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Selfconfirming Equilibrium and Model Uncertainty*

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Abstract

We propose to bring together two conceptually complementary ideas: (1) selfconfirming equilibrium (SCE): at rest points of learning dynamics in a game played recurrently, agents best respond to confirmed beliefs, i.e., beliefs consistent with the evidence they accumulated, and (2) ambiguity aversion: agents, other things being equal, prefer to bet on events with known rather than unknown probabilities and, more generally, distinguish objective from subjective uncertainty, a behavioral trait captured by their ambiguity attitudes.

Using as a workhorse the "smooth ambiguity" model of Klibanoff, Marinacci and Mukerji (2005), we provide a definition of "Smooth SCE" which generalizes the traditional concept of Fudenberg and Levine (1993a,b), here called Bayesian SCE, and admits Waldean (maxmin) SCE as a limit case. We show that *the set of equilibria expands as ambiguity aversion increases*. The intuition is simple: by playing the same strategy in a stable state an agent learns the implied objective probabilities of payoffs, but alternative strategies yield payoffs with unknown probabilities; keeping beliefs fixed, increased aversion to ambiguity makes such strategies less appealing. In sum, by combining the SCE and ambiguity aversion ideas a kind of "status quo bias" emerges: in the long run, the uncertainty related to tested strategies disappears, but the uncertainty implied by the untested ones does not. We rely on this core intuition to show that different notions of equilibrium are nested in a simple way, from finer to coarser: Nash, Bayesian SCE, Smooth SCE and Waldean SCE. We also prove some equivalence results under special assumptions about the information structure.

KEYWORDS: Selfconfirming equilibrium, conjectural equilibrium, model uncertainty, smooth ambiguity.

JEL CLASSIFICATION: C72, D80.

Chi lascia la via vecchia per la via nuova, sa quel che perde ma non sa quel che trova[1]

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[1] Italian proverb "Those who leave the old road for a new one, know what they leave but do not what they will find."

1 Introduction

Uncertainty about variables affecting the consequences of choices is inherent to situations of strategic interaction. This is quite obvious when such situations have been faced only a few times. In this paper, we argue that uncertainty is pervasive also in games played recurrently where agents have had the opportunity to collect a large set of observations and the system has settled into a steady state. Such a situation is captured by the selfconfirming equilibrium concept (also called conjectural equilibrium). In a *selfconfirming equilibrium* (henceforth, SCE) agents best respond to *confirmed* probabilistic beliefs, where "confirmed" means that their beliefs are consistent with the evidence they can collect, given the strategies they adopt. Of course this evidence depends on how everybody else plays.

The SCE concept can be framed within different scenarios. A simple scenario is just a repeated game with a fixed set of players. In this context, the constituent game, which is being repeated, may have sequential moves and monitoring may be imperfect. To avoid repeated game effects, it is assumed that players do not value their future payoffs, but simply best respond to their updated beliefs about the current period strategies of the opponents. Here instead we refer to a scenario that is more appropriate for the ideas we want to explore: there is a large society of individuals who play recurrently a given game Γ, possibly a sequential game with chance moves: for each player/role i in Γ (male or female, buyer or seller, etc.) there is a large population of agents who play in role i. Agents are drawn at random and matched to play Γ. Then, they are separated and re-matched to play Γ with (almost certainly) different co-players, and so on. After each play of a game in which he was involved, an agent obtains some evidence on how the game was played. The accumulated evidence is the data set used by the agent to update his beliefs. Note, there is an intrinsic limitation to the evidence that an agent can obtain: at most he can observe the path (terminal node) realized in the game he just played, but often he can observe even less, e.g., only his monetary payoffs. However, what each agent is really interested about is the statistical distribution of strategies in the populations corresponding to opponents' roles, as such distributions determine (*via* random matching) the objective probabilities of different strategy profiles of the opponents with whom he is matched. Typically, this distribution is not uniquely identified by long-run frequencies of observations. This defines the fundamental inference problem faced by an agent, and explains why uncertainty is pervasive also in steady states. Similar considerations hold for chance moves, when their probabilities are unknown.

The key difference between SCE and Nash equilibrium is that, in a SCE, agents may have incorrect beliefs because many possible underlying distributions are consistent with the empirical frequencies they observe. The following example clarifies this point (see Fudenberg and Levine 1993a, Fudenberg and Kreps 1995).

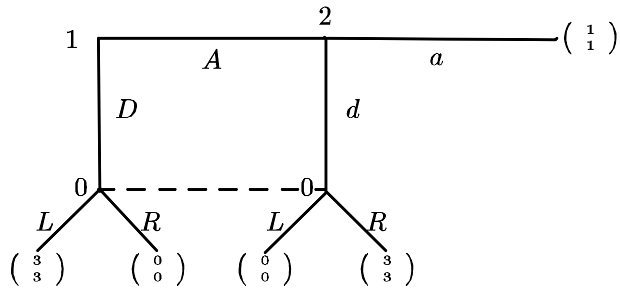

Figure 1. A "horse" game.

In the "horse" game of Figure 1, player 0 is an indifferent player[2] and the dashed line represents his information set. Assuming that agents can observe ex post the terminal node, or their realized payoff, there is an SCE with the following features: (1) all agents of population 1 play A believing that R is very likely and that all agents in 2 play a, (2) and all agents of population 2 play a believing that L is very likely and that all agents in 1 play A. The resulting outcome is always (A, a). Of course, someone must be wrong, but all beliefs are consistent with the evidence, i.e., they are confirmed. On the other hand, in a Nash equilibrium beliefs about player 0 must be correct, hence they must agree: if R is (believed) more likely, 1 and 2 play (A, d); if L is (believed) more likely, 1 chooses D. There is no Nash equilibrium with outcome (A, a).

According to the traditional SCE concept, agents are Bayesian subjective expected utility (SEU) maximizers. Nevertheless, when a set of underlying distributions is consistent with their information, agents face a condition of *model uncertainty*, or *ambiguity*, rather than risk. It is therefore plausible to assume that they have non-Bayesian attitudes toward uncertainty and decide accordingly. Many models of choice capturing such attitudes have been studied in decision theory (see Gilboa and Marinacci, 2011). The decision theoretic work which is more germane to our approach distinguishes between objective and subjective uncertainty. Given a set S of states, there is a set $\Sigma \subseteq \Delta(S)$ of possible probabilistic "models."[3] Each $\sigma \in \Sigma$ specifies the objective probabilities of states and, for each action a of the decision maker (DM), a von Neumann-Morgenstern expected utility evaluation $U(a, \sigma)$; the DM is uncertain about the true model σ (see Cerreia-Vioglio et al, 2011b). In our framework, a is the action, or strategy, of an agent playing in role i, σ is a distribution of strategies in the population of opponents (or a profile of such distributions in n-person games), and Σ is the set of distributions consistent with the database of the agent. Roughly, an agent who dislikes the uncertainty about the expected utility value implied by uncertainty about the model prefers, other things being equal, a strategy a inducing a small range of expected utilities $\{U(a, \sigma) : \sigma \in \Sigma\}$ to a strategy b inducing a large range of expected utilities $\{U(b, \sigma) : \sigma \in \Sigma\}$.

[2]The only reason to have an indifferent player is to simplify the picture, as the payoffs of this player are immaterial for the example. Player 0 may also be interpreted as Chance, assuming that the objective probabilities of L and R are unknown.

[3]In this context, we call "objective probabilities" the possible probability models (distributions) over a state space S. These are not to be confused with the objective probabilities stemming from an Anscombe and Aumann setting. For a discussion, see Cerreia-Vioglio et al (2011b).

We interchangeably refer to such feature of preferences with the expression "aversion to model uncertainty" or the shorter "ambiguity aversion." For example, an extreme form of ambiguity aversion is the *Waldean maxmin criterion:* $\max_a \min_{\sigma \in \Sigma} U(a, \sigma)$.[4] In this paper we span a large set of ambiguity attitudes using the "smooth ambiguity" model of Klibanoff, Marinacci and Mukerji (2005, henceforth KMM). This latter criterion admits the Waldean criterion as a limit case and the Bayesian SEU criterion as a special case. In a SCE, agents in each role best respond to their database choosing actions with the highest value, and their database is the one that obtains under the true data generating process corresponding to the actual strategy distributions. The following example shows how this notion of SCE differs from the traditional, or Bayesian, SCE.

$1+\varepsilon \xleftarrow{O} 1 \longrightarrow$

$1\backslash 2$	h^2	t^2
H^2	4	0
T^2	0	4

$\swarrow$

$1\backslash 2$	h^1	t^1
H^1	2	1
T^1	1	2

$(0 < \varepsilon < 0.5)$

Figure 2. Matching Pennies with increasing stakes

In the zero-sum game[5] of Figure 2, the first player chooses between an outside option O and two Matching-Pennies subgames, say MP^1 and MP^2. Subgame MP^2 has "higher stakes" than MP^1: it has a higher (mixed) maxmin value ($2 > 1.5$), but a lower minimum payoff ($0 < 1$). In this game there is only one Bayesian SCE outcome,[6] which must be the unique Nash outcome: MP^2 is reached with probability 1 and half of the agents in each population play Head. But we argue informally that moderate aversion to uncertainty makes the low-stakes subgame MP^1 reachable, and high aversion to uncertainty makes the outside option O also possible.[7] Specifically, let $\bar{p}^k$ denote the subjective probability assigned by a Bayesian agent in role 1 to h^k, with $k = 1, 2$. Going to the low-stake subgame MP^1 has subjective value $\max\{\bar{p}^1 + 1, 2 - \bar{p}^1)\} \geq 1.5$ and going to the high-stakes subgame MP^2 has subjective value $\max\{4\bar{p}^2, 4(1 - \bar{p}^2)\} \geq 2$. Thus, O is never a Bayesian best reply and cannot be played by a positive fraction of agents in a Bayesian SCE. Furthermore, also the low-stakes subgame MP^1 cannot be played in a Bayesian SCE. For suppose by way of contradiction that a positive fraction of agents in population 1 played MP^1. In the long run, each one of these agents, and all agents in population 2, would learn the relative frequencies of Head and Tail. Since in a SCE agents best respond to confirmed beliefs, the relative frequencies of Head and Tail should be the same in equilibrium, i.e., the agents in population 1 playing MP^1 would learn that its objective expected utility is $1.5 < 2$ and would deviate to MP^2 to maximize their SEU. On the other hand, for agents who are (at least) moderately averse to model uncertainty and keep playing MP^1, having learned the risks involved with the

[4] See Cerreia-Vioglio et al (2011a) on the relations of this criterion with the seminal maxmin model of Gilboa and Schmeidler (1989).

[5] The zero-sum feature simplifies the example, but it is inessential for the main point we are making here.

[6] We call "outcome" a distribution on terminal nodes.

[7] See Section 5 for a rigorous analysis.

low-stakes subgame confers to reduced-form[8] strategies H^1 and T^1 a kind of "status quo advantage": the objective expected utility of the untried strategies H^2 and T^2 is unknown and therefore they are penalized. Thus, the low-stakes subgame MP^1 can be played by a positive fraction of agents if they are sufficiently averse to model uncertainty. Finally, also the outside option O can be played by a positive fraction of agents in a SCE if they are extremely averse to model uncertainty, as represented by the maxmin criterion. If an agent keeps playing O, he cannot learn anything about the opponents' strategy distribution, hence he deems possible every distribution, or model, σ_2. Therefore, the minimum expected utility of H^1 (resp. T^1) is 1 and the minimum expected utility of H^2 (resp. T^2) is zero, justifying O as a maxmin best reply.[9]

The example shows that, by combining the SCE and ambiguity aversion ideas, a kind of "status quo bias" emerges: in the long run, uncertainty about the expected utility of tested strategies disappears, but uncertainty about the expected utility of the untested ones does not. Therefore, ambiguity averse agents have weaker incentives to deviate than Bayesian agents. More generally, higher ambiguity aversion implies a weaker incentive to deviate from an equilibrium strategy. This explains the main result of the paper: the set of SCEs expands as ambiguity aversion increases. We make this precise by adopting the "smooth ambiguity" model of KMM, which conveniently separates the endogenous subjective beliefs about the true strategy distribution from the exogenous ambiguity attitudes, so that the latter can be partially ordered by an intuitive "more ambiguity averse than" relation. With this, we provide a definition of "Smooth SCE" whereby agents "smooth best respond" to beliefs about strategy distributions consistent with their long-run frequencies of observations. The traditional SCE concept is obtained when agents are ambiguity neutral, i.e., Bayesian, while a Waldean (maxmin) SCE concept obtains as a limit case when agents are infinitely ambiguity averse. By our comparative statics result, these equilibrium concepts are intuitively nested from finer to coarser: each Bayesian SCE is also a Smooth SCE, which in turn is also Waldean SCE.

The rest of the paper is structured as follows. In Section 2 we start describing the main idea within recurrent decision problems where the state of nature has an unknown distribution. In Section 3 we analyze games and provide definitions of SCE with non-Bayesian attitudes toward uncertainty. In Section 4, the core of the paper, we carry out some key comparative statics exercises and analyze the relationships between equilibrium concepts. Section 5 illustrates our concepts and results with a detailed analysis of a generalized version of the game of Figure 2. Section 6 explores the consequences of allowing commitment to objective randomization devices, showing that this considerably reduces the status quo bias due to ambiguity aversion. Section 7 concludes the paper with a detailed discussion of some important theoretical issues and of the most related literature. In the main text we provide informal intuitions for our results. All proofs are collected in the Appendix.

[8] H^k (resp. T^k) corresponds to the class of realization-equivalent strategies that choose subgame MP^k and then select H^k (resp. T^k).

[9] Note that we are excluding the possibility of mixing through randomization, an issue addressed in Section 6.

2 Recurrent decisions

We first introduce our concepts in the context of a recurrent decision problem where the outcome of the decision maker's (henceforth, DM) strategy depends on the state (or strategy) of Nature.

2.1 Mathematics

Given any measurable space $(X, \mathcal{X})$, we denote by $\Delta(X)$ the collection of all probability measures $\nu : \mathcal{X} \to [0,1]$. When X is finite, say with cardinality n, we assume that $\mathcal{X} = 2^X$ and we identify $\Delta(X)$ with the simplex of $\mathbb{R}^n$.

We endow $\Delta(X)$ with the smallest σ-algebra that makes the real valued and bounded functions on $\Delta(X)$, of the form $\nu \mapsto \nu(E)$, measurable for all $E \in \mathcal{X}$. When X is finite this σ-algebra coincides with the relative Borel σ-algebra that $\Delta(X)$ inherits as the simplex of $\mathbb{R}^n$. Finally, we also endow any measurable subset Σ of $\Delta(X)$ with the relative σ-algebra inherited from $\Delta(X)$, and we denote by $\Delta(\Sigma)$ the collection of all probability measures defined on such σ-algebra. Among them, δ_x denotes the Dirac measure concentrated on $x \in X$.[10]

Given $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$, a pair of measurable spaces, we endow the Cartesian product $X \times Y$ with the product σ-algebra $\mathcal{X} \times \mathcal{Y}$. We denote by $\Delta(X) \otimes \Delta(Y)$ the collection of all *product probability measures*. Moreover, each measurable function $\varphi : X \to Y$ induces the pushforward map $\hat{\varphi} : \Delta(X) \to \Delta(Y)$ defined by

$$\hat{\varphi}(\nu) = \nu \circ \varphi^{-1} \qquad \forall \nu \in \Delta(X).$$

In other words, we have that $\hat{\varphi}(\nu)(E) = \nu(\varphi^{-1}(E))$ for all $E \in \mathcal{Y}$.

Finally, $\Delta^*(X)$ denotes the collection of all affine functions $\varphi : \Delta(X) \to \mathbb{R}$, that is, $\varphi(\alpha\nu + (1-\alpha)\nu') = \alpha\varphi(\nu) + (1-\alpha)\varphi(\nu')$ for all $\alpha \in [0,1]$ and all $\nu, \nu' \in \Delta(X)$.

2.2 The decision problem

We assume there is a large population of DMs.[11] In each period, each of them faces the same decision problem which consists of a finite state space S_N, a finite strategy space S_{DM}, and a finite outcome space Z. This may be a dynamic decision problem representable by a decision tree with moves by the DM and moves by Nature. In this case S_i, with $i \in \{DM, N\}$, is the set of pure strategies of i. In this paper we do not address dynamic consistency issues: we assume that the DM commits to a particular strategy that is automatically implemented in the decision problem. Hence S_{DM} is the set of *actions* which DMs can choose from and, to ease notation, we write A in place of S_{DM} and Ω in place of S_N.

An *outcome function*

$$\zeta : A \times \Omega \to Z$$

[10] That is, $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 1$ if $x \notin A$.

[11] The existence of many DMs plays an important role only in our game-theoretic analysis. Nevertheless, in order to be consistent, we mention it also in this section even though we mainly focus on a specific DM.

maps actions and states to "outcomes." The set of outcomes Z should be interpreted as the set of terminal nodes of the decision tree.[12] Such terminal nodes give rise to consequences $c \in C$ according to a function[13]

$$\gamma : Z \to C.$$

Ex post, DMs receive a message $m \in M$ according to a *feedback function*

$$f : Z \to M.$$

The function f maps outcomes to messages. A DM that gets message m knows that the realized outcome belongs to the set $f^{-1}(m) \subseteq Z$.

Together the outcome and the feedback functions determine the *message function*

$$F : A \times \Omega \to M$$

given by $F = f \circ \zeta$, that is, $F(a, \omega) = f(\zeta(a, \omega))$ for all action/state profiles $(a, \omega) \in A \times \Omega$. Let $F_a : \Omega \to M$ be the section at strategy a of F defined by $F_a(\omega) = F(a, \omega)$ for all $\omega \in \Omega$. When each F_a is one-to-one, under each action different states of Nature generate different messages. Thus, ex post DMs learn the true state upon receiving the message. This, however, can happen when two conditions are satisfied: (i) each section ζ_a is one-to-one, i.e., in the decision tree Nature has no move following a move by the DM, and (ii) the feedback function f is one-to-one, i.e., there is *perfect feedback*. Note that under perfect feedback it is without loss of generality to set $M = Z$ and $f = \mathrm{Id}_Z$, the identity function on Z.

Elements σ of $\Delta(\Omega)$ are interpreted as the vectors of "objective" probabilities that each state ω obtains, that is, as possible probabilistic *models* for states. If the decision problem is viewed as a game with Nature, then σ is a mixed strategy of Nature. Thus, the true model can be seen as Nature's actual mixed play.

Elements α of $\Delta(A)$ can be seen as mixed actions implemented by a random device,[14] or – in our preferred interpretation – as statistical distributions of actions in the population of DMs. In both cases, $\alpha(a)$ is the "objective" probability that the DM (or a DM chosen at random) selects a.

A distribution on $A \times \Omega$, in particular a product distribution $\alpha \times \sigma$, delivers a random message. Specifically, each message function $F : A \times \Omega \to M$ induces a mixed message function $\hat{F} : \Delta(A) \otimes \Delta(\Omega) \to \Delta(M)$, where

$$\hat{F}(\alpha \times \sigma)(m) = (\alpha \times \sigma)\left(F^{-1}(m)\right) \tag{1}$$

is the probability that a DM observes an ex post message m given the product measure $\alpha \times \sigma$ of a mixed action α and a model σ. We will mostly take the point of view of a DM who chooses a pure action a, but we will also consider the case of randomization, hence our general notation. We denote by μ a generic mixed message in $\Delta(M)$.

