postlewaite (1993) (AMP (1993) henceforth) define two types of bubbles–expected bubbles and strong bubbles–in their finite-agent finite-horizon finite-state trade model, and show that private information about the states and short sales constraints for all agents are another two necessary conditions for the existence of strong bubbles. An *expected bubble* is said to exist if it is mutual knowledge that the price of the asset is higher than the expected dividend. They call it a *strong bubble* if everyone knows that the price is higher than the maximum possible dividend. While the concept of expected bubbles provides a starting point for analysis, economists are more interested in

[4] Among all the bubble phenomena, finite horizon bubbles are probably most puzzling.
[5] For a complete proof, see Tirole (1982).

the concept of strong bubbles.

Combining these three together with a fourth requirement that the agents' trade should not be common knowledge, AMP (1993) presented an example of strong bubbles in a rational expectations equilibrium with three agents and three periods.[6] This model captures the "greater fools" dynamic in the sense that because of asymmetric information, agents may hold a worthless asset at a positive price in the first period (hence a strong bubble), in hopes of selling it in the second period to someone else who thinks it may be worth something. In short, a rational bubble can exist in this setting because even though everyone knows that the asset is overpriced, they may still hold it with the belief that others might think that it is valuable.

Given the success of the Allen, Morris and Postlewaite model, economists are somewhat less than satisfied with the last assumption, the one requiring no common knowledge of trades, since many bubbles do exist in reality with the public information of agents' actions. Conlon (2004) constructed a strong bubble example in a similar setting[7] where there are only two agents. Since trades are automatically common knowledge for the two-agent case, this result has questioned the necessity of the assumption of no common knowledge of trades for the existence of a finite horizon bubble in a rational expectations equilibrium. Another contribution of Conlon (2004) is that the bubble in the model is not

6 It has been shown in that paper that there is no expected bubble in the last two periods under their framework, which will be described in Section 2; hence the minimum number of periods for the existence of a bubble is 3.

7 The setting of Conlon (2004) differs from AMP (1993) in the sense that agents' information structures are determined both by the private signals they receive at the beginning of period 1 and by the public signals they receive at the beginning of every period. The information structures are chosen so that prices reveal no additional information.

only strong but also robust to nth order knowledge, that is (all agents know that)^n the price is higher than any possible dividend agents will receive.

Based on the fact of the existence of nth order bubbles, one may naturally ask whether a bubble can be robust to common knowledge. In this paper, by requiring common knowledge instead of mutual knowledge, I develop two new concepts of bubbles: a common expected bubble and a common strong bubble. A *common expected bubble* is said to exist if it is common knowledge that the price of the asset is higher than the expected dividend. A *common strong bubble* is said to exist if it is common knowledge that the price of the asset is higher than the maximum possible dividend. The concept of the common strong bubble is so "strong" that it can be shown never to exist in any rational expectations equilibrium under the standard assumption of perfect memory. However, I am able to show that within the same framework as the AMP (1993) model but with common knowledge of trades, a strong bubble can exist in the case of two agents, and this bubble can still exist even when it is common knowledge that the price is higher than the expected dividend agents will receive (hence a common expected bubble). Moreover, such a bubble, both a strong bubble and a common expected bubble, is robust to one class of symmetric perturbations in beliefs and another class of symmetric perturbations in dividends, and can exist for any finite number of agents.[8] This positive result itself, on the one hand, weakens the assumptions of the models of bubbles by reducing the four necessary conditions to three, and hence improves these models' applicability and powers in interpretation. On the

[8] I assume that each agent is distinguished from the others in the sense that either their beliefs are heterogeneous or their information structures are different, or both. Otherwise, this result would hold trivially since each agent can be "divided" according to endowments into any finite number of subagents.

other hand, the surprising result of the existence of common expected bubbles is somewhat counterintuitive but captures the idea that agents do not rush in face of bubbles since, given the common knowledge of the heterogeneous beliefs and the information structures, they believe that they can take advantage of it in a later period. Another contribution of this paper lies in the understanding of the structural characteristics of models of bubbles: I show that a couple of structural conditions must be satisfied for a strong bubble to exist in a rational expectations equilibrium in a 2-agent symmetric economy. One of them is that the minimum number of states is 8.

The next section of the paper introduces the basic framework following AMP (1993), gives four concepts of bubbles, and shows the nonexistence of common strong bubbles in any rational expectations equilibrium. Section 3 presents a simple example of a rational bubble with two agents; the bubble is both a strong bubble and a common expected bubble. Section 4 characterizes necessary conditions about the number of states and the structure of information partitions for the existence of strong bubbles and common expected bubbles. Section 5 shows the general results for any finite number of agents. Section 6 offers another example where a second order strong bubble and a common expected bubble can coexist in equilibrium. Section 7 provides concluding remarks and directions for further study.

1.2 The Model

1.2.1 Basic Setup

The same framework is established here as in AMP (1993), except that the requirement that the trades should not be common knowledge is removed.

In the pure exchange economy under study, there are I (≥ 2) risk neutral[9] agents ($i = 1, 2, \cdots, I$), T (≥ 3) periods ($t = 1, 2, \cdots, T$) and N (≥ 2) states of the world represented by $\omega \in \Omega$. Only 2 assets exist in the market: one riskless (money) and the other risky. There is no discount between any two periods. Each share of the risky asset will only pay a state-dependent dividend denoted by $d(\omega)$ at the end of period T.

Agent i is endowed with m_i units of money and e_i shares of the risky asset at the beginning of period 1. In each period t and in each realized state ω, agents can exchange claims on the risky asset at a state-and-period-dependent price $P_t(\omega)$. Agent i's net trade in period t when state ω is realized is denoted by $x_{it}(\omega)$, and we write $x_i = (x_{i1}, x_{i2}, \cdots, x_{iT})$, $x_t = (x_{1t}, x_{2t}, \cdots, x_{It})$, and $x = (x_1, x_2, \cdots, x_I)$. Hence agent i's final consumption in state ω with net trades x_i at price $P(\omega) = (P_1(\omega), P_2(\omega), \cdots, P_T(\omega))$, denoted by $y_i(\omega, P(\omega), x_i)$, is equal to $m_i + e_i P_T(\omega) + \sum_{t=1}^{T} x_{it}(\omega)[P_{t+1}(\omega) - P_t(\omega)]$, where $P_{T+1}(\omega) = d(\omega)$. Let $u_i(\cdot)$ be agent i's utility function. Then agent i's utility in state ω with net trades x_i at price $P(\omega)$, is $u_i(y_i(\omega, P(\omega), x_i))$. For simplicity, assume that $u_i(\cdot)$ is the identity function for all i.

Each agent i has a subjective belief about the probability distribution of the state, denoted by $\pi_i(\omega)$.[10] $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \pi_i(\omega) > 0$.

9 Agents are assumed to be either risk averse or risk neutral in AMP (1993). Here for simplicity, I only consider the case of risk neutrality. All the results will remain valid for the risk averse case as long as the potential gain from trade is high enough.

10 We may either assume same utility function with heterogeneous beliefs, or assume common prior with different utility functions, in order to give agents an incentive to trade. Here we adopt the former one and in the next version we may also consider the latter. For other approaches to induce trade, see AMP (1993) for details.

1.2.2 Information Structure

At the beginning of each period t, before observing the current price and making the trade, agent i's information about the state is represented by S_{it}, a partition of the space Ω, and his price–and-trade-refined information is represented by S_{it}^{PX}.[11] We denote by $s_{it}(\omega)$ ($s_{it}^{PX}(\omega)$) the partition member in S_{it} (S_{it}^{PX}) containing the state ω. In other words, $s_{it}(\omega)$ consists of all the possible states agent i believes he might be in when the state ω is realized in period t. For example, $s_{i1}(\omega_1) = \{\omega_1, \omega_2\}$ means that in period 1 agent i believes he might be either in ω_1 or ω_2 when ω_1 is realized.

S_{it}^{PX} is determined by (S_{it}, P_t, x_t) such that

$$\forall \omega \in \Omega, s_{it}^{PX}(\omega) = s_{it}(\omega) \cap \{\omega' | P_{t'}(\omega') = P_{t'}(\omega) \text{ and } x_{t'}(\omega') = x_{t'}(\omega) \ \forall t' \leq t\}.$$

Obviously $\forall i = 1, 2, \cdots, I, \forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, \{\omega\} \subseteq s_{it}^{PX}(\omega) \subseteq s_{it}(\omega)$.

We assume agents have perfect memory so that

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \forall t > t', s_{it}(\omega) \subseteq s_{it'}(\omega).$$

Obviously this implies that

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \forall t > t', s_{it}^{PX}(\omega) \subseteq s_{it'}^{PX}(\omega).$$

It should be noted that when agents make trades to optimize their payoffs, the information they based on is $s_{it}^{PX}(\omega)$ instead of S_{it}, since it is assumed that rational agents

[11] In the AMP (1993) model, they only focus on the price-refined information S_{it}^{P}. In their model it is assumed that the trades are not common knowledge and hence agents cannot get additional information from trades.

should make use of all the information they can obtain. As we will see, the assumption of perfect memory plays an important role in Proposition 1, which we will state at the end of this section.

1.2.3 Rational Expectations Equilibrium

Before we come to the definition of a rational expectations equilibrium, in order to be consistent with the AMP (1993) model, two concepts have to be introduced first.

Definition 1 (Information Feasibility) *Agent i's net trades x_i are information feasible if in each period t, x_{it} is measurable with respect to player i's price–and-trade-refined information, S_{it}^{PX}. Formally, x_i are information feasible if*

$$\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, s_{it}^{PX}(\omega) \subseteq \{\omega' : x_{it}(\omega') = x_{it}(\omega)\}.$$

The last part of the above expression is equivalent to $\forall \omega', \omega'' \in s_{it}^{PX}(\omega), x_{it}(\omega') = x_{it}(\omega'')$, which might capture more intuition than the one used in the definition. Basically, information feasibility rules out the possibility of acting differently given the same information.

Definition 2 (No Short Sales) *Agent i's net trades x_i satisfy no short sales if in each period t and in each state ω agent i's holdings of the risky asset are non-negative. Formally, x_i satisfy no short sales if*

$$\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, e_i + \sum_{s=0}^{t} x_{it}(\omega) \geq 0.$$

As shown in AMP (1993), this no short sales condition is necessary for the existence of a bubble in a rational expectations equilibrium. It should be noted that there is no constraint on the short sales of money.

Denote by $j_t(\omega)$ the join of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$,[12] and by $m_t(\omega)$ the meet of

[12] The join $j_t(\omega)$ of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$ is such that (1) $\forall i = 1, 2, \cdots, I, j_t(\omega) \subseteq s_{it}(\omega)$ and (2) for all $j_t'(\omega)$ satisfying (1), $j_t'(\omega) \subseteq j_t(\omega)$. It is also called the coarsest common refinement.

$s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$.[13]

Now we are ready to give the definition of a Rational Expectations Equilibrium in this pure exchange economy.

Definition 3 (Rational Expectations Equilibrium) $(P, x) \in R_+^{NT} \times R^{INT}$ *is a Rational Expectations Equilibrium if*

(C1) $\forall i = 1, 2, \cdots, I$, x_i *are information feasible and satisfy no short sales. Denote the set of all such* x_i*'s by* $F_i(e_i, P, x_{-i}, S_i)$*, where* $S_i = (S_{i1}, S_{i2}, \cdots, S_{iT})$*;*[14]

(C2) $\forall i = 1, 2, \cdots, I$, $x_i \in \arg\max_{x_i' \in F_i(e_i, P, x_{-i}, S_i)} \sum_{\omega \in \Omega} \pi_i(\omega) u_i(y_i(\omega, P, x_i'))$*;*[15]

(C3) $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, \sum_{i=1}^{I} x_{it}(\omega) = 0$*;*

(C4) $\forall t = 1, 2, \cdots, T, P_t(\cdot)$ *is measurable with respect to* $j_t(\omega)$*. Formally,* $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, j_t(\omega) \subseteq \{\omega' : P_t(\omega') = P_t(\omega)\}$.

Basically, (C1) describes the feasible set of trade for each agent, (C2) says that each agent maximizes his expected utility given his price-and-trade-refined information, (C3) requires that the market should clear in equilibrium, and (C4) implies that all the information contained in price is from the join of the individual information.

1.2.4 Different Concepts of Bubbles

Different definitions of bubbles will lead to different results even within the same framework. As a base line, we use the concept of an expected bubble, defined in AMP (1993). As we will see, the stronger the concept of a bubble become, the harder for it to

13 The meet $m_t(\omega)$ of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$ is such that (1) $\forall i = 1, 2, \cdots, I, s_{it}(\omega) \subseteq m_{it}(\omega)$ and (2) for all $m_t'(\omega)$ satisfying (1), $m_t(\omega) \subseteq m_t'(\omega)$. It is also called the finest common coarsening.

14 Since $\forall x_i \in F_i$, x_i are information feasible, F_i depends on the information structure S_i, the prices P, and other agents' trades x_{-i}. Since x_i satisfy no short sales, F_i depends on the endowment e_i. That's why it is written as $F_i(e_i, P, x_{-i}, S_i)$.

15 Another perhaps more intuitive way to express (C2) is (C2') $\forall i = 1, 2, \cdots, I, x_i \in \arg\max_{x_i' \in F_i(e_i, P, x_{-i}, S_i)} E_i\left[u_i(y_i(\omega, P, x_i')) | S_{i1}^{PX}\right]$. It is easy to see that (C2') is equivalent to (C2).

exist in equilibrium.

Definition 4 (Expected Bubble) *As in AMP (1993), an expected bubble is said to exist in state ω in period t if in state ω it is mutual knowledge that the price of the risky asset in period t is higher than the expected dividend an agent will receive, that is*

$$\forall i = 1, 2, \cdots, I, P_t(\omega) > \frac{1}{\sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega') \, d(\omega').$$

Definition 5 (Strong Bubble) *As in AMP (1993), a strong bubble is said to exist in state ω in period t if in state ω it is mutual knowledge that the price of the risky asset in period t is higher than the maximum possible dividend an agent will receive, that is*

$$\forall i = 1, 2, \cdots, I, \forall \omega' \in s_{it}^{PX}(\omega), P_t(\omega) > d(\omega').$$

As seen from above, the concept of strong bubbles strengthens the concept of expected bubbles in a way that it requires that the asset price be higher than the maximum possible dividend, not just the expected dividend. As will be seen below, another way to strengthen the concept of expected bubbles is to require common knowledge instead of mutual knowledge. This requirement is reasonable since in the real world people's behaviors do not only depend on their own beliefs, but also depend on others' beliefs, others' beliefs on their own beliefs, and so on. Therefore, we might expect to see something different when common knowledge is introduced into the concept of bubbles.

Definition 6 (Common Expected Bubble) *A common expected bubble is said to exist in state ω in period t if in state ω it is common knowledge that the price of the risky asset in period t is higher than the expected dividend an agent will receive, that is*

$$\forall i = 1, 2, \cdots, I, \forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > \frac{1}{\sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'')} \sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'') \, d(\omega'').$$[16]

.

Definition 7 (Common Strong Bubble) *A common strong bubble is said to exist in state ω in period t if in state ω it is common knowledge that the price of the risky asset in period*

[16] $m_t^{PX}(\omega)$ is the meet of $s_{1t}^{PX}(\omega), s_{2t}^{PX}(\omega), \cdots, s_{It}^{PX}(\omega)$.

t is higher than the maximum possible dividend an agent will receive, that is

$$\forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > d(\omega').$$

1.2.5 Nonexistence of Common Strong Bubbles in Equilibrium

Among the 4 definitions above, clearly the common strong bubble is the strongest one. One may wonder if there exists such a bubble in a rational expectations equilibrium. The answer is NO, due to the following proposition. This nonexistence result is actually an immediate implication from Corollary 4.1 in Morris-Postlewaite-Shin (1995). Here we adopt a different approach to proof.

Proposition 1 *Under the perfect memory assumption, $\forall \omega \in \Omega, \forall t = 1, 2, \cdots, T$,it is impossible for a common strong bubble to exist in state ω in period t in any rational expectations equilibrium.*

Proof. Suppose it is possible and $\exists \omega, \exists t$ such that a common strong bubble exists in state ω in period t in a rational expectations equilibrium. Then $m_t^{PX}(\omega)$ is the set of states where there is common knowledge among agents when ω is realized. Thus we have $\forall \omega' \in m_t^{PX}(\omega), P_t(\omega) = P_t(\omega') > d(\omega')$. By the feature of rational expectations equilibrium, there must exist some agent i for whom buying is at least as good as selling, which implies that $P_t(\omega) \leq E_i\left[P_{t+1}(\omega') \mid \omega' \in s_{it}^{PX}(\omega)\right]$. Therefore, $P_t(\omega) \leq \max_i \max_{\omega' \in s_{it}^{PX}(\omega)} P_{t+1}(\omega') \leq \max_{\omega' \in m_t^{PX}(\omega)} P_{t+1}(\omega')$. Since agents have perfect memory, we have $\forall i = 1, 2, \cdots, I, s_{i(t+1)}^{PX}(\omega) \subseteq s_{it}^{PX}(\omega)$, which implies $m_{t+1}^{PX}(\omega) \subseteq m_t^{PX}(\omega)$. By induction we have $P_t(\omega) \leq \max_{\omega' \in m_t^{PX}(\omega)} P_{T+1}(\omega') = \max_{\omega' \in m_t^{PX}(\omega)} d(\omega')$. Thus $\exists \omega^* \in m_t^{PX}(\omega)$ such that $d(\omega^*) \geq P_t(\omega)$, which causes a contradiction. ■

The intuition behind the nonexistence of common strong bubbles is that if it is common knowledge that the price today is higher than the highest dividend agents may receive, then agents might be better off by selling the asset instead of holding it, no matter what kind of heterogeneous beliefs they may have. Since everyone wants to sell, there cannot be a rational expectations equilibrium any more. It is worth noting that the result of Proposition 1 is independent of the assumption of common knowledge of trades. In the case of no common knowledge of trades, the result is still true. The only modification needed is replacing the price–and-trade-refined information by the price-refined information. It is also worth noting that the result of Proposition 1 crucially depends on the perfect memory assumption. If we allow for agents to forget some information they knew before, a common strong bubble may exist in a rational expectations equilibrium. Such a counterexample is presented in Section 6.

Though under the standard assumption of perfect memory there is no common strong bubble in any rational expectations equilibrium, an expected bubble, which is both strong and common expected, can exist in a rational expectations equilibrium of a three-period two-agent economy, as will be shown in the next section.

1.3 A Simple Example: Strong Bubbles and Common Expected Bubbles with Two Agents

1.3.1 Exogenous Setting

AMP (1993) has constructed a strong bubble in a rational expectations equilibrium of a three-period three-agent economy with the assumption of no common knowledge of trades. In this section, I will provide a simple example of the existence of strong bubbles with two agents where trades become automatically common knowledge. Moreover, as will be shown, the bubble in the example will also be robust to common knowledge in the expected sense, hence a common expected bubble.

There are 2 agents (A and B), 3 periods (1, 2, and 3) and 8 states (ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7 and ω_8). Only 2 assets exist in the market: one is money and the other is called a risky asset. Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state is either ω_1 or ω_4, and will pay nothing otherwise, as shown in the table below.

Table 1.1 Dividend Distribution Accross States

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$d(\omega)$	4	0	0	4	0	0	0	0

Each agent is endowed with m_i unit of money and 1 share of the risky asset at the beginning of period 1. Agents can trade in each of period 1, 2, and 3. In period 3, after the trade is made, the dividend is realized, and then the consumption takes place.

Keeping in mind that the asymmetric information is the key to generating strong bubbles, we achieve this goal by giving agents different information structures. Remind that agent i's ($i = A, B$) information about the state in period t ($t = 1, 2, 3$) is represented by S_{it}, a partition of the space Ω. The specific structures of S_{it}'s are given by

$$
\begin{aligned}
S_{A1} &= \{\{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_8\},\{\omega_6,\omega_7\}\} \\
S_{B1} &= \{\{\omega_1,\omega_2,\omega_4,\omega_5,\omega_6,\omega_8\},\{\omega_3,\omega_7\}\} \\
S_{A2} &= \{\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5\},\{\omega_6,\omega_7\},\{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4,\omega_5,\omega_6\},\{\omega_1,\omega_2\},\{\omega_3,\omega_7\},\{\omega_8\}\} \\
S_{A3} &= S_{B3} = \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\},\{\omega_6\},\{\omega_7\},\{\omega_8\}\}.
\end{aligned}
$$

At first glance, this particular structure of information may seem complicated, but as our analysis goes on, the reason why it is set in this form will become clear. So far, there are at least three observations. First, in period 3, each agent is perfectly informed of what the realized state is and hence there is no asymmetric information then. Second, in period 2, agent A receives more information only when he observed $\{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_8\}$ in period 1, and agent B receives more information only when he observed $\{\omega_1,\omega_2,\omega_4,\omega_5,\omega_6,\omega_8\}$ in period 1. Third, in period 1, if the state ω_7 is realized, each agent knows that he will receive no dividend for sure.[17] Hence if the price is positive in period $t=1$ in state $\omega=\omega_7$, there will be a strong bubble, and that is part of what we are going for. The state where there is a strong bubble is called a *bubble state*.

There are different approaches to generate potential gains from trade. Instead of assuming different marginal utility levels across the states, here we let agents have heterogeneous beliefs, as shown in the table below with weight $W=\frac{1}{16}$.

[17] Take agent A into consideration for example. When ω_7 is realized, agent A will have observed the event $\{\omega_6,\omega_7\}$. Since in either state ω_6 or ω_7, there is no dividend payment, agent A knows that he will receive no dividend with probability 1.

Table 1.2 Agents' Beliefs Accross States

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	2	1	1	1	2	1	1	7
π_B	1	2	1	2	1	1	1	7

Also, the structure of the beliefs may seem complicated for now, but it will become clear why it serves for the existence of a bubble in a rational expectation equilibrium. So far, it is easy to observe that within the two states where there will be a dividend of 4, agent A puts a higher weight on state ω_1, and agent B puts a higher weight on state ω_4. They put the same weight on state ω_7, and state ω_8, respectively. The weights they put on events $\{\omega_1, \omega_2, \omega_3\}$ and $\{\omega_4, \omega_5, \omega_6\}$ are also symmetric.

1.3.2 A Rational Expectations Equilibrium with a Bubble

Recall the standard definition given in the last section, and in our example a rational expectations equilibrium will be a vector $(P, x) \in R_+^{3\times 8} \times R^{2\times 3\times 8}$ such that

(C1) $\forall i = A, B$, net trades x_i are information feasible and satisfy no short sales;

(C2) $\forall i = A, B$, x_i maximize player i's expected payoff with respect to his own price-and-trade-refined information;

(C3) $\forall t = 1, 2, 3, \forall n = 1, \cdots, 8, x_{At}(\omega_n) + x_{Bt}(\omega_n) = 0;$

(C4) $\forall t = 1, 2, 3, \forall n, m = 1, \cdots, 8, j_t(\omega_n) \subseteq \{\omega_m : P_t(\omega_m) = P_t(\omega_n)\}.$

Although there are multiple rational expectations equilibria for this example, the one with the equilibrium prices and trades given in the following two tables is what we are interested in - the one in which there is a strong bubble and a common expected bubble.

