Approximate Revenue Maximization with Multiple Items[*]

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Abstract

Myerson's classic result provides a full description of how a seller can maximize revenue when selling a single item. We address the question of revenue maximization in the simplest possible multi-item setting: *two items* and a *single buyer* who has *independently distributed* values for the items, and an additive valuation. In general, the revenue achievable from selling two independent items may be strictly higher than the sum of the revenues obtainable by selling each of them separately. In fact, the structure of optimal (i.e., revenue-maximizing) mechanisms for two items even in this simple setting is not understood.

In this paper we obtain *approximate* revenue optimization results using two simple auctions: that of selling the items *separately*, and that of selling them as a single *bundle*. Our main results (which are of a "direct sum" variety, and apply to any distributions) are as follows. Selling the items separately guarantees at least half the revenue of the optimal auction; for identically distributed items, this becomes at least 73% of the optimal revenue.

For the case of $k > 2$ items, we show that selling separately guarantees at least a $c/\log^2 k$ fraction of the optimal revenue; for identically distributed items, the bundling auction yields at least a $c/\log k$ fraction of the optimal revenue.

1   Introduction

Suppose that you have one item to sell to a single buyer whose willingness to pay is unknown to you but is distributed according to a known prior (given by a cumulative distribution F). If you offer to sell it for a price p then the probability that the buyer will buy is[1] $1 - F(p)$, and your revenue will be $p \cdot (1 - F(p))$. The seller will choose a price p^* that maximizes this expression.

This problem is exactly the classical monopolist pricing problem, but looking at it from an auction point of view, one may ask whether there are mechanisms for selling the item that yield a higher revenue. Such mechanisms could be indirect, could offer different prices for different probabilities of getting the item, and perhaps others. Yet, Myerson's characterization of optimal auctions (Myerson [1981]) concludes that the take-it-or-leave-it offer at the above price p^* yields the optimal revenue among *all* mechanisms. Even more, Myerson's result also applies when there are multiple buyers, in which case p^* would be the reserve price in a second price auction.

Now suppose that you have two (different) items that you want to sell to a single buyer. Furthermore, let us consider the simplest case where the buyer's values for the items are independently and identically distributed according to F ("i.i.d.-F" for short), and furthermore that his valuation is additive: if the value for the first item is x and for the second is y, then the value for the *bundle* – i.e., getting both items – is[2] $x + y$. It would seem that since the two items are completely independent from each other, then the best we should be able to do is to sell each of them separately in the optimal way, and thus extract exactly twice the revenue we would make from a single item. Yet this turns out to be false.

Example: Consider the distribution taking values 1 and 2, each with probability 1/2. Let us first look at selling a single item optimally: the seller can either choose to price it at 1, selling always[3] and getting a revenue of 1, or choose to price the item at 2, selling it with probability 1/2, still obtaining an expected revenue of 1, and so the optimal revenue for a single item is 1. Now consider the following mechanism for selling both items: bundle them together, and sell the bundle for price 3. The probability that the sum of the buyer's values for the two items is at least 3 is 3/4, and so the revenue is $3 \cdot 3/4 = 2.25$ – larger than 2, which is obtained by selling them separately.

However, that is not always so: bundling may sometimes be worse than selling the

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[1]Assume for simplicity that the distribution is continuous.

[2]Our buyer's demand is *not* limited to one item (which is the case in some of the existing literature; see below).

[3]Since we want to maximize revenue we can always assume without loss of generality that ties are broken in a way that maximizes revenue; this can always be achieved by appropriate small perturbations.

items separately. For the distribution taking values 0 and 1, each with probability 1/2, selling the bundle can yield at most a revenue of 3/4, and this is less than twice the single-item revenue of 1/2. In some other cases neither selling separately nor bundling is optimal. For the distribution that takes values 0, 1 and 2, each with probability 1/3, the unique optimal auction turns out to offer to the buyer the choice between any single item at price 2, and the bundle of both items at a "discount" price of 3. This auction gets revenue of 13/9 revenue, which is larger than the revenue of 4/3 obtained from either selling the two items separately, or from selling them as a single bundle. A similar situation happens for the uniform distribution on $[0, 1]$, for which neither bundling nor selling separately is optimal (Manelli and Vincent [2006]). In yet other cases the optimal mechanism is not even deterministic and must offer lotteries for the items. This happens in the following example from Hart and Reny [2011][4]: Let F be the distribution which takes values 1, 2 and 4, with probabilities 1/6, 1/2, 1/3, respectively. It turns out that the unique optimal mechanism offers the buyer the choice between buying any one good *with probability 1/2* for a price of 1, and buying the bundle of both goods (surely) for a price of 4; any deterministic mechanism has a strictly lower revenue.

So, it is not clear what optimal mechanisms for selling two items look like, and indeed characterizations of optimal auctions even for this simple case are not known. We shorty describe some of the previous work on these type of issues. McAfee and McMillan [1988] identify cases where the optimal mechanism is deterministic. However, Thanassoulis [2004] and Manelli and Vincent [2006] found a technical error in the paper and exhibit counter-examples. These last two papers contain good surveys of the work within economic theory, with more recent analysis by Fang and Norman [2006], Jehiel et al. [2007], Hart and Reny [2010], Hart and Reny [2011], Lev [2011]. In the last few years algorithmic work on these types of topics was carried out. One line of work (e.g. Briest et al. [2010] and Cai et al. [2012]) shows that for discrete distributions the optimal auction can be found by linear programming in rather general settings. This is certainly true in our simple setting where the direct representation of the auction constraints provides a polynomial size linear program. Thus we emphasize that the difficulty in our case is *not computational*, but is rather that of characterization and understanding the results of the explicit computations: this is certainly so for continuous distributions, but also for discrete ones.[5] Another line of work in computer science (Chawla et al. [2007], Chawla et al.

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[4]Previous examples where randomization helps appear in Manelli and Vincent [2006], Manelli and Vincent [2007] and Thanassoulis [2004], but these require *interdependent* distributions of values, rather than independent and identically distributed values.

[5]If we limit ourselves to *deterministic* auctions (and discrete distributions), finding the optimal one is easy computationally in the case of one buyer (just enumerate), in contrast to the general case of multiple buyers with correlated values for which computational complexity difficulty has been established by Papadimitriou and Pierrakos [2011].

[2010a], Chawla et al. [2010b], Daskalakis and Weinberg [2011]) attempts approximating the optimal revenue by simple mechanisms. This was done for various settings, especially unit-demand settings and some generalizations. One conclusion from this line of work is that for many subclasses of distributions (such as those with monotone hazard rate) various simple mechanisms can extract a constant fraction of the expected value of the items[6]. This is true in our simple setting, where for such distributions selling the items separately provides a constant fraction of the expected value and thus of the optimal revenue.

The current paper may be viewed as continuing this tradition of approximating the optimal revenue with simple auctions. It may also be viewed as studying the extent to which auctions can gain revenue by doing things that appear less "natural" (such as pricing lotteries whose outcomes are the items; of course, the better our understanding becomes, the more things we may consider as natural.) We study two very simple and natural auctions that we show do give good approximations: the first simple auction is to sell the items separately and independently, and the second simple auction is to sell all items together as a bundle. We emphasize that our results hold for *arbitrary* distributions and we do not make any assumptions (such as monotone hazard rate).[7] In particular, our approximations to the optimal revenue also hold when the expected value of the items is arbitrarily (even infinitely) larger than the optimal revenue.

We will denote by $\text{REV}(\mathcal{F}) \equiv \text{REV}_k(\mathcal{F})$ the optimal revenue obtainable from selling, to a single buyer (with an additive valuation), k items whose valuation is distributed according to a (k-dimensional joint) distribution $\mathcal{F}$. This revenue is well understood only for the special case of one item ($k = 1$), i.e., for a one-dimensional F, in which case it is obtained by selling at the Myerson price (i.e., $\text{REV}_1(F) = sup_{p \geq 0} p \cdot (1 - F(p))$). The first three theorems below relate the revenue obtainable from selling multiple independent items optimally (which is not well understood) to the revenue obtainable by selling each of them separately (which is well understood).

Our first and main result shows that while selling two independent items separately need not be optimal, it is not far from optimal and always yields at least half of the optimal revenue. We do not know of any easier proof that provides *any* constant approximation bound.[8]

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[6]In our setting ths is true even more generally, for instance whenever the ratio between the median and the expectation is bounded, which happens in particular when the tail of the distribution is "thinner" than $x^{-\alpha}$ for $\alpha > 1$.

[7]One may argue that there is no need for *uniform* approximation results on the ground that the seller knows the distribution of the buyer's valuation. However, as we have shown above, that does not help finding the optimal auction (even for simple distributions) – whereas the approximations are always easy and simple (as they use only optimal prices for one-dimensional distributions).

[8]There is an easy proof for the special case of *deterministic* auctions, which we leave as an exercise to the reader. It does not seem that this type of easy proof can be extended to general auctions since it

The joint distribution of two items distributed independently according to F_1 and F_2, respectively, is denoted by $F_1 \times F_2$.

Theorem 1 *For every one-dimensional distributions F_1 and F_2,*

$$\mathrm{REV}_1(F_1) + \mathrm{REV}_1(F_2) \geq \frac{1}{2} \cdot \mathrm{REV}_2(F_1 \times F_2).$$

This result is quite robust and generalizes to auctions with multiple buyers, using either the Dominant-Strategy or the Bayes-Nash notions of implementation. It also generalizes to multi-dimensional distributions, i.e., to cases of selling two collections of items, and even to more general mechanism design settings (see Theorems 20 and 30).[9] However, as we show in a companion paper Hart and Nisan [2012], such a result does *not* hold when the values for the items are allowed to be *correlated*: there exists a joint distribution of item values such that the revenue obtainable from each item separately is finite, but selling the items optimally yields infinite revenue.

For the special case of two identically distributed items (one-dimensional and single buyer), i.e., $F_1 = F_2$, we get a tighter result.

Theorem 2 *For every one-dimensional distribution F,*

$$\mathrm{REV}_1(F) + \mathrm{REV}_1(F) \geq \frac{e}{e+1} \cdot \mathrm{REV}_2(F \times F).$$

Thus, for two independent items, each distributed according to F, taking the optimal Myerson price for a single item distributed according to F and offering the buyer to choose which items to buy at that price per item (none, either one, or both), is guaranteed to yield at least 73% of the optimal revenue for the two items. This holds for *any* distribution F (and recall that, in general, we do not know what that optimal revenue is; in contrast, the Myerson price is well-defined and immediate to determine).

There is a small gap between this bound of $e/(e+1) = 0.73...$ and the best separation that we have with a gap of of 0.78... (see Corollary 29). We conjecture that the latter is in fact the tight bound.