[12] Thus, this is the same as the "outcome function" in Chapter 6 of Osborne and Rubinstein (1994), generalized to games with imperfect information.

[13] We could simplify the notation identifying terminal nodes and consequences, but we feel that introducing the consequence function improves conceptual clarity.

[14] Random devices (e.g., roulette wheels and dice) feature probabilities that can be computed according to Laplace's classical notion.

2.3 Decision criteria

The stage decision problem features two sources of uncertainty: state uncertainty within each model and model uncertainty across different models.

As to state uncertainty, given a model σ the DMs evaluate actions a through their expected utility

$$U(a,\sigma) = \sum_{\omega\in\Omega} u(\zeta(a,\omega))\,\sigma(\omega) \tag{2}$$

where $u = v \circ \gamma : Z \to \mathbb{R}$ and $v : C \to \mathbb{R}$ is a von Neumann-Morgenstern (henceforth, vNM) utility function that captures the attitudes toward risk of the DM.[15] Our main results rely on the assumption that *payoffs* are *observable*: the payoff function u is f-measurable, that is, in a finite setting, $f(z) = f(z')$ implies $u(z) = u(z')$ for all $z, z' \in Z$. This is a natural assumption in many applications, e.g., when $c \in C$ is DM's consumption.

As to model uncertainty, let $U(a,\cdot) : \Delta(\Omega) \to \mathbb{R}$ be the affine function that, given an action a, associates to each model σ its expected utility (2). Suppose that, based on his information, the DM knows that the true model σ belongs to a compact subset $\Sigma \subseteq \Delta(\Omega)$. In other words, he is able to posit a set Σ of possible models, a standard assumption in classical statistics. The payoff scope of model uncertainty for the DM that chooses action a is thus described by the expected utility profile $\{U(a,\sigma) : \sigma \in \Sigma\}$.

The simplest criterion to deal with both state and model uncertainty is the *Waldean criterion*, which ranks actions according to

$$V(a,\Sigma) = \min_{\sigma\in\Sigma} U(a,\sigma). \tag{3}$$

This criterion, due to Wald (1950), does not assume any knowledge on models other than what is necessary to posit Σ. In particular, no prior probability appears. In contrast, the *classical Bayesian criterion*[16] ranks actions according to

$$V(a,p,\Sigma) = \int_\Sigma U(a,\sigma)\,dp(\sigma) \tag{4}$$

where $p \in \Delta(\Sigma)$ is a prior probability on the possible models, a subjective probability that quantifies the DM's personal information on models. Set

$$\bar{p}(\cdot) = \int_\Sigma \sigma(\cdot)\,dp(\sigma).$$

The reduced probability $\bar{p} \in \Delta(\Omega)$ is the *predictive probability* on states induced by the prior p. Using the predictive probability, we can write

$$\int_\Sigma U(a,\sigma)\,dp(\sigma) = \sum_{\omega\in\Omega} u(\zeta(a,\omega))\,\bar{p}(\omega) = U(a,\bar{p}).$$

The predictive form $U(a,\bar{p})$ is a "reduced form" of (4). Note that $\bar{p}$ is a subjective probability, so that $U(a,\bar{p})$ is a subjective expected utility criterion a la Savage (1954), while $U(a,\sigma)$ was a vNM one.

[15] For example, if $\gamma : Z \to [\underline{r}, \bar{r}] \subseteq \mathbb{R}$ specifies monetary payoffs and u is concave, then the DM is risk averse.

[16] See Cerreia-Vioglio et al (2011b) for a derivation of this model and a discussion of its relations with Savage (1954). The adjective "classical" here and for the smooth model below is due to the classical statistics assumption of a Σ of possible models.

DMs may well be ambiguity averse, that is, they may dislike having a range of possible vNM expected utilities $\{U(a,\sigma) : \sigma \in \Sigma\}$ that is not a singleton because of model uncertainty. For this reason the *classical smooth ambiguity* criterion of KMM ranks actions according to

$$V(a, p, \Sigma; \phi) = \phi^{-1}\left(\int_{\Sigma} \phi(U(a,\sigma))\, dp(\sigma)\right) \tag{5}$$

where $p \in \Delta(\Sigma)$ is a prior probability on Σ and $\phi : \operatorname{Im} U \to \mathbb{R}$ is a strictly increasing continuous function that describes ambiguity attitudes.[17] In particular, a concave ϕ captures ambiguity aversion, while a linear ϕ corresponds to ambiguity neutrality. The smooth criterion thus relaxes the assumption that agents linearly combine attitudes toward state and toward model uncertainty, which are allowed to differ.

The Waldean and classical Bayesian criteria can be viewed as, respectively, a limit case and a special case of the classical smooth criterion. For, under ambiguity neutrality the smooth model reduces to the Bayesian one (4), while under extreme ambiguity aversion – that is, when ambiguity aversion "goes to infinity" – the smooth model reduces to the limit to the Wald criterion (3) provided that $\operatorname{supp}(p) = \Sigma$. We refer the reader to KMM for details. Note that the Bayesian criterion can be written as $V(a, p, \Sigma; \mathrm{Id}_{\mathbb{R}})$ as a special smooth criterion; to ease notation in what follows we will keep writing $V(a, p, \Sigma)$.

We will also consider the possibility that a DM can delegate his choice to a random device, thus implementing a mixed strategy $\alpha \in \Delta(A)$. All the formulas above can be adapted, replacing a with α and $U(a,\sigma)$ with the vNM expected utility

$$U(\alpha,\sigma) = \sum_{a \in A} \alpha(a) U(a,\sigma).$$

2.4 Recurrent decisions and status quo

The decision problem is faced recurrently by a large population of DMs. In each period one of them is drawn at random and faces the stage decision problem. In most of the paper we assume that DMs select pure actions a. Mixed actions α are interpreted as population *action distributions*, that is, $\alpha(a)$ is the fraction of DMs who choose a when selected for the stage decision problem. In a classical approach a la Laplace, this fraction is the "objective" probability that a DM drawn at random chooses a. Formally, action distributions and mixed actions are identical, in the spirit of the Nash mass action interpretation (see, e.g., Weibull, 1996). In Section 6 we will also consider the case where a DM can play a mixed strategy α, therefore our general formulas encompass this case too.

To abstract away from learning issues, we interpret a mixed action also as the empirical frequency of (pure) actions actually chosen by DMs, that is, $\alpha(a)$ is also the long run frequency with which a is chosen by the DMs who have been drawn for the stage decision problem. In a frequentist approach a la von Mises, this frequency is the "objective" probability that a DM drawn at random chooses a.[18] The probabilities that α features thus have a

[17] With a slight abuse of notation, here $\operatorname{Im} U$ is the smallest interval that contains $\bigcup_{a \in A} U(a,\cdot)$, i.e., its convex hull.

[18] This frequentist interpretation actually requires that the stage decisions be independent (an "ergodic" interpretation of probabilities as time averages hold, however, more generally; see Lith, 2001, and the references therein).

dual, classical and frequentist, interpretation as both population distributions and empirical frequencies. Such duality relies, heuristically, on a "long run" that is long and stationary enough for asymptotic ergodic-type results to hold.

A similar dual interpretation applies to σ, where $\sigma(\omega)$ is both the classical probability with which Nature (regarded as a random device) selects state ω at each stage and the empirical frequency of such state.

In view of all this, mixed messages μ of $\Delta(M)$ can be interpreted as long run frequency distributions of messages received by DMs, so that $\mu(m)$ is the empirical frequency of message m. In particular, the pushforward map $\hat{F} : \Delta(A) \otimes \Delta(\Omega) \to \Delta(M)$ associates to each profile (α, σ) the message distribution (1), which, as argued before, is the probability that a DM observes message m given an action distribution α and a model σ. Because of the dual nature of α and σ, such probability is both the "objective" probability that each stage generates message m and the empirical frequency with which DMs receive such message.

Consider the section $\hat{F}_\alpha : \Delta(\Omega) \to \Delta(M)$ defined by $\hat{F}_\alpha(\sigma) = \hat{F}(\alpha \times \sigma)$. Its inverse correspondence $\hat{F}_\alpha^{-1}$ partitions $\Delta(\Omega)$ in classes

$$\hat{F}_\alpha^{-1}(\mu) = \left\{\sigma \in \Delta(\Omega) : \hat{F}(\alpha \times \sigma) = \mu\right\}$$

of models that are observationally equivalent given that α is played and the frequency distribution of messages μ is "observed" in the long run. In other words, $\hat{F}_\alpha^{-1}(\mu)$ is the collection of all models that may have generated μ given α.

The inverse correspondence is compact valued. It is a function if and only if $\hat{F}_\alpha$ is one-to-one; in this case the decision problem is identified under α. Different models generate different message distributions which thus uniquely pin down models. It is the counterpart in our setup of the classic notion of identifiability (see, e.g., Rothenberg, 1971). Accordingly, we only have *partial identification* when $\hat{F}_\alpha$ is not one-to-one. In the extreme case when $\hat{F}_\alpha$ is constant – that is, all models generate the same message distribution – the decision problem is completely unidentified.

For each $\alpha \in \Delta(A)$, consider the correspondence

$$\hat{\Sigma}(\alpha, \cdot) = \hat{F}_\alpha^{-1} \circ \hat{F}_\alpha : \Delta(\Omega) \to 2^{\Delta(\Omega)}.$$

For any fixed $\sigma^* \in \Delta(\Omega)$, the image

$$\hat{\Sigma}(\alpha, \sigma^*) = \left\{\sigma \in \Delta(\Omega) : \hat{F}(\alpha \times \sigma) = \hat{F}(\alpha \times \sigma^*)\right\} \tag{6}$$

is the collection of models that are observationally equivalent given α and the message distribution $\mu = \hat{F}(\alpha \times \sigma^*)$ that α generates along with model σ^*.

We can thus regard $\hat{\Sigma}(\alpha, \cdot)$ as the *identification correspondence* determined by α. It is easily seen to be convex and compact valued, as well as nonempty since $\sigma \in \hat{\Sigma}(\alpha, \sigma)$. It is a function if and only if $\hat{F}_\alpha$ is one-to-one. In this case, $\hat{\Sigma}(\alpha, \cdot)$ is the identity function, with $\hat{\Sigma}(\alpha, \sigma) = \sigma$ for all $\sigma \in \Delta(\Omega)$, and so message distributions identify the true model.

Given a mixed strategy α, its identification correspondence partitions $\Delta(\Omega)$ in sets $\hat{\Sigma}(\alpha,\sigma)$ of observationally equivalent models. Next we show that, under the observable payoff assumption, such models share the same expected utility, that is, the function $U(\alpha,\cdot) : \Delta(\Omega) \to \mathbb{R}$ is measurable with respect to the partition determined by $\hat{\Sigma}(\alpha,\cdot)$.

Lemma 1 *Under observable payoffs, given any $\sigma^* \in \Delta(\Omega)$ and any $\alpha \in \Delta(A)$, it holds*

$$U(\alpha,\sigma) = U(\alpha,\sigma^*) \qquad \forall \sigma \in \hat{\Sigma}(\alpha,\sigma^*).$$

The lemma captures formally the status quo bias mentioned in the Introduction. For, suppose that the agent keeps playing, for example, a pure action a and that σ^* is the true model. Then, the message distribution $\hat{F}_a(\sigma^*)$, with the associated set $\hat{\Sigma}(a,\sigma^*)$, is the relevant evidence for his stage decision. Since payoffs are observable, $\hat{F}_a(\sigma^*)$ determines the frequencies of the values $U(a,\omega)$, with $\omega \in \Omega$. By definition, these frequencies are the same for every model in $\hat{\Sigma}(a,\sigma^*)$. Thus, the function $U(a,\cdot)$ must be constant on $\hat{\Sigma}(a,\sigma^*)$, that is, the agent does not perceive any model uncertainty in his evaluation of the "status quo" action a. In contrast, model uncertainty may affect the evaluation of any alternative action a' since the function $U(a',\cdot)$ might well be nonconstant on $\hat{\Sigma}(a,\sigma^*)$. This kind of status quo bias plays a key role in Theorem 6: the paper main result.[19]

Identification correspondences with larger images exhibit a higher degree of partial identification. We now show that, ultimately, such degree depends on the underlying feedback functions. To this end, given any two feedback functions f and $\bar{f}$, say that f is *coarser* than $\bar{f}$ if it is $\bar{f}$-measurable, that is, $\bar{f}(z) = \bar{f}(z')$ implies $f(z) = f(z')$ for all $z, z' \in Z$.

For example, if $f = u$ (DM only observes his payoffs) and $\bar{f} = \mathrm{Id}_Z$ (DM observes the outcome, or terminal node of the decision tree), then f is trivially coarser than $\bar{f}$. DMs with coarser feedback functions have worse information on outcomes. Clearly, constant feedbacks are the coarsest ones, while perfect feedbacks are the least coarse.

Lemma 2 *If f is coarser than $\bar{f}$, then, for all $\alpha \in \Delta(A)$,*

$$\hat{\bar{\Sigma}}(\alpha,\sigma^*) \subseteq \hat{\Sigma}(\alpha,\sigma^*) \qquad \forall \sigma^* \in \Delta(\Omega).$$

Coarser feedback functions thus determine, for each action, coarser identification correspondences: worse information translates into a higher degree of partial identification.

2.5 Selfconfirming decisions

We begin with the general notion of selfconfirming decision for the smooth classical criterion (5). We focus on the case where DMs can only choose pure actions. Given an action distribution α in the population of DMs, a key issue for this notion is whether DMs have access to some kind of public database (e.g., provided by the media), or obtain only the observations generated when they choose their action a when they are drawn (individual database). If σ is the true model, $\hat{\Sigma}(\alpha,\sigma)$ is the set of models consistent with the message

[19] Note that a can be viewed as a familiar alternative for the agent; accordingly, the bias can be interpreted as a "familiarity effect", a notion sometimes evoked in applications of model uncertainty (e.g., to home bias phenomena in asset allocations).

distribution in the public database, while $\hat{\Sigma}(a,\sigma)$ is the set of models consistent with the individual database of a DM playing a. Though we will focus on individual databases, a similar analysis can be carried out for public ones (see also the discussion in Section 7.3).

That said, here is our general notion of selfconfirming decision with individual databases. The goal is to describe distributions of actions/strategies that are stationary because each DM's beliefs are confirmed.

Definition 3 *Suppose σ^* is the true model. An action distribution α^* is a* smooth selfconfirming decision (SCD) *if, for each $a^* \in \operatorname{supp}\alpha^*$, there is a belief $p_{a^*} \in \Delta(\hat{\Sigma}(a^*,\sigma^*))$ such that*

$$V\left(a^*, p_{a^*}, \hat{\Sigma}(a^*,\sigma^*);\phi\right) \geq V\left(a, p_{a^*}, \hat{\Sigma}(a^*,\sigma^*);\phi\right) \qquad \forall a \in A \tag{7}$$

The "confirmed rationality" condition (7) can be written as

$$a^* \in \arg\max_{a\in A} \int_{\hat{\Sigma}(a^*,\sigma^*)} \phi\left(U(a,\sigma)\right) dp_{a^*}(\sigma)$$

and ensures that a^* is a best reply that takes into account the available evidence. Specifically, each action a^* chosen by a positive fraction $\alpha^*(a^*)$ of DMs must be a best response within A to the statistical distribution of messages $\hat{F}(a^* \times \sigma^*) \in \Delta(M)$ generated by selecting a^* when the true model is σ^*. Since the available evidence depends on the chosen action a^*, also the belief "justifying" a^* as a best response may depend on a^*. Since A is finite, for any given p_{a^*} the maximization problem has a solution.[20]

Extreme ambiguity attitudes determine two notions of selfconfirming decision. Specifically, an action distribution α^* is a:

(i) *Waldean SCD* if, for each $a^* \in \operatorname{supp}\alpha^*$,

$$a^* \in \arg\max_{a\in A} \min_{\sigma\in\hat{\Sigma}(a^*,\sigma^*)} U(a,\sigma);$$

(ii) *Bayesian SCD* if, for each $a^* \in \operatorname{supp}\alpha^*$, there is a prior $p_{a^*} \in \Delta(\hat{\Sigma}(a^*,\sigma^*))$ such that

$$a^* \in \arg\max_{a\in A} \int_{\hat{\Sigma}(a^*,\sigma^*)} U(a,\sigma)\, dp_{a^*}(\sigma).$$

Note that when the decision problem is identified under all actions – that is, $\hat{\Sigma}(a,\sigma) = \{\sigma\}$ for all $a \in A$ and all $\sigma \in \Delta(\Omega)$ – confirmed rationality require consistency with the true model and the smooth criterion reduces to a vNM expected utility with respect to the true model. The confirmed rationality condition (7) becomes $a^* \in \arg\max_{a\in A} U(a,\sigma^*)$.

[20] Note, since we are considering the case of individual databases and σ^* is exogenously given, the equilibrium conditions only restrict the support of α^*. But in the game theoretic extension to follow, σ^* will be the "endogenous" distribution of strategies of a co-player who is also maximizing given confirmed beliefs and the equilibrium conditions will restrict also the probabilities of strategies in the support of α^*.

2.6 An urn example

It may be useful to review the concepts introduced so far by means of an urn example. Consider an urn that contains 90 balls: 30 are red (R), 30 are blue (B), and 30 are yellow (Y). In every period a DM is randomly selected from a population and is asked to bet 1 euro on the color of the ball that will be drawn from the urn in that period. The selected DMs just know that R, B, and Y are the only possible colors. After each draw the DMs are told the result of their bets, that is, whether or not they won 1 euro.

Here the state space is $\Omega = \{R, B, Y\}$, the action space is $A = \{1_R, 1_B, 1_Y\}$, the set of terminal nodes of the decision tree is $Z = A \times \Omega$, hence $\zeta = \mathrm{Id}_Z$, and the message space is $M = \{0, 1\}$. Since $\zeta = \mathrm{Id}_Z$, $f = F : A \times \Omega \to \{0, 1\}$; in particular:

$$F(1_R, R) = F(1_B, B) = F(1_Y, Y) = 1$$

and

$$F(1_R, B) = F(1_R, Y) = F(1_B, R) = F(1_B, Y) = F(1_Y, R) = F(1_Y, B) = 0.$$

Suppose a DM keeps betting on red, that is, his action is 1_R. As a result, he imperfectly observes the draws: he can only observe whether the color is red (message 1: he won) or not (message 0: he lost). Such DM's action prevents him from obtaining evidence on the frequency of B and Y: given his betting he only gets either message 0 or 1. In particular,

$$\hat{F}_{1_R}(\sigma)(1) = \sigma(R), \ \hat{F}_{1_R}(\sigma)(0) = 1 - \sigma(R) \qquad \forall \sigma \in \Delta(\Omega).$$

Since the true model $\sigma^* \in \Delta(R, B, Y)$ is such that

$$\sigma^*(R) = \sigma^*(B) = \sigma^*(Y) = \frac{1}{3},$$

the evidence allowed by 1_R partially identifies the model as an urn where one third of the balls is red:

$$\begin{aligned} \hat{\Sigma}(1_R, \sigma^*) &= \left\{\sigma \in \Delta(\{R, B, Y\}) : \hat{F}_{1_R}(\sigma) = \hat{F}_{1_R}(\sigma^*)\right\} \\ &= \left\{\sigma \in \Delta(\{R, B, Y\}) : \sigma(R) = \frac{1}{3}\right\}. \end{aligned}$$

Turning to selfconfirming decisions, set $u(0) = 0$ and $u(1) = 1$, so that payoffs are observable and $U(1_\omega, \sigma) = \sigma(\omega)$ for each $\omega \in \{R, B, Y\}$. Since

$$\min_{\sigma \in \hat{\Sigma}(1_R, \sigma^*)} U(1_R, \sigma) = \frac{1}{3} \quad \text{and} \quad \min_{\sigma \in \hat{\Sigma}(1_R, \sigma^*)} U(1_B, \sigma) = \min_{\sigma \in \hat{\Sigma}(1_R, \sigma^*)} U(1_Y, \sigma) = 0$$

action 1_R, i.e., betting on red, is a strict Waldean SCD. A similar conclusion can be reached for smooth selfconfirming decisions under enough ambiguity aversion.

