COMPETITION FOR A MAJORITY

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Abstract. We define the class of two-player zero-sum games with payoffs having mild discontinuities, which in applications typically stem from how ties are resolved. For games in this class we establish sufficient conditions for existence of a value of the game and minimax or Nash equilibrium strategies for the players. We prove first that if all discontinuities favor one player then a value exists and that player has a minimax strategy. Then we establish that a general property called payoff approachability implies that the value results from an equilibrium. We prove further that this property implies that every modification of the discontinuities yields the same value; in particular, for every modification, epsilon-equilibria exist.

We apply these results to models of elections in which two candidates propose policies and a candidate wins election if a weighted majority of voters prefer his policy. We provide tie-breaking rules and assumptions on voters' preferences sufficient to imply payoff approachability, hence existence of equilibria, and each other tie-breaking rule yields the same value and has epsilon-equilibria. These conclusions are also derived for the special case of Colonel Blotto games in which each candidate allocates his available resources among several constituencies and the assumption on voters' preferences is that a candidate gets votes from those constituencies allocated more resources than his opponent offers. Moreover, for the case of simple-majority rule we prove existence of an equilibrium that has zero probability of ties.

1. Introduction

Following Downs [5], studies of elections often use models in which two candidates compete for votes via the policies they propose. Each candidate's sole objective is to obtain a majority of votes, where each voter will cast her vote for the candidate whose policy she prefers. Because only one candidate can win a majority of votes, such models induce zero-sum games between the candidates. However, because outcomes depend on how voters resolve ties between candidates' policies, the candidates' payoffs are discontinuous functions of their policies. A major hindrance to studies of such models has been a lack of sufficient conditions for existence of a value of the game, and existence of minimax or equilibrium strategies for

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the candidates.[1] Here we establish such conditions for a large class of games, and then apply them to models in which a candidate must win a weighted- or simple-majority of votes to win election.

Section 2 defines the class of two-player zero-sum games with payoffs with mild discontinuities, as specified by Assumption 2.1, and establishes two general existence theorems.[2] Section 3 and 4 apply these theorems to voting games in which the winner is determined by majority rule. These games typically have mild discontinuities at strategy profiles where voters indifferent between the policies proposed by the candidates are pivotal in determining the outcome of the election.[3]

The general results in Section 2 consider two cases. First we show that if discontinuities invariably favor one player then a value exists and that player has a minimax strategy that assures him at least the value for any strategy of his opponent. This case arises in applications when one candidate wins all ties among voters. Next we identify a general property called payoff approachability that implies the condition called 'better-reply security' by Reny [14]. This condition implies that the players have equilibrium strategies that yield the value. Moreover, we show that in games satisfying payoff approachability this remains the value for every modification of payoffs at discontinuities.

In the applications to models of elections, therefore, we show that the value exists and is independent of tie-breaking rules by verifying that payoff approachability is satisfied by a particular rule that is convenient for the verification. In several cases this is not the standard tie-breaking rule that resolves each tie by tossing a fair coin. Nevertheless, this method suffices to obtain the general result—for every tie-breaking rule, the value exists, and therefore, for every $\varepsilon > 0$, each candidate has a strategy that assures a payoff within ε of the value, and an ε-equilibrium exists.

Section 3 applies these results to models of elections. If one candidate wins all ties then the value exists and that candidate has a minimax strategy. For another tie-breaking rule that is symmetric, we identify assumptions on voters' preferences sufficient to imply payoff

[1]In a two-player zero-sum game, minimax and maximin payoffs are defined in terms of infimum and supremum operators applied to a player's payoffs, and when these two payoffs are the same the game is said to have a (unique) value. If equilibrium strategies exist then the value is player 1's equilibrium payoff. A maximin strategy for player 1 is a strategy that assures him at least the value, and a minimax strategy for player 2 is one that holds player 1's payoff down to the value. More generally, whenever the value exists each player has an ε-optimal strategy for every $\varepsilon > 0$, and a profile of these strategies is an ε-equilibrium.

[2]Other than Dasgupta and Maskin [3] and Parthasarathy [12], who focus on discontinuities along well-behaved curves with zero measure, the prior literature has not restricted the set of strategies where payoffs are discontinuous and therefore must allow for pervasive discontinuities.

[3]Although other games of economic interest, such as auctions and Bertrand-style competition between duopolists, have payoffs with mild discontinuities, we do not address them here because typically the payoffs are not zero-sum.

approachability and thus better-reply security, ensuring that the candidates have equilibrium strategies that yield the value, and any other tie-breaking rule yields the same value. These assumptions are satisfied by generic preferences if the number of voters does not exceed one plus the dimension of the space of policies.

Section 4 obtains stronger results for the special case of majority-rule 'Colonel Blotto' games, which are often used to model election campaigns and lobbying.[4] In these games a candidate's policy allocates his available resources among several constituencies, each of which votes for the candidate offering more. Initially we consider versions in which the winner is the candidate obtaining a weighted majority of votes. As in Section 3, such a game has a value when one of the candidates wins all ties, and this candidate has a minimax strategy. To address other cases we provide a tie-breaking rule that implies payoff approachability. Applying our general results to games with this tie-breaking rule shows that the candidates have equilibrium strategies that yield the value, and games with any other tie-breaking rule inherit this value. Next we strengthen this result for the special case that the winner is the candidate obtaining a simple majority of votes. We show that the value results from an equilibrium that has zero probability of ties.

2. General Existence Theorems

We study two-player zero-sum games with the following general features. In each game, the two players are labeled by $i = 1$ and 2. Given a player i, let j be the other player. For each player i, his set X_i of pure strategies is a compact metric space and his set Σ_i of mixed strategies is the set of Borel probability measures on X_i endowed with the weak-$*$ topology.[5] Since X_i is a compact metric space, so is Σ_i. Let $\delta_{x_i} \in \Sigma_i$ denote the point mass on $x_i \in X_i$.

Let $X = X_1 \times X_2$ and $\Sigma = \Sigma_1 \times \Sigma_2$ denote the sets of profiles of players' pure and mixed strategies. Player i's payoff function from pure strategies is $\pi_i : X \to [-1, +1]$, where $\pi_1 + \pi_2 = 0$, and it is extended to the expected payoff from mixed strategies via $\pi_i(\sigma_1, \sigma_2) = E_{\sigma_1,\sigma_2}[\pi_i(x_1, x_2)]$ for each profile $(\sigma_1, \sigma_2) \in \Sigma$.[6] Recall that when $(\sigma_1^n, \sigma_2^n) \to (\sigma_1, \sigma_2)$, the

[4]The moniker *Colonel Blotto* stems from the paper by Gross and Wagner [9], but studies of such games date to work in 1921 by Borel; cf. Borel [2]. Most analyses of such games assume that each player's objective is to maximize the number of votes won, as in Roberson [15], rather than winning a majority of votes as assumed here.

[5]It is sufficient that the strategy sets be compact Hausdorff spaces. Metrizability is assumed to simplify the exposition of the proofs by allowing us to use sequences rather than nets. See Remark 2.10(5) for more on this.

[6]Of course, the restriction to $[-1, +1]$ as the range is without loss. All that we require is that each π_i is bounded. But even this is unnecessary, since in the unbounded case the transformation $\frac{e^{\pi_i}-1}{1+e^{\pi_i}}$ yields a bounded payoff function with range $[-1, +1]$, with the convention that if the value is ± 1 after the transformation then it is $\pm\infty$ in the original version.

corresponding product measure $\sigma_1^n \otimes \sigma_2^n$ converges to $\sigma_1 \otimes \sigma_2$. So $E_{\sigma_1,\sigma_2}[f(x_1, x_2)]$ is upper semi-continuous (u.s.c.) if $f : X \to \mathbb{R}$ is u.s.c., and l.s.c. if f is l.s.c.

Let $D \subset X$ be the subset consisting of those pure-strategy profiles at which π_1 and π_2 are discontinuous. We focus on games for which D is not empty, although we do not assume it explicitly. For each player i and his pure strategy $x_i \in X_i$, let $D(x_i) \subset X_j$ be the cross-section of D at x_i, i.e. the set of x_j such that $(x_i, x_j) \in D$. Say that a pure strategy $x_i \in X_i$ of player i is a *point of continuity* against the other player j's mixed strategy $\sigma_j \in \Sigma_j$ if σ_j assigns zero probability to the cross section $D(x_i)$. At such a pair of strategies, player i's expected payoff $\pi_i(x_i, \sigma_j)$ is independent of how payoffs are determined at profiles in $D(x_i)$. The phrase "point of continuity" is justified by Lemma 2.3 below. The class of games with *mild discontinuities* consists of those that satisfy the following assumption.

Assumption 2.1 (Mild Discontinuities)**.**

(1) The set D of pure-strategy profiles at which payoffs are discontinuous is closed.[7]
(2) For each player j the set $\{\, \sigma_j \in \Sigma_j \mid \sigma_j(D(x_i)) = 0 \ \forall x_i \in X_i \,\}$ is dense in Σ_j.
(3) For each mixed strategy σ_j of a player j the set $\{\, x_i \in X_i \mid \sigma_j(D(x_i)) = 0 \,\}$ is dense in X_i.

Conditions (1) and (3) of Assumption 2.1 are relatively easy to check, as we show in the examples studied later. The following sufficient condition for Assumption 2.1(2), which is satisfied in many typical games, is also readily verifiable.

Lemma 2.2. *Suppose X_j is (homeomorphic to) the closure of an open subset of a Euclidean space. Condition (2) of Assumption 2.1 holds if for each $x_i \in X_i$, $D(x_i)$ has Lebesgue measure zero in X_j.*

Proof. For each x_j and each integer n, let $\mu^n_{x_j}$ be the uniform distribution over the ball of radius $1/n$ around x_j in X_j. Then $\mu^n_{x_j}$ is absolutely continuous with respect to the Lebesgue measure on X_j. Hence every x_i is a point a continuity against $\mu^n_{x_j}$, since $D(x_i)$ has Lebesgue measure zero in X_j for each x_i. Convex combinations of the $\mu^n_{x_j}$'s are also absolutely continuous, so each x_i is a point of continuity against every σ^n_j in the convex hull Σ^n_j of the set of $\{\, \mu^n_{x_j} \,\}_{x_j \in X_j}$. Let Σ^*_j be the union over n of Σ^n_j and note that $\Sigma^*_j \subset \{\, \sigma_j \in \Sigma_j \mid \sigma_j(D(x_i)) = 0 \ \forall x_i \in X_i \,\}$. Observe that for each $\mu_j \in \Sigma_j$ with finite support there exists a sequence $\{\, \sigma^k_j \,\} \subset \Sigma^*_j$ with $\sigma^k_j \to \mu_j$. It follows that $\Sigma^f_j \subset \mathrm{Cl}(\Sigma^*_j)$, where Σ^f_j is the set of mixed strategies with finite support and $\mathrm{Cl}(\cdot)$ denotes closure. As $\mathrm{Cl}(\Sigma^f_j) = \Sigma_j$, we have $\mathrm{Cl}(\Sigma^*_j) = \Sigma_j$, and the result follows. □

[7]This assumption can be eliminated by using throughout the closures of D and each $D(x_i)$.

We consider a basic game and the corresponding family of games that differ only in their payoffs at profiles in D, which in applications correspond to the possible resolutions of ties. Represent the basic game as $G(\pi)$ where $\pi_1 = \pi$ and $\pi_2 = -\pi$. Variants of this basic game are parameterized by the set Π of payoff functions $\pi' : X \to [-1, 1]$ such that $\pi'(x) = \pi(x)$ for all $x \notin D$. Thus the family of games is $\{G(\pi') \mid \pi' \in \Pi\}$.

Lemma 2.3. *If $\sigma_j(D(x_i)) = 0$ then player i's payoff function π'_i is continuous at $(\delta_{x_i}, \sigma_j) \in \Sigma_i \times \Sigma_j$.*

Proof. Let π_i^+ and π_i^- be u.s.c. and l.s.c. payoff functions in Π defined as follows: both functions agree with π on $X \setminus D$ but $\pi_i^+(x) = 1$, and $\pi_i^-(x) = -1$ for all $x \in D$. That π_i^+ and π_i^- are u.s.c. and l.s.c., respectively, follows from the assumption that D is closed. Let (σ_i^n, σ_j^n) be a sequence converging to (δ_{x_i}, σ_j). By the properties of the weak-$*$ topology:

$$\pi_i^+(\delta_{x_i}, \sigma_j) \geqslant \limsup_n \pi_i^+(\sigma_i^n, \sigma_j^n) \geqslant \limsup_n \pi'_i(\sigma_i^n, \sigma_j^n)$$

and

$$\liminf_n \pi'_i(\sigma_i^n, \sigma_j^n) \geqslant \liminf_n \pi_i^-(\sigma_i^n, \sigma_j^n) \geqslant \pi_i^-(\delta_{x_i}, \sigma_j)\,.$$

Because $\sigma_j(D(x_i)) = 0$, π_i^+ and π_i^- coincide with π'_i at (δ_{x_i}, σ_j), so the above inequalities are equalities, and the lemma follows. □

For each payoff function $\pi' \in \Pi$, player 1's maximin and minimax values are

$$\underline{v}(\pi') = \sup_{\sigma_1 \in \Sigma_1} \inf_{x_2 \in X_2} \pi'(\sigma_1, x_2) \qquad \text{and} \qquad \overline{v}(\pi') = \inf_{\sigma_2 \in \Sigma_2} \sup_{x_1 \in X_1} \pi'(x_1, \sigma_2)\,,$$

where necessarily $\underline{v}(\pi') \leqslant \overline{v}(\pi')$. If $\underline{v}(\pi') = \overline{v}(\pi') \equiv v^*(\pi')$ then $v^*(\pi')$ is called the *value* of the game $G(\pi')$ to player 1.