We next consider the case of more than two items. It turns out that, as the number of items grows, the ratio between the revenue obtainable from selling them optimally to that obtainable by selling them separately is unbounded. In fact, we present an example showing that the ratio may be as large as $O(\log k)$ (see Lemma 8). Our main positive

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would apply also to interdependent item values in which case, as we show in a companion paper Hart and Nisan [2012], there is no finite bound relating the two-item revenue to that of selling them separately.

[9]However, we have not been able to generalize these decomposition results to multiple buyers and multiple items simultaneously.

result for the case of multiple items is a bound on this gap in terms of the number of items. When the k items are independent and distributed according to $F_1, ..., F_k$, we write $F_1 \times \cdots \times F_k$ for their product joint distribution.

Theorem 3 *There exists a constant $c > 0$ such that for every integer $k \geq 2$ and every one-dimensional distributions $F_1, ..., F_k$,*

$$\text{REV}_1(F_1) + \cdots + \text{REV}_1(F_k) \geq \frac{c}{\log^2 k} \cdot \text{REV}_k(F_1 \times \cdots \times F_k).$$

We then consider the other simple single-dimensional auction, the *bundling* auction, which offers a single price for the bundle of all items.[10] We ask how well it can approximate the optimal revenue. We first observe that, in general, the bundling auction may do much worse and only yield a revenue that is a factor of almost k times lower than that of the optimal auction (see Example 15; moreover, we show in Lemma 14 that this is tight up to a constant factor). However, when the items are independent and *identically* distributed, then the bundling auction does much better. It is well known (Armstrong [1999], Bakos and Brynjolfsson [1999]) that for every fixed distribution F, as the number of items distributed independently according to F approaches infinity, the bundling auction approaches the optimal one (for completeness we provide a short proof in Appendix 5.) This, however, requires k to grow as F remains fixed. On the other hand, we show that this is not true uniformly over F: for every large enough k, there are distributions where the bundling auction on k items extracts less than 57% of the optimal revenue (Example 19). Our main result for the bundling auction is that in this case it extracts a logarithmic (in the number of items k) fraction of the optimal revenue. We do not know whether the loss is in fact bounded by a constant fraction. Since the distribution of the sum of k independent and identically distributed according to F items is the k-times convolution $F * \cdots * F$, our result is:

Theorem 4 *There exists a constant $c > 0$ such that for every integer $k \geq 2$ and every one-dimensional distribution F,*

$$\text{REV}_1(\underbrace{F * \cdots * F}_{k}) \geq \frac{c}{\log k} \cdot \text{REV}_k(\underbrace{F \times \cdots \times F}_{k}).$$

Many problems are left open. From the general point of view, the characterization of the optimal auction is still mostly open, despite the many partial results in the cited papers. In particular, it is open to fully characterize when selling separately is optimal;

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[10] By Myerson's result, this is indeed the optimal mechanism for selling the bundle.

when the bundling auction is optimal[11]; or when are deterministic auctions optimal. More specifically, regarding our approximation results, gaps remain between our lower bounds and upper bounds.

The structure of the paper is as follows. In Section 2 we present our notations and the preliminary setup. Section 3 studies the relations between the bundling auction and selling separately; these relations are not only interesting in their own right, but are then also used as part of the general analysis and provide us with most of the examples that we have for gaps in revenue. Section 4 studies the case of two items, gives the main decomposition theorem together with a few extensions; Section 5 gives our results for more than two items. Several proofs are postponed to appendices. Finally, Appendix 5 provides a table summarizing our bounds on the revenue gaps between the separate auction and the optimal auction, and between the bundling auction and the optimal auction.

2 Notation and Preliminaries

2.1 Mechanisms

A mechanism for selling k items specifies a (possibly randomized) protocol for interaction between a seller (who has no private information and commits to the mechanism) and a buyer who has a private valuation for the items. The outcome of the mechanism is an allocation specifying the probability of getting each of the k items and an (expected)[12] payment that the buyer gives to the seller. We will use the following notations:

- **Buyer valuation:** $x = (x_1, \ldots, x_k)$ where $x_i \geq 0$ denotes the value of the buyer for getting item i.

- **Allocation:** $q = (q_1, \ldots, q_k) \in [0,1]^k$, where $q_i = q_i(x)$ denotes the probability that item i is allocated to the buyer when his valuation is x (alternatively, one may interpret q_i as the fractional quantity of item i that the buyer gets).

- **Seller revenue:** $s = s(x)$ denotes the expected payment[13] that the seller receives from the buyer when the buyer's valuation is x.

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[11]We do show that this is the case for a class of distributions that decrease not too slowly; see Theorem 28.

[12]We only consider risk-neutral agents.

[13]In the literature this is also called transfer, cost, price, revenue, and denoted by p, t, c, etc. We hope that using the mnemonic s for the **S**eller's final payoff and b for the **B**uyer's final payoff will avoid confusion.

- **Buyer utility:** $b = b(x)$ denotes the utility of the buyer when his valuation is x, i.e., $b(x) = \sum_i x_i q_i(x) - s(x) = x \cdot q(x) - s(x)$.

We will be discussing mechanisms that are:

- **IR – (Ex-post) Individually Rational**: $b(x) \geq 0$ for all x.

- **IC – Incentive Compatible**: For all x, x': $\sum_i x_i q_i(x) - s(x) \geq \sum_i x_i q_i(x') - s(x')$.

The IC requirement simply captures the notion that the buyer acts strategically in the mechanism. Since we are discussing a single buyer, this is in a simple decision-theoretic sense and in particular there is no distinction between the domainant strategy and the Bayes-Nash implementation notions.

The following lemma gives well known and easily proven equivalent conditions for incentive compatibility.

Lemma 5 *The following three definitions are equivalent for a mechanism with $b(x) = x \cdot q(x) - s(x) = \sum_i x_i q_i(x) - s(x)$:*

1. *The mechanism is IC.*

2. *The allocation q is weakly monotone, in the sense that for all x, x' we have $(x - x') \cdot (q(x) - q(x')) \geq 0$, and the payment to the seller satisfies $x' \cdot (q(x) - q(x')) \leq s(x) - s(x') \leq x \cdot (q(x) - q(x'))$ for all x, x'.*

3. *The buyer's utility b is a convex function of x and for all x the allocation $q(x)$ is a subgradient of b at x, i.e., for all x' we have $b(x') - b(x) \geq q(x) \cdot (x' - x)$. In particular b is differentiable almost everywhere and there $q_i(x) = \partial b(x)/\partial x_i$.*

Proof. • *1 implies 2:* The RHS of the second part is the IC constraint for x, the LHS is the IC constraint for x', and the whole second part directly implies the first part.

• *2 implies 1:* Conversely, the RHS of the second part is exactly the IC constraint for x.

• *1 implies 3:* By IC, $b(x) = \sup_{x'} x \cdot q(x') - s(x')$ is a supremum of linear functions of x and is thus convex. For the second part, $b(x') - b(x) - q(x) \cdot (x' - x) = x' \cdot q(x') + s(x) - s(x') - x' \cdot q(x) \geq 0$, where the inequality is exactly the IC constraint for x'.

• *3 implies 1:* Conversely, as in the previous line, the subgradient property at x is exactly equivalent to the IC constraint for x'. ∎

Note that this in particular implies that any convex function b with $0 \leq \partial b(x)/\partial x_i \leq 1$ for all i defines an incentive compatible mechanism by setting $q_i(x) = \partial b(x)/\partial x_i$ (at non-differentiability points take q to be an arbitrary subgradient of b) and $s(x) = x \cdot q(x) - b(x)$.

When $x_1, \ldots, x_k$ are distributed according to the joint cumulative distribution function $\mathcal{F}$ on[14] $\mathbb{R}_+^k$, the expected revenue of the mechanism given by b is

$$R(b; \mathcal{F}) = \mathbb{E}_{x \sim \mathcal{F}}(s(x)) = \int \cdots \int \left(\sum_{i=1}^k x_i \frac{\partial b(x)}{\partial x_i} - b(x) \right) \, \mathrm{d}\mathcal{F}(x_1, \ldots, x_k).$$

Thus we want to maximize this expression over all convex functions b with $0 \leq \partial b(x)/\partial x_i \leq 1$ for all i. We can also assume

- **NPT – No Positive Transfers**: $s(x) \geq 0$ for all x.

This is without loss of generality as any mechanism can be converted to an NPT one, with the revenue only increasing.[15] This in particular implies that $b(0) = 0$ without loss of generality (as it follows from IR+NPT).

2.2 Revenue

For a cumulative distribution $\mathcal{F}$ on $\mathbb{R}_+^k$ (for $k \geq 1$), we consider the optimal revenue obtainable from selling k items to a (single, additive) buyer whose valuation for the k items is jointly distributed according to $\mathcal{F}$:

- $\mathrm{REV}(\mathcal{F}) \equiv \mathrm{REV}_k(\mathcal{F})$ is the maximal revenue obtainable by any incentive compatible and individually rational mechanism.

- $\mathrm{SREV}(\mathcal{F})$ is the maximal revenue obtainable by selling each item *separately*.

- $\mathrm{BREV}(\mathcal{F})$ is the maximal revenue obtainable by *bundling* all items together.

Thus, $\mathrm{REV}(\mathcal{F}) = sup_b R(b; \mathcal{F})$ where b ranges over all convex functions with $0 \leq \partial b(x)/\partial x_i \leq 1$ for all i and $b(0) = 0$. It will be often convenient to use random variables rather than distributions, and thus we use $\mathrm{REV}(X)$ and $\mathrm{REV}(\mathcal{F})$ interchangeably when the buyer's valuation is a random variable $X = (X_1, \ldots, X_k)$ with values in $\mathbb{R}_+^k$ distributed according to $\mathcal{F}$. In this case we have $\mathrm{SREV}(X) = \mathrm{REV}(X_1) + \cdots + \mathrm{REV}(X_k)$ and $\mathrm{BREV}(X) = \mathrm{REV}(X_1 + \cdots + X_k)$.

This paper will only deal with *independently* distributed item values, that is, $\mathcal{F} = F_1 \times \cdots \times F_k$, where F_i is the distribution of item[16] i. We have[17] $\mathrm{SREV}(\mathcal{F}) = \mathrm{REV}(F_1) +$

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[14]We write this as $x = (x_1, \ldots, x_k) \sim \mathcal{F}$. We use $\mathcal{F}$ for multi-dimensional distributions and F for one-dimensional distributions.

[15]For each x with $s(x) < 0$ redefine $q(x)$ and $s(x)$ as $q(x')$ and $s(x')$ for x' that maximizes $\sum_i x_i q_i(x') - s(x')$ over those x' with $s(x') \geq 0$.

[16]As these are cumulative distribution functions, we have $\mathcal{F}(x_1, \ldots, x_k) = F_1(x_1) \cdot \ldots \cdot F_k(x_k)$.

[17]The formula for SREV holds without independence, with F_i the i-th marginal distribution of $\mathcal{F}$.

