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THE FINANCING OF A PUBLIC UTILITY

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Abstract

The interaction of capital stock with overlapping generations is investigated where the time structures of human capital and other physical capital does not match. We consider the economies with either gold or fiat as the outside money and consider the financing problems that appear in the financing of capital stock. The complexity of the underlying physical structure combined with concern for efficiency and equity help to determine the financial structure.

Key words: capital stock, time scales, fiat, gold, joint product.

JEL Classification: C73, D24,.E22, E41

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1 The Economy: Time, Size and Complexity:

1.1 Dynamics, Finance and Institutions

In this essay we consider a communal financing problem that could describe institutionally the type of problem faced in a rural community towards the end of the nineteenth century with a relatively sparse population, but an opportunity for the construction of a utility such as a grain silo, or a power plant; where the product could be sold to the individuals, but the construction could be achieved by communal ownership.

Our purpose in presenting these extensions of a strategic market game is to show how extra physical facts added to basic economic process models can be used to illustrate the natural extension of the complexity of financial control instruments in the economy. The call for the efficient finance of the economy arises as an economic imperative in the need to deal with the underlying broad physical facts of durable goods and the human life span.

The importance of capital stock was well known and emphasized by the Austrian school of economics and others such as Böhm-Bawerk, Ackerman and Wicksell; however mathematical treatments that cover both capital structure and finance require considerable detail.

In a previous paper [10] we compared economies with gold and with fiat money. We continues this contrast in a more complex setting.

1.2 Comments on Process

The embedding of the economy within the framework of government and society provides both a natural formal and an informal control system. The government provides the formal rules with the laws and their enforcement, and the society and polity on different time scales provide the pressures on the government on rule formation and the direct pressures on the economy to conform to custom as well as law. The price system where it exists provides a perception device where the pressures of disequilibrium are signalled by the shadow prices that develop on both commodities and the price of loans and other financial instruments.

2 The production and exchange economy in a dynamic setting with overlapping generations.

The simple transformation to a fully defined multistage economy immediately exposes the difficulties with the forward looking features of the noncooperative equilibrium solution and calls for the need for coordination. The literature on repeated games (Mailath and Larry Samuelson [6], Fudenberg and Levine [3], for example) illustrates many inadequacies and redundancies in the unqualified concept of the noncooperative equilibrium. Do individuals look ahead and compute infinite horizon strategies that are exceptionally consistent, or do they use simpler rules of thumb such as a local backward looking optimal response strategy? As soon as one writes down some form of difference or differential equation to describe process it is easy to spell out time paths with virtually any trajectory including cycles, bubbles, inflations and deflations of any magnitude (there is a formal mathematical literature on cycles dating from at least Harrod [4] and Domar [1]. The mathematical literature on growth theory includes Frank Ramsey [8], von Neumann [12], Phelps [7] (on the golden rule), and many others. Hicks [5] provides simple difference equation models with easy macroeconomic interpretations that display growth, decline and cyclic behavior in more or less elementary mathematical models.

A way of coping with these situations appears to lie more in the realm of political economy with government providing a control mechanism than in refining the concept of a noncooperative equilibrium.

2.1 Social and political process and jointly owned goods

Even were we to imagine a society with no innovation or evolution, but nevertheless with consumption, production and jointly owned goods we would be required to solve two different types of coordination and control processes. They are the production and distribution of private goods and jointly owned goods. We note that natural problems of scale can easily call forth the need for joint forms of ownership and the needs for financing in even elementary situations.

2.2 Pareto optimality, welfare functions and political process

In our simplifications we have resurrected older somewhat ill-defined concepts such as a social welfare function (SWF) and money measure in a modern garb because we believe that as approximations they are productive and can be justified methodologically. In particular when all individuals have identical preferences a SWF can be defined and efficiency can be measured. This is important because although Pareto optimality (PO) can be well defined in a no-process or cost-free-process world, when the enforcement and coordination mechanism itself absorbs resources PO must be replaced with optima in a cost reduced feasible set. A comparison of mechanisms within an appropriate domain is called for.