2.1. **The Case That One Player Wins All Ties.** Of particular interest are the two games $G(\pi^+)$ and $G(\pi^-)$, as in the proof of Lemma 2.3. In the game $G(\pi^+)$, $\pi_1^+(x) = +1$, and in $G(\pi^-)$, $\pi_1^-(x) = -1$, for each profile $x \in D$. In applications these correspond to the two cases that one player wins all ties. The following theorem establishes existence of values for these games.

Theorem 2.4. *If $\pi' = \pi^+$ or $\pi' = \pi^-$ then the value $v^*(\pi')$ exists. Moreover, in the game $G(\pi^+)$ player 1 has a minimax strategy, and in the game $G(\pi^-)$ player 2 has a minimax strategy.*

Proof. We prove the theorem only for π^+ since the other case is similar. By Assumption 2.1(2) let $\tilde{\Sigma}_2$ be a dense set of strategies σ_2 such that $\sigma_2(D(x_1)) = 0$ for all $x_1 \in X_1$. We can assume without loss of generality that $\tilde{\Sigma}_2$ is a countable set: indeed, for each positive integer k, take a covering of Σ_2 by a finite number of balls of radius $1/k$, pick a point in each of these balls that belongs to $\tilde{\Sigma}_2$ and then take the countable union (over k) of these finite sets. Let $\tilde{\Sigma}_2^1 \subset \tilde{\Sigma}_2^2 \subset \cdots$ be an increasing sequence of subsets of $\tilde{\Sigma}_2$ such that each $\tilde{\Sigma}_2^n$ is a finite set and $\cup_n \tilde{\Sigma}_2^n = \tilde{\Sigma}_2$. For each n let Σ_2^n be the convex hull of $\tilde{\Sigma}_2^n$. Σ_2^n is a compact convex subset of Σ_2 for each n. Also, for each n and $\sigma_2^n \in \Sigma_2^n$, $\sigma_2^n(D(x_1)) = 0$ for all $x_1 \in X_1$; in particular, π^+ is continuous at each (x_1, σ_2^n) by Lemma 2.3.

Define a perturbed game G^n as follows. The strategy set of player 1 is Σ_1 and the strategy set of player 2 is Σ_2^n. The payoff function is the restriction of π^+ to $\Sigma_1 \times \Sigma_2^n$. The payoffs are clearly continuous and bilinear so the game G^n has a value, say v^n, and each player i has an equilibrium strategy σ_i^n that assures this value.

Take a convergent subsequence of equilibria (σ_1^n, σ_2^n) and associated values v^n converging to, say, (σ_1^*, σ_2^*) and v^* as $n \to \infty$. We show first that $\pi^+(\sigma_1^*, x_2) \geqslant v^*$ for all x_2. Indeed, otherwise there exists some x_2 such that $\pi^+(\sigma_1^*, x_2) < v^*$. In this case, we claim that we can assume without loss of generality that x_2 is a point of continuity against σ_1^*. To prove this claim, start with the given x_2 and first decompose σ_1^* into an average of two strategies σ_1^c and σ_1^d where x_2 is a point of continuity against σ_1^c and $\sigma_1^d(D(x_2)) = 1$. By Lemma 2.3, $\lim_{x_2^k \to x_2} \pi^+(\sigma_1^c, x_2^k) = \pi^+(\sigma_1^c, x_2)$. Moreover, $\pi^+(\sigma_1^d, x_2) = 1 \geqslant \pi^+(\sigma_1^d, x_2')$ for all x_2'. Therefore, for all x_2' sufficiently close to x_2, $\pi^+(\sigma_1^*, x_2') < v^*$. By Assumption 2.1(3) we can now choose a point x_2' close to x_2 such that it is a point of continuity against σ_1^* and also $\pi^+(\sigma_1^*, x_2') < v^*$. Thus, the claim is proved and we can assume that x_2 itself is a point of continuity against σ_1^*.

Because π^+ is continuous at $(\sigma_1^*, \delta_{x_2})$, pick $\varepsilon > 0$ and a neighborhood $U = U_1 \times U_2$ of $(\sigma_1^*, \delta_{x_2})$ such that for all $(\tau_1, \tau_2) \in U$, $\pi^+(\tau_1, \tau_2) < v^* - \varepsilon$. For all large n, $\sigma_1^n \in U_n$, and because $\tilde{\Sigma}_2$ is dense in Σ_2, there exists N and $\sigma_2 \in \tilde{\Sigma}_j^N$ that belongs to U_2 and thus $\pi^+(\sigma^n, \sigma_2) < v^* - \varepsilon$ for all large n. But, as the sequence $\tilde{\Sigma}_j^n$ is increasing, we have that for all $n \geqslant N$, σ_2 belongs to Σ_j^n and thus, $\pi^+(\sigma^n, \sigma_2) \geqslant v^n$, which is impossible as v^n converges to v^* and as we just saw $\pi^+(\sigma^n, \sigma_2) < v^* - \varepsilon$ for all large n. Thus $\pi^+(\sigma_1^*, x_2) \geqslant v^*$ for all x_2, which implies that $v^* \leqslant \underline{v}(\pi^+)$. On the other hand, observe that σ_2^n is a feasible strategy in $G(\pi^+)$ for player 2 that holds player 1's payoff down to v^n. Therefore, $v^n \geqslant \overline{v}(\pi^+)$ for all n, which implies that $v^* \geqslant \overline{v}(\pi^+)$. Putting the two inequalities together shows that the game $G(\pi^+)$ has a value and that this value equals v^*. Moreover the fact that $\pi^+(\sigma_1^*, x_2) \geqslant v^*$ for all $x_2 \in X_2$ implies that σ_1^* is a minimax strategy. □

Remark 2.5. Let $\tilde{\pi}^+$ be defined by $\tilde{\pi}^+(x) = \sup_{x^n \to x} \limsup_{x^n} \pi(x^n)$ where the sup is over all sequences in $X \setminus D$ converging to x, and $\tilde{\pi}^-$ is defined analogously. Let $\overline{\Pi}$ (resp. $\underline{\Pi}$) be the set of u.s.c. (l.s.c.) functions in Π that majorize (minorize) $\tilde{\pi}^+$ (resp. $\tilde{\pi}^-$). The minimax strategy of player 1 (resp. player 2) in π^+ (resp. π^-) that we computed above is a minimax strategy in each of these games in $\overline{\Pi}$ (resp. $\underline{\Pi}$). Also, the value v^* is the value of each game in $\overline{\Pi}$ (similarly, the value of π^- is the value of each game in $\underline{\Pi}$).

2.2. **A Sufficient Condition for Existence of an Equilibrium.** A game that has a value has an ε-equilibrium σ^ε for every $\varepsilon > 0$. Also, if it has a value and player i has a minimax strategy σ_i^*, then for every $\varepsilon > 0$, σ_i^ε can be chosen to be σ_i^*. While Theorem 2.4 shows that two variants of a game have a value, ideally one wants an existence result that does not depend on how ties are resolved. To obtain such a result, we invoke the following sufficient condition.[8]

Definition 2.6 (Payoff Approachability)**.** A payoff function $\tilde{\pi} \in \Pi$ is said to satisfy *payoff approachability* if for each player i, his pure strategy $x_i \in X_i$, and the other's mixed strategy $\sigma_j \in \Sigma_j$,

$$\tilde{\pi}_i(x_i, \sigma_j) \leqslant \sup_{x_i^n \to x_i} \limsup_n \tilde{\pi}_i(x_i^n, \sigma_j)\,,$$

where the supremum is over all sequences $\{x_i^n\} \subset X_i$ converging to x_i for which each pure strategy x_i^n in the sequence is a point of continuity against σ_j.

Payoff-approachability requires that a player's payoff cannot be more than the limit of what he can get from nearby points of continuity against any strategy of his opponent.[9] In the applications to voting games we specify tie-breaking rules and assumptions on voters' preferences sufficient to imply payoff approachability.

Theorem 2.7. *If there exists a payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability then:*

1. *$G(\tilde{\pi})$ has an equilibrium that yields the value $v^*(\tilde{\pi})$.*
2. *For each $\varepsilon > 0$, each player i has a strategy σ_j^ε that is ε-optimal in $G(\tilde{\pi})$ and such that $\sigma_j^\varepsilon(D(x_i)) = 0$ for all $x_i \in X_i$.*
3. *For each payoff function $\pi' \in \Pi$, the value $v^*(\pi')$ exists and is the same as $v^*(\tilde{\pi})$.*

[8] Observe that for any payoff function π', $v^*(\pi^-) = \underline{v}(\pi^-) \leqslant \underline{v}(\pi') \leqslant \overline{v}(\pi') \leqslant \overline{v}(\pi^+) = v^*(\pi^+)$, so the value is independent of π' iff $v^*(\pi^-) = v^*(\pi^+)$. Payoff-approachability ensures this last equality.

[9] We use the name payoff approachability to distinguish it from Blackwell's [1] definition for repeated games of approachability of a subset of the players' pairs of possible long-run average payoffs, which requires that for some mixed strategy of one player and any mixed strategy of the other player, eventually the resulting sequence of time-average payoffs is arbitrarily close to the set with arbitrarily high probability. The restriction to nearby points that are points of continuity against σ_j implies that we could have used π_n instead of $\tilde{\pi}_n$ in the right-hand side of the above inequality.

Proof. We divide the proof into intermediate steps. First we prove part (1).

Lemma 2.8. *The game $G(\tilde{\pi})$ has an equilibrium and thus has a value $v^*(\tilde{\pi})$.*

Proof of Lemma. We show that $G(\tilde{\pi})$ satisfies better-reply security, and then existence follows from Reny [14, Corollary 5.2].[10] The players' payoff functions are reciprocally upper semi-continuous because the game is zero-sum, so it remains to show that the game is payoff secure (Reny [14, Definition, p. 1033]). For this, fix a mixed-strategy profile (σ_1, σ_2). For each player i, take a pure strategy x_i in the support of σ_i such that $\tilde{\pi}_i(x_i, \sigma_j) \geqslant \tilde{\pi}_i(\sigma_i, \sigma_j)$. By payoff approachability, for each $\varepsilon > 0$ there exists a point y_i close to x_i such that y_i is a point of continuity against σ_j and $\tilde{\pi}_i(y_i, \sigma_j) > \tilde{\pi}_i(x_i, \sigma_j) - \varepsilon/2$. Then, by Lemma 2.3, there exists a neighborhood U_j^ε of σ_j such that for each $\tau_j \in U_j^\varepsilon$, $\tilde{\pi}_i(y_i, \tau_j) > \tilde{\pi}_i(x_i, \sigma_j) - \varepsilon$, as required. □

Lemma 2.9. *Consider a sequence of games $G(\tilde{\pi}^n)$, where $\tilde{\pi}^n$ is the restriction of $\tilde{\pi}$ to strategies in $\Sigma_1^n \times \Sigma_2^n \subset \Sigma$, and each sequence Σ_i^n converges to Σ_i in the Hausdorff topology on compact subsets of Σ_i. If each game $G(\tilde{\pi}^n)$ has an equilibrium σ^n and a value v^n, then v^n converges to $v^*(\tilde{\pi})$ and every limit point of σ^n is an equilibrium of $G(\tilde{\pi})$.*

Proof of Lemma. Take a convergent subsequence of equilibria σ^n and associated values v^n of $G(\tilde{\pi}^n)$ converging to say σ^* and v^*. We show that $v^* = v^*(\tilde{\pi})$ and that σ^* is an equilibrium of $G(\tilde{\pi})$, which proves the result. Fix a point x_1 for player 1 that is a point of continuity of σ_2^*. Fix $\varepsilon > 0$. Applying Lemma 2.3, there exists a neighborhood of $U_1 \times U_2$ of $(\delta_{x_1}, \sigma_2^*)$ such that $\tilde{\pi}(\sigma) \geqslant \tilde{\pi}(x_1, \sigma_2^*) - \varepsilon$ for all $\sigma \in U_1 \times U_2$. Since the strategy sets Σ_i^n converge to Σ, for all large n, there exists a strategy $\tau_1^n \in \Sigma_1^n \cap U_1$. Also, σ_2^n belongs to U_2 for large n. For such large n, as σ_2^n is an optimal strategy in $G(\tilde{\pi}^n)$, $v^n \geqslant \tilde{\pi}^n(\tau_1^n, \sigma_2^n)$, and thus $\tilde{\pi}(x_1, \sigma_2^*) - \varepsilon \leqslant \tilde{\pi}(\tau_1^n, \sigma_2^n) \leqslant v^n$, which implies that $\tilde{\pi}(x_1, \sigma_2^*) \leqslant \varepsilon + v^*$. Because ε is arbitrary, we conclude that $\tilde{\pi}(x_1, \sigma_2^*) \leqslant v^*$ for any x_1 that is a point of continuity against σ_2^*. Applying payoff approachability to player 1's payoffs shows that $\pi(x_1, \sigma_2^*) \leqslant v^*$ for *all* $x_1 \in X_1$ and thus that $v^* \geqslant v^*(\tilde{\pi})$. A similar argument with the roles of the players reversed shows that $v^* \leqslant v^*(\tilde{\pi})$ and thus $v^* = v^*(\tilde{\pi})$ as required. As shown above, $\tilde{\pi}(x_1, \sigma_2^*) \leqslant v^*$ for all x_1 and $v^* = v^*(\tilde{\pi})$. Thus σ_2^* is an optimal strategy for player 2 in $G(\tilde{\pi})$. Likewise, σ_1^* is optimal for player 1. Hence σ^* is a Nash equilibrium of $G(\tilde{\pi})$. □

Now we conclude the proof of the other parts of the theorem. We show that player 2 has a strategy as specified in part (2) of the theorem and that $\overline{v}(\pi') \leqslant v^*(\tilde{\pi})$ for all $\pi' \in \Pi$. A

[10] One can show further that $G(\tilde{\pi})$ satisfies the conditions in Duggan [6] that are stronger than better-reply security.

similar argument for player 1 completes the proof. As in the proof of Theorem 2.4, consider the perturbed games G^n where the strategies of player 2 are restricted to Σ_2^n. The strategy sets converge to the strategy sets in $G(\tilde{\pi})$ and thus Lemma 2.9 applies. Take a convergent subsequence of equilibria (σ_1^n, σ_2^n) and associated values v^n converging to (σ_1^*, σ_2^*) and v^*. From Lemma 2.9 we know that $v^* = v^*(\tilde{\pi})$.