$\cdots + \text{REV}(F_k)$ and $\text{BREV}(\mathcal{F}) = \text{REV}(F_1 * \cdots * F_k)$, where $*$ denotes convolution. Our companion paper Hart and Nisan [2012] studies general distributions $\mathcal{F}$, i.e., interdependent values.

For $k = 1$ we have **Myerson's characterization** of the optimal revenue:

$$\text{REV}_1(X) = \text{SREV}(X) = \text{BREV}(X) = \sup_{p \geq 0} p \cdot \mathbb{P}(X \geq p)$$

(which also equals $\sup_{p \geq 0} p \cdot \mathbb{P}(X > p) = \sup_{p \geq 0} p \cdot (1 - F(p))$).

Note that for any k, both the separate revenue SREV and the bundling revenue BREV require solving only one-dimensional problems; by Myerson's characterization, the former is given by k item prices $p_1, \ldots, p_k$, and the latter by one price $\bar{p}$ for all items together.

3   Warm up: Selling Separately vs. Bundling

In this section we analyze the gaps between the two simple auctions: bundling and selling the items separately. Not only are these comparisons interesting in their own right, but they will be used as part of our general analysis, and will also provide the largest lower bounds we have on the approximation ratios of these two auctions relative to the optimal revenue.

We start with a particular distribution which will turn out to be key to our analysis. We then prove upper bounds on the bundling revenue in terms of the separate revenue, and finally we prove upper bounds on the separate revenue in terms of the bundling revenue.

3.1   The Equal-Revenue Distribution

We introduce the distribution which we will show is extremal in the sense of maximizing the ratio between the bundling auction revenue and the separate auction revenue.

Let us denote by ER – the *equal-revenue distribution* – the distribution with density function $f(x) = x^{-2}$ for $x \geq 1$; its cumulative distribution function is thus $F(x) = 1 - x^{-1}$ for $x \geq 1$ (and for $x < 1$ we have $f(x) = 0$ and $F(x) = 0$). (This is also called the *Pareto* distribution with parameter $\alpha = 1$.) It is easy to see that, on one hand, $\text{REV}_1(ER) = 1$ and, moreover, this revenue is obtained by choosing any price $p \geq 1$. On the other hand its expected value is infinite: $\mathbb{E}(ER) = \int_1^\infty x \cdot x^{-2} dx = \infty$. We start with a computation of the distribution of the weighted sum of two ER distributions.

Lemma 6 *Let X_1, X_2 be*[18] *i.i.d.-ER and $\alpha, \beta > 0$. Then* [19]

$$\mathbb{P}(\alpha X_1 + \beta X_2 \geq z) = \frac{\alpha\beta}{z^2} \log\left(1 + \frac{z^2 - (\alpha + \beta)z}{\alpha\beta}\right) + \frac{\alpha + \beta}{z}$$

for $z \geq \alpha + \beta$, and $\mathbb{P}(\alpha X_1 + \beta X_2 \geq z) = 1$ for $z \leq \alpha + \beta$.

Proof. Let $Z = \alpha X_1 + \beta X_2$. For $z \leq \alpha + \beta$ we have $\mathbb{P}(Z \geq z) = 1$ since $X_i \geq 1$. For $z > \alpha + \beta$ we get

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\begin{aligned}
\mathbb{P}(Z \geq z) &= \int f(x)\left(1 - F\left(\frac{z - \alpha x}{\beta}\right)\right) dx \\
&= \int_1^{(z-\beta)/\alpha} \frac{1}{x^2}\frac{\beta}{z - \alpha x} \, dt + \int_{(z-\beta)/\alpha}^\infty \frac{1}{x^2} 1 \, dt \\
&= \frac{\beta}{z}\left[\frac{\alpha}{z}\log x - \frac{\alpha}{z}\log\left(\frac{z}{\alpha} - x\right) - \frac{1}{x}\right]_1^{(z-\beta)/\alpha} + \frac{\alpha}{z - \beta} \\
&= \frac{\alpha\beta}{z^2}\left(\log\left(\frac{z}{\beta} - 1\right) + \log\left(\frac{z}{\alpha} - 1\right)\right) - \frac{\alpha\beta}{z(z - \beta)} + \frac{\beta}{z} + \frac{\alpha}{z - \beta} \\
&= \frac{\alpha\beta}{z^2}\log\left(1 + \frac{z^2 - (\alpha + \beta)z}{\alpha\beta}\right) + \frac{\alpha + \beta}{z} \, .
\end{aligned}
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We can now calculate the revenue obtainable from bundling several independent ER items.

Lemma 7 $\mathrm{BREV}(ER \times ER) = 2.5569...$, *where* $2.5569... = 2(w + 1)$ *with w the solution of* [20] $we^w = 1/e$.

Remark. We will see below (Corollary 29) that bundling is optimal here, and so 2.5569... is in fact the *optimal revenue* for two i.i.d.-ER items.

Proof. Using Lemma 6 with $\alpha = \beta = 1$ yields $p \cdot \mathbb{P}(X_1 + X_2 \geq p) = p^{-1}\log(1 + p^2 - 2p) + 2 = 2p^{-1}\log(p - 1) + 2$, which attains its maximum of $2w + 2$ at $p = 1 + 1/w$. ∎

Lemma 8 *There exist constants $c_1, c_2 > 0$ such that for all $k \geq 2$,*

$$c_1 k \log k \leq \mathrm{BREV}(ER^{\times k}) \leq c_2 k \log k.$$

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[18] For a one-dimensional distribution F, "i.i.d.-F" refers to a collection of independent random variables each distributed according to F.

[19] log denotes *natural* logarithm.

[20] Thus $w = W(1/e)$ where W is the so-called "Lambert-W" function.

In particular, this shows that selling separately may yield, as k increases, an arbitrarily small proportion of the optimal revenue: $\text{REV}(ER^{\times k}) \geq \text{BREV}(ER^{\times k}) \geq c_1 k \log k = c_1 \log k \cdot \text{SREV}(ER^{\times k})$.

Proof. Let X be a random variable with distribution ER; for $M \geq 1$ let $X^M := \min\{X, M\}$ be X truncated at M. It is immediate to compute $\mathbb{E}(X^M) = \log M + 1$ and $\mathbb{V}\text{ar}(X^M) \leq 2M$.

- *Lower bound*: Let $X_1, \ldots, X_k$ be i.i.d.-ER; for every $p, M > 0$ we have $\text{REV}(\sum_i X_i) \geq p \cdot \mathbb{P}(\sum_i X_i \geq p) \geq p \cdot \mathbb{P}(\sum_i X_i^M \geq p)$.

When $M = k \log k$ and $p = (k \log k)/2$ we get $(k\mathbb{E}(X^M) - p)/\sqrt{k\mathbb{V}\text{ar}(X^M)} \geq \sqrt{\log k/8}$, so p is at least $\sqrt{\log k/8}$ standard deviations below the mean of $\sum_{i=1}^{k} X_i^M$. Therefore, by Chebyshev's inequality, $\mathbb{P}(\sum_{i=1}^{k} X_i^M \geq p) \geq 1 - 8/\log k \geq 1/2$ for all k large enough, and then $\text{REV}(\sum_{i=1}^{k} X_i) \geq p \cdot 1/2 = k \log k/4$.

- *Upper bound*: We need to bound $\sup_{p \geq 0} p \cdot \mathbb{P}(\sum_{i=1}^{k} X_i \geq p)$. If $p \leq 6k \log k$ then $p \cdot \mathbb{P}(\sum_{i=1}^{k} X_i \geq p) \leq p \leq 6k \log k$.

If $p \geq 6k \log k$ then (take $M = p$)

$$p \cdot \mathbb{P}\left(\sum_{i=1}^{k} X_i \geq p\right) \leq p \cdot \mathbb{P}\left(\sum_{i=1}^{k} X_i^p \geq p\right) + p \cdot \mathbb{P}(X_i > p \text{ for some } 1 \leq i \leq k). \quad (1)$$

The second term is at most $p \cdot k \cdot (1 - F(p)) = k$ (since $F(p) = 1 - 1/p$). To estimate the first term, we again use Chebyshev's inequality. When k is large enough we have $p/(k(\log p + 1)) \leq 2$ (recall that $p \geq 6k \log k$), and so p is at least $\sqrt{p/(8k)}$ standard deviations above the mean of $\sum_{i=1}^{k} X_i^p$. Thus $p \cdot \mathbb{P}(\sum_{i=1}^{k} X_i^p \geq p) \leq p \cdot (8k)/p = 8k$, and so $p \cdot \mathbb{P}(\sum_{i=1}^{k} X_i \geq p) \leq 9k$ (recall (1)).

Altogether, $\text{REV}(\sum_{i=1}^{k} X_i) \leq \max\{6k \log k, 9k\} = 6k \log k$ for all k large enough. ■

Remark. A more precise analysis, based on the "Generalized Central Limit Theorem,"[21] shows that $\text{BREV}(ER^{\times k})/(k \log k)$ converges to 1 as $k \to \infty$. Indeed, when X_i are i.i.d.-ER, the sequence $(\sum_{i=1}^{k} X_i - b_k)/a_k$ with $a_k = k\pi/2$ and $b_k = k \log k + \Theta(k)$ converges in distribution to the Cauchy distribution as $k \to \infty$. Since $\text{REV}_1(\text{Cauchy})$ can be shown to be bounded (by $1/\pi$), it follows that $\text{REV}(\sum_{i=1}^{k} X_i) = k \log k + \Theta(k)$.

3.2 Upper Bounds on the Bundling Revenue

It turns out that the equal revenue distribution exhibits the largest possible ratio between the bundling auction and selling separately. This is a simple corollary from the fact that the equal revenue distribution has the heaviest possible tail.

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[21]See, e.g., Zaliapin et al. [2005].

Let X and Y be one-dimensional random variables. We say that X is *(first-order) stochastically dominated* by Y if for every real p we have $\mathbb{P}(X \geq p) \leq \mathbb{P}(Y \geq p)$. Thus, Y gets higher values than X.

Lemma 9 *If a one-dimensional X is stochastically dominated by a one-dimensional Y then* $\mathrm{REV}_1(X) \leq \mathrm{REV}_1(Y)$.

Proof. $\mathrm{REV}(X) = \sup_p p \cdot \mathbb{P}(X \geq p) \leq \sup_p p \cdot \mathbb{P}(Y \geq p) = \mathrm{REV}(Y)$ (by Myerson's characterization). ∎

It should be noted that this monotonicity of the revenue with respect to stochastic dominance does *not* hold when there are two or more items Hart and Reny [2011].

Lemma 10 *For every one-dimensional X and every $r > 0$:* $\mathrm{REV}_1(X) \leq r$ *if and only if X is stochastically dominated by*[22] $r \cdot ER$.