In a Bayesian setting, each belief $p \in \Delta\left(\hat{\Sigma}(1_R, \sigma^*)\right)$ induces a predictive $\bar{p} \in \hat{\Sigma}(1_R, \sigma^*)$ so that

$$U(1_\omega, \bar{p}) = \bar{p}(\omega) = \int_{\hat{\Sigma}(1_R, \sigma^*)} \sigma(\omega) dp(\sigma), \ \bar{p}(R) = \frac{1}{3}.$$

Therefore 1_R is supported as a Bayesian selfconfirming decision (though not strict, i.e., the $\arg\max$ is nonsingleton) in the "knife edge" case $\bar{p}(B) = \bar{p}(Y)$, which implies $\bar{p}(B) = \bar{p}(Y) = \bar{p}(R) = 1/3$.

By the symmetry of the example, a similar analysis holds for actions 1_B and 1_Y. Thus, every action distribution α is a Waldean, Smooth and Bayesian SCD, but no α is a "strict" Bayesian SCD.

3 Recurrent games

In this section we extend the previous analysis to general games, our main object of interest.

3.1 Games with feedback and ambiguity

We consider finite extensive-form games with perfect recall and no chance moves played recurrently between agents drawn at random from large populations. Since we need not be explicit about all the details of the extensive form, we specify our notation only for some primitive and derived objects. The *rules of the game* directly or indirectly determine the following elements:

- populations, or player roles $i \in I$,
- complete paths, or terminal histories $z \in Z$,
- pure strategies $s_i \in S_i$ for each $i \in I$,[21]
- a path, or outcome function $\zeta : S \to Z$ specifying the terminal history $\zeta(s)$ determined by each strategy profile $s = (s_i)_{i \in I} \in S = \times_{i \in I} S_i$,
- a (onto) consequence function $\gamma : Z \to C$ specifying the material consequences $c \in C$ of each terminal history $z \in Z$,
- a feedback function $f_i : Z \to M$ for each $i \in I$ specifying the message received *ex post* by an agent playing in role i as a function of the terminal history z.

It may be useful to think of the above elements as what can be implemented in a laboratory experiment. In particular, the designer specifies the rules. Such rules include the ex post information feedback $f = (f_i)_{i \in I}$, which typically does not appear in the standard mathematical definition of game, and a function $\gamma = (\gamma_i)_{i \in I} : Z \to C \subseteq \mathbb{R}^I$ that gives the monetary payoff distributions induced by sequences of actions. On the other hand, the designer cannot control subjects' preferences, notably absent from the above list.

But, as analysts, we need to specify preferences over objective and subjective lotteries of consequences. We assume for simplicity that populations are homogeneous, that is, all agents in the same population $i \in I$ have the same preferences (the extension to heterogeneous populations is straightforward but notationally cumbersome):

[21] Of course, S_i is determined by the game tree and the information structure for i.

- $v_i : C \to \mathbb{R}$ is a vNM utility function capturing i's attitudes toward risk,[22]
- $\phi_i : [\min_{c \in C} v_i(c), \max_{c \in C} v_i(c)] \to \mathbb{R}$ is a strictly increasing weighting function capturing i's attitudes toward ambiguity as in KMM.

Like the feedback functions f, the weighting functions $\phi = (\phi_i)_{i \in I}$ do not appear in the standard theory, as they are implicitly assumed to be linear. The standard definition of game is a structure[23]

$$\Gamma = (I, ..., \gamma, (v_i)_{i \in I}),$$

where the dots "..." refer to the details of the game tree and information structure. We keep the parameters specified by Γ fixed throughout our analysis, while we look at the effects of changing f or, more importantly, ϕ.

A *game with feedback* is a pair (Γ, f). We say that such a game features *observable payoffs* when each payoff function $u_i = v_i \circ \gamma : Z \to \mathbb{R}$ is f_i-measurable, that is, $f_i(z) = f_i(z')$ implies $u_i(z) = u_i(z')$ for all $z, z' \in Z$. An important special case is *perfect feedback*, where each i directly observes the path, that is, $M = Z$ and $f_i = \mathrm{Id}_Z$ for each i.[24] To anticipate, some notions of selfconfirming equilibrium, including the standard one which assumes Bayesian agents, refer to a game with feedback. But our main equilibrium concept refers to a *game with feedback and ambiguity attitudes*, that is, a triple (Γ, f, ϕ).

As a matter of interpretation, we have to ascribe some knowledge of (Γ, f, ϕ) to the agents. In order to make sense of the following analysis, we informally assume that each agent in population i knows the extensive form, the feedback function f_i, the random matching structure, and – of course – the vNM utility function v_i and weighting function ϕ_i. No additional knowledge, mutual knowledge or common knowledge of (Γ, f, ϕ) is required except in our discussion of rationalizable selfconfirming equilibrium in Section 7.2. The assumption of no chance moves is made for simplicity, without substantial loss of generality. Chance can be modeled in this framework as a player with a constant utility function, assuming that agents have no knowledge of the objective probabilities of chance moves. The analysis can be easily adapted to incorporate complete or partial knowledge of such probabilities.

The (*strategic-form*) *message function* $F_i : S \to M$ is given by $F_i = f_i \circ \zeta$. In turn, each message function $F_i : S \to M$ induces a message distribution function $\hat{F}_i : \bigotimes_{j \in I} \Delta(S_j) \to \Delta(M)$, where $\hat{F}_i(\sigma_i \times \sigma_{-i})(m)$ is the probability that i observes message m given the strategy distribution σ_i of his role and the strategy distribution σ_{-i} of his opponents' roles.

If i plays the pure strategy s_i and observes the long-run frequency distribution of messages $\mu \in \Delta(M)$, then i can compute the set of (product) strategy distributions of the opponents

[22] For example, if $\gamma = (\gamma_i) : Z \to C \subseteq \mathbb{R}^I$ and i is selfish, $u_i(z) = v_i(\gamma_i(z))$, where v_i is concave if i is risk averse.

[23] Often the consequence function does not appear explicitly. We introduce it here for conceptual clarity.

[24] Perfect feedback was assumed by Fudenberg and Levine (1993a) and Fudenberg and Kreps (1995). It is also reasonable to assume that f_i reflects perfect recall, that is, if the sequence of information sets and actions of i determined by path z' is different from the sequence determined by path z'', then $f_i(z') \neq f_i(z'')$. This, however, does not play an explicit role in our analysis because we assume that an agent takes into account the strategy he is playing when he considers the possible strategy distributions of the opponents consistent with the evidence he observes.

that may have generated μ given s_i:

$$\left\{ \sigma_{-i} \in \bigotimes_{j \neq i} \Delta(S_j) : \hat{F}_i \left(s_i \times \sigma_{-i} \right) = \mu \right\}.$$

If $\sigma^*_{-i} = \times_{j \neq i} \sigma^*_j$ is the true strategy distribution of his opponents' roles, the long-run frequency distribution of messages observed by i is the one induced by the objective distribution $s_i \times \sigma^*_{-i}$, that is, $\mu = \hat{F}_i \left(s_i \times \sigma^*_{-i} \right)$. The set of possible distributions from i's perspective is thus

$$\hat{\Sigma}_{-i}(s_i, \sigma^*_{-i}) = \left\{ \sigma_{-i} \in \bigotimes_{j \neq i} \Delta(S_j) : \hat{F}_i \left(s_i \times \sigma_{-i} \right) = \hat{F}_i \left(s_i \times \sigma^*_{-i} \right) \right\},$$

the game theoretic version of (6). The identification correspondence $\hat{\Sigma}_{-i}(s_i, \cdot)$ is nonempty and compact valued; it is convex valued in two-person games.

We obtain the decision problem analyzed in Section 2 as a special case when $I = \{DM, N\}$ and u_N is constant. In particular, the decision criteria previously introduced are easily extended to this game theoretic setting. Here (2) becomes

$$U_i \left(s_i, \sigma_{-i} \right) = \sum_{s_{-i} \in S_{-i}} u_i(\zeta(s_i, s_{-i})) \sigma_{-i}(s_{-i}),$$

with $u_i = v_i \circ \gamma$, so that the smooth criterion (5) becomes

$$V_i \left(s_i, p_i, \Sigma_{-i}; \phi_i \right) = \phi_i^{-1} \left(\int_{\Sigma_{-i}} \phi_i \left(U \left(s_i, \sigma_{-i} \right) \right) dp_i(\sigma_{-i}) \right)$$

where $p_i \in \Delta(\Sigma_{-i})$ is a belief of i on his opponents' possible strategy distributions. The Waldean and Bayesian criteria are, respectively,

$$V_i \left(s_i, \Sigma_{-i} \right) = \min_{\sigma_{-i} \in \Sigma_{-i}} U_i \left(s_i, \sigma_{-i} \right) \quad \text{and} \quad V_i \left(s_i, p_i, \Sigma_{-i} \right) = \int_{\Sigma_{-i}} U_i \left(s_i, \sigma_{-i} \right) dp_i \left(\sigma_{-i} \right).$$

To ease notation we will sometimes write $V_i \left(s_i, p_i; \phi_i \right)$ and $V_i \left(s_i, p_i \right)$ for the smooth and Bayesian criteria, omitting the set Σ_{-i}.

3.2 Selfconfirming equilibrium

Next we give a general definition of selfconfirming equilibrium for the smooth criterion that restricts agents to choose pure strategies, so that "mixed" strategies arise only as distributions of pure strategies within populations of agents.[25]

[25] At the interpretive level, we are not assuming that agents are prevented from using randomization devices: it may be the case that agents in population i have a set $\hat{S}_i \subset S_i$ of "truly pure" strategies and that S_i includes a finite set of choices that are realization equivalent to randomizations over $\hat{S}_i$. Of course the definition of F_i has to be adapted accordingly, as $F_i(s_i, s_{-i})$ is a random message when s_i is a randomization device.

Definition 4 *A profile of strategy distributions* $\sigma^* = (\sigma_i^*)_{i\in I}$ *is a* smooth selfconfirming equilibrium (SSCE) *of a game with feedback and ambiguity attitudes* (Γ, f, ϕ) *if, for each* $i \in I$ *and each* $s_i^* \in \operatorname{supp}\sigma_i^*$*, there is a prior* $p_{s_i^*} \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ *such that*

$$V_i\left(s_i^*, p_{s_i^*}, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*); \phi_i\right) \geq V_i\left(s_i, p_{s_i^*}, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*); \phi_i\right) \tag{8}$$

for each $s_i \in S_i$*.*

This "confirmed rationality" condition extends condition (7) to general games and can be written as

$$s_i^* \in \arg\max_{s_i\in S_i} \int_{\hat{\Sigma}_{-i}(s_i^*,\sigma_{-i}^*)} \phi_i\left(U_i\left(s_i, \sigma_{-i}\right)\right) dp_{s_i^*}(\sigma_{-i}).$$

It requires that every pure strategy s_i^* that a positive fraction $\sigma_i^*(s_i)$ of agents keeps playing must be a best response within S_i to the "evidence," that is, the statistical distribution of messages $\hat{F}_i(s_i, \sigma_{-i}^*) \in \Delta(M)$ generated by playing s_i^* against the strategy distribution σ_{-i}^*.

A profile $\sigma^* = (\sigma_i^*)_{i\in I}$ is a:

(i) *Waldean selfconfirming equilibrium (WSCE)* if, for each $i \in I$ and each $s_i^* \in \operatorname{supp}\sigma_i^*$,

$$s_i^* \in \arg\max_{s_i\in S_i} \min_{\sigma_{-i}\in\hat{\Sigma}_{-i}(s_i^*,\sigma_{-i}^*)} U_i\left(s_i, \sigma_{-i}\right); \tag{9}$$

(ii) *Bayesian selfconfirming equilibrium (BSCE)* if, for each $i \in I$ and each $s_i^* \in \operatorname{supp}\sigma_i^*$, there is a prior $p_{s_i^*} \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ such that

$$s_i^* \in \arg\max_{s_i\in S_i} \int_{\hat{\Sigma}_{-i}(s_i^*,\sigma_{-i}^*)} U_i\left(s_i, \sigma_{-i}\right) dp_{s_i^*}\left(\sigma_{-i}\right); \tag{10}$$

(iii) *symmetric selfconfirming equilibrium* if σ^* is a pure strategy profile, that is, all agents in the same role play the same pure strategy.

This definition of Bayesian SCE subsumes the definition of conjectural equilibrium due to Battigalli (1987) (see also Battigalli and Guaitoli, 1997) and the definition of SCE of Fudenberg and Levine (1993a) as special cases. Battigalli (1987) allows for general feedback functions f_i, but considers only symmetric equilibria and assumes independent beliefs.[26] Fudenberg and Levine (1993a) assume that f_i is the identity for each i (perfect feedback).

Like these earlier notions of SCE, our more general notion is motivated by a partial identification problem: the mapping from strategy distributions to the distributions of observations available to an agent is not one to one. In fact, if for each agent i identification is full – i.e., $\hat{\Sigma}_{-i}(s_i, \sigma_{-i}) = \{\sigma_{-i}\}$ for all s_i and all σ_{-i} – condition (8) is easily seen to reduce to the standard Nash equilibrium condition $U_i\left(s_i^*, \sigma_{-i}^*\right) \geq U_i\left(s_i, \sigma_{-i}^*\right)$. In other words, if none of the agents features a partial identification problem, we are back to the Nash equilibrium notion (in its mass action interpretation).

[26] Battigalli (1987) also allows for randomization. Equilibria where agents choose randomized strategies are discussed in the next section.

Finally, note that ambiguity aversion may give rise to dynamic inconsistency issues: as the play unfolds, an agent may have incentives to deviate from an ex ante optimal strategy. As we mentioned, we avoid this problem by assuming that i just commits to a strategy that is then automatically implemented. As well known, when agents are assumed to be ambiguity averse it is important whether randomization is explicitly allowed or not; in the latter case mixed strategies σ_i only arise as pure strategy distributions in the population of agents who play in role i. We will see that the definition of SCE where agents cannot randomize is not a special case of the definition of SCE where agents can choose any randomization they like. In Sections 6 and 7 we will say more on this.

4 Comparative statics and relationships

4.1 Main result

Looking at games with feedback and ambiguity attitudes, we can carry out a comparative statics analysis in information and ambiguity. From this we also derive some results about relationships between equilibrium concepts. To this end, we say that:

(i) a feedback profile f is *coarser* than $\bar{f}$ if, for each i, f_i is coarser than $\bar{f}_i$;

(ii) an ambiguity profile ϕ is *more ambiguity averse* than $\bar{\phi}$ if, for each i, ϕ_i is *more ambiguity averse* than $\bar{\phi}_i$.[27]

A coarser feedback characterizes agents with less private ex post information, while higher ambiguity aversion characterizes agents with a higher dislike for model uncertainty. Clearly, any feedback profile is coarser than the perfect feedback profile $f = (\mathrm{Id}_Z, ..., \mathrm{Id}_Z)$, while any ambiguity averse profile ϕ is more ambiguity averse than the ambiguity neutral profile $\phi = (\mathrm{Id}_\mathbb{R}, ..., \mathrm{Id}_\mathbb{R})$.

Accordingly, we say that:

(i) game (Γ, f, ϕ) has *coarser feedback* than $\left(\Gamma, \bar{f}, \phi\right)$ if f is coarser than $\bar{f}$;

(ii) game (Γ, f, ϕ) is *more ambiguity averse* than $\left(\Gamma, f, \bar{\phi}\right)$ if ϕ is more ambiguity averse than $\bar{\phi}$.

Note that the comparison of ambiguity attitudes does not require that the profiles themselves be ambiguity averse. It only matters that one profile be comparatively more ambiguity averse than the other (something that can happen even if both are ambiguity loving).

We can now turn to the comparative static analysis. We begin by studying how equilibria are affected by changes in information feedback.[28]

[27] That is, there is a concave and strictly increasing function $\varphi_i : \mathrm{Im}\,\phi_i \to \mathbb{R}$ such that $\bar{\phi}_i = \varphi_i \circ \phi_i$. On this comparative notion see KMM.

[28] Similar results on information feedbacks are part of the folkore on SCE (see, e.g., Fudenberg and Kamada, 2011). We do not provide such proposition because of its originality, but rather because it sharpens the reader's understanding of the framework and concepts.

Proposition 5 *Suppose* (Γ, f, ϕ) *has coarser feedback than* $\left(\Gamma, \bar{f}, \phi\right)$. *Then, the SSCEs of* $\left(\Gamma, \bar{f}, \phi\right)$ *are also SSCEs of* (Γ, f, ϕ), *while the BSCE and WSCEs of* $\left(\Gamma, \bar{f}\right)$ *are also, respectively, BSCE and WSCEs of* (Γ, f).

Worse feedback enlarges the set of opponents' strategy distributions consistent with any distribution of messages (Lemma 2) and hence enlarges the set of SCEs of each type. A similar monotonicity holds for higher ambiguity aversion. It is the paper main result, which relies on the status quo bias of Lemma 1.