Table 1.3 Equilibrium Prices

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

Table 1.4 Equilibrium Net Trades

$\forall\omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0

1.3.2.1 Price-and-Trade-Refined Information

First, derive the price–and-trade-refined information for each agent in each period. It is easy to observe from the price table that $P_1(\omega) = 1\ \forall\omega \in \Omega$ and from the trade table that $x_{A1}(\omega) = x_{B1}(\omega) = 0\ \forall\omega \in \Omega$. This implies that the prices and trades in period 1 reveal no information. Hence $S_{A1}^{PX} = S_{A1}$, $S_{B1}^{PX} = S_{B1}$. Since in period 3, all agents already have full information about the state before observing the prices and making the trades,[18] the prices and trades in period 3 again, reveal no information. Hence $S_{A3}^{PX} = S_{A3}$, $S_{B3}^{PX} = S_{B3}$. The only new information revealed by prices and trades in period 2 is that agents know where they are for sure when the state ω_7 is realized. Hence agents' price–and-trade-refined information in period 2 is the following, with the original

[18] Actually there is no trade in period 3 in the equilibrium under study.

information structure attached below for comparison.

$$
\begin{aligned}
S_{A2}^{PX} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\} \\
S_{B2}^{PX} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_7\}, \{\omega_8\}\} \\
S_{A2} &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}, \{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_7\}, \{\omega_8\}\}.
\end{aligned}
$$

The following graph may give more intuition about the information structure than the mathematical expression does. In the graph, agent A's information sets are described by the black solid curves; agent B's information sets are described by the blue dotted curves; dividend paying states are emphasized in gray color.

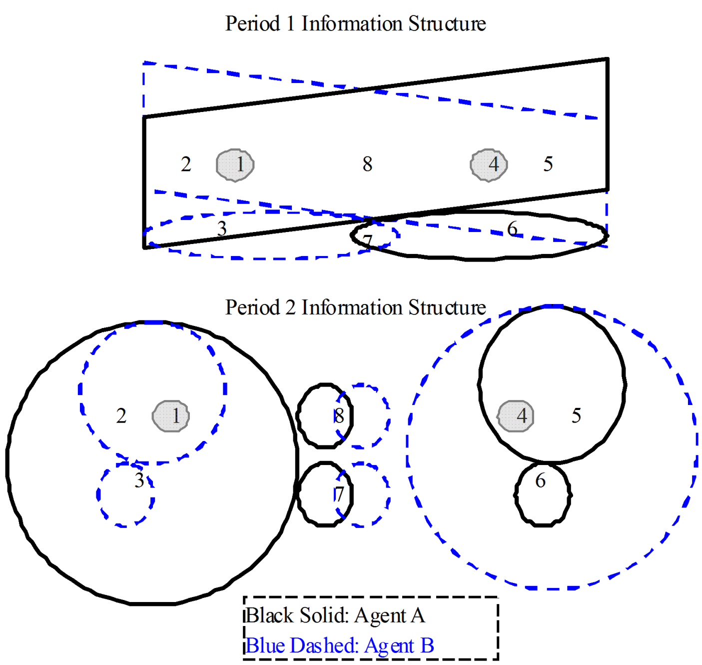

Figure 1.1: 3-Period Information Structure for Agent A and Agent B

It is worth noting that in period 2, with the price-and-trade-refined information, agent

A is better informed than agent B when event $\{\omega_4, \omega_5, \omega_6\}$ happens, and agent B is better informed than agent A when event $\{\omega_1, \omega_2, \omega_3\}$ happens. We will see soon that the subgroup of states $\{\omega_4, \omega_5, \omega_6\}$ is where agent A takes advantage of agent B by selling the asset he believes is overpriced to agent B, and similarly, the subgroup of states $\{\omega_1, \omega_2, \omega_3\}$ is where agent B takes advantage of agent A.

1.3.2.2 The Existence of Strong Bubbles and Common Expected Bubbles

Second, note that there is a strong bubble in period 1 in state ω_7 since for agent A, $s_{A1}^{PX}(\omega_7) = \{\omega_6, \omega_7\}$, $P_1(\omega_7) = 1 > 0 = d(\omega_6) = d(\omega_7)$, and for agent B, $s_{B1}^{PX}(\omega_7) = \{\omega_3, \omega_7\}$, $P_1(\omega_7) = 1 > 0 = d(\omega_3) = d(\omega_7)$. In short, a strong bubble exists in period 1 in ω_7 because in that state every agent knows the asset is worthless but with a positive current price.

In this example, $m_1^{PX}(\omega_7) = \Omega$. To see that this bubble is robust to common knowledge in the expected sense, we need to check that $\forall i = A, B, \forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$. There are four cases:

(1)$\omega = \omega_7$: Agent A observes the event $\{\omega_6, \omega_7\}$, and agent B observes the event $\{\omega_3, \omega_7\}$. Each of them will deduce that the expected dividend in period 3 will be $\frac{1}{2}0 + \frac{1}{2}0 = 0$, which is less than the current price.

(2)$\omega = \omega_6$: Agent A observes the event $\{\omega_6, \omega_7\}$, and his expected dividend in period 3 is 0, less than the current price. Agent B observes $\Omega \backslash \{\omega_3, \omega_7\}$, and his expected dividend in period 3 is $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$, less than the current price.

(3)$\omega = \omega_3$: Agent B observes the event $\{\omega_3, \omega_7\}$, and his expected dividend in period 3 is 0, less than the current price. Agent A observes $\Omega \backslash \{\omega_6, \omega_7\}$, and his expected dividend in period 3 is $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$, less than the current price.

(4)$\omega_n \in \Omega \backslash \{\omega_3, \omega_6, \omega_7\}$, Agent A observes the event $\Omega \backslash \{\omega_6, \omega_7\}$, and agent B observes the event $\Omega \backslash \{\omega_3, \omega_7\}$. Each of them will deduce that the expected dividend in period 3 will be $\frac{3}{14}4 + \frac{11}{14}0 = \frac{6}{7}$, which is less than the current price.

Therefore, the bubble in period 1 in state ω_7 is a common expected bubble. Actually, the reader can check that in our example the common expected bubble exists in period 1, not only in state ω_7, but also in any other state.

1.3.2.3 Check of Equilibrium Conditions

Last, check that the prices and trades described above constitute a rational expectations equilibrium. We check all four conditions step by step.

Check (C1): We observe from the trade table that the minimum amount of trade in period 2 is -1. By the fact that there is no trade in either period 1 or 3 and that each agent is endowed with 1 share of the risky asset, the no short sales condition is satisfied for x_A and x_B. To see if the x_i's are information feasible, it suffices to only look at period 2 since no trade occurs either in period 1 or 3. In period 2, actually each agent's action remains the same given the same price–and-trade-refined information.[19] This implies that x_A and x_B also satisfy the information feasibility condition.

Check (C2): Maximization of the expected payoff at the beginning of period 1 under the constraints of information feasibility and no short sales, is equivalent to maximization of the expected payoff in each period given the current price–and-trade-refined information under the same constraints.

In period 3, each agent has no incentive to trade since the price is exactly equal to the dividend for every state.

[19] Take agent A for example.
$\forall\omega = \omega_6, s_{A2}^{PX}(\omega) = \{\omega_6\} \subset \{\omega_4, \omega_5, \omega_6\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall\omega \in \{\omega_4, \omega_5\}, s_{A2}^{PX}(\omega) = \{\omega_4, \omega_5\} \subset \{\omega_4, \omega_5, \omega_6\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall\omega \in \{\omega_1, \omega_2, \omega_3\}, s_{A2}^{PX}(\omega) = \{\omega_1, \omega_2, \omega_3\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall\omega \in \{\omega_7, \omega_8\}, s_{A2}^{PX}(\omega) = \{\omega\} \subset \{\omega_7, \omega_8\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$.

In period 2, there are in total 4 cases:

(p2-i) $\forall i \in \{A, B\}$, if agent i observes the event $\{\omega_7\}$ or $\{\omega_8\}$, he knows that with probability 1 the price in period 3 will be 0, which is equal to the current price, thus he is indifferent between trading or not in period 2, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p2-ii) If agent A observes the event $\{\omega_1, \omega_2, \omega_3\}$ (or if agent B observes the event $\{\omega_4, \omega_5, \omega_6\}$), he will deduce that the expected price in period 3 will be $\frac{1}{2}4 + \frac{1}{4}0 + \frac{1}{4}0 = 2$, which is equal to the current price, thus he is indifferent between trading or not in period 2, so the equilibrium trade of 1 maximizes his expected payoff in this case.

(p2-iii) If agent A observes the event $\{\omega_4, \omega_5\}$ (or if agent B observes the event $\{\omega_1, \omega_2\}$), he will deduce that the expected price in period 3 will be $\frac{1}{3}4 + \frac{2}{3}0 = \frac{4}{3}$, which is less the current price 2, thus he has an incentive to sell any of the asset he owns in period 2, so under the short sales constraint and given there is no trade in period 1, the equilibrium trade of -1 maximizes his expected payoff in this case.

(p2-iv) If agent A observes the event $\{\omega_6\}$ (or if agent B observes the event $\{\omega_3\}$), he knows that with probability 1 the price in period 3 will be 0, which is less the current price 2, thus he has an incentive to sell any of the asset he owns in period 2, so under the short sales constraint and given there is no trade in period 1, the equilibrium trade of -1 maximizes his expected payoff in this case.

In period 1, there are 2 cases:

(p1-i) If agent A observes the event $\{\omega_6, \omega_7\}$ (or if agent B observes the event $\{\omega_3, \omega_7\}$), he will deduce that the expected price in period 2 will be $\frac{1}{2}2 + \frac{1}{2}0 = 1$, which is equal to the current price, thus he is indifferent between trading or not in period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p1-ii) If agent i observes the event other than the one described in (p1-i), he will deduce that the expected price in period 2 will be $\frac{2\times2+1\times3}{14}2 + \frac{7}{14}0 = 1$, which is equal to the current price, thus he is indifferent between trading or not in period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

The above analysis guarantees that condition (C2) is satisfied.

Check (C3) and (C4): It is seen that the market clears in each period in each state from the table of trades, hence (C3) is satisfied. Note that $P_1(\omega) = 1 \ \forall \omega \in \Omega$, hence $P_1(\cdot)$ is measurable with respect to $j_1(\cdot)$. Also note that $j_3(\omega) = \{\omega\} \ \forall \omega \in \Omega$, hence $P_3(\cdot)$ is measurable with respect to $j_3(\omega)$. To see $P_2(\cdot)$ is measurable with respect to $j_2(\omega)$, note that $\forall n = 1, \cdots, 6$, $j_2(\omega_n) \subseteq \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 2\}$, and $\forall n = 7, 8$, $j_2(\omega_n) \subseteq \{\omega_7, \omega_8\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 0\}$. This completes the check that the prices and trades given in the example constitute a rational expectations equilibrium.

1.3.3 Discussion

We have shown that, in a simple finite horizon model with asymmetric information and short sales constraints, a strong bubble and a common expected bubble can exist in the same period in the same state in a rational expectations equilibrium with common knowledge of trades, under the same basic setting as in AMP (1993).

It is worthwhile to make some remarks about this simple example.

(1) The initial distribution of the asset is not efficient. To see this, with zero-trade, each agent's expected payoff

$$m_i + \sum_{\omega \in \Omega} \pi_i(\omega) \left[e_i P_T(\omega) + \sum_{t=1}^{T} x_{it}(\omega) \left[P_{t+1}(\omega) - P_t(\omega) \right] \right]$$

would have been $m_i + \frac{3}{4}$, while in the equilibrium, each agent's expected payoff is $m_i + 1$. Thus our example does not violate the no-trade theorem and the necessary condition of ex ante inefficiency is satisfied here. In fact, as the analysis has shown, in our example those who gain from the trade are the sellers whenever the trade takes place.

(2) The social welfare is maximized in the rational expectation equilibrium with bubbles if there is no initial endowment of money. Note that in our example the social welfare is maximized when in every state the social planner gives all the assets to the agent who puts the highest weight on that state. Hence the maximum social welfare should be $\frac{9}{8}(m_1 + m_2) + 2$. When either agent has positive endowment of money, the social welfare of the equilibrium outcome is not maximized. However, if each agent is endowed with no money, then the social welfare is maximized in equilibrium. To put it in another way, if the social planner is only allowed to reallocate on the risky asset, then the equilibrium maximizes the sum of the utilities of the agents. This implies a surprising observation that the rational bubbles do not necessarily lead to inefficiency.

(3) The short sale constraints are binding in period 2 for the sellers whenever the trade takes place. In the cases of (p2-iii) and (p2-iv), where agents play the seller's role,

since the expected price for the asset is higher than the current price, agents would like to take advantage of this and sell as much as they can. If there were no short sales constraints, an equilibrium would not have been reached under the current price. This is where the no short sales assumption plays its role.

(4)The asymmetric information functions in such a way that even though all agents know that the asset is overpriced, they are still willing to hold the asset as long as the information on overpricing is not common knowledge in the strong sense. It is this feature that makes a bubble possible in a rational expectations equilibrium.

(5)For simplicity, the example is constructed in such a way that even though trade is common knowledge, it reveals no additional information to either agent.

1.4 Structural Characteristics for the Existence of Bubbles

Assume there are only two agents. There is no trade in the first period and information becomes perfect in the last period. The dividend can only take two values, $\forall\omega, d(\omega) \in \{0, D\}$ where $D > 0$.

Claim 2 *Under the perfect memory assumption, suppose there is a bubble in period t in state ω in a rational expectations equilibrium in economy with state set Ω. Then there is also a bubble in equilibrium in the subeconomy with state set $m_t^{PX}(\omega)$.*

Claim 3 *Under the perfect memory assumption, for a strong bubble to exist in a rational expectations equilibrium in a 2-agent 3-period economy, there must be at least 2 states with positive dividends, that is*

$$|\{\omega \in \Omega | d(\omega) > 0\}| \geq 2.$$

Proof. Suppose a strong bubble exists in period 1 in state ω^*.

Consider agent A first. Since $P_1(\omega^*) > \max_{\omega \in s_{A1}^{PX}(\omega^*)} d(\omega) = 0$ and $P_1(\omega^*) = E_A\left[P_2(\omega') | \omega' \in s_{A1}^{PX}(\omega^*)\right]$, the fact that agent A is willing to hold the asset implies that $\exists\omega^A \in s_{A1}^{PX}(\omega^*)$ such that $P_2(\omega^A) \geq P_1(\omega^*) > 0$. Since $s_{A2}^{PX}(\omega^A) \subseteq s_{A1}^{PX}(\omega^A) = s_{A1}^{PX}(\omega^*)$, when ω^A is realized, in period 2 agent A knows for sure that he will receive nothing. Give $P_2(\omega^A) > 0$, it must be the case that when ω^A is realized, in period 2 agent B's expected return is nonzero. This implies that $\exists\omega^{AB} \in s_{B2}^{PX}(\omega^A)$ such that

$d\left(\omega^{AB}\right) > 0$. Since in equilibrium in period 2 agent A will always sell in state ω^{A} and agent B cannot tell the difference between ω^{A} and ω^{AB}, it must be the case that in equilibrium in period 2 agent A will always sell in state ω^{AB} as well.

Then consider agent B, and we have similar results. $\exists\omega^{B} \in s_{B1}^{PX}(\omega^{*})$ such that $P_2\left(\omega^{B}\right) > 0$ and when ω^{B} is realized, in period 2 agent B knows for sure that he will receive nothing. This implies that $\exists\omega^{BA} \in s_{A2}^{PX}\left(\omega^{B}\right)$ such that $d\left(\omega^{BA}\right) > 0$ and in equilibrium in period 2 agent B will always sell in state ω^{BA}.

Since in equilibrium in period 2 agent A always sells in state ω^{AB} and agent B always sells in state ω^{BA}, $\omega^{AB} \neq \omega^{BA}$. ■

Definition 8 (Symmetry) *The model has a symmetric setting if for any $i, j = 1, 2, \cdots, I$, there exists a bijective mapping L from $\{1, 2, \cdots, N = |\Omega|\}$ to $\{1, 2, \cdots, N\}$ such that for any $t = 1, 2, 3$,*

(1) $S_{it} = S_{jt}|L$, where $S_{jt}|L$ is j's relabelled information partition at t under L;

(2) $\pi_i(\omega_n) = \pi_j\left(\omega_{L(n)}\right)$;

(3) $d(\omega_n) = d\left(\omega_{L(n)}\right)$;

(4) $(m_i, e_i) = (m_j, e_j)$.

Basically equation (1) means that it is information-symmetric. Similarly it is belief-symmetric by (2), dividend-symmetric by (3), and endowment-symmetric by (4).

It should be noted that the symmetry assumption is more than assuming symmetry w.r.t information, symmetry w.r.t. dividend, symmetry w.r.t. belief, and symmetry w.r.t. endowment, respectively. That is because we require the same mapping L for conditions (1)-(3) to be satisfied.

We call $\left(\omega_n, \omega_{L(n)}\right)$ a symmetric pair of states for agent i and j if $L(L(n)) = n$.

Recall that a state where there is a strong bubble is called a *bubble state*, denoted by ω^{*}.

Claim 4 *For a strong bubble to exist in a symmetric rational expectations equilibrium in a 2-agent symmetric economy, there must be at least 2 states with positive dividends, that is*

$$|\{\omega \in \Omega | d(\omega) > 0\}| \geq 2.$$

Proof. By AMP(1993), for a strong bubble to exist in a rational expectations equilibrium, there must be potential gains from trade. And these gains will be distributed to the agents in each trade. But since there is no constraint on the short sales of money, in each trade the agent who is buying the asset won't receive any gains, otherwise he would be buying as much as he can, in which situation there would be no equilibrium. Therefore, the agents receive the gains only if they play the role of sellers. Since it is a symmetric economy, each agent has a positive probability to sell the asset. Consider Agent A first. Suppose he is better off by selling the asset in period t in state ω_A. Then in period t there must be a state with positive dividend, denoted by ω_A^B, from which agent B cannot tell the difference to ω_A. Since agent B is buying in period t in state ω_A^B, this implies that agent A is selling in period t in ω_A^B. By symmetry, in period t, there exists another state ω_B^A with positive dividend, where agent A is buying and agent B is selling. Obviously $\omega_A^B \neq \omega_B^A$. ■

Claim 5 *Under the perfect memory assumption, for a strong bubble to exist in a rational expectations equilibrium in a 2-agent symmetric economy, for each agent, at least one price-and-trade-refined information set contains at least 3 states, including one with positive dividend, that is*

$$\forall i, \exists t, \exists \omega \text{ such that } \left|s_{it}^{PX}(\omega)\right| \geq 3 \text{ and } \max_{\omega' \in s_{it}^{PX}(\omega)} d(\omega') > 0.$$

Proof. Let ω^* be the bubble state. Suppose in period 1 agent A cannot tell difference between ω^* and ω_A, both of which are zero-dividend states. And without loss of generality, suppose in period t in state ω_A agent A can sell the asset at a positive price. This implies that in period t agent B cannot tell difference between ω_A and some positive-dividend state ω_A^B, or $\omega_A \in s_{Bt}^{PX}\left(\omega_A^B\right)$. Since agent B will be buying in period t in state ω_A^B, agent A must be selling, hence in period t there must exist some zero-dividend state ω' such that $\omega' \in s_{At}^{PX}\left(\omega_A^B\right)$. If $\omega' \in s_{Bt}^{PX}\left(\omega_A^B\right)$, we are done. Suppose not, then there must exisit some positive-dividend state ω'' such that $\omega'' \in s_{Bt}^{PX}(\omega')$. And this would again imply that there exists some zero-dividend state ω''' such that $\omega''' \in s_{At}^{PX}(\omega'')$. If $\omega'' \in s_{At}^{PX}(\omega')$ or $\omega''' \in s_{Bt}^{PX}(\omega')$, we are done. If not, we can follow the same logic. Since the number of states is finite, and $s_{At}^{PX}(\omega_A)$ does not contain any positive-dividend states, at the end we will find a price-and-trade-refined information set which contains at least 3 states including one with positive dividend. By symmetry this is also true for agent B. ■

Claim 6 *Under the perfect memory assumption, for a strong bubble to exist in a rational expectations equilibrium in a 2-agent symmetric economy, there must be at least 8 states, that is*

$$|\Omega| \geq 8.$$

Proof. Suppose not and there are only 7 states instead. Assume in period t agent i has a price-and-trade-refined information set $\{\omega_{i1}, \omega_{i2}, \omega_{i3}\}$ and the bubble state is ω^*. This implies $P_t(\omega^*) = 0$ and $P_t(\omega_{ik}) > 0$ for $i = A, B$ and $k = 1, 2, 3$. It is easy to know that in period 1 for agent A, $s_{A1}^{PX}(\omega^*) \subset \{\omega^*, \omega_{B1}, \omega_{B2}, \omega_{B3}\}$. Without loss of generality, assume $\omega_{B1} \in s_{A1}^{PX}(\omega^*)$. Since there is no trade in period 1, the equilibrium price should be equal to agent A's expected price. This implies $P_1(\omega_{B1}) < P_t(\omega_{ik})$ from agent A's perspective.

Now consider agent B. It is easy to know that in period 1 for agent B, $\{\omega_{B1}, \omega_{B2}, \omega_{B3}\} \subseteq s_{B1}^{PX}(\omega_{B1}) \subset \{\omega_{A1}, \omega_{A2}, \omega_{A3}, \omega_{B1}, \omega_{B2}, \omega_{B3}\}$. But this would imply $P_1(\omega_{B1}) = P_t(\omega_{ik})$ from agent B's perspective.

Therefore, there must be at least 8 states. ■

Claim 7 *For a common expected bubble to exist in period t in state ω, it must be the case that the current price is higher than every agent's expected dividend across the meet of the information partition containing ω, that is*

$$\forall i = 1, 2, \cdots, I, P_t(\omega) > E_i\left[d(\omega') \,|\omega' \in m_t^{PX}(\omega)\right].$$

Proof. By the definition of common expected bubbles, $\forall i = 1, 2, \cdots, I, \forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > E_i\left[d(\omega'') \,|\omega'' \in s_{it}^{PX}(\omega')\right]$.

Since $E_i\left[d(\omega') \,|\omega' \in m_t^{PX}(\omega)\right]$ is weighted average of $E_i\left[d(\omega'') \,|\omega'' \in s_{it}^{PX}(\omega')\right]$, immediately we have $P_t(\omega) > E_i\left[d(\omega') \,|\omega' \in m_t^{PX}(\omega)\right]$. ■

It turns out that the example of strong bubbles and common expected bubbles we have presented in the previous section is actually the simplest one with minimum number of states.

1.5 General Results

In Section 3, an example of a rational bubble that is both a strong bubble and a common expected bubble is presented in a rational expectations equilibrium with 2 agents. Furthermore, as will be shown next, this result holds for any finite number of agents.

Let $S^F \equiv \{\{\omega\} \,|\omega \in \Omega\}$, and S^F is called the perfect information structure for Ω.

Before constructing bubble examples, we shall make some restrictions on the agents' information structure so as to avoid trial bubbles from duplications.

Assumption 1 (Different Information Structure) $\forall i, j = 1, \cdots, I$, $\forall t = 1, \cdots, T$, $S_{it}, S_{jt} \neq S^F \Rightarrow S_{it} \neq S_{jt}$.

The assumption of Different Information Structure says that as long as agents don't have perfect information, there must be somewhere their information differs from each other. This assumption rules out the possibility of duplicating identical agents.

Assumption 2 (Distinct Information Everywhere) $\forall i, j = 1, \cdots, I$, $\forall t = 1, \cdots, T$, $\forall \omega \in \Omega$, $s_{it}(\omega), s_{jt}(\omega) \neq \{\omega\} \Rightarrow s_{it}(\omega) \neq s_{jt}(\omega)$.

The assumption of Distinct Information Everywhere says that as long as agents don't have perfect information, their information differs from each other everywhere. It is easy to know that Assumption 2 is much stronger than Assumption 1. Assumption 2 implies Assumption 1, but not vice versa.