Proof. By Myerson's characterization, $\mathrm{REV}(X) \leq r$ if and only if for every p we have $\mathbb{P}(X \geq p) \leq r/p$; but r/p is precisely the probability that $r \cdot ER$ is at least p. ∎

We will thus need to consider sums of "scaled" versions of ER, i.e., linear combinations of independent ER random variables. What we will see next is that equalizing the scaling factors yields stochastic domination.

Lemma 11 *Let X_1, X_2 be i.i.d.-ER and let $\alpha, \beta, a', \beta' > 0$ satisfy $\alpha + \beta = \alpha' + \beta'$. If*[23] *$\alpha\beta \leq \alpha'\beta'$ then $\alpha X_1 + \beta X_2$ is stochastically dominated by $\alpha' X_1 + \beta' X_2$.*

Proof. Let $Z = \alpha X_1 + \beta X_2$ and $Z' = \alpha' X_1 + \beta' X_2$, and put $\gamma = \alpha + \beta = \alpha' + \beta'$. Using Lemma 6, for $z \leq \gamma$ we have $\mathbb{P}(Z \geq z) = \mathbb{P}(Z' \geq z) = 1$, and for $z > \gamma$ we get

$$
\begin{aligned}
\mathbb{P}(Z \geq z) &= \frac{\alpha\beta}{z^2} \log\left(1 + \frac{z^2 - \gamma z}{\alpha\beta}\right) + \frac{\gamma}{z} \\
&\leq \frac{\alpha'\beta'}{z^2} \log\left(1 + \frac{z^2 - \gamma z}{\alpha'\beta'}\right) + \frac{\gamma}{z} = \mathbb{P}(Z' \geq z),
\end{aligned}
$$

since $t \log(1 + 1/t)$ is increasing in t for $t > 0$, and $\alpha\beta/(z^2 - \gamma z) \leq \alpha'\beta'/(z^2 - \gamma z)$ by our assumption that $\alpha\beta \leq \alpha'\beta'$ together with $z > \gamma$. ∎

We note the following useful fact: if for every i, X_i is stochastically dominated by Y_i, then $X_1 + \cdots + X_k$ is stochastically dominated by[24] $Y_1 + \cdots + Y_k$.

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[22]We slightly abuse the notation and write $r \cdot ER$ for a random variable $r \cdot Y$ when Y is distributed according to ER.

[23]Equivalently, $|\alpha - \beta| \geq |\alpha' - \beta'|$.

[24]Think of all the random variables being defined on the same probability space and satisfying $X_i \leq Y_i$ pointwise (which can be obtained by the so-called "coupling" construction), and then $\sum X_i \leq \sum Y_i$ is immediate.

Corollary 12 *Let X_i be i.i.d.-ER and $\alpha_i > 0$. Then $\sum_{i=1}^{k} \alpha_i X_i$ is stochastically dominated by $\sum_{i=1}^{k} \bar{\alpha} X_i$, where $\bar{\alpha} = (\sum_{i=1}^{k} \alpha_i)/k$.*

Proof. If, say, $\alpha_1 < \bar{\alpha} < \alpha_2$, then the previous lemma implies that $\alpha_1 X_1 + \alpha_2 X_2$ is stochastically dominated by $\bar{\alpha} X_1 + \alpha_2' X_2$,where $\alpha_2' = \alpha_1 + \alpha_2 - \bar{\alpha}$, and so $\sum_{i=1}^{k} \alpha_i X_i$ is stochastically dominated by $\bar{\alpha} X_1 + \alpha_2' X_2 + \sum_{i=3}^{k} \alpha_i X_i$. Continue in the same way until all coefficients become $\bar{\alpha}$. ∎

We can now provide our upper bounds on the bundling revenues.

Lemma 13 *(i) For every one-dimensional distributions F_1, F_2,*

$$\mathrm{BRev}(F_1 \times F_2) \leq 1.278... \cdot (\mathrm{Rev}(F_1) + \mathrm{Rev}(F_2)) = 1.278... \cdot \mathrm{SRev}(F_1 \times F_2),$$

where $1.278... = w + 1$ with w the solution of $we^w = 1/e$.

(ii) There exists a constant $c > 0$ such that for every $k \geq 2$ and every one-dimensional distributions $F_1, \ldots, F_k$,

$$\mathrm{BRev}(F_1 \times \cdots \times F_k) \leq c \log k \cdot \sum_{i=1}^{k} \mathrm{Rev}(F_i) = c \log k \cdot \mathrm{SRev}(F_1 \times \cdots \times F_k).$$

Proof. Let X_i be distributed according to F_i, and denote $r_i = \mathrm{Rev}(F_i)$, so X_i is stochastically dominated by $r_i Y_i$ where Y_i is distributed according to ER (see Lemma 10). Assume that the X_i are independent, and also that the Y_i are independent. Then $X_1 + \cdots + X_k$ is stochastically dominated by $r_1 Y_1 + \cdots + r_k Y_k$. By Corollary 12 the latter is stochastically dominated by $\bar{r} Y_1 + \cdots + \bar{r} Y_k$ where $\bar{r} = (\sum_i r_i)/k = (\sum_i \mathrm{Rev}(F_i))/k$. Therefore $\mathrm{BRev}(F_1 \times \cdots \times F_k) \leq \bar{r} \mathrm{BRev}(ER^{\times k})$, and the results (i) and (ii) follow from Lemmas 7 and 8 respectively. ∎

3.3 Lower Bounds on the Bundling Revenue

In general, the bundling revenue obtainable from items that are independently distributed according to different distributions may be significantly smaller than the separate revenue.

Lemma 14 *For every integer $k \geq 1$ and every one-dimensional distributions $F_1, \ldots, F_k$,*

$$\mathrm{BRev}(F_1 \times \cdots \times F_k) \geq \frac{1}{k} \cdot \sum_{i=1}^{k} \mathrm{Rev}(F^i) = \frac{1}{k} \cdot \mathrm{SRev}(F_1 \times \cdots \times F_k).$$

Proof. For every i we have $\mathrm{Rev}(F_i) \leq \mathrm{BRev}(F_1 \times \cdots \times F_k)$, and so $\sum_i \mathrm{Rev}(F_i) \leq k \cdot \mathrm{BRev}(F_1 \times \cdots \times F_k)$. ∎

This is tight:

Example 15 $\text{BREV}(F_1 \times \cdots \times F_k) = (1/k + \epsilon) \cdot \text{SREV}(F_1 \times \cdots \times F_k)$:

Take a large M and let F_i have support $\{0, M^i\}$ with $\mathbb{P}(M^i) = M^{-i}$. Then $\text{REV}(F^i) = 1$ and so $\text{SREV}(F^1 \times \cdots \times F^k) = k$, while $\text{BREV}(F^1 \times \cdots \times F^k)$ is easily seen to be at most $\max_i M^i \cdot (M^{-i} + \cdots + M^{-k}) \leq 1 + 1/(M-1)$.

However, when the items are distributed according to *identical* distributions, the bundling revenue cannot be much smaller than the separate revenue, and this is the case that the rest of this section deals with.

Lemma 16 *For every one-dimensional distribution F,*

$$\text{BREV}(F \times F) \geq \frac{4}{3} \cdot \text{REV}(F) = \frac{2}{3} \cdot \text{SREV}(F \times F).$$

Proof. Let X be distributed according to F; let p be the optimal Myerson price for X and $q = \mathbb{P}(X \geq p)$, so $\text{REV}(F) = pq$. If $q \leq 2/3$ then the bundling auction can offer a price of p and the probability that the bundle will be sold is at least the probability that one of the items by itself has value p, which happens with probability $2q - q^2 = q(2-q) \geq 4q/3$, so the revenue will be at least $4q/3 \cdot p = (4/3)\text{REV}(F)$. On the other hand, if $q \geq 2/3$ then the bundling auction can offer price $2p$, and the probability that it will be accepted is at least the probability that both items will get value of at least p, i.e. q^2. The revenue will be $2q^2 p \geq (4/3)qp = (4/3)\text{REV}(F)$. ∎

This bound is tight:

Example 17 $\text{BREV}(F \times F) = (2/3) \cdot \text{SREV}(F \times F)$:

Let F have support $\{0, 1\}$ with $\mathbb{P}(1) = 2/3$, then $\text{REV}(F) = 2/3$ while $\text{BREV}(F \times F) = 8/9$ (which is obtained both at price 1 and at price 2).[25]

We write F^{*k} for the k-times convolution of F; this is the distribution of the *sum* of k i.i.d. random variables each distributed according to F.

Lemma 18 *For every integer $k \geq 1$ and every one-dimensional distribution F,*

$$\text{BREV}(F^{\times k}) = \text{REV}(F^{*k}) \geq \frac{1}{4}k \cdot \text{REV}(F) = \frac{1}{4} \cdot \text{SREV}(F^{\times k}).$$

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[25]It can be checked that the optimal revenue is attained here by the separate auction, i.e., $\text{REV}(F \times F) = \text{SREV}(F \times F) = 4/3$.

Proof. Let X be distributed according to F; let p be the optimal Myerson price for X and $q = \mathbb{P}(X \geq p)$, so $\text{REV}(F) = pq$. We separate between two cases. If $qk \leq 1$ then the bundling auction can offer price p and, using inclusion-exclusion, the probability that it will be taken is bounded from below by $kq - \binom{k}{2}q^2 \geq kq/2$ so the revenue will be at least $kqp/2 \geq k \cdot \text{REV}(F)/2$. If $qk \geq 1$ then we can offer price $p\lfloor qk \rfloor$. Since the median in a Binomial(k, q) distribution is known to be at least $\lfloor qk \rfloor$, the probability that the buyer will buy is at least $1/2$. The revenue will be at least $p\lfloor qk \rfloor/2 \geq kqp/4 = k \cdot \text{REV}(F)/4$. ∎

We have not attempted optimizing this constant $1/4$, which can be easily improved. The largest gap that we know of is the following example where the bundling revenue is less than 57% than that of selling the items separately, and applies to all large enough k. We suspect that this is in fact the maximal possible gap.

Example 19 *For every k large enough, a one-dimensional distribution F such that* $\text{BREV}(F^{\times k})/\text{SREV}(F^{\times k}) \leq 0.57$:

Take a large k and consider the distribution F on $\{0, 1\}$ with $\mathbb{P}(1) = c/k$ where $c = 1.256...$ is the solution of $1 - e^{-c} = 2(1 - (c + 1)e^{-c})$, so the revenue from selling a single item is c/k. The bundling auction should clearly offer an integral price. If it offers price 1 then the probability of selling is $1 - (1 - c/k)^k \approx 1 - e^{-c} = 0.715...$, which is also the expected revenue. If it offers price 2 then the probability of selling is $1 - (1 - c/k)^k - k(c/k)(1 - c/k)^{k-1} \approx 1 - (c + 1)e^{-c}$ and the revenue is twice that, again $0.715...$. If it offers price 3 then the probability of selling is $1 - (1 - c/k)^k - k(c/k)(1 - c/k)^{k-1} - \binom{k}{2}(c/k)^2(1 - c/k)^{k-2} \approx 1 - (1 + c + c^2/2)e^{-c} \approx 0.13...$, and the revenue is three times higher, which is less than 0.715. For higher integral prices t the probability of selling is bounded from above by $c^t/t!$, the revenue is t times that, and is even smaller. Thus $\text{BREV}(F^{\times k})/\text{SREV}(F^{\times k}) \approx 0.715/1.256 \leq 0.57$ for all k large enough.