Theorem 6 *Suppose* (Γ, f, ϕ) *is more ambiguity averse than* $\left(\Gamma, f, \bar{\phi}\right)$. *If payoffs are observable, then the SSCEs of* $\left(\Gamma, f, \bar{\phi}\right)$ *are also SSCEs of* (Γ, f, ϕ). *Similarly, the SSCEs of any game* (Γ, f, ϕ) *are also WSCEs.*

We provided intuition for this result in the Introduction. Now we can be more precise: let σ^* be a SSCE of $(\Gamma, f, \bar{\phi})$, the *less* ambiguity averse game, and pick any strategy played by a positive fraction of agents, $s_i^* \in \mathrm{supp}\sigma_i^*$; then, there is a justifying confirmed belief $p_{s_i^*} \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ such that s_i^* is a best reply to $p_{s_i^*}$ given $\bar{\phi}_i$, i.e., $V_i(s_i^*, p_{s_i^*}; \bar{\phi}_i) \geq V_i(s_i, p_{s_i^*}; \bar{\phi}_i)$ for each s_i. Each agent playing the "status-quo" strategy s_i^* learns the long-run frequencies of its (observable) payoffs; therefore, the value of s_i^* for this agent is the objective expected utility, $U(s_i^*, \sigma_{-i}^*)$, independently of his ambiguity attitudes (Lemma 1). But the value of an untested strategy $s_i \neq s_i^*$ typically depends on ambiguity attitudes and, keeping beliefs fixed, it is higher when ambiguity aversion is lower, that is, $V_i(s_i, p_{s_i^*}; \bar{\phi}_i) \geq V_i(s_i, p_{s_i^*}; \phi_i)$. Therefore $V_i(s_i^*, p_{s_i^*}; \phi_i) = U(s_i^*, \sigma_{-i}^*) \geq V_i(s_i, p_{s_i^*}; \phi_i)$ for all s_i. This means that it is possible to support σ^* as a SSCE of the more ambiguity averse game (Γ, f, ϕ) using the same profile of beliefs supporting σ^* as a SSCE of $(\Gamma, f, \bar{\phi})$.

In sum, the set of SSCEs increases as either ex post private information becomes coarser or ambiguity aversion increases (or both). In particular, if we fix a game with feedback (Γ, f), Theorem 6 implies that under observable payoffs:

(i) the set of BSCEs of (Γ, f) is contained in the set of SSCEs of every (Γ, f, ϕ) with ambiguity averse players;

(ii) the set of SSCEs of every (Γ, f, ϕ) is contained in the set of WSCEs of (Γ, f).

In other words, under observable payoffs and ambiguity aversion, it holds

$$BSCE \subseteq SSCE \subseteq WSCE. \tag{11}$$

The degree of ambiguity aversion determines the size of the set of selfconfirming equilibria, with the sets of Bayesian and Waldean selfconfirming equilibria being, respectively, the smallest and largest one.

As well known, every Nash equilibrium σ^* of game Γ is a BSCE of (Γ, f); intuitively, a Nash equilibrium is a BSCE with correct beliefs.[29]

Lemma 7 *If* σ^* *is a Nash equilibrium of* Γ, *it is a BSCE of any game* (Γ, f) *with feedback.*

[29] For the sake of completeness, in the Appendix we give a proof in our setup.

Since the set NE of Nash equilibria is nonempty, we can enrich (11) as follows:

$$\emptyset \neq NE \subseteq BSCE \subseteq SSCE \subseteq WSCE.$$

This shows, under observable payoffs and ambiguity aversion, that every game (Γ, f, ϕ) has some SSCE and that Nash equilibria are not only BSCE, but more generally SSCE.

4.2 Payoff observability

The observability of payoff played a key role in establishing the inclusions (11). The following example shows that, indeed, such inclusions need not hold when payoffs are not observable.

Example 8 *Consider the zero-sum game of Figure 2 of the Introduction, but now suppose that player* 1 *cannot observe his payoff ex post (he only remembers his actions). For example, the utility values in Figure 2 could be a negative affine transformation of the consumption of player* 2*, reflecting a psychological preference of player* 1 *for decreasing the consumption of player* 2 *(not observed by* 1*) even if the consumption of* 1 *is independent of the actions taken in this game. Then, even if* 1 *plays one of the Matching Pennies subgames for a long time, he gets no feedback: under this violation of the observable payoff assumption* $\hat{\Sigma}_2(s_1, \sigma_2) = \Delta(S_2)$ *for all* (s_1, σ_2)*. Since* $u_1(O) = 1 + \varepsilon$ *is larger than the pure maxmin payoff of each subgame, the outside option O is the only WSCE choice of player* 1 *at the root. If* ϕ_1 *is sufficiently concave, O is also a SSCE choice. But, as already explained, O cannot be a Bayesian best reply. Furthermore, it can be verified that every strategy* s_1 *is a SSCE strategy.*[30] *Therefore,*

$$BSCE \cap WSCE = \emptyset \quad \text{and} \quad SSCE \nsubseteq WSCE$$

and so the inclusions (11) here do not hold. ▲

The observability of payoff is actually immaterial in two-person games. In fact, for them one can show directly, without assumptions about the feedback functions, that a BSCE is also a SSCE: the proof relies on the convexity of $\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ when $|I| = 2$ that allows to go from a justifying belief $p_i \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ to its predictive $\bar{p}_i \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ to a new justifying belief $\delta_{\bar{p}_i} \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ inducing no subjective uncertainty.[31] This argument

[30] We show that all the strategies of 1 that do not choose O are Bayesian and smooth best replies to some beliefs. Since player 1 only remembers his actions, his beliefs are trivially confirmed. Hence all these strategies have positive measure in some BSCE and SSCE. Let $k.X^1.X^2$ denote the strategy of 1 that selects subgame MP^k, with $k = 1, 2$, and action $X^1 \in \{H^1, T^1\}$ (resp. $X^2 \in \{H^2, T^2\}$) in subgame MP^1 (resp. MP^2). Similarly, $x^1.x^2$ denotes the strategy of 2 that selects action $x^1 \in \{h^1, t^1\}$ (resp. $x^2 \in \{h^2, t^2\}$) in subgame MP^1 (resp. MP^2). Then $1.H^1.X^2$ (resp. $1.T^1.X^2$) is a Bayesian best reply to $p_1^{H^1} = \delta_{\frac{1}{2}h^1.h^2 + \frac{1}{2}h^1.t^2}$ (resp. $p_1^{T^1} = \delta_{\frac{1}{2}t^1.h^2 + \frac{1}{2}t^1.t^2}$) for each $X^2 \in \{H^2, T^2\}$. Of course, each strategy of the form $2.X^1.X^2$ is a Bayesian best reply the Nash prior belief $p_1^{NE} = \delta_{\frac{1}{4}h^1.h^2 + \frac{1}{4}h^1.t^2 + \frac{1}{4}t^1.h^2 + \frac{1}{4}t^1.t^2}$. Since these beliefs are Dirac measures, ambiguity aversion does not affect the corresponding smooth value of strategies. Hence, they justify the same strategies also as smooth best responses. Also note that, with beliefs of the form $p_1^{X^1}$ shown above ($X^1 \in \{H^1, T^1\}$), player 1 is indifferent between the two subgames, as both have subjective value 2. Indeed, a Bayesian agent cannot strictly prefer MP^1 to MP^2; but there are non-degenerate priors that make the move to MP^1 a strict smooth best response.

[31] Given the previous example, one might conclude that there is a lack of upper-hemicontinuity of the equilibrium correspondence as the concavity of some ϕ_i goes to infinite. Such inference would be unjustified: the Waldean criterion is a limit of the smooth ambiguity criterion when p_i has full support in the set of possible models. But p_i may have a smaller support. In particular, it may be a Dirac measure, making ambiguity aversion irrelevant, as in the argument above.

does not work with n-person games when $n \geq 3$.

Example 9 *Consider the three-person, simultaneous-move game in Figure 3, where player* 1 *chooses the row, player* 2 *the column and player* 3 *the matrix. Assume that player 3 has trivial feedback: he only remembers his action, hence* $\hat{\Sigma}_{-3}(s_3, \sigma_{-3}) = \Delta(S_1) \bigotimes \Delta(S_2)$. *In contrast, players* 1 *and* 2 *have perfect feedback.*

ℓ	L	R
T	$1,1,2$	$0,0,0$
B	$0,0,0$	$1,1,0$

r	L	R
T	$1,1,1$	$0,0,1$
B	$0,0,1$	$1,1,1$

Figure 3 *A three-person simultaneous game*

In equilibrium $\sigma_1^*(T) = \sigma_1^*(L) \in \{0, 1, 1/2\}$. *Suppose that player* 3 *is certain that players* 1 *and* 2 *achieve perfect coordination, but he feels completely ignorant as to how. By symmetry, his belief is thus*

$$p_3 = \frac{1}{2}\delta_T \times \delta_L + \frac{1}{2}\delta_B \times \delta_R.$$

If he is a Bayesian (linear ϕ_3*), then both* ℓ *and* r *are best replies. But, if he is strictly uncertainty averse (strictly concave* ϕ_3*), then only* r *is a best reply. The predictive probability*

$$\frac{1}{2}(T, L) + \frac{1}{2}(B, R)$$

is a correlated distribution in $\Delta(S_{-3})$, *hence it does not belong to* $\hat{\Sigma}_{-3}(\ell, \sigma_{-3}^*)$, *and*

$$\delta_{\frac{1}{2}(T,L)+\frac{1}{2}(B,R)} \notin \Delta(\hat{\Sigma}_{-3}(l, \sigma_{-3}^*)).$$

Note, however, that we can support ℓ *as a SSCE choice using different beliefs: this is not an example where some BSCE is not a SSCE.* ▲

4.3 Converses and equivalences

Next we consider assumptions about information implying the equivalence between the different SCE concepts (or at least their symmetric version) and also the equivalence with Nash equilibrium.

We start with a simple, but instructive result that requires a preliminary definition. We say that (Γ, f) has *own-strategy independent feedback* if what each player observes ex post about the strategies of other players is independent of his own strategy, that is, for each $i \in I$, $F_{s_i}^{-1}(\cdot)$ is independent of s_i (recall that $F_{s_i}(\cdot) = f_i(\zeta(s_i, \cdot)) : S_{-i} \to M$).[32] This is a restrictive assumption that is violated by all of our examples.[33] The next result illustrates its strength.

[32] This property is called "non manipulability of information" in Battigalli *et al.* (1992) and Azrieli (2009b), and "own-strategy independence" by Fudenberg and Kamada (2011).

[33] One can show that a game (Γ, f) with perfect feedback has also own-strategy independent feedback if and only if Γ can be reduced to a realization-equivalent simultaneous game Γ' by interchanging simultaneous moves and coalescing sequential moves by the same player.

Proposition 10 *In every game with observable payoffs and own-strategy independent feedback, every type of SCE is equivalent to Nash equilibrium:*

$$NE = BSCE = SSCE = WSCE.$$

The intuition for this result is quite simple: the strategic-form payoff function $U_i(s_i, \cdot) : S_{-i} \to \mathbb{R}$ is constant on each cell $F_{s_i}^{-1}(m)$ of the partition of S_{-i} induced by $F_{s_i} : S_{-i} \to M$ (observability of payoffs), but $F_{s_i}^{-1}$ is independent of s_i (own-strategy independence of feedback). This means that, in the long run, an agent does not only learn the objective probabilities of the payoffs associated to his "status quo" strategy, but also the objective probabilities of the payoffs associated to every other strategy. Hence model uncertainty is irrelevant and he learns to play the best response to the true strategy distributions of the other players/roles even if he does not exactly learn these distributions.[34]

Assuming perfect information and perfect feedback, we can prove another partial converse to the above inclusions (11) and hence another equivalence result.

Proposition 11 *In games with perfect information and perfect feedback, every symmetric WSCE is also a symmetric SSCE and BSCE where each player's strategy is justified by a confirmed deterministic belief.*

Since perfect feedback implies observable payoffs, Theorem 6 and Proposition 11 imply that under perfect information and perfect feedback[35]

$$symBSCE = symSSCE = symWSCE.$$

Note that the result about symSSCE holds with either two players or n ambiguity averse players, as in both cases $symBSCE \subseteq symSSCE \subseteq symWSCE$.

Games with perfect information necessarily feature observable deviators[36] and every predictive belief $\bar{p}_i \in \Delta(S_{-i})$ is realization-equivalent to an uncorrelated belief $\bar{q}_i \in \bigotimes_{j \neq i} \Delta(S_j)$. In games with observable deviators and perfect feedback, every symmetric BSCE supported by uncorrelated beliefs has the same outcome as some mixed Nash equilibrium with a deterministic path and *viceversa*. Therefore, in games with perfect information and perfect feedback the set of terminal histories induced (with probability one) by some mixed Nash equilibrium coincides with the set $\zeta(symWSCE)$ of terminal histories induced by a symmetric WSCE. To summarize, let $detNE$ denote the set of Nash equilibria with a deterministic path; then it makes sense to write $\zeta(detNE)$ for the set of these paths, and the following holds:

Proposition 12 *In games with perfect information and perfect feedback*

$$\zeta(detNE) = \zeta(symBSCE) = \zeta(symWSCE);$$

if, in addition, there are either two players or n ambiguity averse players, then

$$\zeta(detNE) = \zeta(symBSCE) = \zeta(symSSCE) = \zeta(symWSCE).$$

[34] Similar results are part of the folklore on SCE. See, for example, Battigalli (1999) and Fudenberg and Kamada (2011).

[35] The prefix *sym* denotes symmetric selfconfirming equilibria.

[36] For every path z and every information set $\mathbf{h}$ reached with a unilateral deviation from z, the player moving at $\mathbf{h}$ can identify the deviator. See Fudenberg and Levine (1993a).

5 A zero-sum example

In this section we analyze the SCEs of a zero-sum example parametrized by the number of strategies. The game is related to the Matching Pennies example of the Introduction. We show how the SSCE set gradually expands from the BSCE set to the WSCE set as the degree of ambiguity aversion increases.

To help intuition, we first consider a generalization of the game of Figure 2: player 1 chooses between an outside option O and n Matching-Pennies subgames against player 2. Subgames with a higher index k have "higher stakes," that is, a higher (mixed) maxmin value, but a lower minimum payoff (see Figure 4). The game of Figure 2 obtains for $n = 2$.

$$n-1+\varepsilon \quad \overset{O}{\longleftarrow} \quad 1 \qquad (0<\varepsilon<1/2)$$
$$\swarrow \ \dots \quad \downarrow k \quad \dots \ \searrow$$

$1\backslash 2$	h^k	t^k
H^k	$n+2(k-1)$	$n-k$
T^k	$n-k$	$n+2(k-1)$

Figure 4. Fragment of zero-sum game

In this game, player 1 has $(n+1)\times 2^n$ strategies and player 2 has 2^n strategies. To simplify the notation, we instead analyze an equivalent extensive-form game Γ_n obtained by two transformations. First, player 2 is replaced by a team of opponents $2.1, ..., 2.n$, one for each subgame k, each one with the same payoff function $u_{2.k} = -u_1$. Second, the sequence of moves (k, H^k) of player 1 (go to subgame k then choose Head) – which is common to 2^{n-1} realization-equivalent strategies – is coalesced into the single strategy H^k. Similarly (k, T^k) becomes T^k. The new strategy set of player 1 has $2n+1$ strategies: $S_1 = \{O, H^1, T^1, ..., H^n, T^n\}$. If player 1 chooses H^k or T^k, player $2.k$ moves at information set $\{H^k, T^k\}$ (i.e., without knowing which of the two actions was chosen by player 1) and chooses between h^k and t^k; hence $S_{2.k} = \{h^k, t^k\}$. See Figure 4bis.

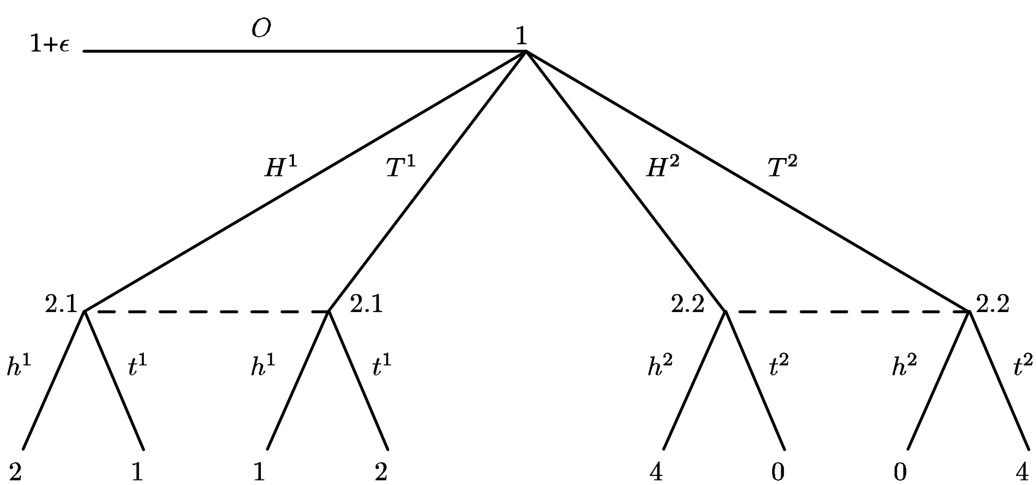

Figure 4bis. The case $n = 2$.

We assume *perfect feedback*,[37] therefore the game with feedback is (Γ_n, f) with $f_i = \mathrm{Id}_Z$ for each $i = 1, ..., n$.

Although there are no proper subgames in Γ_n, we slightly abuse language and informally refer to "subgame k" when player 1 chooses H^k or T^k, giving the move to opponent $2.k$. The game Γ_n and the previously described game have isomorphic sets of terminal nodes (with cardinality $4n + 1$) and the same reduced normal form (once players $2.1, ..., 2.n$ of the second game are coalesced into a unique player 2). By standard arguments, these two games have equivalent sets of Nash equilibria, equivalent BSCE and WSCE sets for every f, and equivalent SSCE sets for every (f, ϕ).[38]

That said, consider the extensive-form game Γ_n (Figure 4bis illustrates the special case Γ_2). It is easily seen that, for every profile of strategy distributions $\sigma_2^* = (\sigma_{2.k}^*)_{k=1}^n$, it holds[39]

$$\hat{\Sigma}_2(O, \sigma_2^*) = \bigotimes_{k=1}^{n} \Delta(S_{2.k}), \tag{12}$$

and

$$\hat{\Sigma}_2(H^k, \sigma_2^*) = \hat{\Sigma}_2(T^k, \sigma_2^*) = \{\sigma_2 : \sigma_{2.k} = \sigma_{2.k}^*\}. \tag{13}$$

As a result, next we provide necessary SCE conditions that partially characterize the equilibrium strategy distribution for player/role 1 and fully characterize the equilibrium strategy distributions for the opponents.

Lemma 13 *For every (Bayesian, Smooth, Waldean) SCE σ^* and every $k = 1, ..., n$,*

$$\sigma_1^*(H^k) + \sigma_1^*(T^k) > 0 \Rightarrow \frac{\sigma_1^*(H^k)}{\sigma_1^*(H^k) + \sigma_1^*(T^k)} = \frac{1}{2} = \sigma_{2.k}^*(h^k). \tag{14}$$

Furthermore, for every σ^* and $\bar{\sigma}^*$, if σ^* is a (Bayesian, Smooth, Waldean) SCE, and $\operatorname{supp} \sigma_1^* = \operatorname{supp} \bar{\sigma}_1^*$, then also $\bar{\sigma}^*$ is a (Bayesian, Smooth, Waldean) SCE.

Note that these necessary conditions do not restrict at all the set of strategies that can be played in equilibrium: for every $s_1 \in \{O, H^1, T^1, ..., H^n, T^n\}$ there is some distribution profile σ^* such that $\sigma_1^*(s_1) > 0$ and (14) holds. The formal proof of the lemma is straightforward and left to the reader. Informally, if subgame k is played with positive probability, then each agent playing this subgame learns the relative frequencies of Head and Tail in the opponent's

[37] We could equivalently assume observable payoffs.