Assumption 3 (Common Knowledge of Trades) $\forall i = 1, \cdots, I$, $\forall t = 1, \cdots, T$, x_{it} *is common knowledge.*

Based on the assumptions above, two propositions can be made on the existence of strong bubbles in a rational expectations equilibrium.

Proposition 8 *Under Assumption 1 and 3, for any $I \geq 2$, there exists an economy under the framework described in Section 2, with I agents, 3 periods and $3I + 2$ states, presenting a bubble, both strong and common expected, in a rational expectations equilibrium.*

Proof. See Appendix 1. ■

Proposition 9 *Under Assumption 2 and 3, for any $I \geq 2$, there exists an economy under the framework described in Section 2, with I agents, 3 periods and $I \cdot \max\{3, I\} + 2$ states, presenting a bubble, both strong and common expected, in a rational expectations equilibrium.*

Proof. See Appendix 2. ■

The strong bubble part of the result is not new, and has been analyzed by AMP (1993) and Conlon (2004). However, by presenting a bubble, not only strong but also common expected, the above propositions provide a new answer to what properties of bubbles we can expect to have in a rational world. The common expected bubble part of the result is surprising since it is somewhat counterintuitive that an expected bubble can be robust to common knowledge in a raitional expectations equilibrium. But actually it is the common knowledge of the heterogeneous beliefs and the information structures that guarantees that agents have no incentive to rush in face of bubbles, because by rational expectations they know that they can take advantage of it in a later period.

It should also be noted that the conclusions above are independent of the assumption of no common knowledge of trade. In Proposition 3 of AMP (1993), the assumption of no common knowledge of trades was argued as a necessary condition for the existence of bubbles in a rational expectations equilibrium. The idea of the argument is the following: Geanakoplos (1992) has argued that with common knowledge of trades, agents would have behaved in the same way without the private part of their information (originally stated as "common knowledge of actions negates asymmetric information about events"), and then there would be no strong bubbles since there is no asymmetric information about the states. However, as pointed out by Conlon (2004), the conclusion that there are no strong bubbles is only true for the new economy where every agent has the same information, which is the common part of their original information. The bubble may still exist in the original economy since in period 1 there is no trade and hence agents still have their private information.

1.6 The Coexistence of Second Order Strong Bubbles and Common Expected Bubbles

1.6.1 Exogenous Setting

In this section an even strong result is provide regarding the higher order uncertain. Here I provide an example for the coexistence of second order strong bubbles and common expected bubbles in a rational expectations equilibrium. The examples for higher order strong bubbles can be constructed similarly. It is checked that the nth order strong bubble model in Conlon (2008) does not have the "common expected" feature.

There are 2 agents (A and B), 4 periods (1, 2, 3, and 4) and 14 states (ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7, ω_8, ω_9, ω_{10}, ω_{11}, ω_{12}, ω_{13} and ω_{14}). Only 2 assets exist in the market: one is money and the other is called a risky asset. Each share of the risky asset will pay a dividend of amount 8 at the end of period 4 if the state is either ω_1 or ω_4, and will pay nothing otherwise, as shown in the table below.

Table 1.5 Dividend Distribution Accross States

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
$d(\omega)$	8	0	0	8	0	0	0	0	0	0	0	0	0	0

Each agent is endowed with m_i unit of money and 1 share of the risk asset at the beginning of period 1. Agents can trade in each of period 1, 2, 3 and 4. At period 4, after the trade is made, the dividend is realized, and then the consumption takes place.

Since the asymmetric information is the key to generate bubbles, we achieve this goal by giving agents different information structures. Recall that agent i's ($i = A, B$) information about the state in period t ($t = 1, 2, 3, 4$) is represented by S_{it}, a partition of the space Ω. The specific structures of S_{it}'s are given below.

$$
\begin{aligned}
S_{A1} &= \{\{\omega_1, \omega_2, \omega_4, \omega_5, \omega_8, \omega_{10}, \omega_{13}, \omega_{14}\}, \{\omega_6, \omega_7, \omega_{12}\}, \{\omega_3, \omega_9, \omega_{11}\}\} \\
S_{B1} &= \{\{\omega_1, \omega_2, \omega_4, \omega_5, \omega_8, \omega_9, \omega_{13}, \omega_{14}\}, \{\omega_3, \omega_7, \omega_{11}\}, \{\omega_6, \omega_{10}, \omega_{12}\}\} \\
S_{A2} &= \{\{\omega_1, \omega_2, \omega_{13}\}, \{\omega_4, \omega_5, \omega_{10}, \omega_{14}\}, \{\omega_3, \omega_9\}\} \cup \{\{\omega_n\} | n = 6, 7, 8, 11, 12\} \\
S_{B2} &= \{\{\omega_4, \omega_5, \omega_{14}\}, \{\omega_1, \omega_2, \omega_9, \omega_{13}\}, \{\omega_6, \omega_{10}\}\} \cup \{\{\omega_n\} | n = 3, 7, 8, 11, 12\} \\
S_{A3} &= \{\{\omega_1, \omega_2\}, \{\omega_4, \omega_5, \omega_{10}\}\} \cup \{\{\omega_n\} | n = 3, 6, 7, 8, 9, 11, 12, 13, 14\} \\
S_{B3} &= \{\{\omega_4, \omega_5\}, \{\omega_1, \omega_2, \omega_9\}\} \cup \{\{\omega_n\} | n = 3, 6, 7, 8, 10, 11, 12, 13, 14\} \\
S_{A4} &= S_{B4} = \{\{\omega_n\} | n = 1, \cdots, 14\}
\end{aligned}
$$

There are different approaches to generate potential gains from trade. Instead of assuming different marginal utility levels accross the states, here we let agents have heterogeneous beliefs, as shown in the table below with weight $W = \frac{1}{38}$.

Table 1.6 Agents' Beliefs Accross States

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
π_A	1	2	1	2	1	2	1	15	1	1	2	1	3	5
π_B	2	1	2	1	2	1	1	15	1	1	1	2	5	3

1.6.2 A Rational Expectations Equilibrium with a Bubble

Recall the standard definition of rational expectations equilibrium, and in our example a rational expectations equilibrium will be a vector $(P, x) \in R_+^{4\times 14} \times R^{2\times 4\times 14}$ such that

(C1) $\forall i = A, B$, x_i are information feasible and satisfy no short sales.

(C2) $\forall i = A, B$, x_i maximizes player i's expected payoff with respect to his own price-and-trade-refined information.

(C3) $\forall t = 1, 2, 3, 4, \forall n = 1, \cdots, 14, x_{At}(\omega_n) + x_{Bt}(\omega_n) = 0$.

(C4) $\forall t = 1, 2, 3, 4, \forall n, m = 1, \cdots, 14, j_t(\omega_n) \subseteq \{\omega_m : P_t(\omega_m) = P_t(\omega_n)\}$.

Although there are multiple rational expectations equilibria for this example, the one with the equilibrium prices and trades given in the following two tables is what we are interested in - the one in which there is a second order strong bubble and a common expected bubble.

Table 1.7 Equilibrium Prices

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
$P_1(\omega)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0	2	2	0	0	2	2
$P_3(\omega)$	4	4	0	4	4	0	0	0	4	4	0	0	0	0
$P_4(\omega)$	8	0	0	8	0	0	0	0	0	0	0	0	0	0

Table 1.8 Equilibrium Net Trades

$$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A4}(\omega) = x_{B4}(\omega) = 0$$

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0	1	-1	0	0	1	-1
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0	-1	1	0	0	-1	1
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$x_{A3}(\omega)$	-2	-2	0	2	2	0	0	0	-2	2	0	0	0	0
$x_{B3}(\omega)$	2	2	0	-2	-2	0	0	0	2	-2	0	0	0	0
$x_{A3}(\omega) + x_{B3}(\omega)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1.6.2.1 Price-and-Trade-Refined Information

First derive the price–and-trade-refined information for each agent in each period. It can be checked that our example is constructed in a way that the price and trade does not reveal any additional information to the agents. So we have $S_{it}^{PX} = S_{it}^{PX}$ for $i = A, B$, $t = 1, 2, 3, 4$.

The following graph may give more intuition about the information structure than the mathematical expression does. In the graph, agent A's information sets are described by the black solid curves, agent B's information sets are described by the blue dotted curves, and dividend paying states are emphasized in gray color.

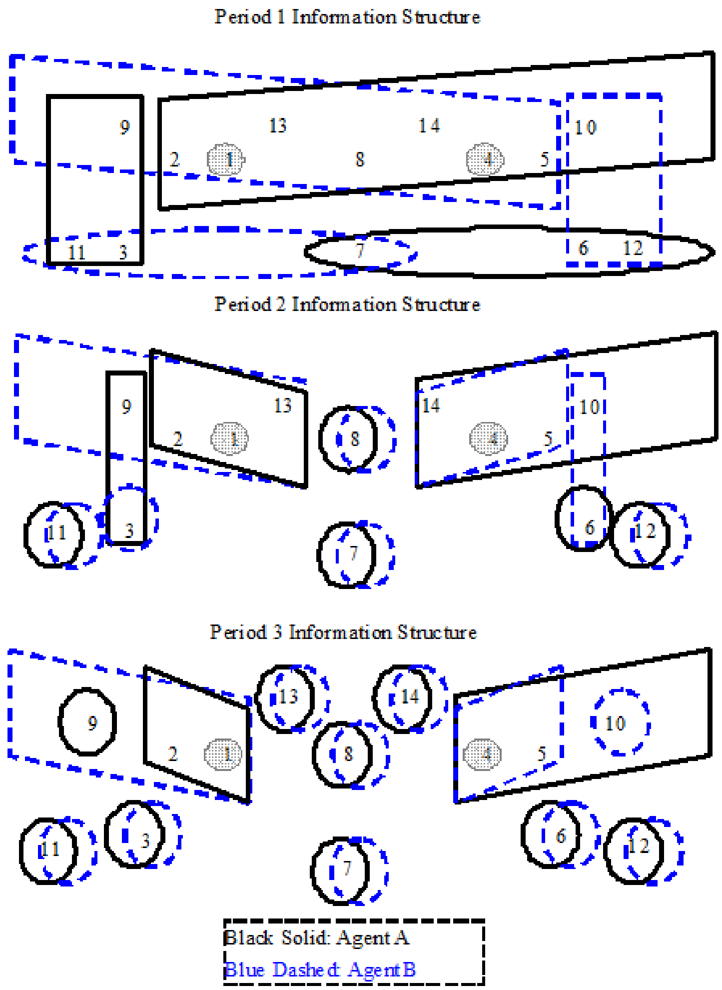

Figure 1.2: 4-Period Information Structure for Agent A and Agent B

1.6.2.2 The Existence of 2nd Order Strong Bubbles and Common Expected Bubbles

Second note that there is a second order strong bubble at period 1 in state ω_7.

For agent A, $s_{A1}^{PX}(\omega_7) = \{\omega_6, \omega_7, \omega_{12}\}$, $P_1(\omega_7) = 1 > 0 = d(\omega_6) = d(\omega_7) == d(\omega_{12})$. This means that at period 1 when the state ω_7 is realized agent A knows sure that the price of the asset is higher than any possible dividend he will receive. Furthermore, $s_{B1}^{PX}(\omega_7) = \{\omega_3, \omega_7, \omega_{11}\}$, $s_{B1}^{PX}(\omega_6) = s_{B1}^{PX}(\omega_{12}) = \{\omega_6, \omega_{10}, \omega_{12}\}$, and

$d(\omega_3) = d(\omega_6) = d(\omega_7) = d(\omega_{10}) = d(\omega_{11}) = d(\omega_{12}) = 0$. This is equivalent to saying that $\forall \omega \in s_{A1}^{PX}(\omega_7)$, $\forall \omega' \in s_{B1}^{PX}(\omega)$,$d(\omega') = 0 < 1 = P_1(\omega)$, which implies that at period 1 in state ω_7 agent A knows that agent B knows that that the price of the asset is higher than any possible dividend he (agent B) will receive. By symmetry, it surffices to check for agent A only.

In this example, $m_1^{PX}(\omega_7) = \Omega$. To see that this bubble is robust to common knowledge in the expected sense, by symmetry it suffices to check that $\forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{A1}^{PX}(\omega)} \pi_A(\omega')} \sum_{\omega' \in s_{A1}^{PX}(\omega)} \pi_A(\omega') d(\omega')$. There are three cases:

(1)$\omega \in \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_8, \omega_{10}, \omega_{13}, \omega_{14}\}$: Agent A will induce that the expected dividend in period 3 will be $\frac{3}{30}8 + \frac{27}{30}0 = \frac{4}{5}$, which is less to the current price 1.

(2)$\omega \in \{\omega_6, \omega_7, \omega_{12}\}$: In this case agent A's expected dividend in period 3 is 0, less than the current price.

(3)$\omega \in \{\omega_3, \omega_9, \omega_{11}\}$: again in this case agent A's expected dividend in period 3 is 0, less than the current price.

Therefore, the bubble at period 1 in state ω_7 is a common expected bubble. Actually, the reader can check that in our example the common expected bubble exists at period 1, not only in state ω_7, but also in any other state.

1.6.2.3 Check of Equilibrium Conditions

Last check that the prices and trades described above constitute a rational expectations equilibrium. We check all the four conditions step by step.

Check (C1): We observe from the trade table that (1) the minimum amount of net trade at period 2 is -1; (2) in any state where an agent's net trade at period 3 is -2 his net trade at period 2 is 1; (3) there is no trade in period 1; and (4) there is no trade in period 4. it is also given that (5) each agent is endowed with 1 share of the risky asset. (4) implies that as long as the short sale constraint is satisfied for period 3, it is satisfied for period 4. (2), (3) and (5) together impliy that as long as the short sale constraint is satisfied for period 2, it is satisfied for period 3. From (1) and (3) we know the no short sale condition is satisfied for period 1 and 2. To see if x_i are information feasible, it suffices to only look at period 2 and 3 since no trade occurs either in period 1 or 4. In period 2, actually each agent's action remains the same given the same price–and-trade-refined information.[20] This is also true for period 3. This implies that x_A and x_B also satisfy the information feasibility condition.

Check (C2): Maximization of the expected payoff at the beginning of period 1 under the constraints of information feasibility and no short sales, is equivalent to maximization of the expected payoff in each period given the current price–and-trade-refined information under the same constraints. By symmetry, it suffices to consider agent A's case. In period 4, agent A has no incentive to trade since the price is exactly equal to the dividend for every state.

In period 3, there are in total 4 cases:

(p3-i) If agent A observes the event $\{\omega_n\}$ where $n \in \{3, 6, 7, 8, 11, 12, 13, 14\}$, he knows that with probability 1 the price in period 4 will be 0, which is equal to the current price, thus he is indifferent between trading or not at period 3, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p3-ii) If agent A observes the event $\{\omega_1, \omega_2\}$, he will induce that the expected price in period 4 will be $\frac{1}{3}8 + \frac{2}{3}0 = \frac{8}{3}$, which is less than the current price 4, thus he has incentive to sell any of the asset he owns at period 3, so under the short sale constraint, the

20 Take agent A for example.
$\forall \omega = \omega_6, s_{A2}^{PX}(\omega) = \{\omega_6\} \subset \{\omega_4, \omega_5, \omega_6, \omega_{10}, \omega_{14}\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall \omega \in \{\omega_4, \omega_5, \omega_{10}, \omega_{14}\}$,
$s_{A2}^{PX}(\omega) = \{\omega_4, \omega_5, \omega_{10}, \omega_{14}\} \subset \{\omega_4, \omega_5, \omega_6, \omega_{10}, \omega_{14}\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall \omega \in \{\omega_3, \omega_9\}, s_{A2}^{PX}(\omega) = \{\omega_3, \omega_9\} \subset \{\omega_1, \omega_2, \omega_3, \omega_9, \omega_{13}\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall \omega \in \{\omega_1, \omega_2, \omega_{13}\}, s_{A2}^{PX}(\omega) = \{\omega_1, \omega_2, \omega_{13}\} \subset \{\omega_1, \omega_2, \omega_3, \omega_9, \omega_{13}\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$,
$\forall \omega \in \{\omega_7, \omega_8, \omega_{11}, \omega_{12}\}, s_{A2}^{PX}(\omega) = \{\omega\} \subset \{\omega_7, \omega_8, \omega_{11}, \omega_{12}\} = \{\omega' : x_{A2}(\omega') = x_{A2}(\omega)\}$.

equilibrium trade of -2 maximizes his expected payoff in this case.

(p3-iii)If agent A observes the event $\{\omega_4, \omega_5, \omega_{10}\}$, he will induce that the expected price in period 4 will be $\frac{2}{4}8 + \frac{1}{4}0 + \frac{1}{4}0 = 4$, which is equal to the current price, thus he is indifferent between trading or not at period 3, so the equilibrium trade of 2 maximizes his expected payoff in this case.

(p3-iv)If agent A observes the event $\{\omega_9\}$, he knows that with probability 1 the price in period 4 will be 0, which is less the current price $P_3(\omega_9) = 4$, thus he has incentive to sell any of the asset he owns at period 3, so under the short sale constraint the equilibrium trade of -2 maximizes his expected payoff in this case.

In period 2, there are in total 5 cases:

(p2-i)if agent A observes the event $\{\omega_n\}$ where $n \in \{7, 8, 11, 12\}$, he knows that with probability 1 the price in period 3 will be 0, which is equal to the current price, thus he is indifferent between trading or not at period 2, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p2-ii)If agent A observes the event $\{\omega_1, \omega_2, \omega_{13}\}$, he will induce that the expected price in period 3 will be $\frac{1}{6}4 + \frac{2}{6}4 + \frac{3}{6}0 = 2$, which is equal to the current price, thus he is indifferent between trading or not at period 2, so the equilibrium trade of 1 maximizes his expected payoff in this case.

(p2-iii)If agent A observes the event $\{\omega_3, \omega_9\}$, he will induce that the expected price in period 3 will be $\frac{1}{2}4 + \frac{1}{2}0 = 2$, which is equal to the current price, thus he is indifferent between trading or not at period 2, so the equilibrium trade of 1 maximizes his expected payoff in this case.

(p2-iv)If agent A observes the event $\{\omega_4, \omega_5, \omega_{10}, \omega_{14}\}$, he will induce that the expected price in period 3 will be $\frac{2}{9}4 + \frac{1}{9}4 + \frac{1}{9}4 + \frac{5}{9}0 = \frac{16}{9}$, which is less the current price 2, thus he has incentive to sell any of the asset he owns at period 2, so under the short sale constraint, the equilibrium trade of -1 maximizes his expected payoff in this case.

(p2-v)If agent A observes the event $\{\omega_6\}$, he knows that with probability 1 the price in period 3 will be 0, which is less the current price 2, thus he has incentive to sell any of the asset he owns at period 2, so under the short sale constraint, the equilibrium trade of -1 maximizes his expected payoff in this case.

In period 1, there are 3 cases:

(p1-i)If agent A observes the event $\{\omega_6, \omega_7, \omega_{12}\}$, he will induce that the expected price in period 2 will be $\frac{2}{4}2 + \frac{1}{4}0 + \frac{1}{4}0 = 1$, which is equal to the current price, thus he is indifferent between trading or not at period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p1-ii)If agent i observes the event $\{\omega_3, \omega_9, \omega_{11}\}$, he will induce that the expected price in period 2 will be $\frac{1}{4}2 + \frac{1}{4}2 + \frac{2}{4}0 = 1$, which is equal to the current price, thus he is indifferent between trading or not at period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

(p1-iii) If agent i observes the event $\{\omega_n\}$ where $n \in \{1, 2, 4, 5, 8, 10, 13, 14\}$, he will induce that the expected price in period 2 will be $\frac{1\times3+2\times2+3+5}{30}2 + \frac{15}{30}0 = 1$, which is equal to the current price, thus he is indifferent between trading or not at period 1, so the equilibrium trade of 0 maximizes his expected payoff in this case.

The above analysis guarantees that the condition (C2) is satisfied.

Check (C3) and (C4): It is seen that the market clears in each period at each state from the table of trades, hence (C3) is satisfied. Note that $P_1(\omega) = 1$ $\forall \omega \in \Omega$ hence $P_1(\cdot)$ is measurable with respect to $j_1(\cdot)$ and that $j_3(\omega) = \{\omega\}$ $\forall \omega \in \Omega$ hence $P_3(\cdot)$ is measurable with respect to $j_3(\omega)$. To see $P_2(\cdot)$ is measurable with respect to $j_2(\omega)$, note that $\forall n = 1, \cdots, 6, 9, 10, 13, 14$, $j_2(\omega_n) \subseteq \{\{\omega_n\} | n = 1, \cdots, 6, 9, 10, 13, 14\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 2\}$ and $\forall n = 7, 8, 11, 12$, $j_2(\omega_n) \subseteq \{\{\omega_n\} | n = 7, 8, 11, 12\} = \{\omega : P_2(\omega) = P_2(\omega_n) = 0\}$. To see $P_3(\cdot)$ is measurable with respect to $j_3(\omega)$, note that $\forall n = 1, 2, 5, 6, 9, 10$, $j_2(\omega_n) \subseteq \{\{\omega_n\} | n = 1, 2, 5, 6, 9, 10\} = \{\omega : P_3(\omega) = P_3(\omega_n) = 4\}$ and $\forall n = 3, 4, 7, 8, 11, \cdots, 14$, $j_2(\omega_n) \subseteq \{\{\omega_n\} | n = 3, 4, 7, 8, 11, \cdots, 14\} = \{\omega : P_3(\omega) = P_3(\omega_n) = 0\}$. This completes the check that the prices and trades given in the example constitute a rational expectations equilibrium.

1.7 Conclusion

Based on the work of Allen, Morris and Postlewaite (1993), Conlon (2004), and many others, this paper develops two new concepts of rational bubbles: a common expected bubble and a common strong bubble, and shows that in a finite-state finite-horizon model the following results hold for any finite number of agents. First, there is no common strong bubble in any rational expectations equilibrium under the perfect memory assumption. Second, there exists a three-period economy with asymmetric information and short sales constraints, where an expected bubble can exist in a rational expectations equilibrium, and moreover this bubble, is not only a strong bubble, but also a common expected bubble. The first result partially answers what properties a bubble cannot have in a rational world, and the second result tells more about what a bubble might look like, given the results in AMP (1993) and Conlon (2004). The necessary structural conditions in Section 4 provide insight into the structural characteristics of models of bubbles. One important condition is that for a strong bubble to exist in equilibrium the minimum number of states is 8.

One direction for future work will be to show the coexistence of common expected bubbles and higher order strong bubbles for any finite number of agents, following Conlon (2004) in which an example of higher order bubbles is constructed for the two-agent case. Another direction will be to introduce some irrational agents into the model and to see whether a common strong bubble can exist in such a setting. Since bubbles modeled in this paper are not robust to perturbations in a general sense, introducing noise into the model might be another good direction. It might also be important and potentially interesting to test the theory on the existence of rational bubbles by conducting experimental work.

Appendix

Appendix 1:

Proof to Proposition 2:

Write $\Omega = \{\omega_n | n = 1, 2, \cdots, 3I+2\}$. Let $\Omega_D \equiv \{\omega_n \in \Omega | n = 3i-2, i = 1, 2, \cdots, I\}$, $\Omega_{2W} \equiv \{\omega_n \in \Omega | n = 3i-1, i = 1, 2, \cdots, I\}$, $\Omega_i \equiv \{\omega_{3i-2}, \omega_{3i-1}, \omega_{3i}\}$, $\Omega_i^- \equiv \Omega_i \backslash \{\omega_{3i}\} = \{\omega_{3i-2}, \omega_{3i-1}\}$, $i = 1, 2, \cdots I$.

Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state $\omega \in \Omega_D$ and will pay nothing otherwise. Each agent is endowed with I units of money and 1 share of the risky asset at the beginning of period 1.

The specific structures of S_{it}'s are given by

$$
\begin{aligned}
S_{11} &= \{\Omega \backslash \{\omega_{3I}, \omega_{3I+1}\}, \{\omega_{3I}, \omega_{3I+1}\}\} \\
S_{i1} &= \{\Omega \backslash \{\omega_{3i-3}, \omega_{3I+1}\}, \{\omega_{3i-3}, \omega_{3I+1}\}\} \ \forall i = 2, \cdots, I \\
S_{12} &= \left\{\Omega_1, \Omega_2, \cdots, \Omega_{I-1}, \Omega_I^-, \{\omega_{3I}\}, \{\omega_{3I+1}\}, \{\omega_{3I+2}\}\right\} \\
S_{i2} &= \left\{\Omega_1, \cdots, \Omega_{i-2}, \Omega_i, \cdots, \Omega_I, \Omega_{i-1}^-, \{\omega_{3i-3}\}, \{\omega_{3I+1}\}, \{\omega_{3I+2}\}\right\} \ \forall i = 2, \cdots, I \\
S_{i3} &= S^F \ \forall i = 1, 2, \cdots, I.
\end{aligned}
$$

The agents' beliefs about the states are given by the following functions.

$$
\pi_i(\omega_n) = \begin{cases} 2W \text{ if } n = 3i-2 \text{ or } \omega_n \in \Omega_{2W} \backslash \{\omega_{3i-1}\} \\ (4I-1)W & \text{if } n = 3I+2 \\ W & \text{otherwise} \end{cases} \quad \forall i = 1, 2, \cdots, I, W = \frac{1}{8I}.
$$

To see that the belief of agent i is well defined, note that the number of elements in

Ω_{2W} is I, hence there are I states which are put with probability $2W$. Since there is only one state with probability $(3I+2)\,W$, the number of the states with probability W is $3I+2-I-1=2I+1$. Thus, $\sum_{\omega\in\Omega}\pi_i(\omega)=I\times 2W+1\times(4I-1)\,W+(2I+1)\times W=8IW=1$.

The equilibrium with the prices and trades given below is what we look for - the one in which there is a strong and common expected bubble in period 1 in state ω_{3I+1}.

$$
\begin{aligned}
P_1(\omega) &= 1\forall\omega\in\Omega. \\
P_2(\omega_n) &= \begin{cases} 0 & \text{if } n=3I+1 \text{ or } n=3I+2 \\ 2 & \text{otherwise} \end{cases}. \\
P_3(\omega_n) &= \begin{cases} 4 & \text{if } n\in\Omega_D \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

$$
\forall i=1,2,\cdots,I,\forall\omega\in\Omega, x_{i1}(\omega)=x_{i3}(\omega)=0.
$$

$$
x_{i2}(\omega_n)=\begin{cases} I-1 & \text{if } \omega_n\in\Omega_i \\ 0 \text{ if } n=3I+1 \text{ or } n=3I+2 & \\ -1 & \text{otherwise} \end{cases}\quad \forall i=1,2,\cdots,I.
$$

Observe that neither the prices nor the trades reveal any addition information with the settings above.

It can be similarly checked following the procedures described in the two-agent example that the above prices and trades constitute a rational expectations equilibrium. And since in period 1 in state ω_{3I+1}, each agent knows that he will receive nothing at the end of period 3, given the positive price of 1 in period 1, there exists a strong bubble in this equilibrium.

Note that $m_1^{PX}\left(\omega_{3I+1}\right) = \Omega$. To see that this bubble is robust to common knowledge in the expected sense, we need to check that $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, 1 > \frac{1}{\sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') d(\omega')$. Note that for agent 1 (or agent i, $i \geq 2$), either he will observe $\{\omega_{3I}, \omega_{3I+1}\}$ (or $\{\omega_{3i-3}, \omega_{3I+1}\}$), or he will observe $\Omega \backslash \{\omega_{3I}, \omega_{3I+1}\}$ (or $\Omega \backslash \{\omega_{3i-3}, \omega_{3I+1}\}$). If it is the first case, his expected dividend will be $\frac{1}{2}0 + \frac{1}{2}0 = 0$; If it is the second case, his expected dividend will be $\frac{I+1}{8I-2}4 + \frac{7I-3}{8I-2}0 = \frac{2I+2}{4I-1}$. In either case, the expected dividend is less than the price. Therefore, the bubble in period 1 in state ω_{3I+1} is a common expected bubble.

However it should noted under the structure above, $\forall \omega_n \in \Omega \backslash \{\omega_{3I+1}, \omega_{3I+2}\}$, in period 2 in state ω_n there are always $(I-1)$ agents who observes the same event $\Omega_i = \{\omega_{3i-2}, \omega_{3i-1}, \omega_{3i}\}$ [21] where i is determined such that $\omega_n \in \Omega_i$. Obviously this violates Assumption 2. In order to ensure that agents' information differs from each other everywhere when there is no perfect information, the number of the states has to be large enough to guarantee the existence of bubbles.

Appendix 2:

Proof to Proposition 3:

The case of 2 agents has already been shown in section 3. Here it suffices to consider the case when $I \geq 3$.

Write $\Omega = \{\omega_n | n = 1, 2, \cdots, I^2 + 2\}$. Let $\Omega_D \equiv \{\omega_n \in \Omega | n = I(i-1) + 1, i = 1, 2, \cdots, I\}$,

[21] Though there is one agent observing $\{\omega_n\}$ or $\Omega_i \backslash \{\omega_n\}$, Ω_i is common knowledge in this case. And this feature holds also for the constructed example under proposition.

$\Omega_{(I-1)W} \equiv \{\omega_n \in \Omega | n = I(i-1)+2, i = 1, 2, \cdots, I\}$, $\Omega_j \equiv \{\omega_n \in \Omega | I(j-1)+1 \leq n \leq Ij\}$, $\Omega_j^{-k} \equiv \Omega_j \backslash \left\{\omega_{I(j-1)+k}\right\}$, $j, k = 1, 2, \cdots I$.

Again, each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state $\omega \in \Omega_D$ and will pay nothing otherwise. Each agent is endowed with I units of money and 1 share of the risky asset at the beginning of period 1.

Let a_{ij} be the ith row and jth column element of the following $I \times I$ matrix. Hence $\omega_{I(j-1)+a_{ij}}$ is the a_{ij}th element in Ω_j.

$$\begin{bmatrix} & 2 & 3 & \cdots & I-1 & I \\ I & & 2 & \cdots & I-2 & I-1 \\ I-1 & I & & 2 & \cdots & I-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 3 & 4 & \cdots & I & & 2 \\ 2 & 3 & \cdots & I-1 & I & \end{bmatrix}$$

The specific structures of S_{it}'s are given by

$$\begin{aligned} S_{i1} &= \left\{\Omega \backslash \left\{\omega_{Ik_i}, \omega_{I^2+1}\right\}, \left\{\omega_{Ik_i}, \omega_{I^2+1}\right\}\right\} \text{ where } k_i \text{ is determined by } a_{ik_i} = I \\ S_{i2} &= \left\{\left\{\omega_{I(j-1)+a_{ij}}\right\} : 1 \leq j \leq I, j \neq i\right\} \cup \left\{\Omega_j^{-a_{ij}} : 1 \leq j \leq I, j \neq i\right\} \cup \left\{\Omega_i, \left\{\omega_{I^2+1}\right\}, \left\{\omega_{I^2+2}\right\}\right\} \\ S_{i3} &= S^F \; \forall i = 1, 2, \cdots, I. \end{aligned}$$

The agents' beliefs about the states are given by the following functions. $\forall i = 1, 2, \cdots, I$,

$$\pi_i(\omega_n) = \begin{cases} (I-1)W \text{ if } n = I(i-1)+1 \text{ or } \omega_n \in \Omega_{(I-1)W} \backslash \left\{\omega_{I(i-1)+2}\right\} \\ (2I(I-1)-1)W \quad \text{if } n = I^2+2 \\ W \quad \text{otherwise} \end{cases}, W = \frac{1}{4I(I-1)}.$$

To see that the belief of agent i is well defined, note that the number of elements in $\Omega_{(I-1)W}$ is I, hence there are I states which are put with probability $(I-1)W$. Since there is only one state with probability $(2I(I-1)-1)W$, the number of the states with probability W is $I^2+2-I-1 = I(I-1)+1$. Thus, $\sum_{\omega\in\Omega}\pi_i(\omega) = I\times(I-1)W + 1\times(2I(I-1)-1)W + (I(I-1)+1)\times W = 4I(I-1)W = 1$.

The equilibrium with the prices and trades given below is what we look for - the one in which there is a strong and common expected bubble in period 1 in state ω_{I^2+1}.

$$
\begin{aligned}
P_1(\omega) &= 1\forall\omega\in\Omega. \\
P_2(\omega_n) &= \begin{cases} 0 & \text{if } n = I^2+1 \text{ or } n = I^2+2 \\ 2 & \text{otherwise} \end{cases}. \\
P_3(\omega_n) &= \begin{cases} 4 & \text{if } n \in \Omega_D \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

$$
\forall i = 1,2,\cdots,I, \forall\omega\in\Omega, x_{i1}(\omega) = x_{i3}(\omega) = 0.
$$

$$
x_{i2}(\omega_n) = \begin{cases} I-1 & \text{if } \omega_n \in \Omega_i \\ 0 \ \text{if } n = I^2+1 \text{ or } n = I^2+2 \\ -1 & \text{otherwise} \end{cases} \ \forall i = 1,2,\cdots,I.
$$

Observe that neither the prices nor the trades reveal any addition information with the settings above.

It can be similarly checked following the procedures described in the two-agent example that the above prices and trades constitute a rational expectations equilibrium. And since in period 1 in state ω_{I^2+1}, each agent knows that he will receive nothing at the end of period 3, given the positive price of 1 in period 1, there exists a strong bubble in this equilibrium.

Note that $m_1^{PX}(\omega_{_{I^2+1}}) = \Omega$. To see that this bubble is robust to common knowledge in the expected sense, we need to check that $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, 1 > \frac{1}{\sum\limits_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega')} \sum\limits_{\omega' \in s_{i1}^{PX}(\omega)} \pi_i(\omega') \, d(\omega')$. Note that for agent 1, either he will observe $\{\omega_{Ik_i}, \omega_{I^2+1}\}$, or he will observe $\Omega \backslash \{\omega_{Ik_i}, \omega_{I^2+1}\}$. If it is the first case, his expected dividend will be $\frac{1}{2}0 + \frac{1}{2}0 = 0$; If it is the second case, his expected dividend will be $\frac{2(I-1)}{4I(I-1)-2}4 + \frac{4I(I-1)-2-2(I-1)}{4I(I-1)-2}0 = \frac{4}{2I - \frac{1}{I-1}}$. In either case, the expected dividend is less than the price. Therefore, the bubble in period 1 in state $\omega_{_{I^2+1}}$ is a common expected bubble.

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Chapter 2 When Can Forgetfulness Make Us Better Off?

2.1 Introduction

Since Akerlof's famous 1970 paper on lemon market, the problem of asymmetric information has been a hot research topic among economists. There is huge literature on the value of information, as well as the cost of information acquisition. Most of the papers in this category build a positive relationship between information and welfare: the more informative players become, the better off they are. Among these few exceptions, Levin (2001) revisits the lemon market and finds the surprising result that greater information asymmetries do not necessarily reduce the gains from trade. According to Levin (2001), better information on the selling side may worsen the welfare while better information on the buying side unambiguously improves trade. In this paper, by making slightly different assumptions, we show in a trade game example that even on the buying side more information does not lead to a better result. Moreover, this surprising result is not restricted to the lemon market; it is true in a more general setting. By studying the situations where rational players choose to remain ignorant even though the information acquisition is free, we can better understand how people behave in the world of incomplete information. Behind some seemingly weird thoughts, there may exist a rational mind. It is not always beneficial to know everything; sometimes being forgetful might make people better off.

The next section of the paper investigates three simple examples where forgetfulness does make players better off. Section 3 presents a general setup of the game where players are allowed to have imperfect memory, and characterizes necessary conditions for the existence of ration ignorance. Conclusions are drawn and Directions for future work are pointed out in the last section.

2.2 Simple Examples

2.2.1 A Trade Game

There are 2 states (ω_1 and ω_2), 2 periods (t_1 and t_2), and 2 players (A and B).

Both players assign equal probability to ω_1 and ω_2. Players receive different utilities from consumption across different states. Player A's marginal utility is 2 for every dollar of consumption made in ω_1 and 1 in ω_2. Player B's marginal utility is 1 in ω_1 and 3 in ω_2. In other words, Player A values the consumption twice as much as player B does in ω_1, and player B values the consumption three times as much as player A does in ω_2.

Player A initially owns an asset and some money m_A. At the end of t_2, the asset pays nothing in ω_1 and \$1 in ω_2. Player B initially owns m_B of money and has no asset.

In period t_1, player A offers a price P at which he is willing to sell the asset to player B. In period t_2, player B decides whether to accept or to reject player A's offer. At the end of period t_2, all the information becomes perfect, and the game ends.

Initially players may have private information on which state is realized, and they can learn additional information from the actions of the other player.

We also assume that players may have an option to be forgetful in a sense that they may not remember the state information they knew before.

2.2.1.1 Case 1: $S_{A0} = S_{B0} = \{\{\omega_1, \omega_2\}\}$

In this case, neither player A nor B has any information about the true state at the beginning of the game. So they will choose the actions which maximize their expected utilities given any information they may possibly have.

At period t_1, since player A cannot tell the difference between ω_1 and ω_2, his action (the price P he offers to player B) reveals no information to player B. Therefore, at period t_2, player B still knows nothing about the true state, and hence his expected value of the asset is $\frac{1}{2} \times 1 \times 0 + \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$. Let P_B be the highest price of the asset at which player B would like to buy. Then we have $\frac{1}{2} \times 1 \times P_B + \frac{1}{2} \times 3 \times P_B = \frac{3}{2}$, or $P_B = \frac{3}{4}$. This tells us that Player B's best response to player A's action is

$$\begin{cases} \mathit{Accept} \text{ if } P \leqslant \frac{3}{4} \\ \mathit{Reject} \text{ if } P > \frac{3}{4} \end{cases}$$

Now let's consider player A's problem. At period t_1, his expected value of the asset is $\frac{1}{2} \times 2 \times 0 + \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$. Let P_A be the lowest price of the asset at which player A would like to sell. Then we have $\frac{1}{2} \times 2 \times P_A + \frac{1}{2} \times 1 \times P_A = \frac{1}{2}$, or $P_A = \frac{1}{3}$. This tells us that as long as $P > \frac{1}{3}$, player A can benefit from the trade, and if the trade happens, the higher the price P is, the better off player A can be. Given player B's best response, player A should set P equal to $\frac{3}{4}$.

The equilibrium outcome will be (1) in period t_1, player A offers that he is willing to sell the asset at price $P = \frac{3}{4}$; (2) in period t_2, player B accepts the offer.

The equilibrium payoff for player A is $\frac{1}{2} \times 2 \times \left(m_A + \frac{3}{4}\right) + \frac{1}{2} \times 1 \times \left(m_A + \frac{3}{4}\right) = \frac{3}{2} m_A + \frac{9}{8}$, and the equilibrium payoff for player B is $\frac{1}{2} \times 1 \times \left(m_B - \frac{3}{4}\right) + \frac{1}{2} \times 3 \times \left(m_B - \frac{3}{4} + 1\right) = 2m_B$.

2.2.1.2 Case 2: $S_{A0} = \{\{\omega_1\}, \{\omega_2\}\}, S_{B0} = \{\{\omega_1, \omega_2\}\}$

In this case, initially player A knows the true state and player B knows nothing. Player

A may choose to be forgetful, then the result will be exactly the same as in case 1. Player A's payoff is $\frac{3}{2}m_A + \frac{9}{8}$ and Player B's payoff is $2m_B$.

Now let's suppose that player A chooses to remember the information he has initially.

If the true state is ω_1, the asset is valueless. In period t_1 player A immediately knows this and hence he is willing to sell the asset at any possible positive price.

If the true state is ω_2, the asset is worth 1 dollar. In period t_1 player A immediately knows this and hence he is willing to sell the asset at any price no less than 1, and won't sell the asset at any price less than 1.

There are two subcases:

(1) If player A offers the same price in both ω_1 and ω_2, then his action reveals no information to player B. Player B is in the same situation as before. Hence player B will reject any price higher than $\frac{3}{4}$. However, from the analysis above, we already know that player A will offer a price no less than 1. Therefore, there will be no trade in this case. In fact this cannot be an equilibrium outcome since in ω_1 player A will have incentive to deviate by offering a price of $\frac{3}{4}$.

(2) If player A offers different prices in different states, then his action reveals full information on states to player B. At period t_2, when player B decides whether to accept or to reject player A's offer, he surely knows the true state, and hence the true value of the asset. Therefore, Player B's best response to player A's action is

$$\begin{cases} \text{In } \omega_1, \textit{Accept} \text{ if } P(\omega_1) \leq 0 \text{ and } \textit{Reject} \text{ if } P(\omega_1) > 0 \\ \text{In } \omega_2, \textit{Accept} \text{ if } P(\omega_2) \leq 1 \text{ and } \textit{Reject} \text{ if } P(\omega_2) > 1 \end{cases}$$

Therefore, the equilibrium outcome will either be no trade, or the asset is sold at the price of its true value, which will not make anyone better off.

The equilibrium payoff for player A is $\frac{1}{2} \times 2 \times m_A + \frac{1}{2} \times 1 \times (m_A + 1) = \frac{3}{2} m_A + \frac{1}{2}$, and the equilibrium payoff for player B is $\frac{1}{2} \times 1 \times m_B + \frac{1}{2} \times 3 \times m_B = 2m_B$.

Comparing player A's equilibrium payoffs whether he chooses to forget or not, we come up with a surprising result: Player A has an incentive to be forgetful in our example. Put it in another way, if we allowed player A to have access to the information about the true state at the first beginning, he would rather not knowing that. In this example, less information makes player A strictly better off and player B as good as before. So the total welfare is improved by player A being forgetful.

Proposition 10 *For the set of trade games* $(\Omega; S_{i,0}; MU_i)$ *with* $|\Omega| \geq 2$ *and* $\pi(\omega) = \frac{1}{|\Omega|}$ $\forall \omega \in \Omega$, *if* $\exists \omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$, $\frac{MU_A(\omega_1)d_1 + MU_A(\omega_2)d_2}{MU_A(\omega_1) + MU_A(\omega_2)} < \frac{MU_B(\omega_1)d_1 + MU_B(\omega_2)d_2}{MU_B(\omega_1) + MU_B(\omega_2)}$, *and* $d_1 \neq d_2$, *then there always exists some information structure under which player* A *chooses to be forgetful in equilibrium.*

Proof. We prove by construction. Let $S_{A0} = \{\{\omega_1\}, \{\omega_2\}\} \cup S_{A0}^-$ and $S_{B0} = \{\omega_1, \omega_2\} \cup S_{B0}^-$, where S_{A0}^- and S_{B0}^- can be any partition over $\Omega \backslash \{\omega_1, \omega_2\}$. It suffices to show that when the event $\{\omega_1, \omega_2\}$ occurs, player A chooses to be forgetful in equilibrium. Similar analysis gives us $P_B = \frac{MU_B(\omega_1)d_1 + MU_B(\omega_2)d_2}{MU_B(\omega_1) + MU_B(\omega_2)}$ and $P_A = \frac{MU_A(\omega_1)d_1 + MU_A(\omega_2)d_2}{MU_A(\omega_1) + MU_A(\omega_2)}$. Without loss of generality, assume $d_1 < d_2$. Then we have $d_1 < P_A < P_B < d_2$.

If player A chooses to forget when the event $\{\omega_1, \omega_2\}$ occurs, the equilibrium outcome will be (1) in period t_1, player A offers that he is willing to sell the asset at price P_B; (2) in period t_2, player B accepts the offer. The equilibrium payoff for player A will be

$\frac{1}{2}\times MU_A(\omega_1)\times(m_A+P_B)+\frac{1}{2}\times MU_A(\omega_2)\times(m_A+P_B)=\frac{MU_A(\omega_1)+MU_A(\omega_2)}{2}(m_A+P_B)$.

If player A chooses to remember when the event $\{\omega_1,\omega_2\}$ occurs, the equilibrium outcome will either be no trade, or the asset is sold at the price of its true value, which will not make anyone better off. The equilibrium payoff for player A will be $\frac{1}{2}\times MU_A(\omega_1)\times(m_A+d_1)+\frac{1}{2}\times MU_A(\omega_2)\times(m_A+d_2)=\frac{MU_A(\omega_1)+MU_A(\omega_2)}{2}m_A+\frac{MU_A(\omega_1)d_1+MU_A(\omega_2)d_2}{2}$.

To show that $\frac{MU_A(\omega_1)+MU_A(\omega_2)}{2}P_B>\frac{MU_A(\omega_1)d_1+MU_A(\omega_2)d_2}{2}$, it suffices to show that $P_B>\frac{MU_A(\omega_1)d_1+MU_A(\omega_2)d_2}{MU_A(\omega_1)+MU_A(\omega_2)}$. This is true since $\frac{MU_A(\omega_1)d_1+MU_A(\omega_2)d_2}{MU_A(\omega_1)+MU_A(\omega_2)}=P_A$ and $P_B>P_A$.

■

2.2.2 A Cooperation Game

There are 2 states (ω_1 and ω_2) and 2 players (A and B).

Player A and B work on a public good project together. Only both of them make positive efforts, can the public good be produced. For each player, the effort e can be any real number between 0 and 1. $y=\begin{cases} y_A+y_B \text{ if } e_A\cdot e_B>0 \\ 0 \quad \text{if } e_A\cdot e_B=0 \end{cases}$

Both players can be good workers or bad workers. If player i is a good worker, his effort e_i will contribute $y_i=2e_i$ to the output of the public good. If player i is a bad worker, his effort e_i will contribute $y_i=\frac{1}{2}e_i$ to the output of the public good. The output $y=y_A+y_B$ is divided between players according to their contribution. A player's payoff will be his share of the public good minus his effort. $u_i=\frac{y_i}{y_i+y_{-i}}y-e_i$.

In ω_1 player A is a good worker and player B is a bad worker. It is the other way round for state ω_2. Both players assign equal probability to ω_1 and ω_1.

Initially players may have private information on which state is realized, and they can learn additional information from the actions of the other player.

We also assume that players may have an option to be forgetful in a sense that they may not remember the state information they knew before.

state ω_1

		B	
		E	N
A	E	$\left(e_A, -\frac{1}{2}e_B\right)$	$(-e_A, 0)$
	N	$(0, -e_B)$	$(0, 0)$

state ω_2

		B	
		E	N
A	E	$\left(-\frac{1}{2}e_A, e_B\right)$	$(-e_A, 0)$
	N	$(0, -e_B)$	$(0, 0)$

2.2.2.1 Case 1: $S_{A0} = S_{B0} = \{\{\omega_1, \omega_2\}\}$

In this case, neither player A nor B has any information about the true state at the beginning of the game. In other words, they don't know they are good workers or bad workers. So they will choose the actions which maximize their expected payoffs.