The distribution of public goods is primarily a socio-political and only secondarily an economic process. Direct representative government, indirect representation and a host of other mechanisms guide both the procurement and allocation of these goods. Jointly owned accumulations of capital assets such as public utilities and privately held corporations lie somewhat closer to the simple world of individually owner-fungible chattels than do many complex public goods.

3 The economic control problem

The distinguished macroeconomist, Jim Tobin regarded macroeconomic analysis as utilizing a short run closed general equilibrium model of the economy open in the longer run to the polity and society so that many key parameters and institutional structures required re-estimation or restructuring frequently to take into account the changes caused by the polity, the society and technology providing feedback of different lengths on an evolving economy that nevertheless for periods ranging from a few months to a few years could be usefully regarded as a closed general equilibrium system for the answering of some economic questions. Although this insight can be easily expressed verbally, the making of the formal connections between the evolving system and the static analysis calls for the structure of fully defined process models with a parsimonious representation of how the economy connects to its polity and society.

In his inaugural address the great mathematical economist Edgeworth

posed the problem that has illustrated the gap between pure abstraction and application, for many years

> It is worth while to consider why the path of applied economics is so slippery; and how it is possible to combine an enthusiastic admiration of theory with the coldest hesitation in practice. The explanation may be partially given in the words of a distinguished logician who has well and quaintly said, that if a malign spirit sought to annihilate to whole fabric of useful knowledge with the least effort and change, it would by no means be necessary that he should abrogate the laws of nature. The links of the chain of causation need not be corroded. Like effects shall still follow like causes; only like causes shall no longer occur in collocation. Every case is to be singular; every species, like the fabled Phoenix, to be unique. Now most of our practical problems have this character of singularity; every burning question is a Phoenix in the sense of being sui generis.
>
> F.Y. Edgeworth, 1891[2]

We are in accord with Edgeworth but do not interpret this as a counsel of despair. Instead it says to us that general theory is no substitute for knowing your business. In application the perceptors need micro-detail. As in military theorizing the selection of goals and grand strategy when applied must be in concord with tactics. And tactics require the appreciation of detail. The cries of the practical businessman against the theorist need to be considered seriously by the theorist.

We argue that the act of converting a timeless static equilibrium model into a playable strategic market game forces us to open the elegant but lifeless static model to its environment. Little details like default rules, inheritance rules, accounting rules emerge even at a minimalist level as necessities in constructing a minimally viable organization. There may be a vast array of minimal organizations reflecting the ecological richness of an economy embedded in a polity and society. But these all still obey the general laws.

The criterion of minimality can be well defined and is at the essence of economics. Any item removed from a minimal model will prevent the performance of some function it is meant to perform. Thus minimality is associated with the level of complexity reflected by the functions.

Money and credit provide not only the possibility for a decentralized system but a part of the need for various levels of coordination and control. They provide the sufficient conditions for the functioning of a loosely coupled system. Any system that remains robust under change must, perforce, be loosely coupled.

In this essay we develop and analyze an OLG model with publicly owned capital stock such as an irrigation system, community silos or a power plant or other utility to demonstrate how the needs for financing arise from the basic physical dynamics. In our investigation we show that the utilization of a fiat monetary system may be both more flexible and more dangerous than one utilizing gold as a money.

4   A production and exchange OLG economy

The models developed below illustrate the need for the introduction of new financing and control features in a system where durables have finite lives larger then one. When construction time, length of life time of the asset and consumption timing all differ, efficiency considerations call forth somewhat sophisticated finance.

4.1   Construction of a playable OLG SMG

Our view of economic process models is that there are myriads of plausible feasible models and in virtually any area of investigation there is a multiplicity of choices. However these can be judiciously pruned in concert with the questions being asked. In the listing below we present a large shopping list of features pertaining to OLG models and indicate by a "*" or a comment the modeling choices we have made in the models that follow.

1. **Time segment:** (a) $[T_1,T_2]$, (b) $[T_1, \infty)$*, or (c) $(-\infty, \infty)$

2. **Number of types of legal persons**, (a)natural persons alone, (b) natural and corporate persons*. The latter are directly or indirectly fully owned by the former.