[38] Each profile $\sigma = (\sigma_1, (\sigma_{2.k})_{i=1}^n)$ of the new n-person game can be mapped to an equivalent profile $(\bar{\sigma}_1, \bar{\sigma}_2)$ of the old two-person game and viceversa while preserving the equilibrium properties. Specifically, $(\sigma_{2.k})_{k=1}^n$ is also a behavioral strategy of player 2 in the two-person game, which corresponds to a realization-equivalent strategy distribution $\bar{\sigma}_2$ for player 2. Similarly, any such distribution $\bar{\sigma}_2$ can be mapped to a realization-equivalent profile $(\sigma_{2.k})_{k=1}^n$. As for σ_1, for each s_1 in the new game, the probability mass $\sigma_1(s_1)$ can be distributed arbitrarily among the pure strategies of the old two-person game that select the corresponding sequence of moves (that is, either (O), or (k, H^k) or (k, T^k)), thus obtaining a realization-equivalent distribution $\bar{\sigma}_1$. In the opposite direction, every $\bar{\sigma}_1$ of the old game yields a unique realization-equivalent σ_1 in the new game, where $\sigma_1(s_1)$ is the $\bar{\sigma}_1$-probability of the set of (realization-equivalent) strategies that select the same sequence of moves as s_1.

[39] For ease of notation, in this section we denote $\hat{\Sigma}_{-1}$ by $\hat{\Sigma}_2$.

population, and the best response conditions imply that an SCE reaching subgame k with positive probability must induce a Nash equilibrium in this Matching-Pennies subgame. Thus, the σ_2^*-value to an agent in population 1 of playing the "status quo" strategy H^k or T^k (with $\sigma_1^*(H^k) + \sigma_1^*(T^k) > 0$) is the mixed maxmin value of subgame k, $n - 1 + k/2$. With this, the value of deviating to another "untested" strategy depends on the exogenous attitudes toward model uncertainty, and on the second-order belief $p_1 \in \Delta(\hat{\Sigma}_2(H^k, \sigma_2^*))$, which is only restricted by $\sigma_{2.k}^*$ (eqs. (12) and (13)). As for the agents in roles $2.1, ..., 2.n$, their attitudes toward uncertainty are irrelevant, because, if they play at all, they learn all that matters to them, that is, the relative frequencies of H^k and T^k.

Suppose that a positive fraction of agents in population 1 play H^k or T^k, with $k < n$. By Lemma 13, in an SCE the value that they assign to their strategy is its vNM expected utility given that opponent $2.k$ mixes fifty-fifty, that is, $n - 1 + k/2$. But, if they are Bayesian, the subjective value of deviating to subgame n is at least the maxmin value $n - 1 + n/2 > n - 1 + k/2$. Furthermore, the outside option O is never a Bayesian best reply.[40] This explains the following:

Proposition 14 *The BSCE set of* $(\Gamma_n, \mathrm{Id}_Z, ..., \mathrm{Id}_Z)$ *coincides with the set of Nash equilibria. Specifically,*

$$BSCE = NE = \left\{ \sigma^* \in \Sigma : \sigma_1^*(H^n) = \sigma_1^*(T^n) = \sigma_n^*(h^n) = \frac{1}{2} \right\}.$$

Next we analyze the SSCEs assuming that agents are ambiguity averse in the KMM sense. The following preliminary result, which has some independent interest, specifies the beliefs about opponents' strategy distributions that minimize the subjective value of deviating from a given strategy s_1 to any subgame j.

Lemma 15 *Let* ϕ_1 *be concave. For all* $j = 1, ..., n$, $p_1, q_1 \in \Delta\left(\bigotimes_{k=1}^{n} \Delta(S_{2.k})\right)$, *if*

$$\mathrm{mrg}_{\Delta(S_{2.j})} q_1 = \frac{1}{2}\delta_{h^j} + \frac{1}{2}\delta_{t^j},$$

then

$$\max\{V_1(H^j, p_1; \phi_1), V_1(T^j, p_1; \phi_1)\} \geq V_1(H^j, q_1; \phi_1) = V_1(T^j, q_1; \phi_1).$$

Intuitively, an ambiguity averse agent dislikes deviating to subgame j the most when his subjective prior assigns positive weight only to the highest and lowest among the possible objective expected utility values, i.e., when its marginal on $\Delta(S_j)$ has the form $x\delta_{h^j} + (1 - x)\delta_{t^j}$. By symmetry of the 2×2 payoff matrix of subgame k, he would pick within $\{H^k, T^k\}$ the strategy corresponding to the highest subjective weight (H^k if $x > 1/2$). Hence the subjective value of deviating to subgame j is minimized when the two Dirac measures δ_{h^j} and δ_{r^j} have the same weight $x = 1/2$.

To analyze how the SSCE set changes with the degree of ambiguity aversion of player 1, we consider the one-parameter family of negative exponential weighting functions

$$\phi_1^\alpha(U) = -e^{-\alpha U},$$

[40] Indeed, O is strictly dominated by every mixed strategy $\frac{1}{2}H^k + \frac{1}{2}T^k$.

where $\alpha > 0$ is the coefficient of ambiguity aversion (see KMM p. 1865). Let $SSCE(\alpha)$ denote the set of SSCEs of $(\Gamma_n, \mathrm{Id}_Z, ..., \mathrm{Id}_Z, \phi_1^\alpha, \phi_2, ...\phi_n)$. To characterize the equilibrium correspondence $\alpha \mapsto SSCE(\alpha)$, we use the following transformation of $\phi_1^\alpha(U)$:

$$M(\alpha, x, y) = (\phi_1^\alpha)^{-1}\left(\frac{1}{2}\phi_1^\alpha(x) + \frac{1}{2}\phi_1^\alpha(y)\right).$$

By Lemma 15, this is the minimum value of deviating to a subgame with payoffs x and y. The following result states that this value is decreasing in the coefficient of ambiguity aversion α, it converges to the mixed maxmin value as $\alpha \to 0$ (approximating the Bayesian case), and it converges to the minimum payoff as $\alpha \to +\infty$.

Lemma 16 *For all $x \neq y$, $M(\cdot; x, y)$ is strictly decreasing, continuous, and satisfies*

$$\lim_{\alpha \to 0} M(\alpha; x, y) = \frac{1}{2}x + \frac{1}{2}y \quad \text{and} \quad \lim_{\alpha \to +\infty} M(\alpha; x, y) = \min\{x, y\}. \tag{15}$$

By Lemma 13, to analyze the $SSCE(\alpha)$ correspondence we only have to determine the strategies s_1 that can be played by a positive fraction of agents in equilibrium, or – conversely – the strategies s_1 that must have measure zero. Let us start from very small values of α, i.e., approximately Bayesian agents. By Lemmas 15 and 16, the subjective value of deviating to the highest-stakes subgame n is approximately bounded below by $n - 1 + n/2 > u_1(O)$. Therefore, the outside option O cannot be a best reply. Furthermore, suppose by way of contradiction that H^k or T^k $(k < n)$ are played by a positive fraction of agents. By Lemma 13, the value of playing subgame k is the vNM expected utility $n - 1 + k/2 < n - 1 + n/2$. Hence all agents playing this game would deviate to the highest-stakes subgame n. Thus, for α small $SSCE(\alpha) = BSCE$. By Lemma 16, as α increases, the minimum value of deviating to subgame n decreases, converging to zero for $\alpha \to +\infty$. More generally, the minimum value $M(\alpha, n - j, n + 2(j - 1))$ of deviating to subgame j converges to $n - j$ for $\alpha \to +\infty$. Since $n - j < u_1(O) < n - 1 + k/2$, this means that, as α increases, it becomes easier to support an arbitrary strategy s_1 as an SSCE strategy. Therefore there must be thresholds $0 < \alpha_1 < ... < \alpha_n$ such that *only* the higher-stakes subgames $k + 1, ...n$ can be played by a positive fraction of agents in equilibrium if $\alpha < \alpha_{n-k}$, and *every* strategy (including the outside option O) can be played by a positive fraction of agents for *some* $\alpha \geq \alpha_{n-k}$. In particular, for α sufficiently large, $SSCE(\alpha)$ coincides with the set of Waldean SCEs, which is just the set

$$\Sigma^* = \{\sigma^* \in \Sigma : \text{eq. (14) holds}\}$$

of distribution profiles satisfying the necessary conditions of Lemma 13.[41] To summarize, by the properties of function $M(\alpha, x, y)$ stated in Lemma 16, we can define strictly positive thresholds $\alpha_1 < \alpha_2 < ... < \alpha_n$ so that the following indifference conditions hold

$$\max_{j \in \{k+1,...,n\}} M(\alpha_{n-k}, n - j, n + 2(j - 1)) = n - 1 + \frac{k}{2}, \qquad k = 1, ..., n - 1, \tag{16}$$

$$\max_{j \in \{k+1,...,n\}} M(\alpha_n, n - j, n + 2(j - 1)) = n - 1 + \varepsilon, \tag{17}$$

[41] This characterization holds for every parametrized family of distributions that satisfies, at every expected utility value $\bar{U}$, properties analogous to those of Lemma 16, with α replaced by the coefficient of ambiguity aversion $-\phi_1''(\bar{U})/\phi'(\bar{U})$.

and $SSCE(\alpha)$ expands as α increases, making subgame k playable in equilibrium as soon as α reaches α_{n-k}, expanding to $WSCE$ and making the outside option O playable as soon as α reaches α_n. Formally:

Proposition 17 *Let $\alpha_1 < ... < \alpha_n$ be the strictly positive thresholds defined by (16) and (17). For every α and $k = 1, ...n-1$,*

$$\alpha < \alpha_{n-k} \Longrightarrow SSCE(\alpha) = \left\{\sigma^* \in \Sigma^* : \sigma_1^*(\{O, L^1, T^1, ..., H^k, T^k\}) = 0\right\}$$

and

$$\alpha < \alpha_n \Longrightarrow SSCE(\alpha) = \{\sigma^* \in \Sigma^* : \sigma_1^*(O) = 0\}.$$

Furthermore

$$\bigcup_{\alpha \geq \alpha_{n-k}} SSCE(\alpha) = \Sigma^* = WSCE,$$

and $SSCE(\alpha) = BSCE = NE$ if $\alpha < \alpha_1$, while $SSCE(\alpha) = WSCE$ if $\alpha \geq \alpha_n$.

6 Randomization

In this section we allow agents to commit to arbitrary objective randomization devices. Credible randomizations require a richer commitment technology than assumed so far. This can be seen by focussing on static games. In such games, playing a pure strategy simply means that an action is irreversibly chosen. But there is a commitment issue in playing mixed strategies. Suppose that a particular mixed strategy σ_i is optimal. If this is true for a Bayesian agent, then also each pure strategy in the support of σ_i is optimal, therefore σ_i can be implemented by mapping each strategy in suppσ_i to the realization of an appropriate roulette spin and then choosing the pure strategy associated to the realization. On the other hand, an ambiguity averse agent who finds σ_i optimal, need not find the pure strategies in suppσ_i optimal (within the simplex $\Delta(S_i)$). Therefore, unlike a Bayesian agent, an ambiguity averse one has to be able to irreversibly delegate his choice to the random device. Despite these reservations, we still find the extension to mixed strategies worth exploring.

When each agent in population i plays a mixed strategy, a mixed strategy distribution $\varsigma_i \in \Delta(\Delta(S_i))$ obtains. To simplify the analysis, we consider only the case in which the ensuing distributions ς_i on mixed strategies have finite support.[42] For each role i, agents in population i are "endogenously" partitioned into k_i "epistemic types," and ς_i^ℓ is the fraction of agents in population i of type $\ell = 1, ..., k_i$. In every period, agents are drawn at random and matched to play the given game. Each type ℓ of agent in population i keeps playing some (possibly degenerate) mixed strategy σ_i^ℓ. This generates a mixture distribution $\sigma_i^* = \sum_{\ell=1}^{k_i} \varsigma_i^\ell \sigma_i^\ell$ on pure strategies in population i. Agents of type ℓ in population i accumulate evidence (a message in every period) that in the long run is summarized by the long-run frequencies $\mu_i^{*,\ell} \in \Delta(M)$, which depend on σ_i^ℓ and σ_{-i}^*: $\mu_i^{\ell,*} = \hat{F}_i(\sigma_i^\ell \times \sigma_{-i}^*)$. Hence the set of possible models for agents of type ℓ is $\hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*) = \{\sigma_{-i} : \hat{F}_i(\sigma_i^\ell \times \sigma_{-i}) = \hat{F}_i(\sigma_i^\ell \times \sigma_{-i}^*)\}$. In the following definitions, we give stability conditions for a profile of strategy distributions $(\sigma_i^*)_{i \in I}$ interpreted as mixtures. We use superscript ρ to denote the equilibrium concepts obtained when randomization is allowed.

[42] We conjecture that, for generic payoffs, this is enough to characterize aggregate equilibrium distributions of pure strategies.

Definition 18 *A profile of strategy distributions* $(\sigma_i^*)_{i\in I}$ *is a* smooth selfconfirming equilibrium with randomization (SSCE$^\rho$) of (Γ, f, ϕ) *if, for each* i*, there are a positive integer* $k_i \in \mathbb{N}$*, a vector of weights* $(\varsigma_i^1, ...\varsigma_i^{k_i}) \in \mathbb{R}_{++}^{k_i}$ *with* $\sum_{\ell=1}^{k_i} \varsigma_i^\ell = 1$ *and a vector of mixed strategies* $(\sigma_i^1, ..., \sigma_i^{k_i}) \in [\Delta(S_i)]^{k_i}$*, such that*

1. *(aggregation)* $\sigma_i^* = \sum_{\ell=1}^{k_i} \varsigma_i^\ell \sigma_i^\ell$,
2. *(*confirmed rationality*) for each* $\ell = 1, ..., k_i$ *there is a belief* $p_i^\ell \in \Delta(\hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*))$ *with*

$$\sigma_i^\ell \in \arg\max_{\sigma_i\in\Delta(S_i)} \int_{\hat{\Sigma}_{-i}(\sigma_i^\ell,\sigma_{-i}^*)} \phi_i(U_i(\sigma_i, \sigma_{-i}))dp_i^\ell(\sigma_{-i})$$

A profile $(\sigma_i^*)_{i\in I}$ *satisfying condition 1 above is a* Waldean selfconfirming equilibrium (WSCE$^\rho$) *of* (Γ, f) *if, for each* i *and* ℓ

$$\sigma_i^\ell \in \arg\max_{\sigma_i\in\Delta(S_i)} \min_{\sigma_{-i}\in\hat{\Sigma}_{-i}(\sigma_i^\ell,\sigma_{-i}^*)} U_i(\sigma_i, \sigma_{-i}),$$

and it is a Bayesian selfconfirming equilibrium with randomization (BSCE$^\rho$) *of* (Γ, f) *if it is a SSCE*$^\rho$ *of the game* (Γ, f, ϕ) *with ambiguity neutral players, where each* ϕ_i *is the identity. A Smooth (Waldean, Bayesian) equilibrium profile is* symmetric *if* $k_i = 1$ *for each* $i \in I$.

Since for Bayesian agents the value of an optimal mixed strategy is the same as the value of any pure strategy in its support, the BSCE concept gives rise to the same strategy distributions as the BSCE$^\rho$ concept:

$$BSCE = BSCE^\rho.$$

Indeed, BSCE is a special case of BSCE$^\rho$ where agents do not randomize, and each BSCE$^\rho$ is equivalent to a BSCE where agents of type ℓ in population i are further divided into "subtypes" playing the pure strategies in the support of σ_i^ℓ.

A symmetric BSCE$^\rho$ is such that every pure strategy s_i^* played with positive probability in equilibrium can be justified by the same confirmed belief p_i. Thus, the symmetric BSCE$^\rho$ concept is equivalent to the SCE with unitary beliefs of Fudenberg and Levine (1993a) generalized to arbitrary feedback.

The comparative statics results and relationships of Section 4 extend seamlessly to equilibria with randomization (the same arguments apply almost *verbatim* with pure strategies replaced by mixed strategies):

- the coarser the feedback, the larger the set of equilibria of each kind;
- under observable payoffs, the set of SSCE$^\rho$s becomes larger as ambiguity aversion increases;
- under observable payoffs and ambiguity aversion,

$$BSCE^\rho \subseteq SSCE^\rho \subseteq WSCE^\rho;$$

  furthermore, the first inclusion holds in every two-person game with any feedback and ambiguity attitudes;

- in games with observable payoffs and own-strategy independent feedback the three kinds of SCE with randomization coincide with Nash equilibrium;
- in games with perfect information and perfect feedback every symmetric WSCE$^\rho$ is also a symmetric SSCE$^\rho$ and BSCE^R where each player's mixed strategy is justified by a confirmed deterministic belief (hence, mixed strategies can only randomize off the equilibrium path).

That said, allowing for randomization considerably reduces the scope for differences between the BSCE, SSCE and WSCE concepts. Indeed, they coincide in every two-person game with observable payoffs and ambiguity aversion.

Proposition 19 *In two-person games, every WSCE$^\rho$ is also a SSCE$^\rho$ and BSCE$^\rho$. Hence, in two-person games with observable payoffs the three kinds of SCE with randomization coincide; since $BSCE = BSCE^\rho$ it holds*

$$BSCE = BSCE^\rho = SSCE^\rho = WSCE^\rho.$$

Intuitively, the wedge between BSCE, SSCE and WSCE is driven by the fact that the uncertainty implied by partial identification of the opponent's strategy distribution affects differently the "status quo" strategy played by an agent in equilibrium, which has a known objective expected utility, and the alternative untested strategies, whose objective expected utility is unknown. But the possibility to randomize reduces the extent to which model uncertainty translates into uncertainty about the objective expected utility of the best deviation. In particular, when strategies are evaluated according to the Waldean criterion, it is easy to see that the incentive to deviate from a strategy s_i^* is stronger when randomization is allowed:

$$\max_{s_i \in S_i} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U_i(s_i, \sigma_{-i}) \leq \max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U_i(\sigma_i, \sigma_{-i}).$$

This point is illustrated by the zero-sum game discussed in the Introduction and carefully analyzed in Section 5.

Example 20 *Consider again the game of Figure 2 of the Introduction, assuming perfect feedback (or observable payoffs). Propositions 14 and 19 imply that $WSCE^\rho = NE$. It is not hard to verify this directly (for simplicity, we refer to the strategies of the reduced normal form: O, H^1, T^1, H^2 and T^2; see Section 5). Mixed strategy $\frac{1}{2}H^2 + \frac{1}{2}T^2$ yields, for every $\sigma_2 \in \Delta(S_2)$,*

$$U_1\left(\frac{1}{2}H^2 + \frac{1}{2}T^2, \sigma_2\right) = 2 = \max_{\sigma_1 \in \Delta(S_1)} \min_{\sigma_2 \in \Delta(S_2)} U_1(\sigma_1, \sigma_2) > 1 + \varepsilon = u_1(O).$$

Therefore the outside option O cannot be a Waldean best reply within the set of mixed strategies, and $\sigma_1^*(O) = 0$ for every $\sigma^* \in WSCE^\rho$. Furthermore, if a positive fraction of agents played the low-stakes subgame MP^1 in an equilibrium σ^*, then they would learn that $\sigma_2^*(h^1) = 1/2$ and $U_1(H^1, \sigma_2^*) = U_1(T^1, \sigma_2^*) = 1.5$, and would deviate to $\frac{1}{2}H^2 + \frac{1}{2}T^2$. In sum, for every $\sigma^* \in WSCE^\rho$, $\sigma_1^*(H^2) = \sigma_1^*(T^2) = \sigma_2^*(h^2) = 1/2$, which implies that σ^* is a Nash equilibrium. ▲

The two-person assumption in Proposition 19 is used to obtain the convexity of $\hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)$, which is crucial in the proof. But we are not aware of counterexamples with more than two players.