4 Two Items

Our main result is an "approximate direct sum" theorem. While the main interest, as stated in Theorem 1, is for selling two independent items, we will state and prove in the next section a more general result that applies to two independent *sets* of items.

4.1 The Main Decomposition Result

In this section X is a k_1-dimensional nonnegative random variable and Y is a k_2-dimensional nonnegative random variable (with $k_1, k_2 \geq 1$). While we will assume that the vectors X

and Y are *independent*, we allow for arbitrary interdependence among the coordinates of X, and the same for the coordinates of Y.

Theorem 20 (Generalization of Theorem 1) *Let X and Y be multi-dimensional random variables. If X and Y are independent then*

$$\textsc{Rev}(X, Y) \leq 2 \cdot (\textsc{Rev}(X) + \textsc{Rev}(Y)).$$

The proof of this theorem is divided into a series of lemmas. The main insights are the "Marginal Mechanism" (Lemma 21) and the "Smaller Value" (Lemma 25).

The first attempt in bounding the revenue from two items, is to fix one of them and look at the induced marginal mechanism on the second. Let us use the notation $\textsc{Val}(X) = \mathbb{E}(\sum_i X_i) = \sum_i \mathbb{E}(X_i)$, the expected total sum of values, for multi-dimensional X's (for one-dimensional X this is $\textsc{Val}(X) = \mathbb{E}(X)$.)

Lemma 21 (Marginal Mechanism) *Let X and Y be multi-dimensional random variables (here X and Y may be dependent). Then*

$$\textsc{Rev}(X, Y) \leq \textsc{Val}(Y) + \mathbb{E}_Y[\textsc{Rev}(X|Y)],$$

where $(X|Y)$ denotes the conditional distribution of X given Y.

Proof. Take a mechanism that obtains the optimal revenue from (X, Y), and fix some value of $y = (y_1, \ldots, y_{k_2})$. The induced mechanism on the X-items, which are distributed according to $(X|Y = y)$, is IC and IR, but also hands out quantities of the Y items. If we modify it so that instead of allocating y_j with probability $q_j = q_j(x, y)$, it pays back to the buyer an additional money amount of $q_j y_j$, we are left with an IC and IR mechanism for the X items. The revenue of this mechanism is that of the original mechanism conditioned on $Y = y$ minus the expected value of $\sum_j q_j y_j$, which is bounded from above by $\sum_j y_j$. Now take expectation over the values y of Y to get $\mathbb{E}_{y \sim Y}[\textsc{Rev}(X|Y = y)] \geq \textsc{Rev}(X, Y) - \textsc{Val}(Y)$. ∎

Remark. When X and Y are *independent* then $(X|Y = y) = X$ for every y and thus $\textsc{Rev}(X, Y) \leq \textsc{Val}(Y) + \textsc{Rev}(X)$.

Unfortunately this does not suffice to get good bounds since it is entirely possible for $\textsc{Val}(Y)$ to be infinite even when $\textsc{Rev}(Y)$ is finite (as happens, e.g., for the equal-revenue distribution ER.) To effectively use the marginal mechanism lemma we will have to carefully cut up the domain of (X, Y), bound the value of one of the items in each of these sub-domains, and then stitch the results together. We will use Z to denote an

arbitrary multi-dimensional nonnegative random variable, but the reader may want to think of it as (X, Y).

Lemma 22 (Sub-Domain Restriction) *Let Z be a multi-dimensional random variable and let S be a set of values of [26] Z. Then*

$$\text{REV}(\mathbb{1}_{Z \in S} Z) \leq \text{REV}(Z).$$

Proof. The optimal mechanism for $\mathbb{1}_{Z \in S} Z$ will extract at least as much from Z. This follows directly from an optimal mechanism having No Positive Transfers (see the end of Section 2.1). ∎

Lemma 23 (Sub-Domain Stitching) *Let Z be a multi-dimensional random variable and let S, T be two sets of values of Z such that $S \cup T$ contains the support of Z. Then*

$$\text{REV}(\mathbb{1}_{Z \in S} Z) + \text{REV}(\mathbb{1}_{z \in T} Z) \geq \text{REV}(Z).$$

Proof. Take the optimal mechanism for Z. $\text{REV}(Z)$ is the revenue extracted by this mechanism, which is at most the sum of what is extracted on S and on T. If you take the same mechanism and run it on the random variable $\mathbb{1}_{Z \in S} Z$, it will extract the same amount on S as it extracted from Z on S, and similarly for T which contains the complement of S. ∎

Our trick will be to choose S so that we are able to bound $\text{VAL}(\mathbb{1}_{(X,Y) \in S} Y)$. This will suffice since the marginal mechanism lemma actually implies:

Lemma 24 (Marginal Mechanism on Sub-Domain) *Let X and Y be multi-dimensional random variables, and let S be a set of values of (X, Y). If X and Y are independent then*

$$\text{REV}(\mathbb{1}_{(X,Y) \in S} \cdot (X, Y)) \leq \text{VAL}(\mathbb{1}_{(X,Y) \in S} Y) + \text{REV}(X).$$

Proof. For every y let $S_y = \{x | (x, y) \in S\}$. Note that $\text{REV}(\mathbb{1}_{(X,Y) \in S} \cdot (X, Y)) = \text{REV}(\mathbb{1}_{(X,Y) \in S} X, \mathbb{1}_{(X,Y) \in S} Y)$, and $\text{REV}(\mathbb{1}_{(X,Y) \in S} X | Y = y) = \text{REV}(\mathbb{1}_{X \in S_y} X)$. The Marginal Mechanism Lemma 21 then yields $\text{REV}(\mathbb{1}_{(X,Y) \in S} \cdot (X, Y)) \leq \text{VAL}(\mathbb{1}_{(X,Y) \in S} Y) + \mathbb{E}_y[\text{REV}(\mathbb{1}_{X \in S_y} X)]$, and by the Sub-Domain Restriction Lemma 22 $\text{REV}(\mathbb{1}_{X \in S_y} X) \leq \text{REV}(X)$ for every y. ∎

We first deal with the case of two items, i.e. one-dimensional X and Y. The set of values S for which we bound $\text{VAL}(\mathbb{1}_{(X,Y) \in S} Y)$ will be the set $\{Y \leq X\}$.

---

[26]If Z is a k-dimensional random variable, then S is a (measurable) subset of $\mathbb{R}_+^k$. We use the notation $\mathbb{1}_{Z \in S}$ for the indicator random variable which takes the value 1 when $Z \in S$ and 0 otherwise.

Lemma 25 (Smaller Value) *Let X and Y be one-dimensional random variables. If X and Y are independent then*

$$\mathbb{E}(\mathbb{1}_{Y \leq X} Y) \leq \text{Rev}(X).$$

Proof. A possible mechanism for X that yields revenue of $\text{Val}(\mathbb{1}_{Y \leq X} Y)$ is the following: choose a random y according to Y and offer this as the price. The expected revenue of this mechanism is $\mathbb{E}_{y \sim Y}(y \cdot \mathbb{P}(X \geq y)) = \mathbb{E}_{y \sim Y}(\mathbb{E}(Y \mathbb{1}_{Y \leq X} | Y = y)) = \mathbb{E}(Y \mathbb{1}_{Y \leq X})$, so this is a lower bound on $\text{Rev}(X)$. ∎

We can now conclude the proof of Theorem 1, which is the special case of Theorem 20 when X and Y are one-dimensional.

Proof of Theorem 20 – one-dimensional case. Using the Sub-Domain Stitching Lemma 23, we will cut the space as follows: $\text{Rev}(X, Y) \leq \text{Rev}(\mathbb{1}_{Y \leq X}(X, Y)) + \text{Rev}(\mathbb{1}_{X \leq Y}(X, Y))$. By the Marginal Mechanism on Sub-Domain Lemma 24, the first term is bounded by $\mathbb{E}(\mathbb{1}_{Y \leq X} Y) + \text{Rev}(X) \leq 2\text{Rev}(X)$, where the inequality uses the Smaller Value Lemma 25. The second term is bounded similarly. ∎

The multi-dimensional case is almost identical. The Smaller Value Lemma 25 becomes:

Lemma 26 *Let X and Y be multi-dimensional random variables. If X and Y are independent then*

$$\text{Val}(\mathbb{1}_{\sum_j Y_j \leq \sum_i X_i} Y) \leq \text{BRev}(X).$$

Proof. Apply Lemma 25 to the one-dimensional random variables $\sum_i X_i$ and $\sum_j Y_j$, and recall that $\text{Rev}(\sum_i X_i) = \text{BRev}(X)$. ∎

From this we get a slightly stronger version of Theorem 20 for multi-dimensional variables (which will be used in Section 5 to get bounds for any fixed number of items).

Theorem 27 *Let X and Y be multi-dimensional random variables. If X and Y are independent then*

$$\text{Rev}(X, Y) \leq \text{Rev}(X) + \text{Rev}(Y) + \text{BRev}(X) + \text{BRev}(Y).$$

Proof. The proof is almost identical to that of the main theorem. We will cut the space by $\text{Rev}(X, Y) \leq \text{Rev}(\mathbb{1}_{\sum_j Y_j \leq \sum_i X_i} \cdot (X, Y)) + \text{Rev}(\mathbb{1}_{\sum_j Y_j \geq \sum_i X_i} \cdot (X, Y))$, and bound the first term by $\text{Val}(\mathbb{1}_{\sum_j Y_j + \text{Rev}(X) \leq \sum_i X_i} Y) \leq \text{BRev}(X) + \text{Rev}(X)$ using Lemmas 24 and 26. The second term is bounded similarly. ∎

Proof of Theorem 20 – multi-dimensional case. Use the previous theorem and $\text{BREV} \leq \text{REV}$. ∎

Remark. The decomposition of this section holds in more general setups than the totally additive valuation of this paper (where the value to the buyer of the outcome $q \in [0,1]^k$ is $\sum_i q_i x_i$). Indeed, consider an abstract mechanism design problem with a set of alternatives A, valuated by the buyer according to a function $v : A \to \mathbb{R}_+$ (known to him, whereas the seller only knows that the function v is drawn from a certain distribution). If the set of alternatives A is in fact a product $A = A_1 \times A_2$ with the valuation additive between the two sets, i.e., $v(a_1, a_2) = v_1(a_1) + v_2(a_2)$, with v_1 distributed according to X and v_2 acording to Y, then Theorem 20 holds as stated. The proof now uses $\text{VAL}(Y) = \mathbb{E}(\sup_{a_2 \in A_2} v_2(a_2))$ (which, in our case, where $A_2 = [0,1]^{k_2}$ and $v_2(q) = \sum_j q_j y_j$, is indeed $\text{VAL}(Y) = \mathbb{E}(\sum_j Y_j)$ since $\sup_q v_2(q) = \sum_j y_j$).