For each $\varepsilon > 0$, choose n such that $v^n \leqslant v^*(\tilde{\pi}) + \varepsilon$. Since v^n is the value of G^n, $\pi(x_1, \sigma_2^n) \leqslant v^n \leqslant v^*(\tilde{\pi}) + \varepsilon$ for all x_1. By construction, $\sigma_2^n(D(x_1)) = 0$ for all x_1, and σ_2^n satisfies the properties specified in part (2) of the theorem. Also, observe that since $\sigma_2^n(D(x_i)) = 0$ for all x_i, no matter how payoffs are defined on D, the strategy σ_2^n holds player 1 down to $v^*(\tilde{\pi}) + \varepsilon$, i.e. $\overline{v}(\pi') \leqslant v^*(\tilde{\pi}) + \varepsilon$ for all π'. Since ε is arbitrary, $\overline{v}(\pi') \leqslant v^*(\tilde{\pi})$, as was to be shown. □

Remark 2.10.

(1) Although part (3) establishes that if some payoff function $\tilde{\pi} \in \Pi$ satisfies payoff approachability then for every $\pi' \in \Pi$ the game $G(\pi')$ has a value $v(\pi') = v(\tilde{\pi})$, it need not be that in $G(\pi')$ a player has a minimax strategy, or if he does then it could depend on the tie-breaking rule; see Remark 4.5 below for an example. Nevertheless, the above proof establishes that for each $\varepsilon > 0$ player 1 has a strategy that assures at least $v^*(\pi') - \varepsilon$ regardless of the tie-breaking rule.

Even if no payoff function in Π satisfies payoff approachability, it is still possible that for every payoff function $\pi' \in \Pi$ the game $G(\pi')$ has an equilibrium and a value, but the value depends on the tie-breaking rule. An example is the "diagonal game" at the end of Section 2.3 below.

(2) Observe that by Lemma 2.9 the strategy profile σ^* constructed in the second part of the proof of Theorem 2.7 by invoking Lemma 2.9 is actually an equilibrium of $G(\tilde{\pi})$. Thus part (1) can be viewed as a corollary to this part.[11]

(3) In some applications, some strategies may be (weakly) dominated and payoff approachability seems irrelevant for these profiles. The hypothesis of Theorem 2.7 can be weakened as follows. Suppose each player i has a compact subset X_i^* of X_i such that for each $x_i \notin X_i$, there exists $X_i^* \in X_i$ such that $\tilde{\pi}_i(x_i^*, \sigma_j) \geqslant \tilde{\pi}_i(x_i, \sigma_j)$ for all σ_j. Then for the conclusion of Theorem 2.7 to hold it is sufficient that payoff approachability holds for all $x_i \in X_i^*$ for each i.

[11]Obtaining part (1) thus as a corollary of Lemma 2.9 relies only on perturbation methods. We present the proof of part (1) separately, using better-reply security, to relate our results to previous literature on existence of equilibria in discontinuous games.

(4) If payoff approachability holds just for just one player i, in the sense that it holds for all (x_i, σ_j) for fixed i and j, then the game has a value and player j has a minimax strategy. For instance, this happens in the games π^+ for $i = 2$ and π^- for $i = 1$.

(5) If we had simply assumed that each X_i is a compact Hausdorff space then we could not have used the sequence $\tilde{\Sigma}_j^1 \subset \tilde{\Sigma}_j^2 \subset \cdots$ to construct a sequence of perturbed games. Rather, we would have needed a net $\{\, \tilde{\Sigma}_j^\alpha \,\}$ where the index α would be a collection of neighborhoods $\{\, U(x_j) \,\}_{x_j \in X_j}$ and $\tilde{\Sigma}_j^\alpha$ would be a finite subset of mixed strategies, one per open subset in a finite subcover of the collection. We would then use the corresponding net of perturbed games and the argument would proceed analogously.

From Lemma 2.9 we obtain the following corollary about finite approximations. Recall that every two-player zero-sum game with finite sets of pure strategies has a value obtained from equilibrium strategies that can be computed by linear programming.

Corollary 2.11. *Suppose that there exists a payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability. Consider a sequence of finite games $G(\tilde{\pi}^n)$, where $\tilde{\pi}^n$ is the restriction of $\tilde{\pi}$ to profiles in the finite set $\Sigma_1^n \times \Sigma_2^n \subset \Sigma$, and each sequence Σ_i^n converges Σ_i in the Hausdorff topology on closed subsets. Then the sequence $v^*(\tilde{\pi}^n)$ of values of $G(\tilde{\pi}^n)$ converges to $v^*(\tilde{\pi})$.*

We conclude this subsection with a sufficient condition for a payoff function $\tilde{\pi}$ to satisfy payoff approachability. The simplification achieved by this result is that in a class of games, which includes our subsequent applications, it is enough to check whether payoff approachability holds against mixed strategies with finite support. More precisely, if a payoff function satisfies condition (1) of Proposition 2.12 below, then payoff approachability is equivalent to condition (2).

Proposition 2.12. *A payoff function $\tilde{\pi} \in \Pi$ satisfies payoff approachability if:*

(1) *For each i, x_i, $D(x_i)$ can be partitioned into finitely many Borel-measurable subsets $D^1(x_i), \ldots, D^n(x_i)$ such that for each $1 \leqslant l \leqslant n$:*
   (a) *$\tilde{\pi}_i(x_i, \cdot)$ is constant on $D^l(x_i)$.*
   (b) *For each closed $A^l \subsetneq D^l(x_i)$, $\tilde{\pi}_i(y_i, \cdot)$ is constant on A^l for an open and dense set of y_i's in a neighborhood U of x_i.*

(2) *The condition in Definition 2.6 of payoff approachability holds for i, x_i and σ_j where the support of σ_j is finite and contained in $D(x_i)$.*

Proof. Suppose that the conditions of the theorem are satisfied by a payoff function $\tilde{\pi}$. We show that $\tilde{\pi}$ satisfies payoff approachability. Fix (x_i, σ_j). We can decompose σ_j into an average of two strategies, σ_j^c and σ_j^d, where $\sigma_j^c(D(x_i)) = 0$ and $\sigma_j^d(D(x_i)) = 1$. For every sequence $x_i^n \to x_i$, we have that $\tilde{\pi}_i(x_i^n, \sigma_j^c) \to \tilde{\pi}_i(x_i, \sigma_j^c)$ as in Lemma 2.3. Thus the condition of Definition 2.6 is really about the property of $\tilde{\pi}_i(x_i, \sigma_j^d)$ and we can therefore assume without loss of generality that $\sigma_j = \sigma_j^d$, i.e. x_i is a point of discontinuity against every pure strategy in the support of σ_j.

Fix $\varepsilon > 0$. For each l choose a closed subset A^l of $D^l(x_i)$ such that $\sigma_j(A) \geqslant 1 - \varepsilon$, where $A = \cup_l A^l$. Let τ_j be the conditional distribution over A. It is sufficient to find a point y_i in the ε-ball around x_i such that y_i is a point of continuity against σ_j and $\tilde{\pi}_i(x_i, \tau_j) \leqslant \tilde{\pi}_i(y_i, \tau_j) + \varepsilon$. Indeed, using the fact that $\tilde{\pi}_i(x_i, x_j) \leqslant \tilde{\pi}_i(y_i, x_j) + 2$ for all x_j, this implies that $\tilde{\pi}_i(x_i, \sigma_j) - \tilde{\pi}_i(y_i, \sigma_j) \leqslant (1-\varepsilon)\varepsilon + 2\varepsilon$, which proves the result.

Pick a point x_j^l in each A^l and define a mixed strategy $\tilde{\tau}_j$ as follows: $\tilde{\tau}_j(x_j^l) = \tau_j(A^l)$. The strategy $\tilde{\tau}_j$ has finite support by construction and also because $\tilde{\pi}_i(x_i, \cdot)$ is constant on each A^l by virtue of condition (1a), $\tilde{\pi}_i(x_i, \tau_j) = \tilde{\pi}_i(x_i, \tilde{\tau}_j)$. By condition (1b), we can choose a neighborhood U contained in the ε-ball around x_i such that $\tilde{\pi}_i(y_i, \cdot)$ is constant on each A^l for an open and dense set of y_i's in U. By condition (2), there exists $\tilde{y}_i$ in U such $\tilde{y}_i$ is a point of continuity against $\tilde{\tau}_j$ and $\tilde{\pi}_i(x_i, \tilde{\tau}_j) \leqslant \tilde{\pi}_i(\tilde{y}_i, \tilde{\tau}_j) + \varepsilon/2$. Because $\tilde{y}_i$ is a point of continuity against $\tilde{\tau}_j$ and using condition (1b) and Assumption 2.1(3) for σ_j, there exists a point y_i in U such that: (i) $\tilde{\pi}_i(y_i, \cdot)$ is constant on each A^l; (ii) y_i is a point of continuity against σ_j; (iii) $\tilde{\pi}_i(\tilde{y}_i, \tilde{\tau}_j) \leqslant \tilde{\pi}_i(y_i, \tilde{\tau}_j) + \varepsilon/2$. By (i), $\tilde{\pi}_i(y_i, \tau_j) = \tilde{\pi}_i(y_i, \tilde{\tau}_j)$. Assembling these inequalities and equalities,

$$\tilde{\pi}_i(x_i, \tau_j) = \tilde{\pi}_i(x_i, \tilde{\tau}_j) \leqslant \tilde{\pi}_i(\tilde{y}_i, \tilde{\tau}_j) + \varepsilon/2 \leqslant \tilde{\pi}_i(y_i, \tilde{\tau}_j) + \varepsilon = \tilde{\pi}_i(y_i, \tau_j) + \varepsilon,$$

which completes the proof. □

2.3. **Relation to Reny's Conditions.** Theorem 2.4 adds to the literature on sufficient conditions for existence of equilibria. To see this consider the following game: the sets of pure strategies are $X_1 = X_2 = [0, 1]$ and player 1's payoff is

$$\pi_1(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 < x_2 \\ 1 - x_1 & \text{if } x_1 > x_2 \\ 1 & \text{if } x_1 = x_2 \end{cases}$$

The set D is the diagonal $x_1 = x_2$, and this is the π^+ version, where player 1 gets +1 on D. Consider the profile $(\delta_{1/2}, \delta_{1/2})$ with associated profile of payoff limits $(1/2, -1/2)$. It is not an equilibrium, as player 2 gets -1 and could get $-1/2$ by any $x_2 \neq 1/2$. Better-reply security fails: If σ_1 is a strategy of player 1, we can choose a point $x_2(\varepsilon)$ in the interval $(1/2 - \varepsilon, 1/2)$ that is a point of continuity against σ_1. It is easily checked that player 1's payoff is less than $1/2 + \varepsilon$ from the profile $(\sigma_1, x_2(\varepsilon))$. Likewise, against $\delta_{1/2}$, player 2 gets

-1 for $x_2 = 1/2$ and $-1/2$ for any $x_2 \neq 1/2$, so $\pi_2(\delta_{1/2}, \sigma_2) \leqslant -1/2$ for all $\sigma_2 \in \Sigma_2$. Thus no strategy of either player can secure strictly more than the corresponding payoff limit. Yet Theorem 2.4 establishes existence of a value and of a minimax strategy for player 1. (It is directly verified that the value of the game is $1/2$, $\sigma_1 = \delta_{1/2}$ is a minimax strategy for player 1, and (σ_1, σ_2), with $\sigma_1 = \delta_{1/2}$ and $\sigma_2 = (1/2)\delta_0 + (1/2)\delta_1$, is an equilibrium.) See Section 4 for another example.

On the other hand, the direction taken by Theorem 2.7 is evidently a specialization of better-reply security. To illustrate, first note that its proof fails in the standard example of a zero-sum game without a value due to Sion and Wolfe [18]. This is so because this game violates payoff approachability. To see this formally, recall that in that game, there are two players, with strategy sets $X_1 = X_2 = [0, 1]$. Player 1's payoff is

$$\pi_1(x_1, x_2) = \begin{cases} -1 & \text{if } x_1 < x_2 < x_1 + 1/2 \\ 0 & \text{if } x_1 = x_2 \text{ or } x_2 = x_1 + 1/2 \\ 1 & \text{otherwise} \end{cases}$$

If we take $x_1 = 0$ and $\sigma_2 = \delta_{1/2}$ then $\pi_1(x_1, \sigma_2) = 0$, while $\pi_1(x_1^n, \sigma_2) = -1$ when we take a sequence of points of continuity. A similar situation holds for $x_1 = 1 = \sigma_2$. The fundamental problem is that these are boundary points for player 1 and one can approach such a point from only one side.