If a player chooses not to make efforts, his payoff is 0. If a player chooses to make effort e, then his payoff will be $2e - e$ if he is a good worker and $\frac{1}{2}e - e$ if he is a bad worker. Therefore, his expected payoff will be $\frac{1}{2}(2e - e) + \frac{1}{2}\left(\frac{1}{2}e - e\right) = \frac{1}{4}e$. Now we can write the payoff matrix as:

		B	
		E	N
A	E	$\left(\frac{1}{4}e_A, \frac{1}{4}e_B\right)$	$(-e_A, 0)$
	N	$(0, -e_B)$	$(0, 0)$

It is easy to see from the above payoff matrix that there are two pure strategy Nash Equilibria: $(e_A = 1, e_B = 1)$ and (N, N). And the mixed strategy Nash Equilibrium is each player making effort $e = 1$ with probability $\frac{3}{4}$ and making no effort with probability $\frac{1}{4}$.

The equilibrium we are interested in is the one where both players are making full efforts. In this case, the public good is produced at the maximum quantity level and each of the players receives a payoff of $\frac{1}{4}$.

2.2.2.2 Case 2: $S_{A0} = \{\{\omega_1\}, \{\omega_2\}\}, S_{B0} = \{\{\omega_1, \omega_2\}\}$

In this case, initially player A knows the true state and player B knows nothing. Player A may choose to be forgetful, then the result will be exactly the same as in case 1. The maximum payoff level each of them can achieve is $\frac{1}{4}$.

Now let's suppose that player A chooses to remember the information he has initially.

If the true state is ω_2, player A knows that he is a bad worker. As a bad worker, he will always receive a negative payoff if he makes positive efforts, no matter what player B's action is. And if he does not make an effort, he will have a payoff of 0. Understanding this, player A will surely choose not to make effort, since this is his dominant strategy.

If the true state is ω_1, player A knows that he is a good worker. As a good worker he will choose to make effort if player B chooses to make effort, and he will choose not to make effort if player B chooses not to. But if player A chooses to make effort, then his action in state ω_1 will be different from his action in state ω_2. Then player B can learn the true state from player A's actions. Once player B knows that the true state is ω_1, he knows he himself is a bad worker, and not making effort will be his dominant strategy. Given that player B's strategy, player A will not make effort from the beginning.

Therefore in both states, player A will not make effort. Given that player A makes no effort, player B best response is not to make effort. The equilibrium outcome will be

(N, N), and both of the players receive 0 payoffs.

Comparing player A's equilibrium payoffs whether he chooses to forget or not, we see that player A can be better off if he chooses to forget his private information about the true state. In this example, less information makes both player A and player B strictly better off. The total welfare is improved by player A being forgetful.

2.2.2.3 Case 3: $S_{A0} = S_{B0} = \{\{\omega_1\}, \{\omega_2\}\}$

In this case, initially both players know the true state.

If player B chooses to be forgetful, the situation will be the same as in case 2. And in this case we know that player A will also choose to be forgetful. By symmetry, the same result holds if player A chooses to be forgetful. This tells us that given the other player being forgetful, a player will be better off by being forgetful. The equilibrium payoff is $\frac{1}{4}$ for both players.

If both players choose to remember, then the payoff matrix is the following:

state ω_1

B

		E	N
A	E	$\left(e_A, -\frac{1}{2}e_B\right)$	$(-e_A, 0)$
	N	$(0, -e_B)$	$(0, 0)$

state ω_2

B

		E	N
A	E	$\left(-\frac{1}{2}e_A, e_B\right)$	$(-e_A, 0)$
	N	$(0, -e_B)$	$(0, 0)$

It is easy to see from the above payoff matrix that (1) in state ω_1 player B has a dominant strategy of making no effort, and (2) in state ω_2 player A has a dominant strategy of making no effort. Therefore, in both states, there will be a unique Nash Equilibrium (N, N), where both players make no efforts. The equilibrium payoff is 0 for both players.

Given the above results, a new payoff matrix regarding forgetfulness can be constructed as below:

		B	
		$Forget$	$Remember$
A	$Forget$	$\left(\frac{1}{4},\frac{1}{4}\right)$	$(0,0)$
	$Remember$	$(0,0)$	$(0,0)$

It is also easy to see from the above new payoff matrix that being forgetful weekly dominates having private information. Hence one Nash Equilibrium is $(Forget, Forget)$, where both players choose to forget private information they initially knew.

Proposition 11 *For the set of cooperation games* $(\Omega; S_{i,0}; u_i)$ *with* $|\Omega| \geq 2$*, if* $\exists \omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2$*,* $y_A(\omega_1) > \frac{\pi(\omega_1)y_A(\omega_1)+\pi(\omega_2)y_A(\omega_2)}{\pi(\omega_1)+\pi(\omega_2)} > e_A > y_A(\omega_2) > 0$ *and* $y_B(\omega_2) > \frac{\pi(\omega_1)y_B(\omega_1)+\pi(\omega_2)y_B(\omega_2)}{\pi(\omega_1)+\pi(\omega_2)} > e_B > y_B(\omega_1) > 0$*, then there always exists some information structure under which both players choose to be forgetful in equilibrium.*

Proof. We prove by construction. Let $S_{A0} = \{\{\omega_1\}, \{\omega_2\}\} \cup S_{A0}^{-}$ and $S_{B0} = \{\{\omega_1\}, \{\omega_2\}\} \cup S_{B0}^{-}$, where S_{A0}^{-} and S_{B0}^{-} can be any partition over $\Omega \backslash \{\omega_1, \omega_2\}$. It suffices to show that when the event $\{\omega_1, \omega_2\}$ occurs, both player A and player B choose to be forgetful in equilibrium. Here is the matrix of the game with perfect state information when the event $\{\omega_1, \omega_2\}$ occurs.

state ω_1

		B	
		E	N
A	E	$(y_A(\omega_1) - e_A, y_B(\omega_1) - e_B)$	$(-e_A, 0)$
	N	$(0, -e_B)$	$(0,0)$

state ω_2

	B	
	E	N
E	$(y_A(\omega_2) - e_A, y_B(\omega_2) - e_B)$	$(-e_A, 0)$
N	$(0, -e_B)$	$(0,0)$

And here is the matrix of the game with no state information when the event $\{\omega_1, \omega_2\}$

occurs.

		B	
		E	N
A	E	$\left(\frac{\pi(\omega_1)y_A(\omega_1)+\pi(\omega_2)y_A(\omega_2)}{\pi(\omega_1)+\pi(\omega_2)} - e_A, \frac{\pi(\omega_1)y_B(\omega_1)+\pi(\omega_2)y_B(\omega_2)}{\pi(\omega_1)+\pi(\omega_2)} - e_B\right)$	$(-e_A, 0)$
	N	$(0, -e_B)$	$(0, 0)$

An analysis similar to the one for the second example gives the result that both player A and player B choose to be forgetful in equilibrium. ■

2.2.3 An Example of Common Strong Bubbles with Agents of Imperfect Memory

2.2.3.1 Common Strong Bubbles

AMP (1993) has shown a strong bubble in a rational expectations equilibrium of a three-period three-agent economy with the assumption of no common knowledge of trades. Conlon (2004) strengthens this result by giving an example of strong bubbles robust to higher order knowledge with two agents where trades become automatically common knowledge. In this section, I will provide a simple example of the existence of strong bubbles robust to common knowledge. The only modification in assumptions I have made is that agents can have imperfect memory now.

Definition 9 (Strong Bubble) *As in AMP (1993), a strong bubble is said to exist in state ω at period t if in state ω it is mutual knowledge that the price of the risky asset at t is higher than the possible dividend agents will receive, that is*

$$\forall i = 1, 2, \cdots, I, \forall \omega' \in s_{it}^{PX}(\omega)\,^{22}, P_t(\omega) > d(\omega')$$

Definition 10 (Common Strong Bubble) *As in Zheng (2009), a common strong bubble is said to exist in state ω at period t if in state ω it is common knowledge that the price of the risky asset at t is higher than the possible dividend agents will receive, that is*

$$\forall \omega' \in m_t^{PX}(\omega)\,^{23}, P_t(\omega) > d(\omega')$$

[22] At the beginning of each period t, before observing the current price and making the trade, agent i's information about the state is represented by S_{it}, a partition of the space Ω, and his price–and-trade-refined information is represented by S_{it}^{PX}. We denote by $s_{it}(\omega)$ ($s_{it}^{PX}(\omega)$) the partition member in S_{it} (S_{it}^{PX}) containing the state ω.

[23] $m_t^{PX}(\omega)$ is the meet of $s_{1t}^{PX}(\omega), s_{2t}^{PX}(\omega), \cdots, s_{It}^{PX}(\omega)$.

2.2.3.2 Exogenous Setting

There are 2 agents (A and B), 3 periods (1, 2, and 3) and 8 states (ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7 and ω_8). There are only 2 assets: money and the risky asset. Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state is either ω_1 or ω_4, and will pay nothing otherwise, as shown in the table below.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$d(\omega)$	4	0	0	4	0	0	0	0

Each agent is endowed with m_i unit of money and 1 share of the risky asset at the beginning of period 1. Agents can trade in each of period 1, 2, and 3. At period 3, after the trade is made, the dividend is realized, and then the consumption takes place. The state-and-period-dependent price of the risky asset is denoted by $P_t(\omega)$. Agent i's net trade at period t in state ω is denoted by $x_{it}(\omega)$, and we write $x_i = (x_{i1}, x_{i2}, \cdots, x_{iT})$, $x_t = (x_{1t}, x_{2t}, \cdots, x_{It})$ and $x = (x_1, x_2, \cdots, x_I)$. Hence agent i's final consumption in state ω with net trades x_i at price $P(\omega) = (P_1(\omega), P_2(\omega), \cdots, P_T(\omega))$, denoted by $y_i(\omega, P(\omega), x_i)$, is equal to $m_i + e_i P_T(\omega) + \sum_{t=1}^{T} x_{it}(\omega)[P_{t+1}(\omega) - P_t(\omega)]$, where $P_{T+1}(\omega) = d(\omega)$. Assume that all agents have utiity function $u(y) = y$. Then agent i's utility in state ω with net trades x_i at price $P(\omega)$, is $y_i(\omega, P(\omega), x_i)$.

Keeping in mind that the asymmetric information is the key to generate bubbles, we achieve this goal by giving agents different information structures. Remind that agent i's ($i = A, B$) information about the state in period t ($t = 1, 2, 3$) is represented by S_{it}, a partition of the space Ω. The specific structures of S_{it}'s are given below.

In period 1, both agents receive the same information, represented by S_{A1} and S_{B1} respectively, where $S_{A1} = S_{B1}$. When it comes to period 2, both agents forget everything they knew in period 1, and then they get to receive some new information, represented by S_{A2} and S_{B2} respectively. In this case, S_{i2} is no longer necessarily a finer partition than S_{i1} is, for $i = A, B$. In period 3, again as before, each agent is perfectly informed of what the realized state is. The structure for the information partitions is shown in the table below.

$$\begin{aligned}
S_{A1} &= S_{B1} = \left\{\left\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\right\}, \left\{\omega_1, \omega_4, \omega_7\right\}\right\} \\
S_{A2} &= \left\{\left\{\omega_1, \omega_2, \omega_3\right\}, \left\{\omega_4, \omega_5\right\}, \left\{\omega_6, \omega_7\right\}, \left\{\omega_8\right\}\right\} \\
S_{B2} &= \left\{\left\{\omega_4, \omega_5, \omega_6\right\}, \left\{\omega_1, \omega_2\right\}, \left\{\omega_3, \omega_7\right\}, \left\{\omega_8\right\}\right\} \\
S_{A3} &= S_{B3} = \left\{\left\{\omega_1\right\}, \left\{\omega_2\right\}, \left\{\omega_3\right\}, \left\{\omega_4\right\}, \left\{\omega_5\right\}, \left\{\omega_6\right\}, \left\{\omega_7\right\}, \left\{\omega_8\right\}\right\}
\end{aligned}$$

The following graph may give more intuition about the information structure than the mathematical expression does. In the graph, agent A's information sets are described by the black solid curves; agent B's information sets are described by the blue dotted curves; dividend paying states are emphasized in gray color.

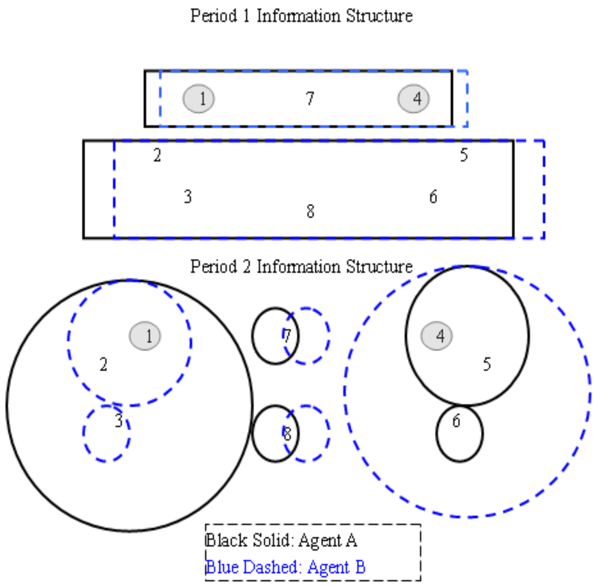

Figure 2.1: 3-Period Information Structure with Impefect Memory

The heterogeneous belief about the probability distribution of the state, for each agent, is shown in the table below with weight $W = \frac{1}{16}$.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	2	1	1	1	2	1	3	5
π_B	1	2	1	2	1	1	3	5

2.2.3.3 A Rational Expectations Equilibrium with Common Strong Bubbles

A rational expectations equilibrium will be a vector $(P, x) \in R_+^{3\times 8} \times R^{2\times 3\times 8}$ such that

(C1) $\forall i = A, B$, x_i are information feasible[24] and satisfy no short sales[25].

(C2) $\forall i = A, B$, x_i maximizes player i's expected payoff with respect to his own

[24] x_i are information feasible if $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, s_{it}^{PX}(\omega) \subseteq \{\omega' : x_{it}(\omega') = x_{it}(\omega)\}$

[25] x_i satisfy no short sales if $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, e_i + \sum_{s=0}^{t} x_{it}(\omega) \geq 0$

price-and-trade-refined information.

(C3) $\forall t = 1, 2, 3, \forall n = 1, \cdots, 8, x_{At}(\omega_n) + x_{Bt}(\omega_n) = 0.$

(C4) $\forall t = 1, 2, 3, \forall n, m = 1, \cdots, 8, j_t(\omega_n) \subseteq \{\omega_m : P_t(\omega_m) = P_t(\omega_n)\}.$

A simple calculation and check procedure will show that the above economy has a rational expectations equilibrium, which is characterized by the price table and the trade table below.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega) + x_{B2}(\omega)$	0	0	0	0	0	0	0	0

Now it is time to look for the common strong bubbles in such an equilibrium. Observe that at period 1 in any state from the set $\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}$, it is common knowledge that the dividend at period 3 will be 0. Given a positive price 1, it is exactly the case that it is common knowledge that the price of the risky asset is higher than the possible dividend agents will receive, and hence there is a common strong bubble at period 1 in any state from the set $\{\omega_2, \omega_3, \omega_5, \omega_6, \omega_8\}$.

This example shows that under the imperfect memory assumption the standard result of nonexistence of common strong bubbles is no longer valid. In the real world, it is arguable that not all people have perfect memory. Therefore, a common strong bubble may exist in an economy of the real life. This seems to be a surprising result, and it provides an alternative explanation of the existence of bubbles by the assumption of imperfect memory, instead of the assumption of noise traders. Another surprising finding with this example is that agents' welfare actually improves when they are assumed forgetful, which is also observed in the previous two examples. The last thing worth pointing out in this example is that if we allowed agents to be forgetful rather than exogenously assume they are forgetful, they would actually choose to be forgetful in equilibrium, where there is a common strong bubble.

2.3 The Model

2.3.1 Basic Setting

There are a finite set of players $I = \{1, 2, \cdots, I\}$ and a finite set of states $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$. The horizon is finite too, denoted by periods: $T = \{t_1, t_2, \cdots, t_T\}$. Player i's action at period t is denoted by $a_{i,t}$. $\forall t \in T, a_t \in A_t = \prod\limits_{i \in I} A_{i,t}$, where A_t is finite. For simplicity, assume that $\forall t \in T, \forall i, j \in I$, $A_{i,t} = A_{j,t} = A_t$. This simply means that all the players share the same action space over time.

Each player i has a subjective belief about the probability distribution of the state, denoted by $\pi_i(\omega)$. $\forall i \in I, \forall \omega \in \Omega, \pi_i(\omega) > 0$. (Better to assume different marginal utilities??)

2.3.1.1 Information Structure

The information structure for player i at period t is represented by a mathematical partition $S_{i,t}$ over the state space Ω. We denote by $s_{it}(\omega)$ the partition member in S_{it}

containing the state ω. $\forall i \in I, t \in T, s_{i,t} : \Omega \rightarrow 2^{\Omega} \backslash \phi$. In other words, $s_{it}(\omega)$ consists of all the possible states player i believes he might be in when the state ω is realized at period t. For example, $s_{i1}(\omega_1) = \{\omega_1, \omega_2\}$ means that at period 1 player i believes he might be either in ω_1 or ω_2 when ω_1 is realized.

The following are some simple features with respect to the information structure:

(1) $\forall \omega, \omega' \in \Omega, s_{i,t}(\omega) \neq s_{i,t}(\omega') \Rightarrow s_{i,t}(\omega) \cap s_{i,t}(\omega') = \phi$

(2) $\bigcup\limits_{\omega \in \Omega} s_{i,t}(\omega) = \Omega$

(3) $S_{i,t} \equiv \{s_{i,t}(\omega) : \omega \in \Omega\}$

At the beginning of period t, player i receives some private information, represented by $S^0_{i,t}$. Hence the information player i has at the beginning of period t is the total of his private information at period t and the information he has at the end of period $t-1$. $S_{i,t} = S^0_{i,t} \bigcap S^R_{i,t-1}$.

A player's strategy consists of two components: information strategy and action strategy.

A player's information strategy, denoted by $P_{i,t}$, is a map from the information profile at the beginning of period t to the set of partitions over the state space Ω. Let $p_{i,t}(\omega)$ be the partition member in $P_{i,t}(S_t)$ containing the state ω. $\forall i \in I, t \in T, \forall \omega \in \Omega, p_{i,t}(\omega) \supseteq s_{i,t}(\omega)$. This simply means that players can choose to forget some information on the states at the beginning of each period of time. Let $P_t = (P_{1,t}, P_{2,t}, \cdots, P_{I,t})$ and $P = (P_1, P_2, \cdots, P_I)$

A player's action strategy, denoted by $\sigma_{i,t}$, is a function from his information strategy to his action space. $\forall i \in I, t \in T, \sigma_{i,t} : P_{i,t} \to \Delta A_i$. Let $\sigma_t = (\sigma_{1,t}, \sigma_{2,t}, \cdots, \sigma_{I,t})$ and $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_I)$.

A player's payoff is a real-valued function dependent on both states and all the players' strategies over the time. $u_i : \Omega \times \prod\limits_{t \in T} \sigma_t \to R$.

2.3.1.2 Strategic Learning

Players are assumed to be smart in a sense that they can actively learn state information from how other players behave and the learning has an immediate effort on players' behaves. We assume that $\forall t \in T$, the additional information revealed from actions is public, denoted by $s_{a,t}$. $s_{a,t} : K_t \to 2^{\Omega} \backslash \phi$, where K_t is the largest subset of $\Omega \times A_t$ such that $\forall \left(\omega, a_t\right), \left(\omega'.a_t'\right) \in K_t, a_t \neq a_t' \Rightarrow \omega \neq \omega'$.

We assume the additional information must satisfy the following conditions:

(4) $\forall \left(\omega, a_t\right), \left(\omega'.a_t'\right) \in K_t, a_t \neq a_t' \Rightarrow s_{a,t}\left(\omega, a_t\right) \cap s_{a,t}\left(\omega', a_t'\right) = \phi$

(5) $\forall \left(\omega, a_t\right), \left(\omega'.a_t'\right) \in K_t, a_t = a_t' \Rightarrow s_{a,t}\left(\omega, a_t\right) = s_{a,t}\left(\omega', a_t'\right)$

(6) $\forall \left(\omega, a_t\right), \left(\omega'.a_t'\right) \in K_t, s_{a,t}\left(\omega, a_t\right) \neq s_{a,t}\left(\omega', a_t'\right) \Rightarrow s_{a,t}\left(\omega, a_t\right) \cap s_{a,t}\left(\omega', a_t'\right) = \phi$

(7) $\bigcup\limits_{(\omega, a_t) \in K_t} s_{a,t}\left(\omega, a_t\right) = \Omega$

The partition based on the additional information at period t is denoted by $S_{a,t}$, where $S_{a,t} \equiv \left\{s_{a,t}\left(\omega, a_t\right) : \left(\omega, a_t\right) \in K_t\right\}$. The refined information for player i at period t, denoted by $S_{i,t}^R$, is the total of his private information updated by the information strategy and the additional information. $S_{i,t}^R = P_{i,t} \bigcap S_{a,t}$. Let $s_{i,t}^R\left(\omega\right)$ be the partition member in $S_{i,t}^R$ containing the state ω.

2.3.1.3 Imperfect Memory

The concept of information strategy captures the key idea of imperfect memory. Player i is forgetful at period t, if $S_{i,t} \prec P_{i,t}(S_t)$ ($P_{i,t}(S_t)$ is coarser than $S_{i,t}$). Under this assumption, a player's information partition can be coarser and coarser over the time when the player chooses to be forgetful, as opposed to what is assumed in the standard literature. Just to put an emphasis on this assumption, we write down these observations below.

$$\forall \omega \in \Omega, \forall t_1, t_2 \in T,$$

$$t_1 < t_2 \nRightarrow p_{i,t_2}(\omega) \subseteq p_{i,t_1}(\omega)$$

$$\forall \omega \in \Omega, \forall t_1, t_2 \in T, \forall (\omega, a_{t_1}) \in K_{t_1}, \forall (\omega, a_{t_2}) \in K_{t_2},$$

$$t_1 < t_2 \nRightarrow s_{a,t_2}(\omega, a_{t_2}) \subseteq s_{a,t_1}(\omega, a_{t_1})$$

2.3.2 Equilibrium

Now we are ready to give the definition of an Equilibrium for this game.

Definition 11 (P, σ) *is an equilibrium of the game* $(I; \Omega; T; A; \pi; S; u)$ *if*

$$\forall i \in I, (P_i, \sigma_i) \in \arg\max_{P_i', \sigma_i'} E_{\pi_i(\omega), \omega \in \Omega}\left[u_i(\omega, \sigma_i', \sigma_{-i}) \,|S_{i,1}^R\right]$$

It is worth noting that in the above expression, the information strategy P_i' affects payoff not only through the action strategy σ_i', but also through the refined information partition $S_{i,1}^R$.

Proposition 12 (Existence) *There exists an equilibrium* (P, σ)*, defined above, for the* *game* $(I;\Omega;T;A;\pi;S;u)$.