4.2 A Tighter Result for Two I.I.D. Items

For the special case of two independent and *identically distributed* items we have a tighter result, namely Theorem 2 stated in the Introduction. The proof is more technical and is relegated to Appendix 1.

4.3 A Class of Distributions Where Bundling Is Optimal

For some special cases we are able to fully characterize the optimal auction for two items. We will show that bundling is optimal for distributions whose density function decreases fast enough; this includes the equal-revenue distribution.

Theorem 28 *Let F be a one-dimensional cumulative distribution function with density function f. Assume that there is $a > 0$ such that for $x < a$ we have $f(x) = 0$ and for $x > a$ the function $f(x)$ is differentiable and satisfies*

$$x f'(x) + \frac{3}{2} f(x) \leq 0. \tag{2}$$

Then bundling is optimal for two items: $\text{REV}(F \times F) = \text{BREV}(F \times F)$.

Theorem 28 is proved in Appendix 2. Condition (2) is equivalent to $\left(x^{3/2} f(x)\right)' \leq 0$, i.e., $x^{3/2} f(x)$ is nonincreasing in x (the support of F is thus either a finite interval $[a, A]$, or the half-line $[a, \infty)$). When $f(x) = c x^{-\gamma}$, (2) holds whenever $\gamma \geq 3/2$. In particular, ER satisfies (2); thus, by Lemma 7, we have:

Corollary 29 $\text{REV}(ER \times ER) = \text{BREV}(ER \times ER) = 2.5569... \, .$

Thus $\text{SREV}(ER \times ER)/\text{REV}(ER \times ER) = 2/2.559... = 0.78...$, which the largest gap we have obtained between the separate auction and the optimal one.

4.4 Multiple Buyers

Up to now we dealt a single buyer, but it turns out that the main decomposition result generalizes to the case of multiple buyers. We consider selling the two items (with a single unit of each) to n buyers, where buyer j's valuation for the first item is X^j, and for the second item is Y^j (with $X^j + Y^j$ being the value for getting both). Let the auction allocate the first item to buyer j with probability q_1^j, and the second item with probability q_2^j; of course, here $\sum_{j=1}^{n} q_1^j \le 1$ and $\sum_{j=1}^{n} q_2^j \le 1$.

Unlike the simple decision-theoretic problem facing the single buyer, we now have a multi-person game among the buyers. Thus, we consider two main notions of incentive compatibility: *dominant-strategy* IC and *Bayes-Nash* IC. Our result below applies equally well to both notions, and with an identical proof.

For either one of these notions, we denote by $\text{REV}^{[n]}(X, Y)$ the revenue that is obtainable by the optimal auction. Similarly, selling the two items separately yields a maximal revenue of $\text{SREV}^{[n]}(X, Y) = \text{REV}^{[n]}(X) + \text{REV}^{[n]}(Y)$.

We allow the values of the different buyers for each single item to be arbitrarily correlated; however, we assume that independence between the two items.

Theorem 30 *Let $X = (X^1, \ldots, X^n) \in \mathbb{R}_+^n$ be the values of the first item to the n buyers, and let $Y = (Y^1, \ldots, Y^n) \in \mathbb{R}_+^n$ be the values of the second item to the n buyers. If X and Y are independent then*

$$\text{REV}^{[n]}(X) + \text{REV}^{[n]}(Y) \ge \frac{1}{2} \cdot \text{REV}^{[n]}(X, Y),$$

where $\text{REV}^{[n]}$ is taken throughout either with respect to dominant-strategy implementation, or with respect to Bayes-Nash implementation.

Thus selling the two items separately yields at least half the maximal revenue, i.e., $\text{SREV}^{[n]}(X, Y) \ge (1/2) \cdot \text{REV}^{[n]}(X, Y)$.

The proof of Theorem 30 is almost identical to the proof of the Theorem 20 and is spelled out in Appendix 3 (we also point out there why we could not extend it to multiple buyers *and* more than 2 items). We emphasize that the proof does not use the characterization of the optimal revenue for a single item and n buyers (just like the proof of Theorem 20 did not use Myerson's characterization for one buyer).

5 More Than Two Items

The multi-dimensional decomposition results of Section 4.1 can be used recursively, by viewing k items as two sets of $k/2$ items each. Using Theorem 20 we can prove by induction that $\text{REV}(F_1 \times \cdots \times F_k) \leq k \sum_{i=1}^{k} \text{REV}(F_i)$, as follows: $\text{REV}(F_1 \times \cdots \times F_k) \leq 2(\text{REV}(F_1 \times \cdots \times F_{k/2}) + \text{REV}(F_{k/2+1} \times \cdots \times F_k)) \leq 2(k/2 \sum_{i=1}^{k/2} \text{REV}(F_i) + k/2 \sum_{i=k/2+1}^{k} \text{REV}(F_i)) = k \sum_{i=1}^{k} \text{REV}(F_i)$, where the first inequality is by Theorem 20, and the second by the induction hypothesis.

However, using the stronger statement of Theorem 27, as well as the relations we have shown between the bundling revenue and the separate revenue, will give us the better bound of $c \log^2 k$ (instead of k) of Theorem 3, stated in the Introduction.

Proof of Theorem 3. Assume first that $k \geq 2$ is a power of 2, and we will prove by induction that $\text{REV}(F_1 \times \cdots \times F_k) \leq c \log^2(2k) \sum_{i=1}^{k} \text{REV}(F_i)$, where c is the constant of Lemma 13. By applying Theorem 27 to $(F_1 \times \cdots \times F_{k/2}) \times (F_{k/2+1} \times \cdots \times F_k)$ we get

$$
\begin{aligned}
\text{REV}(F_1 \times \cdots \times F_k) \leq{} & \text{BREV}(F_1 \times \cdots \times F_{k/2}) + \text{BREV}(F_{k/2+1} \times \cdots \times F_k) \\
& + \text{REV}(F_1 \times \cdots \times F_{k/2}) + \text{REV}(F_{k/2+1} \times \cdots \times F_k). \quad (3)
\end{aligned}
$$

Using Lemma 13 on each of the BREV terms, their sum is bounded by $c \log k \sum_{i=1}^{k} \text{REV}(F_i)$. Using the induction hypothesis on each of the REV terms, their sum is bounded by $c \log^2 k \sum_{i=1}^{k} \text{REV}(F_i)$. Now $\log k + \log^2 k \leq \log^2(2k)$, and so the coefficient of each $\text{REV}(F_i)$ is at most $c \log^2(2k)$ as required.

When $2^{m-1} < k < 2^m$ we can pad to 2^m with items that have value identically zero, and so do not contribute anything to the revenue. This at most doubles k. ∎

As we have seen in Example 15, the bundling auction may, in contrast, extract only $1/k$ fraction of the optimal revenue. This we can show is tight.

Lemma 31 *There exists a constant $c > 0$ such that for every $k \geq 2$ and every one-dimensional distributions $F_1, ..., F_k$,*

$$
\text{BREV}(F_1 \times \cdots \times F_k) \geq \frac{c}{k} \cdot \text{REV}(F_1 \times \cdots \times F_k).
$$

Proof. For k a power of two, we use as in the previous proof the decomposition of (3) to obtain by induction $\text{REV}(F_1 \times \cdots \times F_k) \leq (3k-2)\text{BREV}(F_1 \times \cdots \times F_k)$, where the induction step uses the fact that the bundled revenue from a subset of the items is at most the bundled revenue from all of them. Again, when k is not a power of 2 we can pad to the next power of 2 with items that have value identically zero, which at most doubles k. ∎

However, for identically distributed items the bundling auction does much better, and in fact we can prove a tighter result, with $\log k$ instead of k : Theorem 4, stated in the Introduction.

Proof of Theorem 4. For $k \geq 2$ a power of two we apply Theorem 27 inductively to obtain: $\mathrm{Rev}(F^{\times k}) \leq 2\,\mathrm{BRev}(F^{\times(k/2)}) + 4\,\mathrm{BRev}(F^{\times(k/4)}) + \ldots + (k/2)\,\mathrm{BRev}(F^{\times 2}) + k\,\mathrm{BRev}(F) + k\,\mathrm{Rev}(F)$. Each of the $\log_2 k + 1$ terms in this sum is of the form $(k/m)\,\mathrm{BRev}(F^{\times m}) = (k/m)\,\mathrm{Rev}(F^{*m})$ and is thus bounded from above, using Lemma 18 applied to the distribution F^{*m}, by $4\,\mathrm{Rev}(F^{*k}) = 4\,\mathrm{BRev}(F^{\times k})$. Altogether we have $\mathrm{Rev}(F^{\times k}) \leq 4(\log_2 k + 1)\,\mathrm{BRev}(F^{\times k})$.

When $2^{m-1} < k < 2^m$ we have $\mathrm{Rev}(F^{\times k}) \leq \mathrm{Rev}(F^{\times 2^m}) \leq 4(\log_2 2^m + 1)\,\mathrm{BRev}(F^{\times 2^m}) \leq 4(\log_2 k + 2) \cdot 2 \cdot 1.3 \cdot \mathrm{BRev}(F^{\times 2^{m-1}}) \leq 4(\log_2 k + 2) \cdot 2 \cdot 1.3 \cdot \mathrm{BRev}(F^{\times k})$ (we have used Lemma 13 with $F_1 = F_2 = F^{*2^{m-1}}$ and $1 + w \leq 1.3$). $\blacksquare$

Acknowledgments

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Appendix 1: A Tighter Bound for Two Items

In this appendix we prove Theorem 2 which is stated in the Introduction (see also Section 4.2), which says that selling two i.i.d. items separately yields at least $e/(e + 1) = 0.73...$ of the optimal revenue.

Proof of Theorem 2. Let X and Y be i.i.d.-F. Without loss of generality we will restrict ourselves to symmetric mechanisms, i.e., b such that $b(x, y) = b(y, x)$ (indeed: if $b(x, y)$ is optimal, then so are $\hat{b}(x, y) := b(y, x)$ and their average $\bar{b}(x, y) := (b(x, y) + \hat{b}(x, y))/2$, which is symmetric). Put $R := \text{REV}(X) = \text{REV}(Y) = \sup_{t \geq 0} t \cdot \bar{F}(t)$, where $\bar{F}(t) := \mathbb{P}(X \geq t) = \lim_{u \to t^+}(1 - F(u))$.