By the same logic, there is no payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability. In fact, payoff approachability forces $\tilde{\pi}_1(1, 1) = -1$, as $\pi_1(x_1^n, 1) = -1$ for all sequences $x_1^n \to 1$, and also $-\tilde{\pi}_1(1, 1) = \tilde{\pi}_2(1, 1) = -1$, as $\pi_2(1, x_2^n) = -1$ for all sequences $x_2^n \to 1$. This applies even to the better-reply secure "diagonal game" for which π_1 equals to -1 if $x_2 > x_1$, 1 if $x_1 > x_2$, and 0 if $x_1 = x_2$. More generally, such a game has a pure-strategy equilibrium $(x_1, x_2) = (1, 1)$ yielding the value $v \in [-1, +1]$ when $\pi_1 = v$ on the diagonal $x_1 = x_2$. Because the value v depends on the tie-breaking rule that specifies v, there cannot exist a payoff function $\tilde{\pi} \in \Pi$ satisfying payoff approachability.

3. Models of Elections

In this section we address models of elections, as in Downs [5]. Each candidate proposes a policy and gets votes from those voters who prefer his policy to the policy proposed by the other candidate. First we apply Theorem 2.4 to conclude that if one candidate, say the incumbent, wins all ties then a value exists and the incumbent has a minimax strategy that ensures this value. Then, invoking assumptions on voters' preferences, we show that payoff approachability is satisfied for a specified tie-breaking rule. Therefore, Theorem 2.7 implies existence of an equilibrium that yields the value, and this is also the value for any other tie-breaking rule (so there exists an ε-equilibrium for every $\varepsilon > 0$).[12]

3.1. **Formulation.** The game G is specified as follows. Two candidates compete in an election for the votes of K voters, where $K > 2$, by choosing a policy from a set P of feasible policies. Specifically, each candidate i's set X_i of feasible policies is a subset of P and $X = X_1 \times X_2$.[13] Each voter presumably votes for the candidate whose policy she prefers. If voter k chooses candidate i then i gets w_k votes, where each $0 < w_k < 1/2$ and $\sum_k w_k = 1$. A candidate who gets more than half the votes wins the election and receives the payoff $+1$, and the loser receives the payoff -1. In the case of a draw, in which the candidates get equal numbers of votes, their payoffs are both zero. As in Section 2, the payoff function of candidate i is $\pi_i : X \to [-1, +1]$, which can depend on how voters choose between tied policies.

We represent voter k's preferences by a utility function $u_k : P \to \mathbb{R}$. We impose the following assumptions on the policy space and the preferences of voters.

Assumption 3.1 (Basic Assumptions)**.**

(1) The policy space P is (homeomorphic to) a compact subset of a Euclidean space such that it is the closure of its interior, and for each candidate i, his set X_i of feasible pure strategies is the closure of an open subset of P.
(2) Each voter's utility function is continuous, and each indifference curve has zero Lebesgue measure in P.

Lemma 3.2. *The game G satisfies Assumption 2.1.*

Proof. The set D of points of discontinuity is the set of strategy profiles (x_1, x_2) such that $\sum_{k:u_k(x_i)>u_k(x_j)} w_k \leqslant 1/2$ for each candidate i, with the inequality being strict for at least one i. This set is obviously closed in X, so Assumption 2.1(1) is satisfied. For each policy x_i, the cross-section $D(x_i)$ of D is contained in the intersection of X_j with a finite union of indifference sets in P, one for each voter, each of which is a closed set of Lebesgue measure zero in P, so Assumption 2.1(2) is satisfied using Lemma 2.2. For any given mixed strategy σ_j, the marginal distribution on the utility values of voter k has at most countably many

[12]Plott [13] shows that an equilibrium in *pure* strategies exists only if voters have highly non-generic utility functions. Duggan [6] shows that an equilibrium exists in the case of three voters and the standard tie-breaking rule. Duggan and Jackson [8] show that an equilibrium exists under more general assumptions, but they rely on endogenous tie-breaking as in Simon and Zame [17].

[13]Typically, symmetry is imposed in such models by assuming that $X_1 = X_2$, but our results do not require this assumption. We apply this more general formulation to asymmetric Colonel Blotto games in Section 4.

atoms. Because each indifference curve is closed and has Lebesgue measure zero in P, the set of policies of candidate i that are not points of continuity against σ_j is contained in the intersection of X_i with a countable union of closed sets of measure zero. Therefore Assumption 2.1(3) is satisfied. □

3.2. **The Case That One Candidate Wins All Ties.** Theorem 2.4 implies that if one candidate wins all ties then a value exists and that candidate has a minimax strategy that ensures at least the value.

3.3. **Existence of an Equilibrium.** Now we provide assumptions on voters' preferences and their strategy sets, and a tie-breaking rule, that imply payoff approachability and thus the existence of an equilibrium.

Our first assumption assures a unique winner when there are no ties; see Remark 3.11 for why this assumption matters. For simple majority games, this assumption (which would say that the number of voters is odd) can be omitted; see the next subsection.

Assumption 3.3. For each subset L of voters, $\sum_{k \in L} w_k \neq 1/2$.

Say that a subset of voters L is a minimal subset of voters for whom x_i is Pareto optimal if: (i) x_i is Pareto optimal for voters in L among the policies in X_i; and (ii) there does not exist a strict subset of L for whom it is Pareto optimal. For each policy $x_i \in X_i$, let $\overline{K}(x_i)$ be the set of voters for whom x_i is an *ideal* policy in X_i, i.e. x_i maximizes u_k over X_i. Obviously each voter in $\overline{K}(x_i)$ is a singleton minimal set for whom x_i is optimal among policies in X_i.

Assumption 3.4 (Diversity of Preferences)**.** For each candidate i and policy $x_i \in X_i$:

(1) The policy x_i is Pareto optimal in X_i.[14]
(2) For each minimal subset L of voters for whom the policy x_i is Pareto optimal, each voter $k \in L$, and each neighborhood V of x_i, there exists a policy y_i in V such that $u_{k'}(x_i) < u_{k'}(y_i)$ for every voter k' in $K \setminus \overline{K}(x_i)$ other than voter k, while $u_{k'}(x_i) > u_{k'}(y_i)$ for all voters $k' \in \overline{K}(x_i) \cup \{k\}$.

Observe that there exists at most one minimal subset L of $K \setminus \overline{K}(x_i)$ for whom x_i is Pareto optimal if the assumption is satisfied. Moreover there exists one iff $\overline{K}(x_i)$ is empty. Thus define $K^*(x_i)$ to be $\overline{K}(x_i)$ if the latter is nonempty and otherwise the unique minimal subset L of K for which x_i is optimal.[15]

[14] See Remark 3.10 for a discussion of how to relax this assumption.

[15] If we had assumed that the ideal policies of voters are all different, then each x_i has a unique minimal subset of voters for whom x_i is Pareto optimal. We do not impose this assumption because models with linear preferences over a convex set could admit the robust possibility that the same policy is ideal for multiple voters.

Given a policy $x_i \in X_i$, for every neighborhood $V(x_i)$ of x_i, and every $k \in K^*(x_i)$, from Assumption 3.4 we can choose a policy $y_i(V(x_i), k) \in V(x_i)$ such that $u_{k'}(x_i) < u_{k'}(y_i(V(x_i), k))$ if $k' \neq k$ and belongs to $K \setminus \overline{K}(x_i)$; and $u_{k'}(x_i) > u_{k'}(y_i(V(x_i), k))$ otherwise. To simplify notation, we will use y_i^k to denote $y_i(V(x_i), k)$.

If $x = (x_1, x_2) \in D$ then for each i, define $L^i(x)$ as the set of voters k such that $u_k(x_i) > u_k(x_j)$, and $L^0(x)$ as the set of voters k such that $u_k(x_i) = u_k(x_j)$; $L_i^*(x) \equiv K^*(x_i) \cap L^0(x)$; and $\overline{L}_i(x) = \overline{K}(x_i) \cap L^0(x)$. For all sufficiently small neighborhoods $V(x_i)$ of x_i, for each $y_i \in V(x_i)$, $u_k(y_i) > u_k(x_j)$ for all $k \in L^i(x)$ and $u_k(y_i) < u_k(x_j)$ for all $k \in L^j(x)$. Observe that by construction the payoffs are then well-defined without ties for (y_i^k, x_j) for each $k \in K^*(x_i)$. Thus, by Assumption 3.3, $\pi_i(y_i^k, x_j)$ is ± 1.

We now introduce our next assumption. Suppose $x \in D$ and that either $\overline{L}_i(x_i)$ is nonempty or $|L_i^*(x)| \geqslant 2$. Let $l_i^*(x)$ be a voter in $L_i^*(x)$ with $w_{l_i^*(x)} \leqslant w_{k'}$ for all $k' \in L_i^*(x)$. For each i, let $V(x_i)$ be a neighborhood of x_i such that for all $y_i \in V(x_i)$, $u_k(y_i) > u_k(x_j)$ for all $k \in L^i(x)$ and $u_k(y_i) < u_k(x_j)$ for all $k \in L^j(x)$.

Assumption 3.5 (Relationship Between Candidates' Strategy Sets)**.** If candidate i's policy $y_i^{l_i^*(x)}$ loses to the policy x_j then for each $k \in K^*(x_j)$, candidate j's policy y_j^k beats x_i.

This assumption depends on the neighborhoods only to the extent that voters who are not indifferent between x_i and x_j treat policies in the two neighborhoods the same way. Hence if it holds for some pair of neighborhoods then it holds for all smaller neighborhoods.

Assumptions 3.4 and 3.5 are not restrictive if the dimension N of the policy space P is at least $K-1$. We show later that Colonel Blotto games satisfy these assumptions. For another example, if the voters have Euclidean preferences, say $u_k(p) = -\|p - a^k\|$ where $a^k \in \mathbb{R}^N$ is voter k's ideal policy, then the Pareto set is the convex hull of the a^k's. If $N \geqslant K - 1$, and all the ideal policies are extreme points of the Pareto set, then the assumptions are satisfied if both players have this Pareto set as their strategy set. More generally suppose that P is a convex set and the utility functions are differentiable and strictly quasi-concave. Then generically in the space of such preferences, the rank of the matrix of gradients at a Pareto optimal policy is $K - 1$ and Assumption 3.4 holds. If x_i is an ideal policy of a voter k (and then the only voter, because of the rank condition on the matrix of gradients), then $\overline{L}_i(x)$ is nonempty iff $x_i = x_j$ and then Assumption 3.5 holds vacuously as y_i^k beats x_j. On the other hand if $L_i^*(x)$ has at least two voters, and $y_i^{l_i^*(x)}$ loses to x_j, then it must be that candidate i needs all the voters in $L^0(x)$ to win: $y_i^{l_i^*(x)}$ loses for the tied voter in $l_i^*(x)$ with the least vote and wins for all other tied voters, and yet it loses to x_j, so candidate j needs just one of the tied voters to win the election. And, for each $k \in K^*(x_j)$, it is simple to find a y_j^k that achieves that much, given that Assumption 3.4 holds. So Assumption 3.5 holds as well.

The following example illustrates the ideas involved.

Example 3.6. There are 4 voters and the dimension of the police space is 3. So $K = 4$ and $N = 3$. Voter k's utility function is $u_k(x) = -\sum_{n=1}^{3}(x_n - a_n^k)^2$, where the ideal points are $a^1 = (1, 0, 0)$, $a^2 = (0, 1, 0)$, $a^3 = (0, 0, 0)$ and $a^4 = (0, 0, 1)$. The space of policies P is the tetrahedron obtained as the convex hull in $\mathbb{R}^3$ of the ideal points. Observe that this is the set of Pareto optimal policies, which we assume to be the set of strategies for both candidates. It is simple to come up with weights, e.g. $w_1 = .11$, $w_2 = .2$, $w_3 = .29$ and $w_4 = .4$, to satisfy Assumption 3.3. For a given policy $x_i \in P$, the minimal subset L of voters for which x_i is Pareto optimal is given by the voters whose ideal points span the face at which x_i lies. For instance, for $a, b > 0$ with $a+b < 1$, $L = \{1, 2\}$ if $x_i = (a, 1-a, 0)$; $L = \{1, 4\}$ if $x_i = (a, 0, 0)$; $L = \{1, 2, 4\}$ if $x_i = (a, b, 0)$; and $L = \{1, 2, 3, 4\}$ if x_i is in the interior of P. $\overline{K}(x_i) = \{k\}$ if $x_i = a^k$ and it is empty for policies not equal to an ideal point. Assumption 3.4 is easily verified: if $x_i = a^k$, then $L = \{k\}$ and moving to the interior of P we find the required y_i; if x_i belongs to the face spanned by voters including voter k, then the required y_i for voter k (denoted y_i^k) is found by moving to the interior of P away from a^k.

Consider $x \in D$ given by $x_i = (1/4, 1/4, 0)$ and $x_j = (1/4, 0, 1/4)$. Then $L^i(x) = \{2\}$, $L^j(x) = \{4\}$ and $L^0(x) = \{1, 3\}$. Also $K^*(x_i) = \{1, 2, 3\}$ and $K^*(x_j) = \{1, 3, 4\}$, so $L_i^*(x) = L_j^*(x) = \{1, 3\}$. Hence $l_i^*(x) = l_j^*(x) = \{1\}$.

Observe that $y_i^{l_i^*(x)}$ loses voter 1 (because $l_i^*(x) = \{1\}$), wins voter 3 and does not change the other two voters (relative to x_j – so 2 still prefers i's policy $y_i^{l_i^*(x)}$ over x_j, whereas 4 prefers x_j over $y_i^{l_i^*(x)}$). Given the weights specified above, $y_i^{l_i^*(x)}$ loses to x_j, as it gets $.2 + .29 = .49$ votes, whereas x_j gets .51 votes. To illustrate Assumption 3.5, we must show that y_j^k beats x_i for $k = 1, 3, 4$. It is obvious for $k = 4$, as $u_4(x_j) > u_4(x_i)$, $V(x_j)$ is chosen so that this inequality is preserved for all $y_j \in V(x_j)$, and y_j^4 wins the tied voters 1 and 2. For $k = 3$, y_j^3 loses voter 3, wins voter 1 and does not change the other two voters (relative to x_i), so it gets $.11 + .4 = .51$ votes and beats x_i. Likewise for $k = 1$, as now y_j^1 wins voter 3, so it gets .29 votes on top of the .4 votes already obtained from voter 4.