Proof. The game has finite number of players, states, periods and actions, hence an equilibrium always exists. ■

2.3.3 Necessary Conditions

Proposition 13 *If in equilibrium player i has strong incentive to be forgetful, then there must exist another player j such that i and j have negatively correlated preferences. Here, the negatively correlated preferences between i and j means that, $\exists\omega,\omega'\in\Omega$, $b\in A_i$, $c\in A_j$, $u_i(\omega,a_i=b,a_j=c,\cdots)>u_i(\omega',a_i=b,a_j=c,\cdots)$ and $u_j(\omega,a_i=b,a_j=c,\cdots)<u_j(\omega',a_i=b,a_j=c,\cdots)$.*

Proof. Suppose not. Then for any player j other than i, i and j do not have negatively correlated preferences. This means $\forall\omega,\omega'\in\Omega$, $b\in A_i$, $c\in A_j$, $u_i(\omega,a_i=b,a_j=c,\cdots)>u_i(\omega',a_i=b,a_j=c,\cdots)$ implies $u_j(\omega,a_i=b,a_j=c,\cdots)\geq u_j(\omega',a_i=b,a_j=c,\cdots)$. This indicates that the action profile chosen by players in equilibrium in state ω will be exactly the same as the action profile chosen in equilibrium in state ω'. In this case, no matter how much information players have, the equilibrium remains the same. Therefore, player i will have no strong incentive to be forgetful, which causes the contradiction. ■

Proposition 14 *If in equilibrium player i has strong incentive to be forgetful at period t, then it must be the case that i' actions would reveal additional information if he chose not to be forgetful. That is $S^R_{i,t}\neq P_{i,t}$.*

Proof. We prove by contradiction. Suppose not. Then there exists an equilibrium (P,σ) where $S_{i,t}\prec P_{i,t}(S_t)$ such that

(1) $E_{\pi_i(\omega),\omega\in\Omega}\left[u_i(\omega,\sigma_i,\sigma_{-i})\,|S^R_{i,1}\right]>E_{\pi_i(\omega),\omega\in\Omega}\left[u_i(\omega,\sigma'_i,\sigma_{-i})\,|S^R_{i,1}\right]\ \forall P'_{i,t}\neq P_{i,t}$ and

(2) $S_{j,t}^{R}=P_{j,t}$.

Now consider player i at period t. If he chooses not to be forgetful, then his expected payoff will be

$$E_{\pi_i(\omega),\omega\in\Omega}\left[u_i\left(\omega,\sigma_i\left(S_{i,t}\right),\sigma_{-i}\left(P_{-i,t}\right)\right)|S_{i,t}^{R}\right]=E_{\pi_i(\omega),\omega\in\Omega}\left[u_i\left(\omega,\sigma_i\left(S_{i,t}\right),\sigma_{-i}\left(P_{-i,t}\right)\right)|S_{i,t}\right],$$

since $S_{i,t}^{R}=P_{i,t}\left(S_t\right)=S_{i,t}$.

If he chooses to be forgetful, then his expected payoff will be

$$E_{\pi_i(\omega),\omega\in\Omega}\left[u_i\left(\omega,\sigma_i\left(P_{i,t}\right),\sigma_{-i}\left(P_{-i,t}\right)\right)|S_{i,t}^{R}\right]=E_{\pi_i(\omega),\omega\in\Omega}\left[u_i\left(\omega,\sigma_i\left(P_{i,t}\right),\sigma_{-i}\left(P_{-i,t}\right)\right)|P_{i,t}\right].$$

It is easy to see that

$$E_{\pi_i(\omega),\omega\in\Omega}\left[u_i\left(\omega,\sigma_i\left(S_{i,t}\right),\sigma_{-i}\left(P_{-i,t}\right)\right)|S_{i,t}\right]\geq E_{\pi_i(\omega),\omega\in\Omega}\left[u_i\left(\omega,\sigma_i\left(P_{i,t}\right),\sigma_{-i}\left(P_{-i,t}\right)\right)|S_{i,t}^{R}\right]$$

since $S_{i,t}\prec P_{i,t}$. This gives a contradiction to (1). ■

2.4 Conclusion

The relationship between information and welfare is not necessarily positive. In some situations, people have strong incentive to remain ignorant even though the learning is costless. This paper tries to establish a general model to study the behavior of rational ignorance, and two necessary conditions for rational ignorance in symmetric games are provided. These results will have important applications in policy design: it might be desirable to have social institutions under which some records are destroyed after a period of time.

For future work, it would be both important and interesting to characterize more features of the rational ignorant phenomenon. We also seek to find sufficient conditions for players being forgetful in equilibrium, if there is any. A more accurate and complete answer to the question about the relationship between how much we know and how well off we are would lead to a better understanding of how people behave in the real world.

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Chapter 3 The Robustness of Bubbles in a Finite Horizon Model

3.1 Introduction

The robustness of bubbles has been one of the important and diffcult topics in the study of bubbles in asset markets. On the one hand side, we do see many economic phenomena presenting bubble features (for instance, internet bubbles and housing bubbles) in the real world persist for a long time period. On the other hand side, most economic models of bubbles are not very robust to perturbations. In other word, the existence of bubbles in these models requires strong conditions to be satisfied, and the bubbles will easily disappear if small changes in the parameters occur. Therefore, economists have been looking for a model of bubbles with the robustness feature, which can better interprete the real world phenomena.

Among all the models of bubbles, Allen, Morris and Postlewaite (1993) and Abreu and Brunnermeier (2003) are particularly two different frameworks that well explain the bubble stories based on the assumptions of asymmetric information and short sales constraints. However, these models do not present a roubustness feature. Under a small perturbation of certain parameter in the environment, the bubble equilibrium can crash immediately.

According to the author's knowledge, Doblas-Madrid (2009) is the first to provide a robust model of bubbles with multidimensional uncertainty based on the Abreu and Brunnermeier (2003) framework. In the model, agents observe a noisy price that reflects a mix of noise and sales, and receive signals that indicate that the asset is over priced, but do not know exactly when the bubble crashes. This multidimensional uncertainty leads to a robust bubble equilibrium.

However, the finite horizon setting of Allen, Morris and Postlewaite (1993) is different from the infinite horizonframeworks following Abreu and Brunnermeier (2003), hence it is hard to construct a robust model based on Allen, Morris and Postlewaite (1993) by simply borrowing the tricks in Doblas-Madrid (2009). Conlon (2010) proposes that the introduction of continuum of states can lead to a robust bubble equilibrium where each bad type of the seller pools with some good type of the seller.

In the following, we will first study the bubble examples in Zheng (2011) based on Allen-Morris-Postlewaite model and focus on the symmetric case. We define two class of symmetric perturbations: *strongly symmetric perturbations* and *very symmetric perturbations*. Then we show that these bubbles are robust to both *strongly symmetric perturbations* in beliefs and *very symmetric perturbations* in dividends, but not robust to general perturbations.

In order to have a robust bubble example, we need to assume that the states are continous rather than discrete. We construct a new three-period two-agent robust bubble example where small variations in parameters do not eliminate the bubble equilibria. The key idea is that in equilibrium each bad type of the seller pools with some good type of the seller and hence it is impossible for the buyer to separate the bad states from the good states. This provides a new answer to the question: How robust can rational bubbles be in a finite horizon model?

The next section of the essay introduces Zheng (2011)'s basic setting with discrete states following AMP (1993). Section 3 defines the concept of symmetry and symetric perturbation, and focuses on two certain classes of symmetric perturbations. Section 4 shows that the bubble example we described above is robust to both strongly symmetric perturbations in beliefs and very symmetric perturbations in dividends. Section 5 constructs a continous-state example where the bubble is robust. Section 6 provides concluding remarks and directions for further study.

3.2 The Discrete-State Model

3.2.1 Basic Setup

As in Zheng (2011), there are I (≥ 2) risk neutral[26] agents ($i = 1, 2, \cdots, I$), T (≥ 3) periods ($t = 1, 2, \cdots, T$) and N (≥ 2) states of the world represented by $\omega \in \Omega$. Only 2 assets exist in the market: one riskless (money) and the other risky. There is no discount between any two periods. Each share of the risky asset will only pay a state-dependent dividend denoted by $d(\omega)$ at the end of period T.

Agent i is endowed with m_i units of money and e_i shares of the risky asset at the beginning of period 1. In each period t and in each realized state ω, agents can exchange claims on the risky asset at a state-and-period-dependent price $P_t(\omega)$. Agent i's net trade in period t when state ω is realized is denoted by $x_{it}(\omega)$, and we write $x_i = (x_{i1}, x_{i2}, \cdots, x_{iT})$, $x_t = (x_{1t}, x_{2t}, \cdots, x_{It})$, and

26 Agents are assumed to be either risk averse or risk neutral in AMP (1993). Here for simplicity, I only consider the case of risk neutrality. All the results will remain valid for the risk averse case as long as the potential gain from trade is high enough.

$x = (x_1, x_2, \cdots, x_I)$. Hence agent i's final consumption in state ω with net trades x_i at price $P(\omega) = (P_1(\omega), P_2(\omega), \cdots, P_T(\omega))$, denoted by $y_i(\omega, P(\omega), x_i)$, is equal to $m_i + e_i P_T(\omega) + \sum_{t=1}^{T} x_{it}(\omega) [P_{t+1}(\omega) - P_t(\omega)]$, where $P_{T+1}(\omega) = d(\omega)$. Let $u_i(\cdot)$ be agent i's utility function. Then agent i's utility in state ω with net trades x_i at price $P(\omega)$, is $u_i(y_i(\omega, P(\omega), x_i))$. For simplicity, assume that $u_i(\cdot)$ is the identity function for all i.

Each agent i has a subjective belief about the probability distribution of the state, denoted by $\pi_i(\omega)$.[27] $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \pi_i(\omega) > 0$.

3.2.2 Information Structure

At the beginning of each period t, before observing the current price and making the trade, agent i's information about the state is represented by S_{it}, a partition of the space Ω, and his price–and-trade-refined information is represented by S_{it}^{PX}.[28] We denote by $s_{it}(\omega)$ ($s_{it}^{PX}(\omega)$) the partition member in S_{it} (S_{it}^{PX}) containing the state ω. In other words, $s_{it}(\omega)$ consists of all the possible states agent i believes he might be in when the state ω is realized in period t. For example, $s_{i1}(\omega_1) = \{\omega_1, \omega_2\}$ means that in period 1 agent i believes he might be either in ω_1 or ω_2 when ω_1 is realized.

S_{it}^{PX} is determined by (S_{it}, P_t, x_t) such that

$$\forall \omega \in \Omega, s_{it}^{PX}(\omega) = s_{it}(\omega) \cap \{\omega' | P_{t'}(\omega') = P_{t'}(\omega) \text{ and } x_{t'}(\omega') = x_{t'}(\omega) \ \forall t' \leq t\}.$$

27 We may either assume same utility function with heterogeneous beliefs, or assume common prior with different utility functions, in order to give agents an incentive to trade. Here we adopt the former one and in the next version we may also consider the latter. For other approaches to induce trade, see AMP (1993) for details.

28 In the AMP (1993) model, they only focus on the price-refined information S_{it}^{P}. In their model it is assumed that the trades are not common knowledge and hence agents cannot get additional information from trades.

Obviously $\forall i = 1, 2, \cdots, I, \forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, \{\omega\} \subseteq s_{it}^{PX}(\omega) \subseteq s_{it}(\omega)$.

We assume agents have perfect memory so that

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \forall t > t', s_{it}(\omega) \subseteq s_{it'}(\omega).$$

Obviously this implies that

$$\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \forall t > t', s_{it}^{PX}(\omega) \subseteq s_{it'}^{PX}(\omega).$$

3.2.3 Rational Expectations Equilibrium

Definition 12 (Information Feasibility) *Agent i's net trades x_i are information feasible if in each period t, x_{it} is measurable with respect to player i's price–and-trade-refined information, S_{it}^{PX}. Formally, x_i are information feasible if*

$$\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, s_{it}^{PX}(\omega) \subseteq \{\omega' : x_{it}(\omega') = x_{it}(\omega)\}.$$

Definition 13 (No Short Sales) *Agent i's net trades x_i satisfy no short sales if in each period t and in each state ω agent i's holdings of the risky asset are non-negative. Formally, x_i satisfy no short sales if*

$$\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, e_i + \sum_{s=0}^{t} x_{it}(\omega) \geq 0.$$

Denote by $j_t(\omega)$ the join of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$,[29] and by $m_t(\omega)$ the meet of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$.[30]

Definition 14 (Rational Expectations Equilibrium) $(P, x) \in R_+^{NT} \times R^{INT}$ *is a Rational Expectations Equilibrium if*

(C1) $\forall i = 1, 2, \cdots, I$, x_i are information feasible and satisfy no short sales. Denote the set of all such x_i's by $F_i(e_i, P, x_{-i}, S_i)$, where $S_i = (S_{i1}, S_{i2}, \cdots, S_{iT})$;[31]

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[29] The join $j_t(\omega)$ of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$ is such that (1) $\forall i = 1, 2, \cdots, I, j_t(\omega) \subseteq s_{it}(\omega)$ and (2) for all $j_t'(\omega)$ satisfying (1), $j_t'(\omega) \subseteq j_t(\omega)$. It is also called the coarsest common refinement.

[30] The meet $m_t(\omega)$ of $s_{1t}(\omega), s_{2t}(\omega), \cdots, s_{It}(\omega)$ is such that (1) $\forall i = 1, 2, \cdots, I, s_{it}(\omega) \subseteq m_{it}(\omega)$ and (2) for all $m_t'(\omega)$ satisfying (1), $m_t(\omega) \subseteq m_t'(\omega)$. It is also called the finest common coarsening.

[31] Since $\forall x_i \in F_i$, x_i are information feasible, F_i depends on the information structure S_i, the prices P, and

(C2) $\forall i = 1, 2, \cdots, I$, $x_i \in \arg\max_{x_i' \in F_i(e_i, P, x_{-i}, S_i)} \sum_{\omega \in \Omega} \pi_i(\omega) u_i(y_i(\omega, P, x_i'))$;[32]

(C3) $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, \sum_{i=1}^{I} x_{it}(\omega) = 0$;

(C4) $\forall t = 1, 2, \cdots, T, P_t(\cdot)$ *is measurable with respect to* $j_t(\omega)$. *Formally,* $\forall t = 1, 2, \cdots, T, \forall \omega \in \Omega, j_t(\omega) \subseteq \{\omega' : P_t(\omega') = P_t(\omega)\}$.

Basically, (C1) describes the feasible set of trade for each agent, (C2) says that each agent maximizes his expected utility given his price-and-trade-refined information, (C3) requires that the market should clear in equilibrium, and (C4) implies that all the information contained in price is from the join of the individual information.

3.2.4 Different Concepts of Bubbles

Definition 15 (Expected Bubble) *As in AMP (1993), an expected bubble is said to exist in state* ω *in period* t *if in state* ω *it is mutual knowledge that the price of the risky asset in period* t *is higher than the expected dividend an agent will receive, that is*

$$\forall i = 1, 2, \cdots, I, P_t(\omega) > \frac{1}{\sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega')} \sum_{\omega' \in s_{it}^{PX}(\omega)} \pi_i(\omega') d(\omega').$$

Definition 16 (Strong Bubble) *As in AMP (1993), a strong bubble is said to exist in state* ω *in period* t *if in state* ω *it is mutual knowledge that the price of the risky asset in period* t *is higher than the maximum possible dividend an agent will receive, that is*

$$\forall i = 1, 2, \cdots, I, \forall \omega' \in s_{it}^{PX}(\omega), P_t(\omega) > d(\omega').$$

As seen from above, the concept of strong bubbles strengthens the concept of expected bubbles in a way that it requires that the asset price be higher than the maximum possible dividend, not just the expected dividend. As will be seen below, another way to strengthen the concept of expected bubbles is to require common knowledge instead of mutual

other agents' trades x_{-i}. Since x_i satisfy no short sales, F_i depends on the endowment e_i. That's why it is written as $F_i(e_i, P, x_{-i}, S_i)$.

[32] Another perhaps more intuitive way to express (C2) is (C2') $\forall i = 1, 2, \cdots, I$, $x_i \in \arg\max_{x_i' \in F_i(e_i, P, x_{-i}, S_i)} E_i\left[u_i(y_i(\omega, P, x_i')) | S_{i1}^{PX}\right]$. It is easy to see that (C2') is equivalent to (C2).

knowledge. This requirement is reasonable since in the real world people's behaviors do not only depend on their own beliefs, but also depend on others' beliefs, others' beliefs on their own beliefs, and so on. Therefore, we might expect to see something different when common knowledge is introduced into the concept of bubbles.

Definition 17 (Common Expected Bubble) *A common expected bubble is said to exist in state ω in period t if in state ω it is common knowledge that the price of the risky asset in period t is higher than the expected dividend an agent will receive, that is*

$$\forall i = 1, 2, \cdots, I, \forall \omega' \in m_t^{PX}(\omega), P_t(\omega) > \frac{1}{\sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'')} \sum_{\omega'' \in s_{it}^{PX}(\omega')} \pi_i(\omega'') d(\omega'').$$ [7]

3.3 Symmetric Perturbation

According to AMP (1993), by the nature of the model, such a bubble is not robust, neither to perturbations in beliefs nor to perturbations in dividends. However, for an economy with symmetric structure, we find that the equilibria with these bubbles, though are not robust to perturbations in a general sense, but might be robust to perturbations in a symmetric sense.

In this section, we focus on three-period models with a symmetric setting.

Definition 18 (Symmetry) *The model has a symmetric setting if for any $i, j = 1, 2, \cdots, I$, there exists a bijective mapping L from $\{1, 2, \cdots, N = |\Omega|\}$ to $\{1, 2, \cdots, N\}$ such that for any $t = 1, 2, 3$,*

(1) $S_{it} = S_{jt}|L$, where $S_{jt}|L$ is j's relabelled information partition at t under L;

(2) $\pi_i(\omega_n) = \pi_j\left(\omega_{L(n)}\right)$;

(3) $d(\omega_n) = d\left(\omega_{L(n)}\right)$;

(4) $(m_i, e_i) = (m_j, e_j)$.

7 $m_t^{PX}(\omega)$ is the meet of $s_{1t}^{PX}(\omega), s_{2t}^{PX}(\omega), \cdots, s_{It}^{PX}(\omega)$.

Basically equation (1) means that it is information-symmetric. Similarly it is belief-symmetric by (2), dividend-symmetric by (3), and endowment-symmetric by (4).

It should be noted that the symmetry assumption is more than assuming symmetry w.r.t information, symmetry w.r.t. dividend, symmetry w.r.t. belief, and symmetry w.r.t. endowment, respectively. That is because we require the same mapping L for conditions (1)-(3) to be satisfied.

We call $\left(\omega_n, \omega_{L(n)}\right)$ a symmetric pair of states for agent i and j if $L\left(L\left(n\right)\right) = n$.

Recall that a state where there is a strong bubble is called a *bubble state*, denoted by ω^*.

A zero-dividend state is called a *bubble-related state* for agent i, denoted by $\omega^{*,i}$, if (1) it is not a bubble state and (2) agent i cannot tell the difference between this state and the bubble state in the first period. Note there may be more than one bubble-related state for agent i.

A zero-dividend state is called a *dummy state*, ω^D, if when this state is realized (1) no agents are sure about their future payoff in the first period and (2) all of them know that the asset is worthless in the second period. A dummy state is necessary for a strong bubble to exist in equilibrium in our model because of the equilibrium conditions.

In a three-period model, bubble bursts in the second period, which implies that in a bubble state all agents know that the asset is worthless in the second period. This is the same feature between bubble state and dummy state. The difference is that in the first period in a bubble state all agents know that the asset is worthless while in a dummy state they are not sure about the value of the asset.

For instance, in the example of bubbles in Section 3, the setting is symmetric, and for $i = A, j = B$, we have the relabelling function

$$L(n) = \begin{cases} n+3 \text{ if } n = 1,2,3 \\ n-3 \text{ if } n = 4,5,6 \\ n \text{ if } n = 7,8 \end{cases}.$$

It is easy to see that (ω_1, ω_4), (ω_2, ω_5), (ω_3, ω_6) (ω_7, ω_7), (ω_8, ω_8) are symmetric pairs of states. Here ω_7 is the bubble state, ω_6 is the bubble-related state for agent A, ω_3 is the bubble-related state for agent B, and ω_8 is the dummy state.

Now we are ready to give a definition to the symmetric perturbation.

Definition 19 (Symmetric Perturbation) *For a model with symmetric setting, a perturbation $\eta : \Omega \to R$ is Symmetric if for any symmetric pair of states $(\omega_m, \omega_n), m, n \in \{1, 2, \cdots, N\}$,*

$$\eta(\omega_m) = \eta(\omega_n).$$

Even though mathematically symmetric perturbations are of measure zero when we consider the whole family of perturbations, it does make economic sense to look at this particular type of perturbations. First, economic systems function in a way that same or similar shocks are received in symmetric states. Second, symmetric states may be generated by the same fundamental factor, and hence should be perturbed by the same amount.

In addition to symmetric perturbations, we can have even stronger concepts for perturbations.

Definition 20 (Very Symmetric Perturbation) *For a model with symmetric setting, a perturbation $\eta : \Omega \to R$ is Very Symmetric if (1) it is Symmetric;*

and (2) for the bubble state ω^* and the dummy state ω^D,

$$\eta(\omega^*) = \eta(\omega^D).$$

It is straightforward to see from the definition that a very symmetric perturbation requires same perturbations not only for a symmetric pair of states, but also for a pair of bubble state and dummy state. Since in both bubble state and dummy state the asset is worthless and this becomes agents' mutual knowledge in the second period, it is reasonable to think about the situation where the dividend perturbations for a pair of bubble state and dummy state are the same.

Definition 21 (Strongly Symmetric Perturbation) *For a model with symmetric setting, a perturbation $\eta : \Omega \to R$ is Strongly Symmetric if (1) it is Symmetric;*

(2) for the bubble state ω^* and the dummy state ω^D

$$\eta\left(\omega^*\right) = -\eta\left(\omega^D\right);$$

and (3) for any $i = 1, 2, \cdots, I$, for the bubble state ω^* and all the bubble-related state(s) $\omega^{*,i}$

$$\frac{\eta\left(\omega^*\right)}{\pi_i\left(\omega^*\right)} = \frac{\sum \eta\left(\omega^{*,i}\right)}{\sum \pi_i\left(\omega^{*,i}\right)}.$$

A strongly symmetric perturbation is different from a very symmetric perturbation in two ways. First, for the pair of bubble state and dummy state, the former requires the same amount toward opposite directions while the latter requires the same amount toward the same direction. Second, for the bubble state and all the bubble-related state(s), the former requires the amount proportional on the prior while the latter has no restriction on it. A strongly symmetric perturbation makes sense when we consider a perturbation in beliefs. Condition (2) can be interpreted as the following: if you increase the probability for the bubble state, you have to decrease the probability for the dummy state by the same amount. Condition (3) is reasonable because it requires the perturbation in beliefs does not affect agents' beliefs of having a strong bubble when the bubble state is realized in the first period.

As will be shown next, these two particular types of symmetric perturbations are of our interest because they play an important role in the robust analysis.

3.4 The Robustness Analysis for the Symmetric Case

To illustrate the results, we use the following example of bubbles in Zheng (2011) for perturbation analysis.

There are 2 agents (A and B), 3 periods (1, 2, and 3) and 8 states (ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7 and ω_8). Only 2 assets exist in the market: one is money and the other is called a risky asset. Each share of the risky asset will pay a dividend of amount 4 at the end of period 3 if the state is either ω_1 or ω_4, and will pay nothing otherwise, as shown in the table below.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$d(\omega)$	4	0	0	4	0	0	0	0

Each agent is endowed with m_i unit of money and 1 share of the risky asset at the beginning of period 1. Agents can trade in each of period 1, 2, and 3. In period 3, after the trade is made, the dividend is realized, and then the consumption takes place.

Agent i's ($i = A, B$) information about the state in period t ($t = 1, 2, 3$) is represented by S_{it}, a partition of the space Ω. The specific structures of S_{it}'s are given by

$$
\begin{aligned}
S_{A1} &= \{\{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_8\},\{\omega_6,\omega_7\}\} \\
S_{B1} &= \{\{\omega_1,\omega_2,\omega_4,\omega_5,\omega_6,\omega_8\},\{\omega_3,\omega_7\}\} \\
S_{A2} &= \{\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5\},\{\omega_6,\omega_7\},\{\omega_8\}\} \\
S_{B2} &= \{\{\omega_4,\omega_5,\omega_6\},\{\omega_1,\omega_2\},\{\omega_3,\omega_7\},\{\omega_8\}\} \\
S_{A3} &= S_{B3} = \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\},\{\omega_6\},\{\omega_7\},\{\omega_8\}\}.
\end{aligned}
$$

Agents have heterogeneous beliefs, as shown in the table below with weight $W = \frac{1}{16}$.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	2	1	1	1	2	1	1	7
π_B	1	2	1	2	1	1	1	7

Recall that the equilibrium is characterized by the price table and the trade table below.