3. **Life span of Natural persons**: They live T_1 years. *We select $T_1 = 2$.*

4. **Life span of Corporation:** They live T_2 years; but if they are extant at the end of a finite game they (including the government) are liquidated on the day of final settlement. *We select $T_2 = 3$.*

5. **Agents: (a)** Representative* or (b) individual agents* (c) both. *When there is no exogenous uncertainty the distinction between representative agents and type-symmetric individual agents may not matter.*

6. **Price formation:** The economy may be modeled as (a) sell-all, (b) buy-sell* or (c) a bid-offer strategic market game.

7. **Number of types of goods and services**:. They are: (a) labor/leisure*; (b) services; (c) perishable consumables*; (d) storable consumables; (e) reproducible durables*; (f) non-reproducible durables or land.

8. **Depreciation rates:** There are many *but as an extreme case simplification rather than dealing with various discount rates we give all entities specific lives (the one-hoss shay phenomenon)*.*

9. **Length of production time:** For simplicity we may assume that the length of production for all producible items is 1 period. *We note however that for many services such as supermarkets, electric plants one may build the durable, it then supplies services for may years and new construction may not be needed for many periods. This is central to our models**

10. **The role of the banks:** *We model a dummy inside bank that makes loans or accepts deposits of gold at a fixed rate of interest ρ, or an outside bank that stands ready to make one-period loans of fiat. The interest rate ρ is determined by policy objectives of the banks and properties of solutions determined by the markets.*

11. **The default condition** is given as part of the rules of the game. For simplicity it is introduced as a quasi linear term that connects money with utility.

12. **Type of money:** (a) barley, (b) gold*, (c) fiat* An official government money is specified.. All trades are made in government money. All borrowing and lending is via the bank. *We consider both gold and fiat in different models.*

13. **Initial conditions** consist of a vector of initial physical resources and financial instruments owned by all the n natural persons together with a set of production transformation sets that are owned by the k existing firms (here $k = 1$). We assume that there is a vector of estimated or predicted initial first period prices that exists for all real and financial assets. This enables us to place an estimated monetary value on the initial bundle of assets.

14. **Terminal conditions:** In full generality terminal conditions should be full algorithms dependent on the path down the tree. *We make a great simplification by defining models for which terminal conditions do not propagate more than a finite number of individual life-cycles into the interior solutions of the OLG, so that steady-state solutions we compute in the interior are independent of a large class of changes in detail of the terminal conditions.*

15. **The process of liquidation:** At the end of the game this would require that all non-real corporate legal persons be liquidated at **the day of final settlement.** The order of settlement is that all firms pay back their loans and if they have negative money this is flowed through to the real persons. The firms are then liquidated at the initial prices assigned in the first period. Any profits or losses of the central bank are flowed through; all real and financial assets are then liquidated at the initial prices. At this point the real persons must settle their accounts. *We avoid these complications here.*

16. **The behavior of the firms:** firms can be modeled as (a) strategic dummies* (b) price taking agents, (c) power strategic players. *Here there is only a communal enterprise that mechanically converts nonconsumable inputs into consumable service streams. The proceeds from the sale of these streams are distributed in the manner of dividends or partnership shares. The conversion efficiency or payout is a parameter of the system. Here for simplicity we assume it is 100%.*

17. **The definition of short term profits** is another free parameter of the system as it is in accounting systems. We define short term profits as the revenues from sales minus the direct costs of the directly relevant inputs.

18. **Inheritance conditions:** There are many mixed monetary and non-monetary ways this can be defined. *We want the inheritance conditions to reflect the condition "I want the next generation to be as well off as I am". To define minimal models that introduce the fewest new ad hoc parameters, we will use only the utility functions already defined for agents, and we will introduce inter-generational transfers only when these are needed to overcome constraints of the production functions that would lead to zero consumption of some essential quantities and thus to singular solutions.*

19. **Preferences and utility:** The utility function, together with complete preferences is a hard pill to swallow; but without enormous complication it appears to be about as good a crude economic approximation as one can produce.[1] Edgeworth included a quasi-concave term for concern of others and one can consider his "coefficient of concern" (conventionally denoted θ) with $\theta = 1$ to be the equivalent of your children should have at least the chance you had. For purposes of our producing a model for the financing of a capital good that can be explicitly analyzed we select a specific simple form for the utility function as is noted below.