Next we turn to the relationship between selfconfirming equilibria with randomization and Nash equilibrium. We know that $NE \subseteq BSCE = BSCE^\rho$ in all games, $BSCE^\rho \subseteq SSCE^\rho \subseteq WSCE^\rho$ in two-person games with observable payoffs, and analogous relations hold for symmetric equilibria (in particular, every Nash equilibrium is also a symmetric BSCE$^\rho$). Of course, these results hold under perfect feedback, which implies observable payoff. Battigalli (1987) proved that, in two-person games with perfect feedback, every symmetric BSCE$^\rho$ (a selfconfirming equilibrium with unitary beliefs) is realization-equivalent to a mixed Nash equilibrium. With this, Proposition 19 yields the following equivalence result (consistently with our notation, we let $\hat{\zeta}(\sigma)$ denote the random path induced by the product measure $\sigma = \times_{i \in I} \sigma_i$):

Proposition 21 *In two-person games with perfect feedback, every symmetric WSCE$^\rho$ is realization-equivalent to some mixed Nash equilibrium; therefore*

$$\hat{\zeta}(symWSCE^\rho) = \hat{\zeta}(symSSCE^\rho) = \hat{\zeta}(symBSCE^\rho) = \hat{\zeta}(NE).$$

The two-person condition is tight. For instance, consider the "horse" game with three players and imperfect information depicted in Figure 1. Path (A, a) (for any feedback structure) is supported by (Waldean, Bayesian, Smooth) symmetric selfconfirming equilibria with randomization, but it is not a pure or mixed Nash equilibrium path.[43]

7 Discussion and related literature

Let us take stock of what we did. We analyzed a notion of SCE with agents who have non-Bayesian attitudes toward uncertainty on the true data generating process they are facing. We argued that this uncertainty comes from a partial identification problem: the mapping from strategy distributions to the distributions of observations available to an agent is not one to one. This is discussed and illustrated in the context of decision problems (Section 2) before we move to the game theoretic analysis. We used as our workhorse the KMM smooth-ambiguity model, which separates endogenous beliefs from exogenous ambiguity attitudes. This makes our setup particularly well suited to connect with the previous literature on SCE and to analyze how the set of equilibria changes with the degree of ambiguity aversion. According to our main result, Theorem 6, the set of SSCE expands when agents become more ambiguity averse (assuming observability of payoffs). The reason is that agents learn the expected utility values of the strategies played in equilibrium, but not of the strategies they can deviate to, which are thus penalized by ambiguity aversion. This allows us to derive intuitive relationships between different versions of SCE and show that Nash equilibrium is a refinement of all of them, which guarantees existence. Equivalence results are provided

[43] The same holds for symmetric Bayesian selfconfirming equilibria supported by beliefs that satisfy independence across opponents. This can happen because the "horse" does not feature observable deviators. On the other hand, in all games with perfect feedback and observable deviators, every such equilibrium is realization equivalent to some Nash equilibrium. This can be shown directly. Kamada (2010) proves it as a corollary of another result.

under specific assumptions on the information structure. In the core sections of the paper (Sections 3.1 and 4) we assume that agents play the strategic form of a dynamic game and can only choose pure strategies. Mixed strategies are considered in Section 6 where we put forward notions of equilibrium with randomization: on the one hand, our main results carry over to these equilibrium concepts; on the other hand, we obtain equivalence results under much weaker conditions.

We developed our theoretical insights in the framework of population games played recurrently, but similar intuitions apply to different strategic contexts, such as dynamic games with a Markov structure. Our insights are likely to have consequences for more applied work. For example, the SCE and ambiguity aversion ideas have been applied in macroeconomics to analyze, respectively, learning in policy making (see Sargent, 1999, and the references in Cho and Sargent, 2008) and robust control (Sargent and Hansen, 2008). Our analysis suggests that these two approaches can be fruitfully merged. Fershtman and Pakes (2011) put forward a concept of "experienced based equilibrium" akin to SCE to provide a framework for the theoretical and empirical analysis of dynamic oligopolies. They argue that equilibrium conditions are, in principle, testable when agents beliefs are determined (if only partially) by empirical frequencies, as in their equilibrium concept and SCE. Their model features observable payoffs (firms observe profits), therefore a version of our main result should apply: ambiguity aversion expands the set of equilibria.

We now discuss the following points: (1) We avoided dynamic consistency issues assuming that agents play the strategic form of the game. How should we define our equilibrium concepts without this commitment assumption? How are our results going to be affected? (2) We clarified that our notions of SCE make sense under minimal assumptions about agents' knowledge of the game. But under common knowledge of the game, stronger notions of equilibrium should arguably be considered. (3) Like Fudenberg and Levine (1993a,b), we assume that each agent has only access to his "personal database" of experiences in the games he played. How is the SSCE concept affected if agents have access to a "public database"? (4) Fudenberg and Levine (1993b) give a steady state learning foundation to the SCE concept, is it possible to do the same for SSCE? We close with a discussion of the related literature.

7.1 Dynamic consistency

When agents really play an extensive-form game, not its strategic form, they cannot commit to any strategy. A strategy for an agent is just a plan that allows him to evaluate the likely consequences of taking actions at any information set. The plan is credible and can be implemented only if it prescribes, at each possible information set, an action that has the highest value, given the agent's belief and planned continuation. Plans with this *unimprovability* property can be obtained by means of a backward induction procedure on the subjective decision tree implied by the agent's beliefs. Such a procedure may have to break ties. How ties are broken is immaterial for a Bayesian agent, but matters for a non-Bayesian one. Consider two information sets of player i, viz. $\bar{h}_i$ and h_i, so that $\bar{h}_i$ immediately precedes h_i in the tree of information sets of i, and let $\alpha_i(\bar{h}_i, h_i)$ be the action at $\bar{h}_i$ that leads to h_i.[44] If ties at

[44] Perfect recall implies that function $\alpha_i(\cdot,\cdot)$ is well defined, and that the collection of information sets of i (plus the singleton containing the root of the game, and with the obvious precedence relation derived from precedence between nodes) is a tree.

h_i are broken so as to maximize the value of action $\alpha_i(\bar{h}_i, h_i)$ (and further ties are broken to maximize the value of $\alpha_i(\hat{h}_i, \bar{h}_i)$, where $\hat{h}_i$ is the immediate predecessor of $\bar{h}_i$, and so on), the resulting strategy satisfies *consistent planning.* We can make this precise in the context of the smooth-ambiguity model, and thus provide notions of SSCE assuming unimprovability, or consistent planning. For brevity, we write SSCE$^\delta$ to refer to such smooth selfconfirming equilibria with dynamic consistency. What difference does dynamic consistency make?

It is well known that the strategies that an ambiguity averse agent would commit to, if he could, need not be unimprovable.[45] Therefore, SSCE$^\delta$ is not equivalent to SSCE. Furthermore, it is not obvious whether our main comparative statics result is valid with this definition. We can offer examples and intuitions based on our preliminary analysis of this issue.

Example 22 *The game* Γ_w *given by*

1 $\xrightarrow{\text{in}}$ 0 $\xrightarrow{\text{c}\ (\frac{1}{2})}$

$\downarrow$ out $\quad\quad$ $\downarrow$ s $(\frac{1}{2})$

1.7 $\quad\quad$ 2

$1\backslash\{2,3\}$	L, l	R, l	L, r	R, r
T	4	0	0	0
M	0	0	0	4
B	w	w	w	w

Figure 6. *Game* Γ_w

is a three-person, common interest game with a chance move.[46] *The objective* $\frac{1}{2}:\frac{1}{2}$ *probabilities of this move are known. Parameter* $w > 1$ *is the "Waldean value" of the subgame with root* (in, c)*. Assume that the agent in role* 1 *has the following correlated belief:*

$$p_1 = \delta_{\frac{1}{2}\mathrm{c}+\frac{1}{2}\mathrm{d}} \times \left(\frac{1}{2}\delta_{\mathrm{L}} \times \delta_{\mathrm{l}} + \frac{1}{2}\delta_{\mathrm{R}} \times \delta_{\mathrm{r}}\right).$$

Intuitively, he thinks that all the agents playing in roles 2 *and* 3 *"attended the same school" and hence are doing the same thing, but he does not know what. Suppose his ambiguity attitudes are represented by* $\phi_1(\cdot) = \sqrt{\cdot}$*. On the one hand, actions* T *and* M*, once the matrix game is reached, have value* $\left(\frac{1}{2}\sqrt{4} + \frac{1}{2}\sqrt{0}\right)^2 = 1$*. On the other hand, from the perspective of the agent at the root of* Γ_w*, the value of committing to strategy* in.T *(or* in.M*) is* $\left(\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{1}\right)^2 = 1 + \frac{\sqrt{3}}{2} > 1.7$*,*[47] *and strategy* in.B *has value* $\frac{1}{2}2 + \frac{1}{2}w = 1 + \frac{w}{2}$*. If* $1 < w < 1.4$*, the unique dynamically consistent strategy is* out.B*, which is also an SSCE*$^\delta$ *supported by belief* p_1*, for any* f*,* ϕ_2*, and* ϕ_3*. But, under perfect feedback, there is no SSCE (with commitment) supported by such beliefs: the set of best responses to* p_1 *is* {in.T, in.M}*, but playing* in *reveals the true strategy distributions of agents playing in roles* 2 *and* 3 *(*out *is supported as an SSCE with commitment under different beliefs).* ▲

The following example shows that a naive extension of our main result, Theorem 6, to SSCE$^\delta$ does not hold. Specifically, there are games (Γ, f, ϕ) and $(\Gamma, f, \bar{\phi})$, with the former more ambiguity averse than the latter, and a profile of strategy distributions σ^* that is a SSCE$^\delta$ of $(\Gamma, f, \bar{\phi})$ but not a SSCE$^\delta$ of (Γ, f, ϕ).

[45] Siniscalchi (2011) reports examples and provides an in-depth analysis of dynamic consistency under ambiguity aversion.

[46] Numbers at terminal nodes, including the boxes in the matrix subgame, give the common payoff; chance is labeled 0.

[47] Here is where the numbers under $\sqrt{\cdot}$ come from: $3 = U_1(\text{in.T}, \sigma_0 \times \delta_{\mathrm{L}} \times \delta_{\mathrm{l}})$, $1 = U_1(\text{in.T}, \sigma_0 \times \delta_{\mathrm{r}} \times \delta_{\mathrm{r}})$, with $\sigma_0 = \frac{1}{2}\mathrm{c} + \frac{1}{2}\mathrm{s}$.

Example 23 *The game* Γ *given by*

1 $\xrightarrow{\text{in}}$

$\downarrow$ out

2

$1\backslash\{2,3\}$	L,ℓ	R,ℓ	L, r	R, r
T	4	0	0	0
M	0	0	0	4
B	2	2	2	2

Figure 7. *Game* Γ

is a three-person, common interest game. It can be checked that out.T *is an SSCE*$^\delta$ *strategy if player 1 is Bayesian. There is exactly one belief supporting* out.T *as an unimprovable strategy for a Bayesian agent:*

$$p_1^* = \frac{1}{2}(\delta_{\mathrm{L}} \times \delta_\ell) + \frac{1}{2}(\delta_{\mathrm{R}} \times \delta_{\mathrm{r}}).$$

To see this, first note that out.T *is unimprovable under any belief* p_1 *if and only if* $\bar{p}_1(\mathrm{L},\ell) = \frac{1}{2} = \bar{p}_1(\mathrm{R},\mathrm{r})$*; indeed, if* $\max\{\bar{p}_1(\mathrm{L},\ell), \bar{p}_1(\mathrm{R},\mathrm{r})\} > \frac{1}{2}$*, then the best response at the root is* in*, if* $\max\{\bar{p}_1(\mathrm{L},\ell), \bar{p}_1(\mathrm{R},\mathrm{r})\} \leq \frac{1}{2}$ *and* $\min\{\bar{p}_1(\mathrm{L},\ell), \bar{p}_1(\mathrm{R},\mathrm{r})\} < \frac{1}{2}$*, then the unique best reply in the subgame is* B*. Next observe that player* 1 *is certain that the objective distribution of strategy pairs of* $\{2,3\}$ *is a product measure, i.e.,* $p_1 \in \Delta\left(\Delta(S_2) \bigotimes \Delta(S_3)\right)$*, implying that* p_1^* *is the unique belief with the predictive* $\frac{1}{2} : \frac{1}{2}$ *on* $(\mathrm{L},\ell) : (\mathrm{R},\mathrm{r})$*. Now, consider* ϕ *and* $\bar{\phi}$ *with* $\bar{\phi}_1 = \mathrm{Id}_{\mathbb{R}}$*,* ϕ_1 *strictly concave and* $\phi_j = \bar{\phi}_j$ *for* $j = 2, 3$*. Strategy* out.T *is not unimprovable for* (p_1^*, ϕ_1) *because the value of* T *is* $\phi_1^{-1}\left(\frac{1}{2}\phi_1(4) + \frac{1}{2}\phi_1(0)\right) < 2$ *(by the strict concavity of* ϕ_1*). More generally, for every* p_1 *such that* $\bar{p}_1(L,l) \leq \frac{1}{2}$*, the* ϕ_1*-value of* T *is strictly less than 2 (let* $\Lambda = \sigma_2(L)$*,* $\lambda = \sigma_3(l)$*, then the value of* T *is*

$$\phi_1^{-1}\left(\int_{[0,1]^2} \phi_1(4\Lambda\lambda)p_1(\mathrm{d}\Lambda \times \mathrm{d}\lambda)\right) < \int_{[0,1]^2} 4\Lambda\lambda p_1(\mathrm{d}\Lambda \times \mathrm{d}\lambda) = 4\bar{p}_1(\mathrm{L},\ell)$$

by the strict concavity of ϕ_1*, whenever* p_1 *is not a Dirac measure; if* $\bar{p}_1(\mathrm{L},\ell) \leq \frac{1}{2}$ *then the value of* T *is less than* 2*). Therefore,* out.T *is a SSCE*$^\delta$ *strategy of* $(\Gamma, f, \bar{\phi})$*, with* $\bar{\phi} = (\mathrm{Id}_{\mathbb{R}}, \phi_2, \phi_3)$*, but not of the more ambiguity averse game* (Γ, f, ϕ)*. Letting* $\phi_2 = \phi_3 = \mathrm{Id}_{\mathbb{R}}$*, this also shows that a BSCE*$^\delta$ *need not be a SSCE*$^\delta$*.*

Although out.T *is not a SSCE*$^\delta$ *strategy in the more ambiguity averse game* (Γ, f, ϕ)*, the realization-equivalent strategy* out.B *is. We conjecture that the following version of the comparative statics result holds: if* σ^* *is a SSCE*$^\delta$ *of* $(\Gamma, f, \bar{\phi})$ *and* (Γ, f, ϕ) *is more ambiguity averse than* (Γ, f, ϕ)*, then there is a SSCE*$^\delta$ *of* (Γ, f, ϕ) *realization-equivalent to* σ^**.* ▲

There is another problem related to dynamic consistency: as in Fudenberg and Levine (1993a), the notion of best reply used in this paper allows for suboptimal behavior at unexpected information sets. For example, suppose that, in the game of Figure 1, it is strictly dominant for player 0 (whose payoffs are not specified) to choose action R if his information set is reached. A refined dynamically consistent notion of SCE should require that the strategy (or plan) of 0 is R even if his information set is not reached in equilibrium. To deal with this problem we can represent players' beliefs as *conditional probability systems*: an agent in role i has an initial belief p_i, for each information set h_i he would hold a corresponding conditional belief $p_i(\cdot|h_i)$ over the strategy distributions of the opponents, and these beliefs are related to each other *via* Bayes rule whenever possible (see, for example, Battigalli and Siniscalchi, 1999). With this, we can give stronger versions of unimprovability and consistent planning to obtain a *refined* notion of SSCE$^{\delta}$. It can be shown that this refinement does not change the set of equilibrium outcomes. The reason is simple: agents are not assumed to know the preferences of others and may have incorrect beliefs about the choices of others at off-equilibrium-path information sets. The refinement has bite when mutual or common knowledge of the game is assumed, as discussed in the following subsection.

7.2 Rationalizable selfconfirming equilibrium

In a selfconfirming equilibrium agents are rational and their beliefs are confirmed. If the game[48] is common knowledge, it is interesting to explore the implications of assuming, on top of this, that there is common certainty (probability-one belief) of rationality and confirmation of beliefs. Interestingly, the set of *rationalizable SCE*s thus obtained may be a strict subset of the set of SCEs consistent with common certainty of rationality, which in turn may be a strict subset of the set of SCEs.[49] The separation between ambiguity attitudes and beliefs in the smooth-ambiguity model allows a relatively straightforward extension of this idea to obtain a notion of rationalizable SSCE. Of course, to take dynamic consistency issues into account, this notion of rationalizable SSCE should rely on a definition of sequential best response based on refined unimprovability, or consistent planning. For example, if in the game of Figure 1 action R of player 0 is strictly dominant (conditional on his information set being reached), then the only refined rationalizable SSCE$^{\delta}$ is (A, d, R).

7.3 Selfconfirming equilibrium with a public database

The SSCE concept defined in this paper, like Fudenberg and Levine's SCE, rests on the assumption that agents have only access to their personal experiences. How does this affect our results? The answer is quite straightforward if, like Fudenberg and Levine, we assume perfect feedback. In this case, the "public database" induced a by profile of strategy distributions σ is $\hat{\zeta}(\sigma) \in \Delta(Z)$. The key observation is that the "personal database" of an agent playing $s_i \in \operatorname{supp} \sigma_i$ is subsumed by the "public database": personal experiences allow to learn the conditional frequencies of opponents' actions at the opponents' information sets visited with positive frequency under (s_i, σ_{-i}), a sub-collection of the collection of opponents' information sets visited with positive frequency under (σ_i, σ_{-i}). Therefore σ^* is a *public SSCE* if for each i and $s_i^* \in \operatorname{supp} \sigma_i^*$ there is some $p_i \in \hat{\Sigma}_{-i}(\sigma^*) = \{\sigma_{-i} : \exists \sigma_i \in \Delta(S_i), \hat{\zeta}(\sigma_i, \sigma_{-i}) = \hat{\zeta}(\sigma^*)\}$ such that s_i^* is a smooth best response to p_i. (It can be shown that this is equivalent to a definition of *SSCE with unitary beliefs* whereby the justifying belief p_i must be the same for all strategies in $\operatorname{supp} \sigma_i^*$.) Our results can be adapted to this definition of public SSCE. If perfect feedback does not hold, the analysis is more complex. First, it is not obvious what a "public database" is supposed to be. Second, for some definitions of "public database," the personal database need not be subsumed by the public one. Different definitions of SSCE look plausible and it is not clear to us which of our main results can be adapted to such definitions.