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$P_1(\omega)$	1	1	1	1	1	1	1	1
$P_2(\omega)$	2	2	2	2	2	2	0	0
$P_3(\omega)$	4	0	0	4	0	0	0	0

$$\forall \omega \in \Omega, x_{A1}(\omega) = x_{B1}(\omega) = x_{A3}(\omega) = x_{B3}(\omega) = 0$$

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$x_{A2}(\omega)$	1	1	1	-1	-1	-1	0	0
$x_{B2}(\omega)$	-1	-1	-1	1	1	1	0	0
$x_{A2}(\omega)+x_{B2}(\omega)$	0	0	0	0	0	0	0	0

There are potentially two ways to make perturbations: one is through belief distribution and the other is through dividend distribution.

3.4.1 Belief Perturbation

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
π_A	$2+\varepsilon_{A,1}$	$1+\varepsilon_{A,2}$	$1+\varepsilon_{A,3}$	$1+\varepsilon_{A,4}$	$2+\varepsilon_{A,5}$	$1+\varepsilon_{A,6}$	$1+\varepsilon_{A,7}$	$7+\varepsilon_{A,8}$
π_B	$1+\varepsilon_{B,1}$	$2+\varepsilon_{B,2}$	$1+\varepsilon_{B,3}$	$2+\varepsilon_{B,4}$	$1+\varepsilon_{B,5}$	$1+\varepsilon_{B,6}$	$1+\varepsilon_{B,7}$	$7+\varepsilon_{B,8}$

Suppose the original equilibrium is the one with a bubble in period 1 in state ω_7, which was shown previously. Now for each state ω_n, the associated belief $\pi_i(\omega_n)$ (or denoted by $\pi_{i,n}$ for simplicity) for agent $i(i = A, B)$ is perturbed by a very small amount $\varepsilon_{i,n}$, where $\sum_{1\leq n\leq 8} \varepsilon_{i,n} = 0$, $i = A, B$. Suppose the information structure remains the same and the agents trade the same way in the new equilibrium as before, then it suffices to have the new equilibrium prices satisfy the following equations, denoted by BP.

$$
\begin{aligned}
P_3(\omega_n) &= d_n, n = 1, 2, \cdots 8. \\
P_2(\omega_{n=1,2,3}) &= \frac{d_1(\pi_{A,1}+\varepsilon_{A,1}) + d_2(\pi_{A,2}+\varepsilon_{A,2}) + d_3(\pi_{A,3}+\varepsilon_{A,3})}{\pi_{A,1}+\pi_{A,2}+\pi_{A,3}+\varepsilon_{A,1}+\varepsilon_{A,2}+\varepsilon_{A,3}}. \\
P_2(\omega_{n=4,5,6}) &= \frac{d_4(\pi_{B,4}+\varepsilon_{B,4}) + d_5(\pi_{B,5}+\varepsilon_{B,5}) + d_6(\pi_{B,6}+\varepsilon_{B,6})}{\pi_{B,4}+\pi_{B,5}+\pi_{B,6}+\varepsilon_{B,4}+\varepsilon_{B,5}+\varepsilon_{B,6}}. \\
P_2(\omega_n) &= d_n, n = 7, 8. \\
P_1(\omega_{n,1\leq n\leq 8}) &= \frac{P_2(\omega_6)(\pi_{A,6}+\varepsilon_{A,6}) + P_2(\omega_7)(\pi_{A,7}+\varepsilon_{A,7})}{\pi_{A,6}+\pi_{A,7}+\varepsilon_{A,6}+\varepsilon_{A,7}} \\
&= \frac{P_2(\omega_3)(\pi_{B,3}+\varepsilon_{B,3}) + P_2(\omega_7)(\pi_{B,7}+\varepsilon_{B,7})}{\pi_{B,3}+\pi_{B,7}+\varepsilon_{B,3}+\varepsilon_{B,7}} \\
&= \frac{\sum_{1\leq n\leq 8, n\neq 6,7} P_2(\omega_n)(\pi_{A,n}+\varepsilon_{A,n})}{\sum_{1\leq n\leq 8, n\neq 6,7}(\pi_{A,n}+\varepsilon_{A,n})} \\
&= \frac{\sum_{1\leq n\leq 8, n\neq 3,7} P_2(\omega_n)(\pi_{B,n}+\varepsilon_{B,n})}{\sum_{1\leq n\leq 8, n\neq 3,7}(\pi_{B,n}+\varepsilon_{B,n})}.
\end{aligned}
$$

3.4.1.1 Strongly Symmetric Perturbations

If the perturbation is strongly symmetric, then by definition we have the following conditions.

$$\begin{aligned}
\varepsilon_{A,1} &= \varepsilon_{B,4}, \varepsilon_{B,1} = \varepsilon_{A,4}, \\
\varepsilon_{A,2} &= \varepsilon_{B,5}, \varepsilon_{B,2} = \varepsilon_{A,5}, \\
\varepsilon_{A,3} &= \varepsilon_{B,6}, \varepsilon_{B,3} = \varepsilon_{A,6}, \\
\varepsilon_{A,7} &= \varepsilon_{B,7}, \varepsilon_{A,8} = \varepsilon_{B,8}, \\
\varepsilon_{A,6} &= \varepsilon_{A,7} = -\varepsilon_{A,8}.
\end{aligned}$$

Keep in mind that $P_2(\omega_7) = P_2(\omega_8) = 0$ in our example, which means that the price of the asset is zero in both bubble state and dummy state in period 2. Note that $P_2(\omega^*) = P_2\left(\omega^D\right) = 0$ is not necessarily true in general, but it always holds for a three-period model with a strong bubble in equilibrium.

Consider the prices specified below. It is easy to check that these prices automatically satisfy the set of equations BP.

$$\begin{aligned}
P_3(\omega_n) &= d_n, n = 1, 2, \cdots 8. \\
P_2(\omega_{n,1\leq n\leq 6}) &= \begin{cases} \frac{4\left(2+\varepsilon_{A,1}\right)}{4+\varepsilon_{A,1}+\varepsilon_{A,2}+\varepsilon_{A,3}} & \text{if } 1 \leq n \leq 6 \\ 0 & \text{if } n = 7, 8 \end{cases} . \\
P_1(\omega_{n,1\leq n\leq 8}) &= \frac{2\left(2+\varepsilon_{A,1}\right)}{4+\varepsilon_{A,1}+\varepsilon_{A,2}+\varepsilon_{A,3}}.
\end{aligned}$$

This implies that we have found a new equilibrium with the above equilibrium prices.

The last step is to check whether the coexistence of a strong bubble and a common expected bubble is still true in period 1 in state ω_7. The answer is yes as long as the perturbation is sufficiently small such that $P_1 > \frac{6}{7}$,[33] or $-4\varepsilon_{A,1} + 3\varepsilon_{A,2} + 3\varepsilon_{A,3} < 2$. This can be guaranteed by assuming $\max_{i=A,B,1\leq n\leq 8} |\varepsilon_{i,n}| < \frac{1}{5}$. Therefore, the bubble in our example is robust to any strongly symmetric perturbations in beliefs if $\max_{i=A,B,1\leq n\leq 8} |\varepsilon_{i,n}| < \frac{1}{5}$.

It is also worth noting that this result can also be applied to a more general case where the overpriced asset is not necessarily worthless. As long as the dividend in the bubble state and dummy state are the same ($d_7 = d_8$), hence $P_2(\omega_7) = P_2(\omega_8)$, then we still have the same result.

3.4.2 Dividend Perturbation

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$d(\omega)$	$4+\delta_1$	δ_2	δ_3	$4+\delta_4$	δ_5	δ_6	δ_7	δ_8

Suppose the original equilibrium is the one we studied before. Now for each state ω_n, the associated dividend d_n is perturbed by a very small amount δ_n. Suppose the information structure remains the same and the agents trade the same way in the new equilibrium as before, then it suffices to have the new equilibrium prices satisfy the following equations, denoted by DP.

[33] The number $\frac{6}{7}$ was obtained when we check the existence of a common expected bubble in Section 3.

$$
\begin{aligned}
P_3\left(\omega_n\right) &= d_n+\delta_n, n=1,2,\cdots,8. \\
P_2\left(\omega_{n=1,2,3}\right) &= \frac{\pi_{A,1}\left(d_1+\delta_1\right)+\pi_{A,2}\left(d_2+\delta_2\right)+\pi_{A,3}\left(d_3+\delta_3\right)}{\pi_{A,1}+\pi_{A,2}+\pi_{A,3}}. \\
P_2\left(\omega_{n=4,5,6}\right) &= \frac{\pi_{B,4}\left(d_4+\delta_4\right)+\pi_{B,5}\left(d_5+\delta_5\right)+\pi_{B,6}\left(d_6+\delta_6\right)}{\pi_{B,4}+\pi_{B,5}+\pi_{B,6}}. \\
P_2\left(\omega_n\right) &= d_n+\delta_n, n=7,8. \\
P_1\left(\omega_{n,1\leq n\leq 8}\right) &= \frac{\pi_{A,6}P_2\left(\omega_6\right)+\pi_{A,7}P_2\left(\omega_7\right)}{\pi_{A,6}+\pi_{A,7}} \\
&= \frac{\pi_{B,3}P_2\left(\omega_3\right)+\pi_{B,7}P_2\left(\omega_7\right)}{\pi_{B,3}+\pi_{B,7}} \\
&= \frac{\sum_{1\leq n\leq 8, n\neq 6,7}\pi_{A,n}P_2\left(\omega_n\right)}{\sum_{1\leq n\leq 8, n\neq 6,7}\pi_{A,n}} \\
&= \frac{\sum_{1\leq n\leq 8, n\neq 3,7}\pi_{B,n}P_2\left(\omega_n\right)}{\sum_{1\leq n\leq 8, n\neq 3,7}\pi_{B,n}}.
\end{aligned}
$$

3.4.2.1 Very Symmetric Perturbations

If the perturbation is very symmetric, then by definition we have the following equations.

$$
\begin{aligned}
\delta_1 &= \delta_4, \\
\delta_2 &= \delta_5, \\
\delta_3 &= \delta_6, \\
\delta_7 &= \delta_8.
\end{aligned}
$$

Consider the prices specified below. It is easy to check that these prices automatically satisfy the set of equations DP.

$$\begin{aligned} P_3(\omega_n) &= d_n + \delta_n, n = 1, 2, \cdots 8. \\ P_2(\omega_{n,1\leq n\leq 6}) &= \begin{cases} 2 + \frac{2\delta_1+\delta_2+\delta_3}{4} & \text{if } 1 \leq n \leq 6 \\ \delta_n & \text{if } n = 7, 8 \end{cases} . \\ P_1(\omega_{n,1\leq n\leq 8}) &= 1 + \frac{2\delta_1 + \delta_2 + \delta_3 + 4\delta_7}{8}. \end{aligned}$$

This implies that we have found a new equilibrium with the above equilibrium prices. The last step is to check whether the coexistence of a strong bubble and a common expected bubble is still true in period 1 in state ω_7. Similarly, the answer is yes as long as the perturbation is sufficiently small such that $P_1 > \frac{6}{7}$, or $2\delta_1 + \delta_2 + \delta_3 + 4\delta_7 > -\frac{8}{7}$. This can be guaranteed by assuming $\max_{1\leq n\leq 8} |\delta_n| < \frac{1}{7}$. Therefore, the bubble in our example is robust to any very symmetric perturbations in dividends if $\max_{1\leq n\leq 8} |\delta_n| < \frac{1}{7}$.

Similarly here we don't necessarily require that $d_7 = d_8 = 0$. The result holds as long as $d_7 = d_8$, which implies $P_2(\omega_7) = P_2(\omega_8)$.

3.5 A Robust Bubble with Continuous States

3.5.1 Exogenous Setting

There are 2 agents (Ellen and Frank), 3 periods (1, 2, and 3) and continua of states $\Omega = \left\{b_\alpha, e^B_\alpha, f^B_\alpha, e^G_{1\beta}, e^G_{2\beta}, e^G_{3\beta}, f^G_{1\beta}, f^G_{2\beta}, f^G_{3\beta} | \alpha, \beta \in [0,1]\right\}$. Only 2 assets exist in the market: one is money and the other is called a risky asset. Each share of the risky asset will pay a dividend of amount 1 at the end of period 3 if the state is either $e^G_{3\beta}$ or $f^G_{3\beta}$ for all $\beta \in [0,1]$, and will pay nothing otherwise.

Each agent is endowed with m_i units of money and 1 share of the risky asset at the beginning of period 1. Agents can trade in each of periods 1, 2, and 3. At period 3, after the trade is made, the dividend is realized, and consumption takes place.

Ellen and Frank have common priors with probability density function $\pi(\omega) = \frac{1}{9}$ for all $\omega \in \Omega$. It is easy to check that $\int_{\Omega} \pi(\omega)\, d\omega = 1$.

Their marginal utility levels are given by the table below.

State	b_α	e^B_α	f^B_α	$e^G_{1\beta}$	$e^G_{2\beta}$	$e^G_{3\beta}$	$f^G_{1\beta}$	$f^G_{2\beta}$	$f^G_{3\beta}$
$MU_E(\omega)$	α	α^2	$\frac{1+2\alpha-\sqrt{1+4\alpha}}{2\sqrt{1+4\alpha}}$	β	β^2	β^3	2β	β^2	$2\beta^3$
$MU_F(\omega)$	α	$\frac{1+2\alpha-\sqrt{1+4\alpha}}{2\sqrt{1+4\alpha}}$	α^2	2β	β^2	$2\beta^3$	β	β^2	β^3

Let Ellen's period 1 information sets be

$$E^B_\alpha = \left\{b_\alpha, e^B_\alpha\right\}$$

$$E^G_\beta = \left\{e^G_{1\beta}, e^G_{2\beta}, e^G_{3\beta}\right\}$$

$$E_{Buyer} = \left\{f^B_\alpha, f^G_{1\beta}, f^G_{2\beta}, f^G_{3\beta} | \alpha, \beta \in [0,1]\right\}$$

And Ellen's period 2 informations sets are

$$\begin{aligned} E^0_{\alpha 12} &= \{b_\alpha\}, E^0_{\beta 22} = \left\{e^G_{1\beta}\right\}, E^0_{\beta 32} = \left\{f^G_{1\beta}\right\} \\ E^B_{\alpha 2} &= \left\{e^B_\alpha\right\}, E^G_{\beta 2} = \left\{e^G_{2\beta}, e^G_{3\beta}\right\} \\ E_{Buyer2} &= \left\{f^B_\alpha, f^G_{2\beta}, f^G_{3\beta} | \alpha, \beta \in [0,1]\right\} \end{aligned}$$

By symmetry, Frank's period 1 information sets are

$$F_{\alpha}^{B} = \left\{ b_{\alpha}, f_{\alpha}^{B} \right\}$$

$$F_{\beta}^{G} = \left\{ f_{1\beta}^{G}, f_{2\beta}^{G}, f_{3\beta}^{G} \right\}$$

$$F_{Buyer} = \left\{ e_{\alpha}^{B}, e_{1\beta}^{G}, e_{2\beta}^{G}, e_{3\beta}^{G} | \alpha, \beta \in [0,1] \right\}$$

And Frank's period 2 informations sets are

$$\begin{aligned}
F_{\alpha 12}^{0} &= \{b_{\alpha}\}, F_{\beta 22}^{0} = \left\{ f_{1\beta}^{G} \right\}, F_{\beta 32}^{0} = \left\{ e_{1\beta}^{G} \right\} \\
F_{\alpha 2}^{B} &= \left\{ f_{\alpha}^{B} \right\}, F_{\beta 2}^{G} = \left\{ f_{2\beta}^{G}, f_{3\beta}^{G} \right\} \\
F_{Buyer2} &= \left\{ e_{\alpha}^{B}, e_{2\beta}^{G}, e_{3\beta}^{G} | \alpha, \beta \in [0,1] \right\}
\end{aligned}$$

At period 3, all the information becomes perfect.

3.5.2 An Equilibrium with a Bubble

We are interested in the equilibrium where there is a strong bubble. The equilibrium can be decomposed into 3 cases according to which states of the world occur:

(1) Nature chooses α. (this is the pooling case, where the buyer cannot distinguish the bad seller identifed by α from the good seller identified by $\beta = \frac{-1+\sqrt{1+4\alpha}}{2}$), the equilibrium prices and trades are given in the following tables.

State	b_α	e^B_α	f^B_α
$P_1(\omega)$	$P_1(\alpha)$	$P_1(\alpha)$	$P_1(\alpha)$
$P_2(\omega)$	0	$P_2(\alpha)$	$P_2(\alpha)$
$P_3(\omega)$	0	0	0

$$P_1(\alpha) = \frac{1+2\alpha-\sqrt{1+4\alpha}}{2(1+\alpha)}, P_2(\alpha) = \frac{1+2\alpha-\sqrt{1+4\alpha}}{2\alpha},$$

$$P_1(\alpha) = P_1^L(\beta), P_2(\alpha) = P_2^L(\beta) \text{ where } \alpha = \beta+\beta^2$$

$$\forall\omega\in\Omega, x_1^E(\omega) = x_1^F(\omega) = x_3^E(\omega) = x_3^F(\omega) = 0$$

State	b_α	e^B_α	f^B_α
$x_2^E(\omega)$	0	-1	1
$x_2^F(\omega)$	0	1	-1
$x_2^E(\omega)+x_2^F(\omega)$	0	0	0

(2) Nature chooses $\beta \in \left[0, \frac{\sqrt{5}-1}{2}\right]$ (this is the pooling case, where the buyer cannot distinguish the bad seller identifed by $\alpha = \beta+\beta^2$ from the good seller identified by β), the equilibrium prices and trades are given in the following tables.

State	$e^G_{1\beta}$	$e^G_{2\beta}$	$e^G_{3\beta}$	$f^G_{1\beta}$	$f^G_{2\beta}$	$f^G_{3\beta}$
$P_1(\omega)$	$P_1^L(\beta)$	$P_1^L(\beta)$	$P_1^L(\beta)$	$P_1^L(\beta)$	$P_1^L(\beta)$	$P_1^L(\beta)$
$P_2(\omega)$	0	$P_2^L(\beta)$	$P_2^L(\beta)$	0	$P_2^L(\beta)$	$P_2^L(\beta)$
$P_3(\omega)$	0	0	1	0	0	1

$$P_1^L(\beta) = \frac{\beta^2}{1+\beta+\beta^2}, P_2^L(\beta) = \frac{\beta}{1+\beta}$$

$$\forall\omega\in\Omega, x_1^E(\omega) = x_1^F(\omega) = x_3^E(\omega) = x_3^F(\omega) = 0$$

State	$e^G_{1\beta}$	$e^G_{2\beta}$	$e^G_{3\beta}$	$f^G_{1\beta}$	$f^G_{2\beta}$	$f^G_{3\beta}$
$x_2^E(\omega)$	0	-1	-1	0	1	1
$x_2^F(\omega)$	0	1	1	0	-1	-1
$x_2^E(\omega)+x_2^F(\omega)$	0	0	0	0	0	0

(3) Nature chooses $\beta \in (\frac{\sqrt{5}-1}{2}, 1]$ (in this case the good seller can identify himself), the equilibrium prices and trades are given in the following tables.

State	$e_{1\beta}^G$	$e_{2\beta}^G$	$e_{3\beta}^G$	$f_{1\beta}^G$	$f_{2\beta}^G$	$f_{3\beta}^G$
$P_1(\omega)$	$P_1^H(\beta)$	$P_1^H(\beta)$	$P_1^H(\beta)$	$P_1^H(\beta)$	$P_1^H(\beta)$	$P_1^H(\beta)$
$P_2(\omega)$	0	$P_2^H(\beta)$	$P_2^H(\beta)$	0	$P_2^H(\beta)$	$P_2^H(\beta)$
$P_3(\omega)$	0	0	1	0	0	1

$$P_1^H(\beta) = \frac{2\beta^2(1+\beta)}{(1+2\beta)\left(1+\beta+\beta^2\right)}, P_2^H(\beta) = \frac{2\beta}{1+2\beta}$$

$$\forall \omega \in \Omega, x_3^E(\omega) = x_3^F(\omega) = 0$$

State	$e_{1\beta}^G$	$e_{2\beta}^G$	$e_{3\beta}^G$	$f_{1\beta}^G$	$f_{2\beta}^G$	$f_{3\beta}^G$
$x_1^E(\omega)$	1	1	1	-1	-1	-1
$x_1^F(\omega)$	-1	-1	-1	1	1	1
$x_1^E(\omega)+x_1^F(\omega)$	0	0	0	0	0	0
$x_2^E(\omega)$	0	-2	-2	0	2	2
$x_2^F(\omega)$	0	2	2	0	-2	-2
$x_2^E(\omega)+x_2^F(\omega)$	0	0	0	0	0	0

It is easy to see that in period 1 at state b_α, there is a strong bubble, since the price $P_1(\alpha) = \frac{1+2\alpha-\sqrt{1+4\alpha}}{2(1+\alpha)}$ is positive while the dividend is zero.

The following analysis will make it more clear why the price and trade tables above constitute an equilibrium. By symmetry, it suffices to only consider Ellen's case.

In period 1 good Ellen's confidence level is a function of the signal β:

$$h_G(\beta) = \frac{M_E\left(e_{2\beta}^G, e_{3\beta}^G\right)}{M_E\left(e_{1\beta}^G, e_{2\beta}^G, e_{3\beta}^G\right)} = \frac{\beta+\beta^2}{1+\beta+\beta^2}$$

And bad Ellen's confidence level is a function of the signal α:

$$h_B(\alpha) = \frac{M_E\left(e_\alpha^B\right)}{M_E\left(b_\alpha, e_\alpha^B\right)} = \frac{\alpha}{1+\alpha}$$

Thus, as long as β is not too big, the bad Ellen of type α might be able to pool with some good Ellen of type β, where $\alpha = \beta + \beta^2$. Since $\alpha \in [0,1]$, for pooling to be possible, there must be some solution to the inequality $\beta + \beta^2 \leq 1$. And this together with condition $\beta \in [0,1]$ gives the range of β for the pooling case: $\beta \in \left[0, \frac{\sqrt{5}-1}{2}\right]$.

3.5.2.1 Pooling States

Let's consider the pooling case first (The bad Ellen of type α poolING with some good Ellen of type $\beta \in \left[0, \frac{\sqrt{5}-1}{2}\right]$ where $\alpha = \beta + \beta^2$).