The tie-breaking rule is specified in terms of the implied payoff function $\tilde{\pi} \in \Pi$.

Definition 3.7 (Tie-Breaking Rule $\mathcal{T}$)**.** Suppose the profile x is in D.

($T1$) For each i, let $V(x_i)$ be as in Assumption 3.5. If for some i, $\overline{L}_i(x)$ is nonempty or $L_i^*(x)$ has at least two voters, and if $y_i^{l_i^*(x)}$ loses to x_j, then $\tilde{\pi}_i(x_i, x_j) = -1$ and $\tilde{\pi}_j(x_i, x_j) = +1$.

($T2$) In all other cases, $\tilde{\pi}_i(x_i, x_j) = 0$ for each candidate i.[16]

As above, when $y_i^{l_i^*(x)}$ loses to x_j, candidate j is in a very advantageous situation. For instance, at the pair (x_i, x_j) described in Example 3.6, candidate j has .4 votes already, and capturing any of the two tied voters (1 and 3) would suffice for j to win the election, whereas candidate i has to get the votes from both voters 1 and 3 to win the election. In such situations, the tie-breaking rule $\mathcal{T}$ awards the election to j. This tie-breaking rule has the following convenient property.

Lemma 3.8. *The payoff function $\tilde{\pi}$ induced by rule $\mathcal{T}$ satisfies condition (1) of Proposition 2.12.*

Proof. We can partition $D(x_i)$ into a finite number of subsets, each indexed by a triple (L^0, L^1, L^2) where, as above, candidate i gets the votes of L^i and there are ties in L^0. These sets are further decomposed by whether ($T1$) or ($T2$) applies, which proves (1a). To prove property (1b), fix a closed subset A^L of one of the elements of this partition with index (L^0, L^1, L^2). Then there exists $\varepsilon > 0$ such that for each $x_j \in A^L$, $|u_k(x_i) - u_k(x_j)| > \varepsilon$ for all $k \notin L^0$. Choose a ball V around x_i such that $|u_k(x_i) - u_k(y_i)| < \varepsilon$ for all $y_i \in V$. Then $\tilde{\pi}(y_i, \cdot)$ is constant on A^L for an open and dense subset of V, i.e. those y_i's for which $u_k(x_i) \neq u_k(y_i)$ for all $k \in L^0$, which verifies condition (1b). □

Theorem 3.9. *The game $G(\tilde{\pi})$ has an equilibrium and its value is the same as the value of $G(\pi')$ for all $\pi' \in \Pi$.*

Proof. We check that $\tilde{\pi}$ satisfies payoff approachability for an arbitrary profile (x_i, σ_j) and then apply Theorem 2.7.

By the above lemma and Proposition 2.12, we can assume that σ_j has finite support, say $x_j^1, \ldots, x_j^n$. Choose $\bar{\varepsilon} > 0$ such that for all x_j^l in the support of σ_j, and each k, $|u_k(x_i) - u_k(x_j^l)| > \bar{\varepsilon}$ if $u_k(x_i) \neq u_k(x_j^l)$. Fix a neighborhood $V(x_i)$ of x_i such that $|u_k(x_i) - u_k(x_i')| < \bar{\varepsilon}$ for all $x_i' \in V(x_i)$. By our choice of $\bar{\varepsilon}$, $V(x_i)$ is one of the neighborhoods that could be used in defining the tie-breaking rule. (In particular, for each k there are no ties between y_i^k and the x_j^l's and the former is a point of continuity against σ_j.) We show that there exists some k such that $\tilde{\pi}_i(y_i^k, \sigma_j) \geqslant \tilde{\pi}_i(x_i, \sigma_j)$, which proves payoff approachability.

[16]We could also use fair coin tosses for each tied voter.

For each $k \in K^*(x_i)$, $\tilde{\pi}_i(y_i^k, x_j) = \tilde{\pi}_i(x_i, x_j) = 1$ if $(T1)$ resolves the tie between x_i and x_j in favor of i, and $\tilde{\pi}_i(y_i^k, x_j) \geqslant -1 = \tilde{\pi}_i(x_i, x_j)$ if $(T1)$ resolves the ties in favor of j. Therefore, if $(T1)$ applies to every x_j^l then we are done. Otherwise, let $\hat{X}_j$ be the set of x_j such that $(T2)$ applies to (x_i, x_j) and let $\hat{\sigma}_j$ be the conditional distribution over $\hat{X}_j$. We now show that there exists k such that $\tilde{\pi}_i(y_i^k, \hat{\sigma}_j) \geqslant 0 = \tilde{\pi}_i(x_i, \hat{\sigma}_j)$, which finishes the proof.

If $\overline{K}(x_i)$ is nonempty then $\tilde{\pi}_i(y_i^k, x_j) = 1$ for each $k \in \overline{K}(x)$ and each x_j in $\hat{X}_j$: indeed this is obviously true if $\overline{L}_i(x_i, x_j) = \emptyset$ since y_i^k would win each of the ties in L^0; if $\overline{L}_i(x_i, x_j)$ is nonempty, this is true since otherwise $(T1)$ applies. Therefore, we are done in this case.

Suppose $\overline{K}(x_i)$ is empty. Consider the policy $y_i \equiv y_i(V(x_i), k^*)$, where k^* minimizes w_k over $K^*(x_i)$. If $k^* \notin L_i^*(x_i, x_j)$ for some x_j in $\hat{X}_j$, then obviously $\tilde{\pi}_i(y_i, x_j) = +1$; if $k^* \in L_i^*(x_i, x_j)$ and $L_i^*(x_i, x_j)$ has at least two voters, then too $\tilde{\pi}_i(y_i, x_j) = +1$, since $(T1)$ would apply otherwise. Thus among the policies in $\hat{X}_j$, y_i beats every x_j^l except, possibly, the subset $\hat{A}_j$ of those x_j's in $\hat{X}_j$ for which $L_i^*(x_i, x_j)$ is just the singleton k^*. If the probability of this subset under $\hat{\sigma}_j$ is no more than half, then $\tilde{\pi}_i(y_i, \hat{\sigma}_j) \geqslant 0$ and we are done.

Finally, suppose that the probability of $\hat{A}_j$ under $\hat{\sigma}_j$ is greater than half. Observe that $K^*(x_i)$ contains some other voter, say $\tilde{k}$, since we have assumed that $\overline{K}(x_i)$ is empty. As we argued above, for any $k \in K^*(x_i)$, y_i^k beats any x_j that x_i beats under $(T1)$ and does at least as well when x_i loses because of $(T1)$. On the set $\hat{X}_j$ we now have that $y_i(V(x_i), \tilde{k})$ beats every policy in $\hat{A}_j$, which has a probability at least half, and thus it gets a weakly higher payoff against $\hat{\sigma}_j$ than x_i, which completes the proof. □

Remark 3.10. Suppose X_i includes policies that are not optimal in X_i. Let X_i^* be the set of optimal policies in X_i. If our assumptions hold on the sets X_i^* then our results apply to obtain existence of a value over $X_1^* \times X_2^*$. We could specify payoffs at ties involving non-optimal points to extend this to an equilibrium over the bigger strategy space. But even simpler, the game over X inherits the value from the game over X^*: for each ε, our perturbation technique yields for each player i an ε-optimal strategy σ_i^ε that assigns zero probability to any voter's indifference curves—indeed, this follows if we use for the restricted strategy sets, the sets identified by the proof of Lemma 2.2 which have the property that each element of these sets assigns zero probability to cross-sections. The same strategy is ε-optimal in X.

Remark 3.11. A key feature of the tie-breaking rule $\mathcal{T}$ is $(T1)$. When it is invoked to resolve a tie between x_i and x_j, it ensures that each player can achieve the payoff from the tie by all choices of the form y_i^k. Indeed, if it is resolved in i's favor, it is guaranteed by Assumption 3.5. On the other hand, if it is resolved against i, then it is obvious. The assumption that there are no draws (Assumption 3.3) means that $\pi_i(y_i^k, x_j) \neq 0$. If we allow this possibility, complications can arise. For simplicity suppose $L_i^*(x) = L_j^*(x) = L^0(x)$ and this set contains two voters with unequal weights. It could be that $\pi_i(y_i^{l_i^*(x)}, x_j) = 0$ for i but $\pi_i(y_i^k, x_j) = -1$ for the other voter k in $L_i^*(x)$. Thus, if we set $\pi_i(x_i, x_j) > -1$, the strategy y_i^k cannot guarantee this payoff. On the other hand if $\pi_i(x_i, x_j) = -1$, then j cannot guarantee payoff $+1$ with the strategy y_j^k. The problem here is the combination of the possibility that the game could end in a draw (each candidate gets half of the votes) with the fact that it is a weighted-majority game.

Remark 3.12. Payoff-approachability can fail when $K > N+1$. For example, suppose there are seven voters and the policy space of each candidate is the set of lotteries over three outcomes $o_1, o_2, o_3 \in P$, so $K = 7$ and $N = 2$. The seven voters' utilities $(u_k(o_1), u_k(o_2), u_k(o_3))$ for the three outcomes are $(1, .6, 0)$, $(1, .5, 0)$, $(1, 0, .6)$, $(0, 1, .6)$, $(.6, 1, 0)$, $(.6, 0, 1)$, and $(0, .6, 1)$, and for each voter his expected utility is linear, $u_k(p) = \sum_\ell u_k(o_\ell)p_\ell$. Payoff-approachability is violated at the profile where both candidates offer the policy that yields o_1 for sure. However, that profile is an equilibrium.

3.4. **Simple-Majority Games.** For the case of simple-majority games we specify a slightly different tie-breaking rule that implies the same result even if the number of voters is even. We use the notation from the previous section, except that each $w_k = 1/K$.

Obviously, Assumption 3.3 cannot hold when the number of voters is even, so it is dropped. Assumption 3.4 on diversity of preferences remains the same. Assumption 3.5 relating strategy sets has to be changed. In the following assumption and definition, we retain the notation from the previous subsection.

Assumption 3.13 (Relationship Between Candidates' Strategy Sets—The Simple-Majority Version)**.** Fix $x = (x_i, x_j) \in D$.

(1) If $\overline{L}_i(x)$ is nonempty then $|L^0(x)| \geqslant 2$.

(2) If $L_i^*(x)$ is nonempty and $|L^0(x)| \geqslant 2$ then:

  (a) If $\pi_i(y_i^k, x_j) = 0$ for some (and then all) $k \in L_i^*(x)$, then for all $k \in K^*(x_j)$, $\pi_j(y_j^k, x_i) \in \{0, 1\}$ and in fact equals $+1$ if $|L^0(x)| \geqslant 3$.

  (b) If $\pi_i(y_i^k, x_j) = -1$ for some (and then all) $k \in L_i^*(x)$, then for all $k \in K^*(x_j)$, $\pi_j(y_j^k, x_i) = +1$.

Example 3.14. In the setting of Example 3.6, set the weights to $w_k = 1/4$ for every k. Condition (1) of Assumption 3.13 holds because $\overline{L}_i(x)$ is nonempty iff $x_i = x_j$ and then

$|L^0(x)| = K \geqslant 3$. Condition (2)(a) is illustrated by the policy pair (x_i, x_j) described in Example 3.6: in fact $\pi_i(y_i^k, x_j) = 0$ for $k = 1, 3$, as y_i^1 (resp. y_i^3) wins voter 3 (resp. 1) and loses voter 1 (resp. 3), so each such policy gets 2/4 votes against x_j. So we must show that $\pi_j(y_j^k, x_i) \geqslant 0$ for $k = 1, 3, 4$. And this is true, as it is equal to zero for $k = 1, 3$ (both y_j^1 and y_j^3 win one and lose one of the tied voters, so each gets 2/4 votes against x_i) and it is equal to +1 for $k = 4$, as y_j^4 wins both tied voters 1 and 3 and retains voter 4, so j gets 3/4 votes against x_i.

Example 3.15. To illustrate the second part of condition (2)(a), modify Example 3.6 by adding two voters and two dimensions, $K = 6$, $N = 5$, continuing with Euclidean preferences having ideal points $a^1 = (1, 0, 0, 0, 0)$, $a^2 = (0, 1, 0, 0, 0)$, $a^3 = (0, 0, 0, 0, 0)$, $a^4 = (0, 0, 1, 0, 0)$, $a^5 = (0, 0, 0, 1, 0)$ and $a^6 = (0, 0, 1, 0, 1)$. Again the strategy sets are the Pareto set, the convex hull of the ideal policies. For simple majority rule, the weights are $w_k = 1/6$ for every k. Consider $x_i = (1/4, 1/4, 0, 0, 0)$ and $x_j = (1/4, 0, 1/4, 0, 0)$. Now $L^0(x) = \{1, 3, 5\}$, $L^i(x) = \{2\}$ and $L^j(x) = \{4, 6\}$. We have $\pi_i(y_i^k, x_j) = 0$ for $k \in \{1, 3\} = L_i^*(x)$, as y_i^1 (resp. y_i^3) wins voters 3 and 5 (resp. 1 and 5) and loses voter 1 (resp. 3), totaling 3/6 votes from voters 2, 3 and 5 (resp. 1, 2 and 5). We must show that $\pi_j(y_j^k, x_i) = +1$ for $k = 1, 3, 4$, and this follows because y_j^k for $k = 1, 3, 4$, wins at least two of the tied voters and retains voters 4 and 6 (relative to x_i), so j gets at least 4/6 votes.