Define $\varphi(x) := q_1(x, x) = q_2(x, x)(= b_x(x, x))$ and $\Phi(x) := b(x, x)/2$, then $\Phi(x) = \int_0^x \varphi(t)\,dt$ (recall that $b(0, 0) = 0$ by IR and NPT).

We will consider the two regions $X \geq Y$ and $Y \geq X$ separately; by symmetry, the expected revenue in the two regions is the same, and so it suffices to show that

$$\mathbb{E}(s(X, Y)\mathbb{1}_{X \geq Y}) \leq \left(1 + \frac{1}{e}\right) R.$$

As in Lemma 21, fix y and define a mechanism $(\tilde{q}^y, \tilde{s}^y)$ for X by $\tilde{q}^y(x) := q_1(x, y)$ and $\tilde{s}^y(x) := s(x, y) - yq_2(x, y)$ for every x; note that the buyer's payoff is $\tilde{b}^y(x) = b(x, y)$. The mechanism $(\tilde{q}^y, \tilde{s}^y)$ is IC and IR for X, since (q, s) was IC and IR for (X, Y). Now apply the mechanism $(\tilde{q}^y, \tilde{s}^y)$ to the random variable X conditional on $[X \geq y]$, which we write X_y for short. Since $X_y \geq y$ we have $\tilde{b}^y(X_y) = b(X_y, y) \geq b(y, y) = 2\Phi(y)$ and $\tilde{q}^y(X_y) = q_1(X_y, y) \geq q_1(y, y) = \varphi(y)$, and so applying Lemma 32 below to X_y yields

$$\mathbb{E}(\tilde{s}^y(X)|X \geq y) = \mathbb{E}(\tilde{s}^y(X_y)) \leq (1 - \varphi(y))\text{REV}(X_y) + y\varphi(y) - 2\Phi(y). \qquad (4)$$

Since $\mathbb{P}(X_y \geq t) = \mathbb{P}(X \geq t)/\mathbb{P}(X \geq y) = \bar{F}(t)/\bar{F}(y)$ for all $t \geq y$, we get

$$\text{REV}(X_y) = \sup_{z \geq 0} z \cdot \mathbb{P}(X_y \geq z) = \sup_{z \geq y} z \cdot \frac{\bar{F}(z)}{\bar{F}(y)} \leq \sup_{z \geq 0} z \cdot \frac{\bar{F}(z)}{\bar{F}(y)} = \frac{R}{\bar{F}(y)}.$$

Multiply (4) by $\mathbb{P}(X \geq y) = \bar{F}(y)$ to get

$$\mathbb{E}(\tilde{s}^y(X)\mathbb{1}_{X \geq y}) \leq (1 - \varphi(y))R + (y\varphi(y) - 2\Phi(y))\bar{F}(y),$$

and then take expectation over $Y = y$:

$$\mathbb{E}(\tilde{s}^Y(X)\mathbb{1}_{X \geq Y}) \leq R\mathbb{E}(1 - \varphi(Y)) + \mathbb{E}((Y\varphi(Y) - 2\Phi(Y)\mathbb{1}_{X \geq Y}).$$

Since $s(x, y) = \tilde{s}^y(x) + yq_2(x, y) \leq \tilde{s}^y(x) + yq_2(x, x) = \tilde{s}^y(x) + y\varphi(x)$ (use $y \geq 0$ and the monotonicity of $q_2(x, y)$ in y), we finally get

$$
\begin{aligned}
\mathbb{E}(s(X, Y)\mathbb{1}_{X \geq Y}) &= \mathbb{E}(\tilde{s}^Y(X)\mathbb{1}_{X \geq Y}) + \mathbb{E}(Y\phi(X)\mathbb{1}_{X \geq Y}) \\
&\leq R - R\mathbb{E}(\varphi(Y)) + \mathbb{E}(W\mathbb{1}_{X \geq Y}),
\end{aligned}
\tag{5}
$$

where
$$W := Y\varphi(X) + Y\varphi(Y) - 2\Phi(Y).$$

The expression (5) is affine in φ (recall that $\Phi(x) = \int_0^x \varphi(s)\,\mathrm{d}s$), and φ is a nondecreasing function with values in $[0, 1]$. The set of such functions φ is the closed convex hull of the functions $\varphi(x) = \mathbb{1}_{[t, \infty)}(x)$ for $t \geq 0$. Therefore, in order to bound (5), it suffices to consider these extreme functions.

When $\varphi(x) = \mathbb{1}_{[t, \infty)}(x)$ we get $\Phi(x) = \max\{x - t, 0\}$ and

$$
W = \begin{cases}
2Y - 2(Y - t) = 2t, & \text{if } X \geq Y \geq t, \\
Y - 0 = Y, & \text{if } X \geq t > Y, \\
0, & \text{if } t > X \geq Y.
\end{cases}
$$

Thus

$$
\begin{aligned}
\mathbb{E}(\mathbb{1}_{X \geq Y}) &= 2t\mathbb{P}(X \geq Y \geq t) + \mathbb{E}(Y\mathbb{1}_{X \geq t > Y}) \\
&= t\mathbb{P}(X \geq t)\mathbb{P}(Y \geq t) + \mathbb{P}(X \geq t)\mathbb{E}(Y\mathbb{1}_{t > Y}) \\
&= \mathbb{P}(X \geq t)\mathbb{E}(\min\{Y, t\}) = \bar{F}(t)\mathbb{E}(\min\{Y, t\})
\end{aligned}
$$

(we have used the fact that X, Y are i.i.d., and $\min\{Y, t\} = t\mathbb{1}_{Y \geq t} + Y\mathbb{1}_{t > Y}$). Together

with $\mathbb{E}(\varphi(Y)) = \mathbb{P}(Y \geq t) = \bar{F}(t)$, (5) becomes

$$R - R\bar{F}(t) + \bar{F}(t)\mathbb{E}(\min\{Y, t\}) = R + \bar{F}(t) \left(\mathbb{E}(\min\{Y, t\}) - R \right). \tag{6}$$

Let $r(t)$ denote the expression in (6). When $t \leq R$ we have $\mathbb{E}(\min\{Y, t\}) \leq R$, and so $r(t) \leq R$. When $t > R$ we have

$$
\begin{aligned}
\mathbb{E}(\min\{Y, t\}) &= \int_0^\infty \mathbb{P}(\min\{Y, t\} \geq u) \, \mathrm{d}u = \int_0^t \mathbb{P}(Y \geq u) \, \mathrm{d}u \\
&= \int_0^t \bar{F}(u) \, \mathrm{d}u \leq \int_0^R 1 \, \mathrm{d}u + \int_R^t \frac{R}{u} \, \mathrm{d}u = R + R\log\left(\frac{t}{R}\right),
\end{aligned}
$$

where we have used $\bar{F}(u) \leq \min\{R/u, 1\}$ (which follows from $R = \sup_{u \geq 0} u\bar{F}(u)$). Therefore in this case

$$r(t) \leq R + \frac{R}{t}\left(R + R\log\left(\frac{t}{R}\right) - R \right) = R\left(1 + \frac{\log \tau}{\tau} \right),$$

where $\tau := t/R > 1$. Since $\max_{\tau \geq 1} \tau^{-1} \log \tau = 1/e$ (attained at $\tau = e$), it follows that $r(t) \leq (1 + 1/e)R$ for all $t > R$, and thus also for all $t \geq 0$. Recalling (5) and (6) therefore yields $\mathbb{E}(s(X, Y)\mathbb{1}_{X \geq Y}) \leq (1 + 1/e)R$, and so $\mathbb{E}(s(X, Y)) \leq 2(1 + 1/e)R = (1 + 1/e) \cdot \text{SREV}(F \times F)$. ∎

Lemma 32 *Let X be a one-dimensional random variable whose support is included in $[x_0, \infty)$ for some $x_0 \geq 0$, and let $b_0 \geq 0$ and $0 \leq q_0 \leq 1$ be given. Then the maximal revenue the seller can obtain from X subject to guaranteeing to the buyer a payoff of at least b_0 and a probability of getting the item of at least q_0 (i.e., $b(x) \geq b_0$ and $q(x) \geq q_0$ for all $x \geq x_0$) is*

$$(1 - q_0)\text{REV}(X) + q_0 x_0 - b_0.$$

Proof. A mechanism satisfying these constraints is plainly seen to correspond to a one-dimensional convex function b with $q_0 \leq b'(x) \leq 1$ and $b(x_0) = b_0$. When $q_0 < 1$ (if $q_0 = 1$ the result is immediate) put $\tilde{b}(x) := (b(x) - q_0(x - x_0) - b_0)/(1 - q_0)$, then $\tilde{b}$ is a convex function with $0 \leq \tilde{b}(x) \leq 1$ and $\tilde{b}(x_0) = 0$, and so $\text{REV}(F) \geq R(\tilde{b}; F) = (R(b; F) - q_0 x_0 + b_0)/(1 - q_0)$. ∎

Appendix 2: When Bundling Is Optimal

In this appendix we prove Theorem 28 which is stated in Section 4.3: for two i.i.d. items, if the one-item value distribution satisfies condition (2), then bundling is optimal.

Proof of Theorem 28. Let b correspond to a two-dimensional IC and IR mechanism; assume without loss of generality that b is symmetric, i.e., $b(x,y) = b(y,x)$ (cf. the proof of Theorem 2 in Appendix 1 above). Thus $\mathbb{E}(s) = \mathbb{E}(xb_x + yb_y - b) = \mathbb{E}(2xb_x - b)$, and so

$$R(b, F \times F) = \int_a^\infty \int_a^\infty (2xb_x(x,y) - b(x,y))\, f(x)\, \mathrm{d}x\, f(y)\, \mathrm{d}y = \sup_{M>a} r_M(b),$$

where

$$r_M(b) := \int_a^M \int_a^M (2xb_x(x,y) - b(x,y))\, f(x)\, \mathrm{d}x\, f(y)\, \mathrm{d}y. \tag{7}$$

For each y we integrate by parts the $2xb_x(x,y)f(x)$ term:

$$
\begin{aligned}
\int_a^M 2b_x(x,y)xf(x)\, \mathrm{d}x &= [2b(x,y)xf(x)]_a^M - \int_a^M 2b(x,y)\,(f(x) + xf'(x))\, \mathrm{d}x \\
&= 2b(M,y)Mf(M) - 2b(a,y)af(a) \\
&\quad - \int_a^M 2b(x,y)\,(f(x) + xf'(x))\, \mathrm{d}x.
\end{aligned}
$$

Substituting this in (7) yields

$$
\begin{aligned}
r_M(b) &= 2Mf(M)\int_a^M b(M,y)f(y)\, \mathrm{d}y - 2af(a)\int_a^M b(a,y)f(y)\, \mathrm{d}y \\
&\quad + 2\int_a^M \int_a^M b(x,y)\left(-\frac{3}{2}f(x) - xf'(x)\right) f(y)\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}
$$

Define $\tilde{b}(x,y) := b(x+y-a, a) = b(a, x+y-a)$ for every (x,y) with $x, y \geq a$, then $\tilde{b}$ is a convex function on $[a,\infty) \times [a,\infty)$ with $0 \leq \tilde{b}_x, \tilde{b}_y \leq 1$, and so it corresponds to a two-dimensional IC & IR mechanism. Moreover, since b is convex we have for every $x, y \geq a$

$$b(x,y) \leq \lambda\, b(x+y-a, a) + (1-\lambda)\, b(a, x+y-a) = \tilde{b}(x,y).,$$

where $\lambda = (x-a)/(x+y-2a)$. Therefore replacing b with $\tilde{b}$ can only increase r_M, i.e., $r_M(b) \leq r_M(\tilde{b})$; indeed, in the first and third terms the coefficients of $b(x,y)$ are nonnegative (recall our assumption (2)); and in the second term, $b(a,y) = \tilde{b}(a,y)$. Hence $R(b, F \times F) = \sup_M r_M(b) \leq \sup_M r_M(\tilde{b}) = R(\tilde{b}, F \times F)$.