Since at period 1 the bad Ellen of type $\alpha = \beta + \beta^2$ can pool with the good Ellen of type β, Frank in this case only knows he is in some state of set $\left\{e_\alpha^B, e_{1\beta}^G, e_{2\beta}^G, e_{3\beta}^G\right\}$ where $\alpha = \beta + \beta^2$. Since $\frac{M_F\left(e_\alpha^B\right)(1+2\beta)+M_F\left(e_{2\beta}^G, e_{3\beta}^G\right)}{M_F\left(e_\alpha^B\right)(1+2\beta)+M_F\left(e_{1\beta}^G, e_{2\beta}^G, e_{3\beta}^G\right)}$[34] $= \frac{\beta^2+\beta^2+2\beta^3}{\beta^2+2\beta+\beta^2+2\beta^3} = \frac{\beta+\beta^2}{1+\beta+\beta^2} = h_G(\beta) = h_B(\alpha)$, where $\alpha = \beta + \beta^2$, Frank has the same expected price as the good Ellen of type β and the bad Ellen of type $\alpha = \beta + \beta^2$. As a result, there is no trade at period 1. And from the analysis we know the equilibrium prices must satisfy the following expressions:

$$\begin{aligned}
P_1^L(\beta) &= \frac{\beta+\beta^2}{1+\beta+\beta^2} P_2^L(\beta) \\
P_1(\alpha) &= \frac{\alpha}{1+\alpha} P_2(\alpha) \\
P_1(\alpha) &= P_1^L(\beta), P_2(\alpha) = P_1^L(\beta) \text{ where } \alpha = \beta + \beta^2
\end{aligned}$$

At period 2, if the state is either b_α, $e_{1\beta}^G$, or $f_{1\beta}^G$, the equilibrium price is 0 since there is

34 Here $\alpha'(\beta) = 1 + 2\beta$.

no private information in that case. If the state is either e_{α}^{B}, $e_{2\beta}^{G}$ or $e_{3\beta}^{G}$, Frank cannot tell difference between them, so he will form an "expected" price equal to

$$P_2^L(\beta) = \frac{M_F\left(e_{3\beta}^G\right)}{M_F\left(e_\alpha^B\right)(1+2\beta) + M_F\left(e_{2\beta}^G, e_{3\beta}^G\right)} d\left(e_{3\beta}^G\right) = \frac{\beta}{1+\beta}$$

or

$$P_2(\alpha) = \frac{M_F\left(e_{3\beta}^G\right)\frac{1}{\sqrt{1+4\alpha}}}{M_F\left(e_\alpha^B\right) + M_F\left(e_{2\beta}^G, e_{3\beta}^G\right)\frac{1}{\sqrt{1+4\alpha}}} d\left(e_{3\beta}^G\right)^{35} = \frac{1+2\alpha-\sqrt{1+4\alpha}}{2\alpha}$$

where

$$P_2(\alpha) = P_2^L(\beta) \text{ for } \alpha = \beta + \beta^2$$

And in this case Ellen actually knows whether she is a good type or a bad type. If she is a good type, her "expected" price would be $\frac{M_E\left(e_{3\beta}^G\right)}{M_E\left(e_{2\beta}^G, e_{3\beta}^G\right)} d\left(e_{3\beta}^G\right) = \frac{\beta}{1+\beta}$. Since $\frac{\beta}{1+\beta} = P_2^L(\beta)$, the good Ellen does not feel bad to sell all of her asset to Frank. If she is a bad type, her "expected" price would be 0, and she would be happy to sell. This situation is shown in the trade table.

Given $P_2^L(\beta) = \frac{\beta}{1+\beta}$ and $P_2(\alpha) = \frac{1+2\alpha-\sqrt{1+4\alpha}}{2\alpha}$, we can get

3.5.2.2

$$\begin{aligned} P_1^L(\beta) &= \frac{\beta+\beta^2}{1+\beta+\beta^2} P_2^L(\beta) = \frac{\beta^2}{1+\beta+\beta^2} \\ P_1(\alpha) &= \frac{\alpha}{1+\alpha} P_2(\alpha) = \frac{1+2\alpha-\sqrt{1+4\alpha}}{2(1+\alpha)} \end{aligned}$$

where

[35] Here $\beta'(\alpha) = \frac{1}{\sqrt{1+4\alpha}}$.

$$P_1(\alpha) = P_1^L(\beta) \text{ for } \alpha = \beta + \beta^2$$

3.5.2.3 Nonpooling States

Now let's consider the nonpooling case (the good seller of type $\beta \in (\frac{\sqrt{5}-1}{2}, 1]$ can identify himself).

At period 1 the good Ellen of type $\beta \in (\frac{\sqrt{5}-1}{2}, 1]$ actually won't be pooled with any bad Ellen. And this refines Frank's information sets. Now Frank knows he is in some state of set $\left\{e_{1\beta}^G, e_{2\beta}^G, e_{3\beta}^G\right\}$, rather than $\left\{e_\alpha^B, e_{1\beta}^G, e_{2\beta}^G, e_{3\beta}^G\right\}$ in the previous case. Since $\frac{M_F\left(e_{2\beta}^G, e_{3\beta}^G\right)}{M_F\left(e_{1\beta}^G, e_{2\beta}^G, e_{3\beta}^G\right)} = \frac{\beta^2+2\beta^3}{2\beta+\beta^2+2\beta^3} = \frac{\beta+2\beta^2}{2+\beta+2\beta^2} < h_G(\beta)$,[36] Frank has a lower "expected" price than the good Ellen of type β. As a result, Ellen will be buying and Frank will be selling, the equilibrium price will be determined from Ellen's side. And from the analysis we know the equilibrium prices must satisfy the following expression:

$$P_1^H(\beta) = \frac{\beta+\beta^2}{1+\beta+\beta^2} P_2^H(\beta)$$

At period 2, the price of asset for state $e_{1\beta}^G$ crashes. For the remaining two states ($e_{2\beta}^G$ and $e_{3\beta}^G$), since $\frac{M_F\left(e_{3\beta}^G\right)}{M_F\left(e_{2\beta}^G, e_{3\beta}^G\right)} = \frac{2\beta}{1+2\beta} > \frac{\beta}{1+\beta} = \frac{M_E\left(e_{3\beta}^G\right)}{M_E\left(e_{2\beta}^G, e_{3\beta}^G\right)}$, the equilibrium price will be determined from Frank's side. And this price is equal to

$$P_2^H(\beta) = \frac{M_F\left(e_{3\beta}^G\right)}{M_F\left(e_{2\beta}^G, e_{3\beta}^G\right)} d\left(e_{3\beta}^G\right) = \frac{2\beta}{1+2\beta}$$

Given $P_2^H(\beta) = \frac{2\beta}{1+2\beta}$, we can get

36 $\frac{1}{1+\beta/2+\beta^2} > \frac{1}{1+\beta+\beta^2} \Rightarrow \frac{2}{2+\beta+2\beta^2} > \frac{1}{1+\beta+\beta^2} \Rightarrow \frac{\beta+2\beta^2}{2+\beta+2\beta^2} < \frac{\beta+\beta^2}{1+\beta+\beta^2}$

3.5.2.4

$$P_1^H(\beta) = \frac{\beta+\beta^2}{1+\beta+\beta^2} P_2^H(\beta) = \frac{2\beta^2(1+\beta)}{(1+2\beta)\left(1+\beta+\beta^2\right)}$$

3.5.2.5 Remarks

It is worth noting that:

(1) $P_2^H(\beta) = \frac{2\beta}{1+2\beta}$ is increasing in β, and for $\beta \in (\frac{\sqrt{5}-1}{2}, 1]$, $P_2^H(\beta) \in (\frac{5-\sqrt{5}}{5}, \frac{2}{3}]$.

(2) $P_2^L(\beta) = \frac{\beta}{1+\beta}$ is increasing in β, and for $\beta \in [0, \frac{\sqrt{5}-1}{2}]$, $P_2^L(\beta) \in [0, \frac{3-\sqrt{5}}{2}]$

(3) there is no overlapping range for $P_2^H(\beta)$ and $P_2^L(\beta)$, which makes it impossible that the good types to pool with each other. And our equilibrium depends on this fact crucially.

(4) It makes sense that both P_2^H and P_2^L are increasing in β because the good type takes the seller's role at period 2.

(5) Both P_1^H and P_1^L are also increasing in β.

3.6 Conclusions

This essay studies the robustness problem for models of rational bubbles based on the Allen-Morris-Postlewaite (1993) framework. It is shown that with the discrete-state assumption the bubbles are only robust to both *strongly symmetric perturbations* in beliefs and *very symmetric perturbations* in dividends, but not robust to general perturbations. However, it is possible to have a continous-state robust model of bubbles where prices reveals the state information and small perturbations in states do not ruin the equilibrium in general. This provides a new answer to the question: How robust can rational bubbles be in a finite horizon model?

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Chapter 4 A Note on "Depth of Knowledge and the Effect of Higher Order Uncertainty"

Morris, Shin and Postlewaite (1993) show an upper bound of asset prices in Rational Expectations Equilibrium in a finite horizon model. This note strengthens their result by providing a tighter upper bound and hence offers a better answer to the question: How large can a bubble be in equilibrium?

4.1 Basic Setup

There are I (≥ 2) agents ($i = 1, 2, \cdots, I$), T (≥ 3) periods ($t = 1, 2, \cdots, T$) and N (≥ 2) states of the world represented by $\omega \in \Omega$.

Only 2 assets exist in the market: one riskless (money) and the other risky. There is no discount between any two periods. Each share of the risky asset will only pay a state-dependent nonnegative dividend denoted by $d(\omega)$ at the end of period T. Let $d^* \equiv \max_{\omega \in \Omega} d(\omega)$ be the maximal value of dividends. Let $q_t(\omega)$ be the price of the risky asset in state ω at period t.

Each agent i has a subjective belief about the probability distribution of the state, denoted by $\pi_i(\omega)$. $\forall i = 1, 2, \cdots, I, \forall \omega \in \Omega, \pi_i(\omega) > 0$.

Each agent i's information about the state at time t is represented by a partition of the space Ω. We denote by $P_{it}(\omega)$ the partition member containing the state ω. In other words, $P_{it}(\omega)$ consists of all the possible states agent i believes he might be in when the state ω is realized at time t.

4.2 Definitions and Notations

Definition 22 $B_{it}^{p}F \equiv \{\omega \in \Omega | \pi_i [F|P_{it}(\omega)] \geq p\}$ *is the set of states where event* F *is believed with at least probability* p *by player* i *at time* t*.*

Definition 23 $B_{*t}^{p}F \equiv \bigcap_{i=1}^{I} B_{it}^{p}F$ *is the set of states where event* F *is believed with at least probability* p *by every player at time* t*.*

Definition 24 $K_{it}F \equiv \{\omega \in \Omega | P_{it}(\omega) \subseteq F\}$ *is the set of states where event* F *is known to be true by player* i *at time* t*.*

Definition 25 $K_{*t}F \equiv \bigcap_{i=1}^{I} K_{it}F$ *is the set of states where event* F *is known to be true by every player at time* t*. We will say that the event* F *is mutual knowledge in the event* $K_{*t}F$ *at time* t*.*

Definition 26 $K_{*t}^{m}F \equiv K_{*t}(K_{*t}(\cdots K_{*t}(F)\cdots))$ *is the set of states where everyone knows that everyone knows that* $\cdots$ *that everyone knows* F *(to the* m*th order) at time* t*. We will say that the event* F *is* m*th order mutual knowledge in the event* $K_{*t}^{m}F$ *at time* t*.*

Definition 27 $CK_tF \equiv \lim_{m\to\infty} K_{*t}^{m}F$ *is the set of states where everyone knows that everyone knows that* $\cdots$ *that everyone knows* F *(to any finite order) at time* t*. We will say that the event* F *is common knowledge in the event* CK_tF *at time* t*.*

We say an event F is evident at time t if $F \subseteq CK_tF$.

We say an event F is evident p-belief at time t if $F \subseteq B_{*t}^{p}F$.

4.3 New Result for Theorem 4.2 in MPS(1995)

Theorem 15 (Theorem 4.2 in MPS(1995)) *If every player* p*-believes at time* t *that every player will* p*-believe at time* $t+1$ *that every player will* p*-believe at time* $t+2$ *that ... every player will* p*-believe at time* $T-1$ *that the asset is worthless, then the price of the asset at time* t *is no more than* $(1-p)(T-t)\,d^{*}$*. That is*
$B_{*t}^{p}B_{*(t+1)}^{p}\cdots B_{*(T-1)}^{p}\Omega_T(p) \subseteq \Omega_t(p)$, *where* $\Omega_t(p) = \{\omega \in \Omega | q_t(\omega) \leq (1-p)(T-t)\,d^{*}\}$

Theorem 16 (Stronger Result for Theorem 4.2 in MPS(1995)) *If every player* p*-believes at time* t *that every player will* p*-believe at time* $t+1$ *that every player will* p*-believe at time* $t+2$ *that ... every player will* p*-believe at time* $T-1$ *that the asset is worthless, then the price of the asset at time* t *is no more than* $\left(1-p^{T-t}\right)d^{*}$*. That is*
$B_{*t}^{p}B_{*(t+1)}^{p}\cdots B_{*(T-1)}^{p}\Omega_T(p) \subseteq \Omega_t(p)$, *where* $\Omega_t(p) = \left\{\omega \in \Omega | q_t(\omega) \leq \left(1-p^{T-t}\right)d^{*}\right\}$

Proof. It suffices to show that for all t, $B^p_{*t}\Omega_{t+1}(p) \subseteq \Omega_t(p)$. Suppose $\omega \in B^p_{*t}\Omega_{t+1}(p)$. Then each player i assigns at most probability $1-p$ to states not in $\Omega_{t+1}(p)$, where the price is at most d^*. At states in $\Omega_{t+1}(p)$, the price is at most $\left(1-p^{T-t-1}\right)d^*$. Thus, for each i, the expectation of the price in the next period is no more than $(1-p)\,d^* + p\left(1-p^{T-t-1}\right)d^* = \left(1-p+p-p^{T-t}\right)d^* = \left(1-p^{T-t}\right)d^*$. So $\omega \in B^p_{*t}\Omega_t(p)$. ■

4.3.1 Remarks:

4.3.1.1 The Comparison with MPS (1995) Result

Let $a_t = \left(1-p^{T-t}\right)d^*$. a_t can be viewed as an upper bound of the price at time t. Now let us compare a_t with the value of the upper bound in Morris-Postlewaite-Shin (1995), denoted by $c_t = \min\{(1-p)(T-t), 1\}\,d^*$.

We know the following mathematical facts:

(1) $1-p^{T-t} \leq 1$ with inequality being strict for $p \in (0,1]$; and

(2) $1-p^{T-t} \leq (1-p)(T-t)$ with inequality being strict for $p \in [0,1)$ and $t < T-1$. (To see why it is true, $1-p^{T-t} = (1-p)\sum_{\tau=0}^{T-t-1} p^\tau \leq (1-p)\sum_{\tau=0}^{T-t-1} 1 = (1-p)(T-t)$.)

Thus we have $a_t \leq c_t$ with inequality being strict for $p \in (0,1)$ and $t < T-1$.

A complete comparison is shown by the following table:

Table 4.1 The Comparison between a_t and c_t

	$t < T-1$	$t = T-1$
$p = 0$	$a_t < c_t$	$a_t = c_t$
$p \in (0,1)$	$a_t < c_t$	$a_t = c_t$
$p = 1$	$a_t = c_t$	$a_t = c_t$

Therefore we say $a_t = \left(1 - p^{T-t}\right) d^*$ is a tighter upper bound of the bubble price than $c_t = \min\{(1-p)(T-t), 1\} d^*$.

4.3.1.2 The Tightness of the Result

Actually the upper bound we have found is the best possbile upper bound of the price. To see this, suppose $\{b_t\}_{t=1}^{T}$ is any set of upper bounds that satisfies

$$B_{*t}^{p} B_{*(t+1)}^{p} \cdots B_{*(T-1)}^{p} \Omega_T(p) \subseteq \Omega_t(p)\text{, where } \Omega_t(p) = \{\omega \in \Omega | q_t(\omega) \leq b_t\}.$$

It is obvious that $b_t \leq d^* \forall t = 1, 2, \cdots, T$ and $b_T = 0$.

Let $t = T-1$, then we have $B_{*(T-1)}^{p} \Omega_T(p) \subseteq \Omega_{T-1}(p)$. This implies that for each i, the expectation of the price in the next period is no more than $pb_T + (1-p) d^* = (1-p) d^*$. It is easy to construct an example where the expectation of the price at time T is actually equal to $(1-p) d^*$ (Consider the 2-period case where $\exists A \subset \Omega$ such that $A \equiv \left\{\omega \in \Omega | \pi_i \left[\Omega_T(p) | P_{i(T-1)}(\omega)\right] = p, \forall i\right\}$ and $d(\omega) = \begin{cases} 0 \text{ if } \omega \in A \\ d^* \text{ otherwise} \end{cases}$). Therefore, we must have $b_{T-1} \geq (1-p) d^* = a_{T-1}$.

Let $t = T-2$, similarly we must have $pb_{T-1} + (1-p) d^* \leq b_{T-2}$. Since $b_{T-1} \geq (1-p) d^*$, this implies that $b_{T-2} \geq p(1-p) d^* + (1-p) d^* = (1+p)(1-p) d^* = (1-p^2) d^* = a_{T-2}$.

Let $t = T - 3$, similarly we must have $pb_{T-2} + (1-p)\,d^* \leq b_{T-3}$, which implies that $b_{T-3} \geq p\,(1+p)\,(1-p)\,d^* + (1-p)\,d^* = (1+p+p^2)\,(1-p)\,d^* = (1-p^3)\,d^* == a_{T-3}$.

Now if we suppose $b_t \geq \left(1 - p^{T-t}\right) d^*$, then we can easily have $b_{t-1} \geq \left(1 - p^{T-t+1}\right) d^*$. By mathematical induction, we know

$$\forall t = 1, 2, \cdots, T, b_t \geq \left(1 - p^{T-t}\right) d^* = a_t$$

4.3.1.3 The Single Player Case

The theorem actually works for every single player.

Theorem 17 (The Single Player Case) *For given player i, if he p-believes at time t that he will p-believe at time $t+1$ that he will p-believe at time $t+2$ that ... he will p-believe at time $T-1$ that the asset is worthless, then the price of the asset at time t is no more than $(1-p)\,(T-t)\,d^*$. That is*
$\forall i, B^p_{it} B^p_{i(t+1)} \cdots B^p_{i(T-1)} \Omega_T\,(p) \subseteq \Omega_t\,(p)\,,$ *where* $\Omega_t\,(p) = \left\{\omega \in \Omega | q_t\,(\omega) \leq \left(1 - p^{T-t}\right) d^*\right\}$

Proof. It suffices to show that for all t, $B^p_{it}\Omega_{t+1}\,(p) \subseteq \Omega_t\,(p)$. Suppose $\omega \in B^p_{it}\Omega_{t+1}\,(p)$. Then player i assigns at most probability $1-p$ to states not in $\Omega_{t+1}\,(p)$, where the price is at most d^*. At states in $\Omega_{t+1}\,(p)$, the price is at most $\left(1 - p^{T-t-1}\right) d^*$. Thus, player i's expectation of the price in the next period is no more than $(1-p)\,d^* + p\left(1 - p^{T-t-1}\right) d^* = \left(1 - p + p - p^{T-t}\right) d^* = \left(1 - p^{T-t}\right) d^*$. So $\omega \in B^p_{it}\Omega_t\,(p)$. ■

4.4 New Result for Theorem 4.3 in MPS (1995)

Similarly, we can have a stronger result for Theorem 4.3 in MPS (1995).

Let $\Omega_T = \{\omega \in \Omega | d\,(\omega) = 0\}$.

Theorem 18 (Theorem 4.3 in MPS(1995)) *Suppose that, at state ω^*, (i) it is kth order* *mutual knowledge at time t that the asset is worthless ($\omega^* \in [K_{*t}]^k \, \Omega_T$),*[37] *(ii) there exists a subset of the state space, E, such that E is evident p-belief at every date following t ($E \subseteq B^p_{*t'} E$, for all $t' \geq t$), (iii) E is true ($\omega^* \in E$) and (iv) the depth of knowledge conditional on E is less than or equal to k. Then the price of the asset at state ω^* at time t is at most $(1-p)(T-t)\,d^*$.*

Theorem 19 (Stronger Result for Theorem 4.3 in MPS(1995)) *Suppose that, at state ω^*, (i) it is kth order mutual knowledge at time t that the asset is worthless ($\omega^* \in [K_{*t}]^k \, \Omega_T$), (ii) there exists a subset of the state space, E, such that E is evident p-belief at every date following t ($E \subseteq B^p_{*t'} E$, for all $t' \geq t$), (iii) E is true ($\omega^* \in E$) and (iv) the depth of knowledge conditional on E is less than or equal to k. Then the price of the asset at state ω^* at time t is at most $\left(1 - p^{T-t}\right) d^*$.*

Proof. by (i) $\omega^* \in [K_{*t}]^k \, \Omega_T$ and (iii) $\omega^* \in E$, we have $\omega^* \in [K_{*t}]^k \, \Omega_T \cap E$.

It has been proved by mathematical induction in MPS (1995) (Lemma 3.1) that

$$[K_*]^k \, F \cap E \subseteq [K_*(\cdot|E)]^k \, F, \text{ for all events } E, F \text{ and integers } k$$

Given the above result, let $F = \Omega_T$. We immediately get $\omega^* \in [K_{*t}(\cdot|E)]^k \, \Omega_T$. This means that it is kth order E conditional mutual knowledge at state ω^* at time t that the asset is valueless.

It has also been proved in MPS (1995) (Theorem 2.1) that "if Ω has depth of knowledge n and $A \in \mathcal{F}$, where $\mathcal{F}$ is a partition of Ω generated by the fundamentals, then for all integers $m > n$, the event $K^m_* A$ is evident." Given this result, condition (iv) and $\omega^* \in [K_{*t}(\cdot|E)]^k \, \Omega_T$ imply that $\omega^* \in CK_t(\cdot|E)\Omega_T$. This means that it is E conditional common knowledge at state ω^* at time t that the asset is valueless. In particular, it must be $(T-t)$th order E conditional mutual knowledge at state ω^* at time t that the asset is

[37] There was a typo in the original paper that Ω_t should be Ω_T instead.

valueless, that is, $\omega^* \in [K_{*t}(\cdot|E)]^{T-t}\,\Omega_T$.

Since $K_{*s}F \subseteq K_{*t}F$ and $K_{*t}F \subseteq K_{*t}G$ for all $s \leq t$ and events $F \subseteq G$, we know that

$$[K_{*t}(\cdot|E)]^{T-t}\,\Omega_T \subseteq K_{*t}(\cdot|E)K_{*(t+1)}(\cdot|E)\cdots K_{*(T-1)}(\cdot|E)\Omega_T$$

Lemma 3.2 in MPS (1995) states that "if E is an evident p-belief event, then $K_*\,(F|E) \subseteq B_*^p F$." Thus we have

$$K_{*t}(\cdot|E)K_{*(t+1)}(\cdot|E)\cdots K_{*(T-1)}(\cdot|E)\Omega_T \subseteq B_{*t}^p B_{*(t+1)}^p \cdots B_{*(T-1)}^p \Omega_T$$

By our new theorem 4.2, we have

$B_{*t}^p B_{*(t+1)}^p \cdots B_{*(T-1)}^p \Omega_T(p) \subseteq \Omega_t\,(p)$, where $\Omega_t\,(p) = \left\{\omega \in \Omega | q_t\,(\omega) \leq \left(1 - p^{T-t}\right) d^*\right\}$

Since $\Omega_T(p) = \Omega_T = \{\omega \in \Omega | d\,(\omega) = 0\}$, we have $\omega^* \in \Omega_t\,(p)$. This means that the price of the asset at state ω^* at time t is at most $\left(1 - p^{T-t}\right) d^*$. ■

4.5 Conclusion

This note provides a new answer to the following specific question: What is a good upper bound of the prices of an asset in an equlbirium in a finite horizon model? Compared with the upper bound found in Morris, Shin and Postlewaite (1993), the one shown in this note is smaller and actually tight. This result helps to improve our understanding of the size of rational bubbles. For future research, It would be nice and worthwhile to figure out whether a strong bubble, defined in Allen, Morris and Postlewaite (1993), with the price exactly equal to the upper bound, can exist in equilibrium or not.

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