Example 3.16. To illustrate condition (2)(b) of Assumption 3.13, again modify Example 3.6, but now add only one voter and one dimension ($K = 5$, $N = 4$), with ideal policies $a^1 = (1, 0, 0, 0)$, $a^2 = (0, 1, 0, 0)$, $a^3 = (0, 0, 0, 0)$, $a^4 = (0, 0, 1, 0)$, and $a^5 = (0, 0, 1, 1)$, and $w_k = 1/5$ for all k. For the pair $x_i = (1/4, 1/4, 0, 0)$ and $x_j = (1/4, 0, 1/4, 0)$, we have $L^0(x) = \{1, 3\}$, $L^i(x) = \{2\}$, and $L^j(x) = \{4, 5\}$. Now $\pi_i(y_i^k, x_j) = -1$ for $k = 1, 2$, for the same reason as above, as x_j retains voters 4 and 5 and wins one more voter (voter 1 for $k = 1$ and voter 3 for $k = 3$), so it gets 3/5 votes relative to y_i^k. So we have to verify that $\pi_j(y_j^k, x_i) = +1$ for $k = 1, 3, 4$. This follows, as y_j^k for $k = 1, 3, 4$ wins at least one voter, plus voters 4 and 5 that are already won (relative to x_i).

Again, the tie-breaking rule is specified in terms of the implied payoff function $\tilde{\pi} \in \Pi$.

Definition 3.17 (Modified Tie-Breaking Rule $\mathcal{T}^S$)**.** Suppose $x \in D$.

- ($T1$) For each i, let $V(x_i)$ be as in Assumption 3.13. Suppose for some i, $L_i^*(x)$ is nonempty and $L^0(x)$ has at least two voters. For this i:
  - (a) If $\tilde{\pi}_i(y_i^k, x_j) = 0$ for some $k \in L_i^*(x)$, then $\tilde{\pi}_i(x_i, x_j)$ is zero if $|L^0(x)| = 2$ and -1 if $|L^0(x)| \geqslant 3$.

(b) If $\tilde{\pi}_i(y_i^k, x_j) = -1$ for some $k \in L_i^*(x)$, then $\tilde{\pi}_i(x_i, x_j) = -1$.

$(T2)$ Suppose $L_i^*(x)$ is empty for each i or $L^0(x) = \{ k \}$ for some k. If $\sum_{k' \in L^j(x)} w_{k'} = 1/2$, then $\tilde{\pi}_i(x_i, x_j) = -1/2$.

$(T3)$ In all other cases, $\tilde{\pi}_i(x_i, x_j) = 0$ for each i.[17]

The rule $\mathcal{T}^S$ differs from the rule $\mathcal{T}$ used in the previous subsection only in that provisions $(T1)$(a) and $(T2)$ are added—and the condition that $L_i^*(x)$ has at least two voters if $\overline{L}_i(x)$ is empty, when invoking $(T1)$, is relaxed—to accommodate the fact that with an even number of voters the game could end in a draw. Without these changes, $\mathcal{T}^S$ is the same as $\mathcal{T}$.

From Example 3.16 we see that provision $(T1)$(b) is analogous to provision $(T1)$ of tie-breaking rule $\mathcal{T}$: candidate j is in a very advantageous situation when $\tilde{\pi}_i(y_i^k, x_j) = -1$ for all $k \in L_i^*(x)$, as winning a single one of the tied voters guarantees a victory, whereas candidate i has to win all of the tied voters. In such a situation, $\mathcal{T}^S$ awards the election to j. Provision $(T1)$(a) handles draws: from Example 3.14, we see that $\pi_i(y_i^k, x_j) = 0$ and $|L^0(x)| = 2$ for all $k \in L_i^*(x)$ is a symmetric situation, so the rule $\mathcal{T}^S$ declares it a draw; from Example 3.15 we see that candidate j is in an advantageous situation when $\pi_i(y_i^k, x_j) = 0$ and $|L^0(x)| \geqslant 3$ for all $k \in L_i^*(x)$, as j has the upper hand in the non-tied battles, so $\mathcal{T}^S$ awards the election to j.

Example 3.18. Return to the setting of Example 3.14. Consider the pair (x_i, x_j) with $x_i = (0, 0, 0)$ and x_j in the intersection of 1's indifference surface and the face spanned by voters 1, 2 and 4, in such a way that voter 4 prefers x_j to x_i (for instance, $x_j = (\frac{3-\sqrt{5}}{4}, \frac{1}{2}, \frac{\sqrt{5}-1}{4})$). Then $L^0(x) = \{1\}$ and $L^j(x) = \{2, 4\}$, so the premise of condition $(T2)$ of the rule $\mathcal{T}^S$ applies, and the rule then says that $\tilde{\pi}_i(x_i, x_j) = -1/2$. We see that candidate j is in a stronger position because he has already secured 2/4 votes. But y_j^1 loses voter 1, so it fails to beat x_i. The relatively stronger position of candidate j is then captured by awarding the election to him with probability 3/4 rather than 1/2.

The tie-breaking rule $\mathcal{T}^S$ also satisfies payoff approachability. To prove this, as in Subsection 3.3, it is easy to show that the payoff function $\tilde{\pi}$ satisfies condition 1 of Proposition 2.12 and thus it is sufficient to show that payoff approachability is satisfied by (x_i, σ_j) where σ_j has finite support in $D(x_i)$. This property is verified by Lemma A.1 in the Appendix. Hence, $\tilde{\pi}$ satisfies payoff approachability and we have the following theorem.

Theorem 3.19. *The game $G(\tilde{\pi})$ has an equilibrium and its value is the value of every variant $G(\pi')$ with $\pi' \in \Pi$.*

[17] Again, we could use fair coin tosses for each tied voter.

4. Majority Games of Resource Allocation

This section addresses a special case of the formulation and results in Sections 2 and 3. The two players compete for votes in several constituencies, called battlegrounds. The winner of the game is again the player who wins more votes. The key feature now is that a player wins a battle if he allocates more of his available resources to that battle than his opponent does. Thus the game is a majority-rule version of a Colonel Blotto game.[18]

4.1. **Formulation.** The game G is a weighted-majority game specified as follows. Each player i has an amount R_i of a resource that he allocates among the battles. Assume that $R_1 \geqslant R_2 > 0$ and that the number of battles is an integer $K > 2$. A pure strategy $x_i = (x_{i,k})_{k=1,\dots,K}$ for player i allocates a nonnegative amount $x_{i,k}$ of his resource to battle k. Thus his set of pure strategies is $X_i \equiv \{\, x_i \in \mathbb{R}^K_+ \mid \sum_{k=1}^K x_{i,k} = R_i \,\}$. For each profile $x \equiv (x_1, x_2) \in X_1 \times X_2 \equiv X$ of pure strategies for the two players, player i wins battle k, and the other player j loses, if $x_{i,k} > x_{j,k}$. If $x_{i,k} = x_{j,k}$ then a tie-breaking rule determines the winner of battle k.

For each battle k, the winner of the battle obtains w_k votes, where $0 < w_k < 1/2$ and $\sum_k w_k = 1$. We assume that $\sum_{k \in L} w_k \neq 1/2$ for each subset L of K, except when we consider simple-majority games.[19] Player i wins the game and gets payoff $+1$ if $\sum_{k \in W_i} w_k > 1/2$, where W_i is the set of battles he wins; similarly, he loses and gets payoff -1 if $\sum_{k \in W_i} w_k < 1/2$. The players' payoffs are both zero if both win $1/2$ votes.[20] Thus, if there are no tied battles or the resolutions of ties are inconsequential, then a player's payoff is either $+1$ if he wins a weighted majority of votes, or -1 if he loses. If resolutions of tied battles affect the outcome of the game then his expected payoff is some number in the interval $[-1, +1]$. Either way, player i's payoff function is $\pi_i : X \to [-1, +1]$, and $\pi_1(x) + \pi_2(x) = 0$ for every profile $x \in X$.

[18]Duggan [7] proves existence of an equilibrium of this game for the case of simple-majority rule and symmetric resources. The other literature on Colonel Blotto games assumes that each player maximizes the number of battles won, rather than winning a majority. This case is sometimes interpreted as relevant to plurality rule but the connection is not exact when the number of battles exceeds three. This literature culminates in the article by Roberson [15], who provides a complete analysis of such games, and in Hart [10] for the case that resources are allocated in discrete amounts.

[19]This assumption is not needed if we consider the case where one player wins all ties.

[20]Our results, except those in Section 4.2 for simple-majority games, go through if we use a plurality rule, so that player i's payoff is $\sum_{k \in W_i} w_k$. This makes the game a constant-sum game that is strategically equivalent to a zero-sum game. In fact the proofs are simpler since then we can work with the standard tie-breaking rule in which the winner of each tied battle is chosen by the toss of a fair coin. For more general non-constant-sum games our basic existence theorem—which shows the existence of an equilibrium for the game $G(\tilde{\pi})$ when it satisfies payoff approachability—goes through; such games are studied by Kvasov [11], Roberson [15] and Kvasov and Roberson [16].

This model is a special case of those in Section 3. The policy space P is the union of X_1 and X_2. Each battleground represents a voter whose utility function is $u_k(x_i) = x_{i,k}$.[21]

The following theorem extends a result obtained by Duggan [7], who proves existence of an equilibrium for simple-majority rule with symmetric resources and the standard tie-breaking rule.

Theorem 4.1. *If the tie-breaking rule is $\mathcal{T}$ (or $\mathcal{T}^S$ in the case of simple majority) then the game has an equilibrium that yields the value, and any other tie-breaking rule yields the same value.*

Proof. We verify the assumptions stated in Sections 3.3 and 3.4 and apply Theorems 3.9 and 3.19, respectively. Assumption 3.3 is stated in the formulation. To check the other assumptions, remark first that $K^*(x_i)$ is the set of battles whose coordinates are positive. In particular, $\overline{K}(x_i)$ is a singleton for a vertex (the voter corresponding to the battlefield getting all the resources) and empty elsewhere. With this feature, Assumption 3.4 is easily verified. In fact, for each coordinate that is positive, we can reduce it by an arbitrarily small amount and assign a strictly higher amount to all other battlefields.

Regarding Assumption 3.5, suppose x_i is tied with x_j, $L_i^*(x)$ is nonempty, with $|L_i^*(x)| \geqslant 2$ if x_i is not a vertex, and $y_i^{l_i^*(x)}$ loses to x_j. Since $L_i^*(x)$ is nonempty, if x_i is a vertex then it must be that $L^0(x)$ contains this one non-zero coordinate of player i. Moreover, $i = 2$ and $R_2 < R_1$: indeed as j must assign R_i to this battlefield as well, $R_j \geqslant R_i$, but if $R_j = R_i$, then $x_i = x_j$ and $y_i^{l_i^*(x)}$ would beat x_j. Since $R_j > R_i$, x_j is not a vertex, i.e. $K^*(x_j)$ has at least two nonzero coordinates. As a result, each y_j^k beats x_i on all coordinates except possibly for the one corresponding to the vertex, and thus it wins the game (recall that $w_k < 1/2$ for all k).

If x_i is not a vertex then $L_i^*(x)$ has at least two elements. When $y_i^{l_i^*(x)}$ loses to x_j it means that j could win the game by winning any of the battles in $L_i^*(x)$. Since $L_i^*(x)$ equals $L_j^*(x)$ and has at least two non-zero coordinates, every y_j^k would accomplish this as it would reduce at most one of the nonzero coordinates in $L^0(x)$.

Finally we check Assumption 3.13 for the simple-majority case (with even or odd number of battlefields). Suppose x_i is a vertex. If x_i ties with x_j just on the one non-zero coordinate of x_i, x_j wins, as $K \geqslant 3$. Thus, condition (1) holds. As for condition (2), suppose $L_i^*(x)$ is nonempty and $|L^0(x)| \geqslant 2$. If $\tilde{\pi}_i(y_i^k, x_j) = 0$ for some k, then K is even and $|L^j(x)| = K/2-1$.

[21]Rather than viewing electoral competition as occurring in the space of proposed policies, as the strategy space one can equivalently use the space of voters' utility profiles generated by policies. In this framework, Colonel Blotto games are the special case in which the strategy spaces are simplices.

Each strategy $y_j^{k'}$ of player j would lose battlefield k' if $k' \in L^0(x)$ but win every other battle in $L^0(x)$. Thus, $\tilde{\pi}_j(y_j^k, x_i) = 0$ if $y_j^k \in L^*(x)$ and $|L^0(x)| = 2$; otherwise, it equals $+1$, as required by condition (2a). If $\tilde{\pi}_i(y_i^k, x_j) = -1$, then $|L^j(x)|$ is the greatest integer not more than $K/2$. Each y_j^k can win at least one of the battles in $L^0(x)$ and thus win the war, giving us condition (2b).

□

Remark 4.2. We need something stronger than the standard rule if payoff approachability is to hold. To see the problems with the standard rule, suppose $K = 3$, we have simple majority rule, $R_1 > R_2$, $i = 2$, and x_i allocates zero resources to the first battle and $R_2/2$ to each of the other two. Suppose x_j is the pure strategy of player $j = 1$ that allocates a positive amount to the first battle and ties with player i on the other two battles. Then using tosses of a fair coin for each of the ties gives player i a probability 1/4 of winning. Every nearby strategy loses.

4.2. **Existence of an Equilibrium With Zero Probability of Ties.** The results above can be strengthened for simple-majority games. For this class of games we use the existence result from Section 3.4, under the tie-breaking rule $\mathcal{T}^S$.