It only remains to observe that $\tilde{b}(x,y)$ is a function of $x+y$, and so it corresponds to a bundled mechanism. Formally, put $\beta(t) := \tilde{b}(t-a, a)$, then $\beta : [2a, \infty) \to \mathbb{R}_+$ is a one-dimensional convex function with $0 \leq \beta'(t) \leq 1$. For all $x, y \geq a$ with $x+y = t$ we have $\tilde{b}(x,y) = \beta(t)$ and $x\tilde{b}_x(x,y) + y\tilde{b}_y(x,y) - \tilde{b}(x,y) = t\beta'(t) - \beta(t)$, and so $R(\tilde{b}, F \times F) = R(\beta, F * F) \leq \mathrm{REV}(F * F) = \mathrm{BREV}(F \times F)$. $\blacksquare$

Appendix 3: Multiple Buyers

We prove here Theorem 30 (see Section 4.4): selling separately two independent items to n buyers yields at least one half of the optimal revenue.

Let $X^{\max} = \max_{1 \leq j \leq n} X^j$ and $Y^{\max} = \max_{1 \leq j \leq n} Y^j$ be the highest values for the two items. Define $\text{VAL}^{[n]}(X) = \mathbb{E}(X^{\max})$ and $\text{VAL}^{[n]}(Y) = \mathbb{E}(Y^{\max})$ (these are the values obtained by always allocating each item to the highest-value buyer).

We proceed along the same lines as the proof of Theorem 20 in Section 4.1. In the lemmas below, X and Y are independent n-dimensional random variables, Z is a $2n$-dimensional random variable (for instance, (X, Y)), and S and T are sets of values of Z.

Lemma 33 $\text{REV}^{[n]}(X, Y) \leq \text{VAL}^{[n]}(Y) + \text{REV}^{[n]}(X)$.

Proof. The proof is similar to the case of a single buyer (Lemma 21), except that the amount of money we need to return to compensate for the y's is exactly $\text{VAL}^{[n]}(Y)$ since if each buyer j gets q_2^j units (or probability) of the y item then we have $\sum_j q_2^j \leq 1$ and thus $\sum_j q_2^j y^j \leq y^{\max}$. We emphasize that if the original mechanism for (X, Y) was a dominant-strategy mechanism, so will be the conditional-on-y mechanism for X; and the same for Bayes-Nash mechanisms. ∎

Lemma 34 $\text{REV}^{[n]}(\mathbb{1}_{Z \in S} \cdot Z) \leq \text{REV}^{[n]}(Z)$.

The proof is identical to the case $n = 1$ (Lemma 22).

Lemma 35 *If $S \cup T$ contains the support of Z then $\text{REV}^{[n]}(\mathbb{1}_{Z \in S} \cdot Z) + \text{REV}^{[n]}(\mathbb{1}_{Z \in T} \cdot Z) \geq \text{REV}^{[n]}(Z)$.*

The proof is identical to the case $n = 1$ (Lemma 23).

Lemma 36 $\text{REV}^{[n]}(\mathbb{1}_{(X,Y) \in S} \cdot (X, Y)) \leq \text{VAL}^{[n]}(\mathbb{1}_{(X,Y) \in S} \cdot Y) + \text{REV}^{[n]}(X)$.

The proof is identical to the case $n = 1$ (Lemma 24).

The set according to which we will cut our space will be the following one:

Lemma 37 $\text{VAL}^{[n]}(\mathbb{1}_{Y^{\max} \leq X^{\max}} \cdot Y) \leq \text{REV}^{[n]}(X)$.

Proof. Here is a possible mechanism for X: choose a random $y = (y^1, \ldots, y^n)$ according to Y and offer $y^{\max}$ as the take-it-or-leave-it price to the buyers sequentially (the first one in lexicographic order to accept gets it). The expected revenue of this mechanism is exactly $\text{VAL}^{[n]}(\mathbb{1}_{Y^{\max} \leq X^{\max}} \cdot Y)$ so this is a lower bound on $\text{REV}^{[n]}(X)$. ∎

We can now complete our proof.

Proof of Theorem 30. Using lemma 35 we cut the space into two parts, $\text{REV}^{[n]}(X, Y) \leq \text{REV}^{[n]}(\mathbb{1}_{Y^{\max} \leq X^{\max}} \cdot (X, Y)) + \text{REV}^{[n]}(\mathbb{1}_{X^{\max} \leq Y^{\max}} \cdot (X, Y))$, and bound the revenue in each one. By Lemma 36, the revenue on the first part is bounded by $\text{VAL}^{[n]}(\mathbb{1}_{Y^{\max} \leq X^{max}} \cdot Y) + \text{REV}^{[n]}(X)$ which using Lemma 37 is bounded from above by $2\,\text{REV}^{[n]}(X)$. The revenue in the second part is bounded similarly by $2\,\text{REV}^{[n]}(Y)$. ∎

Remark. The problem when trying to extend this method to more than 2 items is that when Y is a *set* of items we do not have a "Smaller Value" counterpart to Lemma 37 (recall also Lemma 25).

Appendix 4: Many I.I.D. Items

It turns out that when the items are independent and identically distributed, and their number k tends to infinity, then the bundling revenue approaches the optimal revenue. Even more, essentially all the buyer's surplus can be extracted by the bundling auction. The logic is quite simple: the law of large numbers tells us that there is almost no uncertainty about the sum of many i.i.d. random variables, and so the seller essentially knows this sum and may ask for it as the bundle price. For completeness we state this result and provide a short proof, which also covers the case where the expectation $\mathbb{E}(F)$ is infinite.

Theorem 38 (Armstrong [1999], Bakos and Brynjolfsson [1999]) *For every one-dimensional distribution F,*

$$\lim_{k \to \infty} \frac{\text{BREV}(F^{\times k})}{k} = \lim_{k \to \infty} \frac{\text{REV}(F^{\times k})}{k} = \mathbb{E}(F).$$

Proof. We always have $\text{BREV}(F^{\times k}) \leq \text{REV}(F^{\times k}) \leq k\,\mathbb{E}(F)$ (the second inequality follows from NPT). Let us first assume that our distribution F has finite expectation and finite variance. In this case if we charge price $(1 - \epsilon)k\,\mathbb{E}(F)$ for the bundle then by Chebyshev's inequality the probability that the bundle will not be bought is at most $\mathbb{V}\text{ar}(F)/(\epsilon^2 \mathbb{E}(F)\sqrt{k})$, where $\mathbb{V}\text{ar}(F)$ is the variance of F, and this goes to zero as k increases.

If the expectation or variance are infinite, then just consider the truncated distribution where values above a certain M are replaced by M, which certainly has finite expectation and variance. We can choose the finite M so as to get the expectation of the truncated

distribution as close as we desire to the original one (including as high as we desire, if the original distribution has infinite expectation). ∎

Despite the apparent strength of this theorem, it does not provide any approximation guarantees for any fixed value of k. In particular, for $k = 2$ we have already seen an example where the bundling auction gets only $2/3$ of the revenue of selling the items separately (Example 17), and for every large enough k we have seen an example where the bundling auction's revenue is less than 57% than that of selling the items separately (Example 19); of course, as a fraction of the optimal revenue this can only be smaller. The results of Section 4 provide approximation bounds for each fixed k.

Appendix 5: Summary of Approximation Results

The table below summarizes the approximation results of this paper. The four main results are in bold, and the arrows $[\rightarrow]$ and $[\leftarrow]$ indicate that the result in that box is a special case of the one in the next box to the right or left, respectively.

$\mathcal{F} =$	$F_1 \times F_2$	$F \times F$	$F_1 \times \cdots \times F_k$	$F^{\times k}$
$\forall \mathcal{F} \quad \dfrac{\text{SREV}(\mathcal{F})}{\text{REV}(\mathcal{F})} \geq$	$\mathbf{\dfrac{1}{2}}$ [Th 1]	$\mathbf{\dfrac{e}{e+1} \approx 0.73}$ [Th 2]	$\mathbf{\Omega\left(\dfrac{1}{\log^2 k}\right)}$ [Th 3]	$\Omega\left(\dfrac{1}{\log^2 k}\right)$ $[\leftarrow]$
$\exists \mathcal{F} \quad \dfrac{\text{SREV}(\mathcal{F})}{\text{REV}(\mathcal{F})} \leq$	$\dfrac{1}{1+w} \approx 0.78$ $[\rightarrow]$	$\dfrac{1}{1+w} \approx 0.78$ [Co 29]	$O\left(\dfrac{1}{\log k}\right)$ $[\rightarrow]$	$O\left(\dfrac{1}{\log k}\right)$ [Le 8]
$\forall \mathcal{F} \quad \dfrac{\text{BREV}(\mathcal{F})}{\text{REV}(\mathcal{F})} \geq$	$\dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4}$ [Th 1 + Le 14]	$\dfrac{2}{3} \cdot \dfrac{e}{e+1}$ [Th 2 + Le 16]	$\Omega\left(\dfrac{1}{k}\right)$ [Le 31]	$\mathbf{\Omega\left(\dfrac{1}{\log k}\right)}$ [Th 4]
$\exists \mathcal{F} \quad \dfrac{\text{BREV}(\mathcal{F})}{\text{REV}(\mathcal{F})} \leq$	$\dfrac{1}{2} + \varepsilon$ [Ex 15]	$\dfrac{2}{3}$ [Ex 17]	$\dfrac{1}{k} + \varepsilon$ [Ex 15]	$\approx 0.57 + o(1)$ [Ex 19]