[48] Or, at least, a part of the game such as (Γ, f).

[49] See the references on rationalizable SCE in the literature review (subsection 7.5).

7.4 Learning and steady states

Fudenberg and Levine (1993,b) provide a learning foundation of selfconfirming equilibrium within an overlapping generation model with stationary aggregate distributions. The stationarity assumption is a clever trick that allows to use consistency and convergence results in Bayesian statistics about sampling from a "fixed urn" of unknown distribution. It would be important to extend such results to the smooth-ambiguity model and SSCE, and to use results on updating of sets of priors to obtain an analogous learning foundation for the WSCE concept proposed here. Our conjecture is that, with observable payoffs, ambiguity averse agents stop experimenting sooner than Bayesian agents. The intuition is as follows: as an ambiguity averse agent comes close to learning the probabilities of payoffs associated to strategies played many times, the strategies tried only a few times and hence involving higher uncertainty become relatively less appealing (see Anderson, 2012). Fudenberg and Levine (2006) obtain a refinement of SCE based on the idea of steady-state learning with extremely patient and long-lived agents and illustrate it with an application. Similar ideas are worth exploring in the current framework.

7.5 Related literature

Several papers analyze notions of equilibrium in games where agents have non-Bayesian attitudes toward uncertainty (e.g., Dow and Werlang 1994, Klibanoff, 1996, Lo 1996, Eichberger and Kelsey 2000, Marinacci 2000, and more recently Bade, 2011, and Riedel and Sass, 2011 and the references therein). These papers adopt the Nash equilibrium idea that players best respond to beliefs that are, in some sense, correct. We differ from this literature, because we only require beliefs to be confirmed, allowing them to be incorrect. The reason is that we point to a specific rationale for equilibrium, i.e., that it represents a rest point of a learning process, and we take its consequences into account. Another important difference with several papers in this literature is that we model beliefs as measures over strategy distributions rather than pure strategies. This follows from our population game scenario which implies the mass action interpretation of mixed strategies.

To the best of our knowledge, the papers most related to our idea of combining SCE with non-Bayesian attitudes toward uncertainty are Lehrer (2011) and Lehrer and Teper (2009). In these papers a decision maker (DM) is endowed with a "partially specified probability" (PSP), that is, a list of random variables defined on a probability space. The DM knows only the expected values of the random variables, hence he is uncertain about the true underlying probability measure within the set of all measures that give rise to such values. Lehrer (2011) axiomatizes a decision criterion related to the maximization of the minimal expected utility with respect to such set of probability measures consistent with the PSP. Then he defines a notion of equilibrium with partially specified probabilities for a game in strategic form. Under his assumptions on information feedback, Lehrer's equilibrium is similar the one we obtain in the "Waldean" case.[50] To see this, note that in our approach, for each i and s_i, we have a PSP: the probability space is (S_{-i}, σ_{-i}), the random variables are the indicator

[50] Lehrer assumes that each player obtains separate feedback about each opponent's strategy and this feedback is independent of the strategy he plays. In subsection 4, we show that when this independence assumption is coupled with observability of payoffs, every type of SCE is equivalent to Nash equilibrium (Proposition 10). However, it is clear that these specific assumptions on feedback are not essential to Lehrer's approach.

functions of the different messages (ex post observations), and their expectations are the objective probabilities of the messages.[51] Lehrer and Teper (2009) explicitly relate their work to the SCE literature, but the decision criterion they axiomatize for a given PSP, a kind of dual of Bewley's (2002) Knightian criterion featuring intransitivity, is quite different from those considered here.

We are not going to thoroughly review the vast literature on uncertainty and ambiguity aversion, which is covered by a comprehensive recent survey (Gilboa and Marinacci, 2011). We only mention that in the paper we rely on the decision theoretic framework of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011b) that makes formally explicit the DM's uncertainty about the true probabilistic model, or data generating process.

We briefly review the literature on SCE and related ideas, which is not covered by any comprehensive survey.[52] Hahn (1973) intuitively, but quite clearly articulates the idea that an economic system is in equilibrium when agents' strategies are best replies to beliefs confirmed by the evidence generated by the actual strategy profile. Later, Hahn called such states "conjectural equilibria," but he did not provide the game-theoretic formalization that is necessary to express his original idea properly (see Hahn 1977, 1978). Battigalli (1987) provided a precise game theoretic definition of such equilibria, explicitly, but only intuitively motivating it as a characterization of rest points of learning dynamics in recurrent games.[53] Similarly to our paper, information partitions of the set of terminal nodes (complete paths) are introduced among the primitives of the framework, and it is noted that strategies in non-trivial extensive-form games cannot be perfectly observable ex post, even under the assumption of perfect feedback, i.e., of perfect ex post observability of the path. The simple reason is that contingent choices at off-path information sets cannot be observed. Hence, the opponents' mixed strategies consistent with observed frequencies are only partially identified, leaving players in a state of uncertainty. This makes "conjectural equilibrium" a coarser concept than Nash equilibrium, although a path-equivalence result is established for two-person games. Battigalli (1987) studied conjectural equilibria satisfying two properties: (i) *unitary beliefs*: all the pure strategies in the support of a player's mixed strategy are best replies to the same beliefs about others, and (ii) such beliefs satisfy stochastic *independence*. Although worth exploring, such properties are not warranted by a rigorous learning analysis, which was first provided in independent work by Fudenberg, Kreps and Levine (see in particular Fudenberg and Kreps 1995, and Fudenberg and Levine, 1993a,b). While exploring the learning foundations for the Nash equilibrium concept, these authors realized that non-Nash states – called "selfconfirming equilibria" – can be stationary and also asymptotically stable under learning. Roughly, what they call "selfconfirming equilibria" correspond to Battigalli's conjectural equilibria under the perfect feedback assumption, once the unitary belief and independence assumptions are dropped. Since then, "selfconfirming equilibrium" has been used for models with perfect feedback, and "conjectural equilibrium" for models

[51] That is, the random varibles are the functions $\mathbf{1}_{\{s'_{-i}:F_{s_i}(s'_{-i}))=m\}}(s_{-i})$ $(m \in F_{s_i}(S_{-i}) \subseteq M)$, and their expectations are the frequencies $\hat{F}_i(s_i \times \sigma_{-i})(m)$. Clearly, $\hat{\Sigma}_{-i}(s_i, \sigma_{-i})$ is the set of distributions consistent with such PSP on (S_{-i}, σ_{-i}).

[52] For partial surveys, see Battigalli et al (1992) and chapter 6 of Fudenberg and Levine (1998). Cho and Sargent (2008) briefly survey the macroeconomic applications of the selfconfirming equilibrium idea.

[53] The first work in English on this notion of equilibrium is Battigalli and Guaitoli (1988), eventually published in 1997.

where feedback may be imperfect.[54] But this is like using different names for equilibria of games with different rules, as old oligopoly theory used to do with the Nash equilibria of different models. We think a unified terminology is called for, and hence we adopted the more self-explanatory "selfconfirming equilibrium."

Kalai and Lehrer (1993a,b, 1995) analyzed a similarly motivated notion of equilibrium, called "subjective equilibrium," in the context of repeated games, where patient players have to take into account the impact of their current actions on the opponents' future behavior. They showed that when players best respond to beliefs about their opponents' repeated-game strategies and if beliefs and strategies are such that players cannot be "surprised," then continuation-strategies and beliefs converge to a subjective equilibrium, which turns out to be realization-equivalent to a Nash equilibrium under perfect monitoring.

The set of SCE is even larger than the set of Nash equilibria, but it can be refined in conceptually meaningful ways. Eyster and Rabin (2005), Jehiel (2005), Esponda (2008) and Azrieli (2009a) proposed equilibrium notions that can be interpreted as refinements of SCE based on the assumption that agents have simple, or naive, subjective models of the behavior of others. As mentioned in Section 7.4, Fudenberg and Levine (2006) analyze a refinement of SCE (for a simple class of perfect information games) based on steady-state learning with extremely patient and long-lived agents. Battigalli (1987) refined the set of SCE by looking at those that, intuitively, satisfy a form of common belief in extensive-form rationality (see also Battigalli and Guaitoli, 1988, 1997). More interestingly, and independently, Rubinstein and Wolinsky (1994) analyzed *rationalizable SCE*, i.e., states where (1) players are rational, (2) their beliefs are confirmed (but possibly incorrect), and there is common belief of (1) *and* (2).[55] Similar ideas have been explored in the context of extensive-form games by Battigalli (1999), Dekel et al (1999), Fudenberg and Kamada (2011), and in the context of incomplete-information games by Battigalli and Siniscalchi (2003), Esponda (2011) and Letina (2011). Building on Battigalli and Siniscalchi (2002), Battigalli and Siniscalchi (2003) provide a refined, forward-induction version of the rationalizable SCE concept.[56]

8 Appendix

8.1 Proofs for Section 2

Proof of Lemma 1 Since payoffs are observable, there exists a function $\bar{u} : \operatorname{Im} f \to \mathbb{R}$ such that $u = \bar{u} \circ f$. For each $\sigma \in \Delta(\Omega)$ it holds

$$\begin{aligned} U(\alpha,\sigma) &= \sum_{a\in A}\sum_{\omega\in\Omega} (\bar{u}\circ f)(\zeta(a,\omega))\,\sigma(\omega)\,\alpha(a) \\ &= \sum_{a\in A}\sum_{\omega\in\Omega} (\bar{u}\circ F_a)(\omega)\,\sigma(\omega)\,\alpha(a) = \sum_{m\in M} \bar{u}(m)\,\hat{F}_\alpha(\sigma)(m). \end{aligned}$$

[54] An exception to this terminology is Fudenberg and Kamada (2011): they analyze a generalization of Battigalli's conjectural equilibrium that allows for correlated beliefs and call it "partition confirmed equilibrium."

[55] See Section 7.2. In static games, this refines Battigalli's (1987) equilibria, who only rest on the assumptions of rationality, confirmed beliefs, and common belief in rationality (but not in confirmed beliefs).

[56] Esponda and Letina provide rigorous, epistemic and algorithmic characterizations rationalizable selfconfirming equilibrium. Battigalli (1999) and Battigalli and Siniscalchi (2003) also provide algorithmic characterizations, but they refer to the extensive-form epistemic analysis of Battigalli and Siniscalchi (1999, 2002) for a rigorous epistemic characterization.

This implies $U(\alpha,\sigma) = U(\alpha,\sigma^*)$ if $\hat{F}_\alpha(\sigma) = \hat{F}_\alpha(\sigma^*)$, that is, if $\sigma \in \hat{\Sigma}(\alpha,\sigma^*)$. ■

Proof of Lemma 2 Since f is coarser that $\bar{f}$, there is a function $\varphi : \operatorname{Im}\bar{f} \to M$ such that $f = \varphi \circ \bar{f}$. Hence, $F = \varphi \circ \bar{F}$. Fix $\sigma^* \in \Delta(\Omega)$ and $\sigma \in \hat{\bar{\Sigma}}(\alpha,\sigma^*)$ (partially identified set with finer feedback), so that $\hat{\bar{F}}_\alpha(\sigma) = \hat{\bar{F}}_\alpha(\sigma^*)$. We show that $\sigma \in \hat{\Sigma}(\alpha,\sigma^*)$ (partially identified set with coarser feedback). We have

$$\begin{aligned}
\sigma \quad &\in \quad \hat{\Sigma}(\alpha,\sigma^*) \Longleftrightarrow \hat{F}_\alpha(\sigma) = \hat{F}_\alpha(\sigma^*) \\
&\Longleftrightarrow \quad \sigma(F_\alpha = m) = \sigma^*(F_\alpha = m) \qquad \forall m \in M \\
&\Longleftrightarrow \quad \sigma\left(\varphi \circ \bar{F}_\alpha = m\right) = \sigma^*\left(\varphi \circ \bar{F}_\alpha = m\right) \qquad \forall m \in M \\
&\Longleftrightarrow \quad \sigma\left(\bar{F}_\alpha \in \varphi^{-1}(m)\right) = \sigma^*\left(\bar{F}_\alpha \in \varphi^{-1}(m)\right) \qquad \forall m \in M \\
&\Longleftrightarrow \quad \hat{\bar{F}}_\alpha(\sigma)\left(\varphi^{-1}(m)\right) = \hat{\bar{F}}_\alpha(\sigma^*)\left(\varphi^{-1}(m)\right) \qquad \forall m \in M
\end{aligned}$$

Since $\hat{\bar{F}}_\alpha(\sigma) = \hat{\bar{F}}_\alpha(\sigma^*)$, we conclude that $\sigma \in \hat{\Sigma}(\alpha,\sigma^*)$. ■

8.2 Proofs for Section 4

Proof of Proposition 5 Let σ^* be a SSCE for the less coarse game $\left(\Gamma, \bar{f}, \phi\right)$. By Lemma 2, $\hat{\bar{\Sigma}}_{-i}(s_i^*, \sigma_{-i}^*) \subseteq \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ for each i and $s_i^* \in \operatorname{supp}\sigma_i^*$, which easily implies that σ^* be a SSCE for the coarser game (Γ, f, ϕ). A similar argument holds for WSCE. ■

Proof of Theorem 6 By definition, there exists a profile $(\varphi_i)_{i\in I}$ of strictly increasing and concave functions such that $\phi_i = \varphi_i \circ \bar{\phi}_i$ for each i. Let σ^* be a SSCE of $\left(\Gamma, f, \bar{\phi}\right)$. Fix $i \in I$, and pick $s_i^* \in \operatorname{supp}\sigma_i^*$, $p_{s_i^*} \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ such that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_{s_i^*}; \bar{\phi}_i)$. We want to show that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_{s_i^*}; \phi_i)$, which implies the first claim. Since payoffs are observable, by Lemma 1 $U_i(s_i^*, \sigma_{-i}) = U_i(s_i^*, \sigma_{-i}^*)$ for each $\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$. Thus

$$V_i(s_i^*, p_{s_i^*}; \phi_i) = U_i(s_i^*, \sigma_{-i}^*) = V_i(s_i^*, p_{s_i^*}; \bar{\phi}_i). \tag{18}$$

Next observe that, for any $s_i \in S_i$,

$$\begin{aligned}
V_i(s_i, p_{s_i^*}; \bar{\phi}_i) &= \bar{\phi}_i^{-1}\left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \bar{\phi}_i(U_i(s_i, \sigma_{-i})) dp_{s_i^*}(\sigma_{-i})\right) \\
&= (\bar{\phi}_i^{-1} \circ \varphi_i^{-1}) \circ \varphi_i \left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \bar{\phi}_i(U_i(s_i, \sigma_{-i})) dp_{s_i^*}(\sigma_{-i})\right) \\
&\geq (\bar{\phi}_i^{-1} \circ \varphi_i^{-1}) \left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} (\varphi_i \circ \bar{\phi}_i)(U_i(s_i, \sigma_{-i})) dp_{s_i^*}(\sigma_{-i})\right) \\
&= \phi_i^{-1}\left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i(U_i(s_i, \sigma_{-i})) dp_{s_i^*}(\sigma_{-i})\right) = V_i(s_i, p_{s_i^*}; \phi_i)
\end{aligned}$$

where we used Jensen's inequality and $\phi_i = \varphi_i \circ \bar{\phi}_i$. Hence,

$$V_i(s_i, p_{s_i^*}; \phi_i) \leq V_i(s_i, p_{s_i^*}; \bar{\phi}_i) \leq V_i(s_i^*, p_{s_i^*}; \bar{\phi}_i) = V_i(s_i^*, p_{s_i^*}; \phi_i)$$

for each $s_i \in S_i$, which shows that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_{s_i^*}; \phi_i)$.

We now prove that SSCEs are WSCEs. Let σ^* be a SSCE of a game (Γ, f, ϕ). Fix $i \in I$, and pick $s_i^* \in \operatorname{supp} \sigma_i^*$, $p_{s_i^*} \in \Delta(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ such that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, p_{s_i^*}; \phi_i)$. We show that $s_i^* \in \arg\max_{s_i \in S_i} V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*))$ where

$$V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) = \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U(s_i, \sigma_{-i})$$

Since payoffs are observable, by (18) it holds

$$V_i(s_i^*, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) = U_i(s_i^*, \sigma_{-i}^*) = V_i(s_i^*, p_{s_i^*}; \phi_i) \geq V_i(s_i, p_{s_i^*}; \phi_i) \tag{19}$$

for each $s_i \in S_i$. Next observe that for, each $\sigma'_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ and each $s_i \in S_i$,

$$\begin{aligned} & U_i(s_i, \sigma'_{-i}) \geq \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U(s_i, \sigma_{-i}) \\ \Longrightarrow \quad & \phi_i\left(U_i(s_i, \sigma'_{-i})\right) \geq \phi_i\left(\min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U(s_i, \sigma_{-i})\right). \end{aligned}$$

This implies that, for each $\sigma'_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$ and each $s_i \in S_i$,

$$\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(U_i(s_i, \sigma'_{-i})\right) dp_{s_i^*}(\sigma'_{-i}) \geq \int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(\min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U(s_i, \sigma_{-i})\right) dp_{s_i^*}(\sigma'_{-i})$$

which in turn implies

$$\begin{aligned} V_i(s_i, p_{s_i^*}; \phi_i) &= \phi_i^{-1}\left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(U_i(s_i, \sigma'_{-i})\right) dp_{s_i^*}(\sigma'_{-i})\right) \\ &\geq \phi_i^{-1}\left(\int_{\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} \phi_i\left(\min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)} U(s_i, \sigma_{-i})\right) dp_{s_i^*}(\sigma'_{-i})\right) \\ &= V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)). \end{aligned}$$

This latter inequality paired with (19) delivers that

$$V_i(s_i^*, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) \geq V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) \qquad \forall s_i \in S_i$$

proving the statement. ■

Proof of Lemma 7 Pick any i and pure strategy $s_i^* \in \operatorname{supp} \sigma_i^*$. Then $U_i\left(s_i^*, \sigma_{-i}^*\right) \geq U_i\left(s_i, \sigma_{-i}^*\right)$ for each $s_i \in S_i$. Given any feedback function f it holds $\sigma_{-i}^* \in \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)$. Hence, $\delta_{\sigma_{-i}^*} \in \Delta\left(\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)\right)$. Since $V_i\left(s_i, \delta_{\sigma_{-i}^*}\right) = U_i\left(s_i, \sigma_i^*\right)$ for each $s_i \in S_i$, it follows that σ^* is a BSCE of (Γ, f). ■

Proof of Proposition 10 Given the previous results, we only have to show that every WSCE is a Nash equilibrium. Fix a WSCE σ^*, any player i and any $s_i^* \in \operatorname{supp} \sigma_i^*$. Then, for each s_i

$$V_i(s_i^*, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)) \geq V_i(s_i, \hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*)).$$

By Lemma 1, observability of payoffs implies $U_i(s_i, \sigma_{-i}) = U_i(s_i, \sigma^*_{-i})$ for all s_i and $\sigma_{-i} \in \hat{\Sigma}_{-i}(s_i, \sigma^*_{-i})$. Own-strategy independence of feedback implies that, for every s_i, $F^{-1}_{s_i}(\cdot) = F^{-1}_{s^*_i}(\cdot)$, hence

$$\begin{aligned}\hat{\Sigma}_{-i}(s_i, \sigma^*_{-i}) &= \left\{\sigma_{-i} : \forall m, \sigma_{-i}\left(F^{-1}_{s_i}(m)\right) = \sigma^*_{-i}\left(F^{-1}_{s_i}(m)\right)\right\} \\ &= \left\{\sigma_{-i} : \forall m, \sigma_{-i}\left(F^{-1}_{s^*_i}(m)\right) = \sigma^*_{-i}\left(F^{-1}_{s^*_i}(m)\right)\right\} = \hat{\Sigma}_{-i}(s^*_i, \sigma^*_{-i}).\end{aligned}$$

From the above equalities and inequalities we obtain, for each s_i,

$$U_i(s^*_i, \sigma^*_{-i}) = V_i(s^*_i, \hat{\Sigma}_{-i}(s^*_i, \sigma^*_{-i})) \geq V_i(s_i, \hat{\Sigma}_{-i}(s^*_i, \sigma^*_{-i})) = V_i(s_i, \hat{\Sigma}_{-i}(s_i, \sigma^*_{-i})) = U_i(s_i, \sigma^*_i).$$

This shows that σ^* is a Nash equilibrium. ■

Proof of Proposition 11 As in Chapter 6 of Osborne and Rubinstein (1994), nodes of the game tree are histories (sequences of action profiles) h, with $\prec$ denoting the strict precedence "prefix of" relation, H_i is the set of histories where i moves, H_{-i} is the complement of H_i within the set of non-terminal histories, the action selected by strategy s_i at $h \in H_i$ is denoted by $s_i(h)$.