Permutations of the battles induce a symmetry group, and therefore among the equilibria there are some that inherit the symmetries of the game. We show that these equilibria have zero probability of ties except for a single critical value of R_1/R_2.

Assume that $w_k = 1/K$ for all k, so that G is a simple-majority game. Thus a player winning $1 + \lfloor K/2 \rfloor$ battles wins the game.[22] Let $r^* = K/\lceil K/2 \rceil$. Diermeier and Myerson [4] call r^* the hurdle factor and prove the following.

Proposition 4.3. *If $R_1/R_2 > r^*$ then player 1 has a strategy that wins for sure independently of the tie-breaking rule.*

Sketch of Proof. The pure strategy of player 1 that allocates his resources uniformly across all the battles wins the game against every strategy of player 2, and no ties occur that could affect whether player 1 wins. □

In the most relevant case that R_1/R_2 is strictly below the hurdle factor, there exists an equilibrium in which the tie-breaking rule is invoked with zero probability, as we now verify.

Because the game G uses a simple majority to decide the winner, it treats battles symmetrically.[23] Every permutation $\phi : \{1, \ldots, K\} \to \{1, \ldots, K\}$ of the battles defines a

[22] $\lfloor K/2 \rfloor$ is the greatest integer not more than $K/2$, and $\lceil K/2 \rceil$ is the least integer not less than $K/2$.

[23] This feature is also exploited by Hart [10] for the discrete case.

homeomorphism of X_i with itself that sends each x_i to x_i^ϕ where $x_{i,k}^\phi = x_{i,\phi(k)}$ for each k. Obviously $\pi_i(x_i, x_j) = \pi_i(x_i^\phi, x_j^\phi)$ for all ϕ. Let Φ be the set of all permutations. Define $H_i : X_i \to \Sigma_i$ by mapping each x_i to the uniform mixture over the set $\{ x_i^\phi \}_{\phi \in \Phi}$. This map extends to a function from Σ_i to Σ_i. Define $\tilde{\Sigma}_i \equiv H_i(\Sigma_i)$ and let $\tilde{\Sigma} = \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$. There exists an equilibrium σ^* of the game G with the tie-breaking rule $\mathcal{T}^S$ such that $\sigma^* \in \tilde{\Sigma}$ and $\sigma_1^* = \sigma_2^*$ if $R_1 = R_2$. To see this, apply the perturbation method in the proof of Theorem 2.7 but now choosing the strategy sets to be symmetric with respect to the battlefields and perturbing the strategies of both players simultaneously. These perturbed games have an equilibrium that is invariant under all the symmetries of the game and hence the limit of these equilibria as the perturbations shrink inherit the same properties. The following result, proved in the Appendix, shows that ties occur with zero probability in equilibrium.[24]

Theorem 4.4. *If $R_1/R_2 < r^*$ then $(\sigma_1^* \times \sigma_2^*)(D) = 0$. That is, at the equilibrium σ^* the probability is zero that the tie-breaking rule $\mathcal{T}^S$ is invoked.*

Remark 4.5. In the knife-edge case that R_1/R_2 is exactly equal to the hurdle factor r^*, ties can occur in an equilibrium, and minimax strategies can depend on the tie-breaking rule. The uniform strategy described in the proof of Proposition 4.3 continues to be a minimax strategy of player 1 under rule $\mathcal{T}$, or if he wins all ties then again he can assure the value $+1$. But player 1 does not have a minimax strategy if the tie-breaking rule is the standard rule that tosses a fair coin to resolve each tied battle. For simplicity, we illustrate the case $K = 3$, $R_1 = 3/2$, $R_2 = 1$, and $r^* = 3/2$. Let π be the expected payoff function induced by the standard rule. As argued above, because of the symmetry of the battles, if player 1 has a minimax strategy then he has one that is invariant under all permutations of the coordinates. Thus fix a strategy $\tilde{\sigma}_1$ that is invariant under the symmetries of the game. For each x_i, denote the rank order by $(x_{i,k_1}, x_{i,k_2}, x_{i,k_3})$, with $x_{i,k_1} \leqslant x_{i,k_2} \leqslant x_{i,k_3}$ for distinct battles k_1, k_2, k_3. Let $\tilde{\sigma}_1(\{x_1 : x_{1,k_2} \leqslant 1/2\}) = \alpha \geqslant 0$. Observe that the probability of $x_2^1 = (1/2, 1/2, 0)$ winning is bounded below by $(1/6)\alpha$. Thus $\pi_1(\tilde{\sigma}_1, x_2^1) \leqslant 1 - \alpha + 5\alpha/6 = 1 - \alpha/6$. For each $b \in (1/2, 3/4]$, let $\tilde{\sigma}_1(\{x_1 : b/2 + 1/4 \leqslant x_{1,k_2} \leqslant b\}) = \beta(b) \geqslant 0$. Observe that we can find b and $\beta(b) > 0$ when $\alpha = 0$ and that $\alpha = 1$ when $\beta(b) = 0$ for all such b. Now, because for each x_1 with $b/2 + 1/4 \leqslant x_{1,k_2} \leqslant b$, we necessarily have $x_{1,k_1} \leqslant 1 - b$, the probability of the strategy $x_2^b = (b, 1-b, 0)$ winning is bounded below by $(1/6)\beta(b)$. So $\pi_1(\tilde{\sigma}_1, x_2^b) \leqslant 1 - \beta(b)/6$. Combining the two bounds, we must have $\inf_{x_2 \in X_2} \pi_1(\tilde{\sigma}_1, x_2) \leqslant \min\{1 - \alpha/6, 1 - \beta(b)/6\}$.

[24]Zero probability of ties does not imply irrelevance of the tie-breaking rule, since it still has a role in deterring deviations from the equilibrium strategies. We conjecture (at least in the symmetric case where both candidates have equal resources, but possibly also more generally except for a single critical ratio of resources) that the game has an equilibrium that remains an equilibrium for every tie-breaking rule.

Theorem 4.1 above ensures that the game has a value, and the value is independent of the tie-breaking rule. Because the value of the game with payoff function π^+ is $+1$ (the minimax strategy for player 1 assigns $1/2$ to each battlefield), the value of the game with payoff function π is $+1$. So a minimax strategy $\tilde{\sigma}_1$ for player 1 must satisfy $\inf_{x_2 \in X_2} \pi_1(\tilde{\sigma}_1, x_2) = +1$. But this requires that α and $\beta(b)$ are zero for every b, which is impossible. So player 1 does not have a minimax strategy, and a Nash equilibrium cannot exist. Note that this implies that the game with payoff function π violates better-reply security even though the value exists.

5. Concluding Remarks

The absence of general theorems establishing existence of values, minimax strategies, and equilibria of zero-sum majority games has long impeded applications to electoral competition and redistributive politics. In studies of elections, reliance on one-dimensional policy spaces has limited the relevance to practical affairs. In studies of resource allocation in electoral campaigns and lobbying, the absence of general existence results has impaired conclusions about effects of asymmetries in resources available to the candidates. The technical difficulties stem from discontinuities in payoffs at ties, and therefore hinge on how ties are resolved.

Our two general results in Section 2 provide alternative tools. Theorem 2.4 shows that when all ties are resolved in favor of one player then the value exists and that player has a minimax strategy that ensures the value. This conclusion is especially useful in models of elections, where otherwise assumptions about voters' preferences are required. Theorem 2.7 shows that tie-breaking rules satisfying payoff approachability imply better-reply security and therefore equilibria exist that yield the value; and importantly, any other tie-breaking rule yields the same value, so ε-equilibria exist.

This result applies to the models of elections addressed in Section 3, where a tie-breaking rule and the assumed diversity of voters' preferences implies payoff approachability (Theorems 3.9 and 3.19). And it applies to the weighted-majority games of resource allocation addressed in Section 4, where again a particular tie-breaking rule implies payoff approachability (Theorem 4.1), and further, for simple-majority games it implies existence of an equilibrium with zero probability of ties (Theorem 4.4).

Appendix A. Proof of Theorem 4.4

We begin with a preliminary lemma about the payoff function $\tilde{\pi}$ that describes the tie-breaking rule $\mathcal{T}^S$, introduced in Section 3.4 for simple-majority games. In this game, fix

(x_i, σ_j) such that the support of σ_j is finite and contained in $D(x_i)$. Choose $\bar{\varepsilon}$ as in the proof of Theorem 3.9 and fix a neighborhood $V(x_i)$ also as there. The following lemma then proves payoff approachability for (x_i, σ_j) and, additionally, yields properties that we use to prove Theorem 4.4.

Lemma A.1. *There exists $k \in K^*(x_i)$ such that $\tilde{\pi}_i(x_i, \sigma_j) \leqslant \tilde{\pi}_i(y_i^k, \sigma_j)$. Moreover the inequality is strict if one of the following conditions holds:*

1. *$\overline{K}(x_i)$ is nonempty and there is a positive probability of $(T2)$ or $(T3)$ being used.*
2. *$K^*(x_i)$ has at least three coordinates and there is a positive probability of $(T2)$ or $(T3)$ being used.*
3. *$K^*(x_i)$ has two coordinates and $(T2)$ or $(T3)$ is used in resolving a tie (x_i, x_j) for which $L^0(x_i, x_j) \neq \{ k \}$ for one of the k's in $K^*(x_i)$.*
4. *$\overline{K}(x_i)$ is empty and $(T1)$ is invoked for some (x_i, x_j) because i satisfies the conditions for the rule and either: $|L^0(x)| \geqslant 3$ and $\tilde{\pi}_i(y_i^{k'}, x_j) = 0$ for some $k' \in K^*(x_i)$; or $u_{k''}(x_i) \neq u_{k''}(x_j)$ for some $k'' \in K^*(x_i)$.*

Proof. The proof becomes transparent once we compare the payoffs to x_i and y_i^k against x_j for each k and x_j, which we now do.

If $(T1)$ is invoked and $\tilde{\pi}_i(y_i^k, x_j)$ is 0 (resp. -1) for some $k \in L^*(x)$, then $\tilde{\pi}_i(y_i^{k'}, x_j)$ is 0 (resp. -1) for all k' in $L_i^*(x)$, because of simple-majority scoring, and $\tilde{\pi}_i(y_i^{k'}, x_j)$ is 1 (resp. non-negative) for $k' \in K^*(x_i) \setminus L_i^*(x)$. Thus in this case $\tilde{\pi}_i(x_i, x_j) \leqslant \tilde{\pi}_i(y_i^k, x_j)$ for all $k \in K^*(x_i)$, with strict inequality if $\overline{K}(x_i)$ is empty and either: (i) $|L^0(x)| \geqslant 3$ and $\tilde{\pi}_i(y_i^{k'}, x_j) = 0$ for some $k' \in K^*(x_i)$; or (ii) $u_k(x_i) \neq u_k(x_j)$.

If $(T1)$ is invoked because $\tilde{\pi}_j(y_j^k, x_i)$ is 0, then $\tilde{\pi}_i(x_i, x_j)$ is zero if $|L^0(x)| = 2$ and $+1$ if $|L^0(x)| \geqslant 3$. By Assumption 3.13, $\tilde{\pi}_i(y_i^k, x_j)$ is nonnegative in the former case and is $+1$ in the latter. Likewise, if $(T1)$ is invoked because $\tilde{\pi}_j(y_j^k, x_i)$ is -1, then $\tilde{\pi}_i(y_i^k, x_j) = +1$ by Assumption 3.13. In short, $\tilde{\pi}_i(y_i^k, x_j) \geqslant \tilde{\pi}_i(x_i, x_j)$ for all k. Thus, y_i^k does at least as well as x_i against every x_j for which $(T1)$ is applied.

There remains to consider x_j's for which $(T2)$ or $(T3)$ is invoked.

Suppose $L_i^*(x)$ is empty for each i. If $|L^j(x)| = K/2$ then $\tilde{\pi}_i(x_i, x_j) = -1/2$ from $(T2)$. For any $k \in K^*(x_i)$, because $k \notin L^0(x)$, $u_{k'}(y_i^k) > u_{k'}(x_j)$ for all $k' \in L^0(x)$, so $|L^i(y_i^k, x_j)| = K/2$ as well, and $\tilde{\pi}_i(y_i^k, x_j) = 0$. Likewise, if $|L^i(x)| = K/2$, then $\tilde{\pi}_i(x_i, x_j) = 1/2$ from $(T2)$, and because $k \notin L^0(x)$, $\tilde{\pi}_i(y_i^k, x_j) = 1$. Summing up, $\tilde{\pi}_i(y_i^k, x_j) - \tilde{\pi}_i(x_i, x_j) = 1/2$ if either $|L^i(x)|$ or $|L^j(x)|$ equals $K/2$. This difference is equal to $+1$ otherwise (i.e. if neither of the candidates has half the votes outside of $L^0(x)$). Thus all y_i^k's do strictly better against all these x_j's.

Suppose $L^0(x)$ contains just one voter, say k. If $k \notin K^*(x_i)$, then the payoff difference is as in the previous paragraph. If $k \in K^*(x_i)$, then $\overline{K}(x_i)$ is empty (by point (1) of Assumption 3.13) and $\tilde{\pi}_i(y_i^k, x_j) - \tilde{\pi}_i(x_i, x_j) = -1 + 1/2 = -1/2$ (resp. $0 - 1/2 = -1/2$) if $|L^j(x)| = K/2$ (resp. $|L^i(x)| = K/2$). This difference is equal to -1 if neither of the candidates has half of the voters outside of $L^0(x)$. But, observe that for every $k' \neq k$ in $K^*(x_i)$, $\tilde{\pi}_i(y_i^{k'}, x_j) - \tilde{\pi}_i(x_i, x_j) = \tilde{\pi}_i(x_i, x_j) - \tilde{\pi}_i(y_i^k, x_j)$, as $u_k(y_i^{k'}) > u_k(x_j) > u_k(y_i^k)$. Thus all $y_i^{k'}$'s do strictly better against all these x_j's.