5 Production and Exchange OLG Economies with Gold or Fiat

In a previous essay [10] we considered the microstructure of production, trade, and consumption within a single generation, modeled with a large number of symmetric periods. Here we consider multiple timescales created by lifetimes of institutions or capital goods with sunk costs, which may be longer than a generation for agents. The relation of periods in the lifecycle of goods to the lifecycle periods or generations of agents may also be heterogeneous, creating a mismatch with agent preferences in situations where the latter are time-symmetric. The stress on the market system comes from the need to smooth over such mismatches between material constraints and agent preferences. A further source of stress that we introduce comes from non-convexity in the production process, particularly in the form of capacity

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[1]We need to seek some form of sensitivity analysis to help to justify this simplification that is difficult in the extreme to measure.

constraints to exploit higher-efficiency production methods. Thresholds for feasible production together with long lifetimes are common characteristics of goods produced by firms or publicly held utilities.

5.1 Introduction to particular OLG models

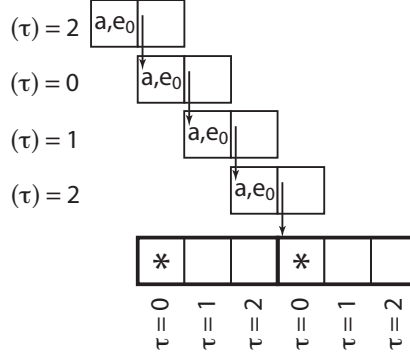The time structure we consider is shown in Fig. 1.

Figure 1: Structure of the overlapping generations model with episodic construction of capital stock. Boxes represent periods. Columns represent time, indexed τ, and rows indicate birth-time, indexed (τ). Cascading two-period rows indicate generations of farmers or prospectors, with the endowment indicated by the parameters a or e_0 in the boxes representing the young period. Three-period heavy boxes at the bottom indicate the service cycle of capital stock, with the period of production indicated by $*$. Vertical arrows show the times at which intergenerational transfers of gold may be made by farmers.

We abstract to two types of production functions: one for non-durable consumables which we consider in aggregate and refer to as "food", and the other for a durable that we call "gold", which may be used as money but is also an input to production. As in Ref. [10], the choice of a production function remains a commitment over an agent's lifetime, so agents have two types, again termed *farmers* and *prospectors*.

5.1.1 Time structure for capital stock as a "One-Hoss Shay"

Capital stock introduces the new longer timescale into the model. A minimal model of agents is a standard two-period overlapping-generations (OLG)

model, for which initial conditions must be specified, but which may be indefinitely repeated thereafter.[2] We term the two periods "young" and "old" for any generation of agents.

We suppose that a production/retirement cycle for capital stock requires three periods, and we distinguish the generations that are young in each period with superscript $\tau \in \{0, 1, 2\}$ cyclic, where period 0 is the period in which capital stock is built. We consider an endless sequence of cycles of capital-stock production and retirement. Capital stock consumes a finite quantity of gold to be constructed, delivers utilitarian services at a fixed rate over its life, and disappears entirely at the end of its third period. (It is a "One-Hoss Shay" rather than a depreciating asset.) It therefore provides the mechanism by which non-depreciating gold exits the system.[3]

5.1.2 Production with thresholds

We wish to consider the general class of cases in which efficiency gains from scale are possible, but a threshold unit size is required to capture them. This problem is similar to the problem of exploiting the gains from specialization considered in Ref. [10], but in an OLG setting.

Publicly owned (government or large-scale corporate) works are often of this kind, including dams, power plants, mass-production assemblies, etc. Typically a unit capacity $\mathcal{C}$ exists for a minimal unit, and these units can then be replicated in integer numbers. We avoid the complexities of integer programming as far as possible by focusing on the threshold for production of the first unit, and considering production thereafter to be linear in the invested amount.