To ease notation, for every pure strategy profile s, we write $\hat{\Sigma}_{-i}(s_i, \delta_{s_{-i}}) = \hat{\Sigma}_{-i}(s)$. We show that, if s^* is a symmetric WSCE, then for each i there is pure strategy profile $\bar{s}_{-i} \in \hat{\Sigma}_{-i}(s^*)$ such that $s^*_i \in \arg\max_{s_i \in S_i} U_i(s_i, \bar{s}_{-i})$; this implies the claim. Let $s^* = (s^*_i)_{i \in I}$ be a symmetric WSCE. By perfect information, the mixed strategy profiles of i's opponents are realization-equivalent to correlated strategy profiles of the coalition of players $-i = I \backslash \{i\}$. Given perfect feedback, for each i, $\hat{\Sigma}_{-i}(s^*)$ is therefore realization-equivalent to the convex hull of set of pure strategy profiles

$$\hat{S}_{-i}(s^*) := \{s_{-i} \in S_{-i} : \forall h \in H_{-i}, h \prec \zeta(s^*) \Rightarrow s_{-i}(h) = s^*_{-i}(h)\}.$$

This is the set of strategies of coalition $-i$ in a modified zero-sum game between i and $-i$ where $-i$ has only one feasible action, $s^*_{-i}(h)$, at each $h \in H_{-i}$ preceding $\zeta(s^*)$, and the payoff function of i is the restriction of u_i on the smaller set of terminal histories thus obtained. Like the original game, this auxiliary zero-sum game has perfect information, hence it has a saddle point in pure strategies. Taking into account that s^* is a WSCE:[57]

$$s^*_i \in \arg\max_{s_i \in S_i} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(s^*)} U_i(s_i, \sigma_{-i}) = \arg\max_{s_i \in S_i} \min_{s_{-i} \in \hat{S}_{-i}(s^*)} U_i(s_i, s_{-i}).$$

Any strategy profile

$$\bar{s}_{-i} \in \arg \min_{s_{-i} \in \hat{S}_{-i}(s^*)} U_i(s^*_i, s_{-i}))$$

is a confirmed degenerate belief justifying s^*_i. ■

Proof of Proposition 12 Let $s^* = (s^*_i)_{i \in I}$ be a BSCE supported by belief profile $(p^*_i)_{i \in I}$. Consider the set of opponents of any player i as a unique pseudo-player $-i$. By perfect information, such pseudo-player has perfect recall. Therefore the predictive belief $\bar{p}^*_i \in \Delta(S_{-i})$ is realization-equivalent to a behavioral strategy of $\beta_{-i} \in \times_{h \in H_{-i}} \Delta(A(h))$ ($A(h)$ is the set of

[57] It does not matter how a strategy s_i is specified at histories outside the modified game.

possible actions at h), which in turn is realization-equivalent to an uncorrelated predictive belief (a profile of mixed strategies) $\bar{q}_{-i} \in \bigotimes_{j \neq i} \Delta(S_j)$. Thus, $s^*$is a symmetric BSCE supported by uncorrelated beliefs. Perfect information implies observable deviators. Kamada (2010) proved a generalization of the following statement: in games with observable deviators and perfect feedback every symmetric BSCE supported by uncorrelated beliefs induces the same path as some mixed Nash equilibrium. To see that every Nash equilibrium σ^* with a deterministic path is realization-equivalent to a symmetric BSCE s^*, take a representation of σ^* with behavioral strategies, say β^*; by assumption β^* is pure on the equilibrium path. Modify β^* off the path so as to obtain a pure strategy profile s^*. Like σ_i^*, also s_i^* is a best reply to product measure σ_{-i}^*, because the off-the-path actions of i do not affect his expected utility. Furthermore, s^* and σ^* induce the same path; therefore, for each i, $\hat{\Sigma}_{-i}(s_i^*, \sigma_{-i}^*) = \hat{\Sigma}_{-i}(s_i^*, s_{-i}^*)$. It follows that s^* is a symmetric BSCE supported by belief profile $(\sigma_{-i}^*)_{i \in I}$. This establishes that $\zeta(\det NE) = \zeta(symBSCE)$. Then Proposition 11 implies that $\zeta(\det NE) = \zeta(symBSCE) = \zeta(symWSCE)$, and $\zeta(\det NE) = \zeta(symBSCE) = \zeta(symSSCE) = \zeta(symWSCE)$ if there are two players or n ambiguity averse players. ■

8.3 Proofs for Section 5

Proof of Proposition 14. For any prior p_1, the Bayesian subjective value of playing any Matching Pennies subgame k is

$$\begin{aligned} & \max\{V_1(H^k, p_1), V_1(T^k, p_1)\} \\ = \; & \max\{[\bar{p}_1^k(h^k)(n+2(k-1)) + (1-\bar{p}_1^k(h^k))(n-k), [\bar{p}_1^k(h^k))(n-k) + (1-\bar{p}_1^k(h^k))(n+2(k-1))\} \\ \geq \; & n - 1 + \frac{k}{2} > n - 1 + \varepsilon = u_1(O), \end{aligned}$$

where $n-1+k/2$ is the mixed maxmin value of subgame k, $\bar{p}_1^k = \mathrm{mrg}_{S_{2.k}} \bar{p}_1$ and $\bar{p}_1$ is the predictive belief. Therefore O cannot be played by a positive fraction of agents in a BSCE because it cannot be a best response to any predictive belief $\bar{p}_1$. Furthermore, no strategy H^k or T^k with $k < n$ can have positive measure in a BSCE. Indeed, by (14), if $s_1^k \in \{H^k, T^k\}$ has positive probability in an equilibrium σ^*, then for every belief $p_1 \in \Delta(\hat{\Sigma}_{-1}(s_1^k, \sigma_2^*))$, the value of s_1^k is

$$V_1(s_1^k, p_1) = U_1\left(s_1^k, \sigma^*_{-\{1,2.k\}} \times \left(\frac{1}{2}h^k + \frac{1}{2}t^k\right)\right) = n - 1 + \frac{k}{2},$$

while the Bayesian value of deviating to subgame n is

$$\max\{V_1(H^n, p_1), V_1(T^n, p_1)\} \geq n - 1 + \frac{n}{2}.$$

Therefore, eq. (14) implies $\sigma_1^*(H^n) = \sigma_1^*(T^n) = \sigma_n^*(h^n) = \frac{1}{2}$ in each BSCE σ^*. It is routine to verify that every such σ^* is also a Nash equilibrium. Therefore $BSCE = NE$. ■

The proof of Lemma 15 is based on the following lemma, where **I** is the unit interval $[0,1]$ endowed with the Borel σ-algebra.

Lemma 24 *Let $\varphi : \mathbf{I} \to \mathbb{R}$ be increasing and concave. For each Borel probability measure p on $\mathbf{I}$*

$$\max\left\{\int_{\mathbf{I}} \varphi(x)\, dp(x), \int_{\mathbf{I}} \varphi(1-x)\, dp(x)\right\} \geq \frac{1}{2}\varphi(1) + \frac{1}{2}\varphi(0). \tag{20}$$

Proof. Let

$$\begin{array}{rccc} \tau: & \mathbf{I} & \to & \mathbf{I} \\ & x & \mapsto & 1-x \end{array}.$$

Then

$$\int_{\mathbf{I}} \varphi(1-x)\, dp(x) = \int_{\mathbf{I}} \varphi(\tau(x))\, dp(x) = \int_{\mathbf{I}} \varphi(y)\, dp_\tau(y)$$

where $p_\tau = p \circ \tau^{-1}$. In particular, for $\varphi = \mathrm{id}_{\mathbf{I}}$ it follows that $1 - \int_{\mathbf{I}} x dp(x) = \int_{\mathbf{I}} y dp_\tau(y)$. Thus (20) becomes

$$\max\left\{\int_{\mathbf{I}} \varphi(x)\, dp(x), \int_{\mathbf{I}} \varphi(x)\, dp_\tau(x)\right\} \geq \frac{1}{2}\varphi(1) + \frac{1}{2}\varphi(0)$$

and either $\int_{\mathbf{I}} x dp(x) \geq 1/2$ or $\int_{\mathbf{I}} y dp_\tau(y) \geq 1/2$. Next we show that for each Borel probability measure q on $\mathbf{I}$ such that $\int_{\mathbf{I}} x dq(x) \geq 1/2$

$$\int_{\mathbf{I}} \varphi(x)\, dq(x) \geq \frac{1}{2}\varphi(1) + \frac{1}{2}\varphi(0). \tag{21}$$

Denote by $F(x) = q([0,x])$ and by $G(x) = \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)([0,x])$. In particular, F and G are increasing, right continuous, and such that $F(1) = G(1) = 1$, moreover $G(x) = 1/2$ for all $x \in [0,1)$. Moreover, there exists $\bar{x} \in (0,1)$ such that $F(\bar{x}) \leq 1/2$. By contradiction, assume $F(x) > 1/2$ for all $x \in (0,1)$, then

$$\frac{1}{2} \leq \int_{\mathbf{I}} x dq(x) = \int_0^1 (1 - F(x))\, dx < \frac{1}{2}.$$

if $F(0) = q(\{0\}) > 1/2$, then

$$\int_{\mathbf{I}} x dq(x) \leq 0q(0) + \int_{(0,1]} x dq(x) \leq 0q(0) + \int_{(0,1]} 1 dq(x) \leq 0q(\{0\}) + (1 - q(\{0\})) < \frac{1}{2}$$

contradicting $\int_{\mathbf{I}} x dq(x) \geq 1/2$. Let $x^* = \inf\{x \in \mathbf{I} : F(x) > 1/2\}$, then $0 < \bar{x} \leq x^* \leq 1$.

Therefore $F(1) = G(1) = 1$ and for each $y \in (x^*, 1)$, $F(y) \geq F(x^*) \geq 1/2 \geq G(y)$. For each $y \in [0, x^*)$, $F(y) \leq 1/2 \leq G(y)$. Finally, by the classic Karlin-Novikoff result F second-order stochastically dominates G, that is (21) holds for all increasing and concave φ. ■

Proof of Lemma 15 Let $x = \sigma_{2.k}(h^k)$. Clearly $U_1(H^k, \sigma_2)$ depends only on x and we can write $U_1(H^k, x)$, and similarly for T^k. Let $\varphi(x) = \phi_1(U_1(H^k, x))$. By symmetry of the payoff matrix, $\varphi(1-x) = \phi_1(U_1(T^k, x))$. Note that φ is strictly increasing and concave. Let $p \in \Delta(\mathbf{I})$ be the marginal belief about $x = \sigma_{2.k}(h^k)$ derived from p_1. Recall that q_1 is a prior such that $\mathrm{mrg}_{\Delta(S_{2.j})} q_1 = \frac{1}{2}\delta_{h^j} + \frac{1}{2}\delta_{t^j}$. With this,

$$\begin{aligned} \max\{V_1(H^j, p_1; \phi_1), V_1(T^j, p_1; \phi_1)\} &= \max\left\{\phi_1^{-1}\left(\int_{\mathbf{I}} \varphi(x)\, dp(x)\right), \phi_1^{-1}\left(\int_{\mathbf{I}} \varphi(1-x)\, dp(x)\right)\right\} \\ &= \phi_1^{-1}\left(\max\left\{\int_{\mathbf{I}} \varphi(x)\, dp(x), \int_{\mathbf{I}} \varphi(1-x)\, dp(x)\right\}\right) \end{aligned}$$

and

$$V_1(H^j, q_1; \phi_1) = V_1(T^j, q_1; \phi_1) = \phi_1^{-1}\left(\frac{1}{2}\varphi(1) + \frac{1}{2}\varphi(0)\right).$$

Hence, the thesis is implied by Lemma 24. ■

Proof of Lemma 16 By definition of ϕ_1^{α} and M

$$M(\alpha; x, y) = -\frac{1}{\alpha}\log\left(\frac{1}{2}e^{-\alpha x} + \frac{1}{2}e^{-\alpha y}\right).$$

The result follows from known properties of the negative exponential. ■

Proof of Proposition 17 By Lemma 13, $SSCE(\alpha)$ is determined by the set of pure strategies of player 1 that can be played by a positive fraction of agents in equilibrium. Fix $\sigma^* \in \Sigma^*$, i.e., a distribution profile that satisfies the necessary SCE conditions, and a strategy s_1; $\sigma_1^*(s_1) > 0$ is possible in equilibrium if and only if there no incentives to deviate to any subgame j. We rely on Lemma 15 to specify a belief $p_1^{s_1} \in \Delta(\hat{\Sigma}_2(s_1, \sigma_2^*))$ that minimizes the incentive to deviate. Thus, s_1 can be played in equilibrium if and only if it is a best reply to $p_1^{s_1}$. Specifically,

$$p_1^O = \times_{j=1}^n \left(\frac{1}{2}\delta_{h^j} + \frac{1}{2}\delta_{t^j}\right) \in \Delta(\hat{\Sigma}_2(O, \sigma_2^*)) = \Delta\left(\bigotimes_{j=1}^n \Delta(S_{j.k})\right),$$

for each $k = 1, ..., n-1$ and $s_1^k \in \{H^k, T^k\}$,

$$p_1^k = \delta_{\frac{1}{2}h^k + \frac{1}{2}t^k} \times \left(\times_{j\neq k}\left(\frac{1}{2}\delta_{h^j} + \frac{1}{2}\delta_{t^j}\right)\right) \in \Delta(\hat{\Sigma}_2(s_1^k, \sigma_2^*)) = \Delta\left(\left\{\sigma_2 : \sigma_{2,k} = \frac{1}{2}h^k + \frac{1}{2}t^k\right\}\right).$$

Given such beliefs, the value of deviating from s_1 to subgame j is $M(\alpha, n-j, n+2(j-1)$. Therefore, O is a best reply to p_1^O, and can have positive measure in equilibrium, if and only if

$$n - 1 + \varepsilon \geq \max_{j\in\{1,\ldots,n\}} M(\alpha, n-j, n+2(j-1)). \tag{22}$$

By Lemma 16 there is a unique threshold $\alpha_n > 0$ that satisfies (22) as an equality so that (22) holds if and only if $\alpha \geq \alpha_n$. Similarly, $s_1^k \in \{H^k, L^k\}$ $(k = 1, ..., n-1)$ is a best reply to p_1^k, and can have positive measure in equilibrium, if and only if

$$n - 1 + \frac{k}{2} \geq \max_{j\in\{1,\ldots,n\}} M(\alpha, n-j, n+2(j-1)), \tag{23}$$

where

$$\max_{j\in\{1,\ldots,n\}} M(\alpha, n-j, n+2(j-1)) = \max_{j\in\{k+1,\ldots,n\}} M(\alpha, n-j, n+2(j-1))$$

because, for all α and $j \leq k$

$$M(\alpha, n-j, n+2(j-1)) \leq n - 1 + \frac{j}{2} < n - 1 + \frac{k}{2}.$$

By Lemma 16 there is a unique threshold $\alpha_{n-k} > 0$ that satisfies (23) as an equality so that (23) holds if and only if $\alpha \geq \alpha_{n-k}$. Since $M(\cdot, x, y)$ is strictly decreasing if $x \neq y$, the thresholds are strictly ordered: $\alpha_1 < \alpha_2 < ... < \alpha_n$. It follows that, for each $k = 1, ..., n-1$, $\sigma^*(\{O, H^1, T^1, ..., H^k, T^k\}) = 0$ for every $\sigma^* \in SSCE(\alpha)$ if and only if $\alpha < \alpha_{n-k}$, and every strategy has positive measure in some SSCE if α is large enough (in particular if $\alpha \geq \alpha_n$). Since the equilibrium set in this case is Σ^*, which is defined by necessary SCE conditions, this must also be the WSCE set. If $\alpha < \alpha_1$, then $\sigma^*(\{O, H^1, T^1, ..., H^{n-1}, T^{n-1}\}) = 0$ for each $\sigma^* \in SSCE(\alpha)$; by Proposition 14, $SSCE(\alpha) = BSCE = NE$ in this case. ■

8.4 Proofs for Section 6

Proof of Proposition 19 Fix a WSCE$^\rho$ σ^*. For each $i \in I$ and $\ell \in \{1, ..., k_i\}$, $\hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)$ is nonempty and compact; since $-i$ is a single player, it is also convex. Choose

$$\bar{\sigma}_{-i}^\ell \in \arg \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)} U_i(\sigma_i^\ell, \sigma_{-i}).$$

By definition of WSCE$^\rho$

$$\sigma_i^\ell \in \arg \max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \hat{\Sigma}_{-i}(\sigma_i^\ell, \sigma_{-i}^*)} U_i(\sigma_i, \sigma_{-i}).$$

Hence, the maxmin theorem implies

$$\sigma_i^\ell \in \arg \max_{\sigma_i \in \Delta(S_i)} U_i(\sigma_i, \bar{\sigma}_{-i}^\ell).$$

Thus, $\bar{\sigma}_{-i}^\ell$ is a confirmed degenerate belief justifying σ_i^ℓ. This shows that σ^* is also a BSCE$^\rho$ and a SSCE$^\rho$ (ambiguity attitudes do not matter if p_i^ℓ is a Dirac measure). Thus, $WSCE^\rho \subseteq BSCE^\rho, SSCE^\rho$. In every two-person game $BSCE^\rho \subseteq SSCE^\rho$. Assuming observable payoffs, $BSCE^\rho \subseteq WSCE^\rho$. Therefore $BSCE^\rho = SSCE^\rho = WSCE^\rho$. ■

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