Finally suppose $L^0(x)$ contains at least two voters, either $L_i^*(x)$ or $L_j^*(x)$ is nonempty but each player for whom it is nonempty that he can achieve $+1$ rather than 0 or -1 specified there. Then if $L_i^*(x)$ is nonempty $\tilde{\pi}_i(y_i^k, x_j) = 1$ for some $k \in L_i^*(x)$ (otherwise $(T1)$ would apply) and it holds for all k while $\tilde{\pi}_i(x_i, x_j) = 0$; on the other hand if $L_i^*(x)$ is empty, then trivially each y_i^k achieves $+1$.

We now complete the proof of the lemma as follows. Obviously if $\overline{K}(x_i)$ is nonempty, then y_i^k does at least as well as x_i against each x_j in the support of σ_j and strictly better against all x_j's for which $(T1)$ is not invoked, proving the first statement and points (1-3) of the second, with point (4) being vacuously true. Assume from now on that $\overline{K}(x_i)$ is empty.

Each y_i^k does as well against all x_j to which $(T1)$ applies and strictly better against those x_j's for which the condition of point (4) of the lemma holds. If $(T2)$ or $(T3)$ is not used with positive probability then the first statement of the lemma holds as does point (4), while points (1-3) are vacuous.

Suppose $(T2)$ or $(T3)$ is invoked with positive probability. If there is one k for which no tie is just on this voter's utility, then y_i^k does strictly better than x_i as the calculations above show. Thus, the inequality holds, regardless of the conditions of points (2)-(4), if there is such a k. Suppose then that for each $k \in K^*(x_i)$ there is an x_j that ties with x_i just on k. It is clear that at least one of the y_i^k's would do as well as x_i against σ_j. Moreover, if there are at least three coordinates in $K^*(x_i)$, one of them would do strictly better, proving point (2). Also, if there are only two such k's then one of them would do strictly better than x_i against σ_j unless each tie involves exactly one of these k's, which proves point (3). Observe that when there are two such k's, and x_i is not inferior to some y_i^k against σ_j, then x_i and each y_i^k give the same payoff against the conditional distribution over the x_j's for which $(T2)$ or $(T3)$ is used.

Coming to ties involving $(T1)$ it is clear now that if there is a tie with an x_j where the rule is invoked because of i, then for x_i to do at least as well as all y_i^k, we must have $K^*(x_i) \subset L^0(x)$ and $\tilde{\pi}_i(y_i^k, x_j) = -1$ for each $k \in K^*(x_i)$ if $|L^0(x)| > 2$. If this is violated for some x_i and if x_i is already not dominated by some y_i^k against the conditional over x_j's where $(T1)$ is not used, then $K^*(x)$ has two coordinates and as we argued at the end of the last paragraph each k would do equally well against those not involving $(T1)$, with the result that it would do strictly better against σ_j, proving point (4). $\square$

We now recall and prove Theorem 4.4 for simple-majority Colonel-Blotto games.

Theorem 4.4. Let σ^* be an equilibrium that is invariant under all the symmetries of the game. If $R_1/R_2 < r^*$ then $(\sigma_1^* \times \sigma_2^*)(D) = 0$, that is, at the equilibrium σ^* the tie-breaking rule $\mathcal{T}^S$ has zero probability of being invoked.

We set up some notation and prove a number of preliminary claims before proving the theorem. Suppose x_i is a strategy in X_i such that $\sigma_j^*(D(x_i)) > 0$. We can decompose σ_j^* into σ_j^{c,x_i} and σ_j^{d,x_i}, where the former puts zero probability on $X_j \setminus D(x_i)$ and the latter puts probability one on it. Let $\mathcal{L}(x_i)$ be the set of quadruples $L = (L^0, L^i, L^j, Tn)$ such that there is a positive probability under σ_j^* of the set $D^L(x_i)$ consisting of x_j's such that $(L^0, L^i, L^j) = (L^0(x_i, x_j), L^i(x_i, x_j), L^j(x_i, x_j))$ and provision (Tn) of rule $\mathcal{T}^S$ is used, where $n \in \{1, 2, 3\}$. For simplicity, from here on we suppress Tn in the notation. For each L choose a point $x_j(L) \in D^L(x_i)$ and consider the conditional distribution $\tilde{\sigma}_j^{x_i}$ over the $x_j(L)$'s given by $\tilde{\sigma}_j^{x_i}(x_j(L)) = \left(\sum_{L'} \sigma_j^*(D^{L'}(x_i))\right)^{-1} \sigma_j^*(D^L(x_i))$. Choose a neighborhood $V(x_i)$ such that for each $y_i \in V(x_i)$ and L, $y_{i,k} > x_{j,k}(L)$ if $k \in L^i$, and $y_{i,k} < x_{j,k}(L)$ if $k \in L^j$.

Claim A.2. $\tilde{\pi}_i(x_i, \sigma_j^*) = \sigma_j^*(X_j \setminus D(x_i))\tilde{\pi}_i(x_i, \sigma_j^{c,x_i}) + \sigma_j^*(D(x_i))\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i})$.

Proof. As the payoff $\tilde{\pi}_i(x_i, \cdot)$ is constant on each $D^L(x_i)$, $\tilde{\pi}_i(x_i, \sigma_j^{d,x_i}) = \tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i})$ and the result follows. $\square$

Claim A.3. *If x_i is a best reply to σ_j^*, then $\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i}) \geqslant \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i})$ for all $k \in K^*(x_i)$.*

Proof. Assume to the contrary that $\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i}) < \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i})$ for some $k \in K(x_i)$. For each $\varepsilon > 0$ let $W^\varepsilon(x_i)$ be the set of y_i such that $|y_{i,k} - x_{i,k}| < \varepsilon$. For each L, let $D^{\varepsilon,L}(x_i)$ be the set of x_j in $D^L(x_j)$ such that $|x_{i,k} - x_{j,k}| > \varepsilon$ for $k \notin L^0$ and let $D^\varepsilon(x_i)$ be the union of the $D^{\varepsilon,L}(x_i)$'s. Choose ε small enough such that each $x_j(L)$ belongs to $D^\varepsilon(x_i)$. Define $\tilde{\sigma}_j^{\varepsilon,x_i}$ to be the distribution over $x_j(L)$ that assigns probability $\sigma_j^{d,x_i}(D^{\varepsilon,L}(x_i))/\sum_{L'} \sigma_j^{d,x_i}(D^{\varepsilon,L'}(x_i)$ to $x_j(L)$. By construction $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \cdot)$ is constant on the set $D^{\varepsilon,L}(x_i)$ for each L and $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), x_j) \in [-1, 1]$ for all x_j. Hence,

$$\tilde{\pi}_i(y_i(W^\varepsilon(x_i), \sigma_j^{d,x_i}) \in (\sigma_j^{d,x_i}(D^\varepsilon(x_i)))\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \tilde{\sigma}_j^{\varepsilon,x_i}) \pm \sigma_j^{d,x_i}(D(x_i) \setminus D^\varepsilon(x_i)) \,.$$

Obviously $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), x_j(L)) = \tilde{\pi}_i(y_i^k, x_j(L))$ for all $x_j(L)$. Moreover, $\tilde{\sigma}_j^{\varepsilon,x_i}$ converges to $\tilde{\sigma}_j^{x_i}$ and $D^\varepsilon(x_i)$ converges to $D(x_i)$. Therefore, $\lim_{\varepsilon\downarrow 0} \tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \sigma_j^{d,x_i}) = \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i}) >$

$\tilde{\pi}_i(x_i, \sigma_j^{d,x_i})$. Since $\lim_{\varepsilon\downarrow 0} \tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \sigma_j^{c,x_i}) = \tilde{\pi}_i(x_i, \sigma_j^{c,x_i})$, we then have that $\tilde{\pi}_i(x_i, \sigma_j^*) < \lim_{\varepsilon\downarrow 0} \tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \sigma_j^*)$ and σ_j^* is not a best reply to σ_j^*, a contradiction. □

The next three claims argue directly about points $(x_i, x_j) \in D$.

Claim A.4. *If x_i is a vertex, then there exists x_j' obtained by permuting the coordinates of x_j such that ($T1$) does not apply to (x_i, x_j').*

Proof. Let x_i be a strategy that assigns R_i to a battle, say $k = 1$. Observe first that for ($T1$) to be used in deciding a tie between x_i and x_j's, this battle must belong to $L^0(x)$. If $R_1 = R_2$, this means that $x_i = x_j$ and ($T3$) is operative. If $R_1 > R_2$, then $i = 2$ and $\tilde{\pi}_i(y_i^1, x_j) = -1$. Since $R_1 < r^* R_2$, there exists some $k' \neq 1$ such that $0 < x_{j,k'} < R_2$. There exists some x_j' that swaps these two coordinates and now ($T3$) applies to (x_i, x_j'). □

Claim A.5. *Suppose x_i is not a vertex, and ($T1$) applies to $(x_i, x_j) \in D$. If $\tilde{\pi}_i(y_i^k, x_j)$ is either 0 or -1 for some $k \in L^*(x)$, then either: (i) there exists $k' \in K^*(x_i)$ such that $x_{i,k'} \neq x'_{j,k'}$ for some x_j' obtained from permuting the coordinates of x_j; or (ii) $|L^0(x)| \geqslant 3$ and $\tilde{\pi}_i(y_i^k, x_j) \geqslant 0$ for some $k \in L_i^*(x)$.*

Proof. If $R_1 = R_2$, conclusion (i) is valid, since otherwise $x_i = x_j$ and ($T3$) would apply. If $R_1 > R_2$ and $i = 1$, then conclusion (i) is obvious.

Assume now that $i = 2$, $R_1 > R_2$ and conclusion (i) of the claim is violated. Then $x_{i,k} = x_{j,k}$ for each positive coordinate of x_i. If $\tilde{\pi}_i(x_i, x_j) = 0$ for some k, then K is even, $|L^0(x)| = 2$, and $|L^j(x)| = K/2 - 1$, while if $\tilde{\pi}_i(x_i, x_j) = -1$, then either $|L^j(x)| = \lfloor K/2 \rfloor$ (K can be odd or even) or $|L^0(x)| \geqslant 3$ K is even and $|L^j(x)| = K/2 - 1$. If $|L^0(x)| = 2$, then $|L^j(x)| = K - |L^0(x)| = K - 2 > \lfloor K/2 \rfloor - 1$. Thus, when $|L^j(x)| = K/2 - 1$, $|L^0(x)| \geqslant 3$.

If $|L^j(x)| = \lfloor K/2 \rfloor$, then $|L^0(x)| = \lceil K/2 \rceil$. Therefore, there exists k' such that $x_{i,k'} \geqslant R_2/\lceil K/2 \rceil$. Moreover, since $|L^j(x)| = \lfloor K/2 \rfloor$, and $R_1 < r^* R_2$, there exists a coordinate k'' such that $x_{i,k''} = 0 < x_{j,k''} < R_1 - R_2 < R_2/\lceil K/2 \rceil$. There exists x_j' that swaps these two coordinates and $(x_i, x_j') \in D$. Now there is a coordinate, namely k', for which $x_{i,k'} > x'_{j,k'}$, a contradiction. So (i) must hold.

If $|L^j(x)| = K/2 - 1$ then, as we saw above, $|L^0(x)| \geqslant 3$. Therefore, $\tilde{\pi}_i(y_i^k, x_j) = 0$ for each $k \in L_i^*(x)$, which proves (ii). □

Claim A.6. *Suppose $(x_i, x_j) \in D$, both x_i and x_j have two positive coordinates, $L^*(x_i)$ is nonempty, and ($T2$) or ($T3$) applies. There exists another x_j' obtained by a permutation of coordinates from x_j where ($T2$) or ($T3$) applies as well but where (x_i, x_j') are either tied in two or more coordinates or in a zero coordinate.*

Proof. Suppose x_i and x_j are tied in just coordinate, say $k = 1$, and that this coordinate is positive for both players. Then $K = 3$ and i wins, say, $k = 2$ and j wins $k = 3$. Derive x'_j from x_j by permuting coordinates 2 and 3. x'_j ties with x_i in coordinates 1 and 3. □

Proof of Theorem 4.4. Fix $x_1 \in D$ such that $\sigma_j^*(D(x_i)) > 0$. We show that x_i is not a best reply to σ_j^*, which proves the result.

Fix x_j in $D(x_i)$. Let $L = (L^0(x), L^i(x), L^j(x))$. Observe that if x'_j is obtained by permuting coordinates of x_j, then there exists $x'_j(L')$ in the support of $\tilde{\sigma}_j^{x_i}$ where $L' = (L^0(x_i, x'_j), L^i(x_i, x'_j), L^j(x_i, x'_j))$. Using this fact, the proof of the theorem follows quite easily. If x_i is a vertex, by Claim A.4, point (1) of Lemma A.1 holds for $\tilde{\sigma}_j^{x_i}$, and by Claim A.3, x_i is not a best reply to σ_j^*.

The other cases work similarly. If x_i is not a vertex, but $(T1)$ applies to (x_i, x_j), then combining Claim A.5, point (4) of Lemma A.1 and Claim A.3 proves the result.

If $(T2)$ or $(T3)$ applies to (x_i, x_j), then by point (2) of Lemma A.1, x_i has only two non-zero coordinates. Claim A.6, point (3) of Lemma A.1 and Claim A.3 finish the proof. □

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