Fig. 2 shows a rationale for this model of production. Three production functions are shown on a log-log scale. For a society large enough to far exceed the capacity $\mathcal{C}$ for investment, the steps of integer production are minor perturbations. For a society that cannot reach the capacity at any

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[2]While initial conditions are required for a well-defined game, we typically regard them as an aggregate representation for the legacy of a long past. For the models we will introduce, even though agent generations repeat indefinitely, for a wide range of terminal conditions, cross-generation interactions truncate over a finite number (here, zero or one) of generations. Therefore, the precise statement of the role of terminal boundary conditions is that among a wide range of specifications (all those that do not involve divergent salvage values for any gold held by agents in the last period), differences do not affect strategic choices sufficiently prior to the last generation.

[3]It may be still physically present but in a form that makes its reclaim uneconomic.

allocation of labor, only linear production with a reduced efficiency ε can be attained. This low-yielding production serves the same function as a fallback position that autarchy served in Ref. [10].

Our interest is in the intermediate range, where the society under a strained labor allocation can meet the capacity constraint, but sufficient strain makes this no more favorable for some class of agents than autarchy. Our model of production (the blue curve in Fig. 2) treats production above capacity $\mathcal{C}$ with the same linear form as autarchy, but higher efficiency. This upper-semicontinuous function permits an invariant utility when agents can meet the threshold at the non-cooperative equilibrium of the OLG game, because all production and consumption are homogeneous of order one in population size. At an expanding population equilibrium the extra production provides enough to sustain a constant living standard.
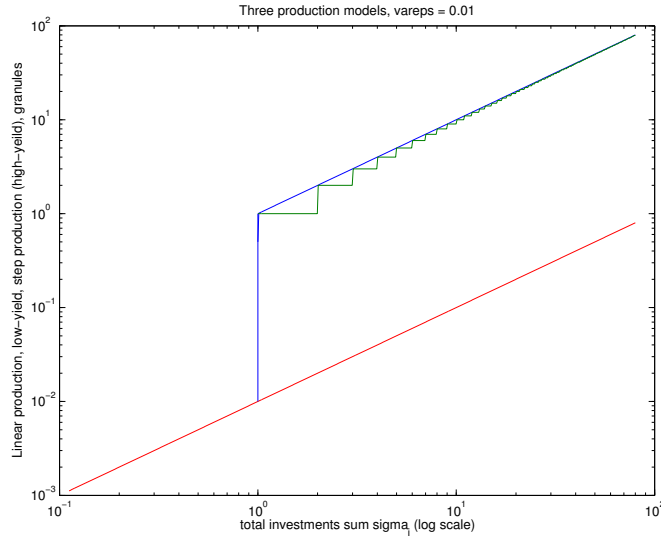

Figure 2: Three production functions. Red is the low-yield proportional production, with $\varepsilon = 1/100$, as used in the later numerics. Green is the granular production we might consider realistic, with a fixed unit size, resulting in a stairstep. Blue is our model, which keeps the first step – the most important, in relation to the low-yield fall-back – but replaces the subsequent steps with a linear production function for ease of handling. The range shown for labor below the capacity constraint – between 10^{-1} and 10^{0} – is the range of stress that we model in numerical simulations.

5.1.3 Production efficiency for the low-yield capital stock

The utility value of capital stock is determined by its rate of delivery of a service which we denote S. We measure the service in the same units as the gold invested to build the capital stock, to avoid introducing a distinct type of unit. The important feature of capital stock is that, once built, the rate of services it yields is constant over three periods until its service life ends and it must be replaced. Therefore, for each amount of capital stock shown in this and later sections, the total service stream delivered multiplies that amount by three.

We represent the quantity of capital stock built as an upper-semicontinuous function of the investment level, and we distinguish low-yielding from high-yielding production by labeling these functions $C_<$ and $C_>$, respectively.

The amount of capital stock formed with the low-yielding production process corresponds to the linear functional form

$$C_< \left(\sum_i \sigma_i \right) \equiv \varepsilon \sum_i \sigma_i, \tag{1}$$

where $\sum_i \sigma_i$ represents the investments of all agents who can invest in period $\tau = 0$. The sum takes the value $\sum_i \sigma_i = n \left(\sigma_0 + \sigma_1 \right) + n_0 \left(\hat{\sigma}_0 + \hat{\sigma}_1 \right)$ where:

$\sigma_i \equiv$ investment by farmers,

$\hat{\sigma}_i \equiv$ investment by prospectors.

There is considerable notation associated with the model; we define the new symbols as they are introduced, but for convenience at the end of the appendices a full listing is given.

5.1.4 Production efficiency for the high-yield capital stock with threshold

The amount of capital stock formed with the high-yielding production process defines the function $C_>$ in terms of a threshold function c,

$$C_> \left(\sum_i \sigma_i \right) \equiv c \left(\sum_i \sigma_i \right), \tag{2}$$

where $c(y)$ approximates the discrete threshold function

$$c(y) \approx y \, \Theta(y - \mathcal{C}) . \tag{3}$$

In Eq. (3) Θ is the Heaviside function and $\mathcal{C}$ the threshold to produce the first unit.[4] The nature of this approximation, and the relation to the low-yield production process, are shown in Fig. 3, and discussed in App. B.
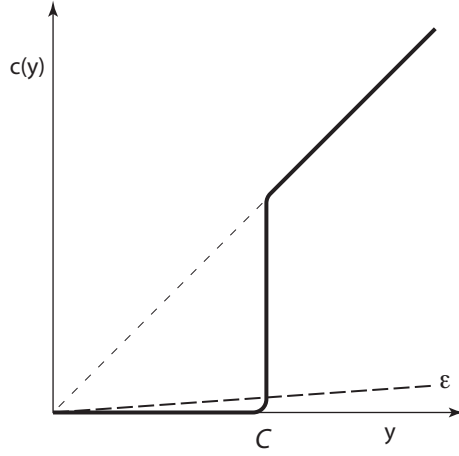

Figure 3: Low-yielding (heavy-dashed) and high-yielding (heavy solid) production functions. Low-yielding function $C_<(y) = \varepsilon y$, while the high-yielding function has form $c(y) \approx y \, \Theta(y - C)$, with derivatives at transitions smoothed to make optimization criteria well-defined.

5.1.5 Time-symmetric and type-symmetric consumption utilities

All agents are given identical functional forms of consumption utility for both food and the services delivered by capital stock. The endowment a for food to farmers and the endowment e_0 for gold to prospectors set the scales for consumption. A fully-specified consumption bundle is a quantity A or S of food or services, a subscript index i for the agent, superscript (τ) for the generation in which the agent was born, and further subscript 0 or 1 to indicate whether the agent is in the young or old period of life. A

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[4]The Heaviside function $\Theta(x) \equiv 0$ if $x < 0$ and $\Theta(x) \equiv 1$ if $x \geq 0$.

Cobb-Douglas consumption utility[5] for agent i then becomes

$$\mathcal{U}_i^{(\tau)} = \log\left(\frac{A_{i0}^{(\tau)} A_{i1}^{(\tau)}}{(a/2)^2}\right) + s\log\left(\frac{S_{i0}^{(\tau)} S_{i1}^{(\tau)}}{e_0^2}\right) + \theta s\log\left(\frac{S_{i0}^{(\tau+1)} S_{i1}^{(\tau+1)}}{e_0^2}\right). \quad (4)$$

where:

$A_{i0}^{(\tau)}$ = food consumption of the focal generation-(τ) agent i when young.

$A_{i1}^{(\tau)}$ = food consumption of the focal generation-(τ) agent i when old.

$S_{i0}^{(\tau)}$ = services consumption of the focal generation-(τ) agent i when young.

$S_{i1}^{(\tau)}$ = services consumption of the focal generation-(τ) agent i when old.

$S_{i0}^{(\tau+1)}$ = services consumption of the equivalent offspring generation-$(\tau+1)$ to agent i, when young.

$S_{i1}^{(\tau+1)}$ = services consumption of the equivalent offspring generation-$(\tau+1)$ to agent i, when old.

Our parameter θ is one exemplar of Edgeworth's "coefficient of concern" of one generation for the next. Our instantiation is mediated by the *particular* concern about the offspring-generation's consumption of services from capital stock, which we choose because it addresses the most important potential allocation failure in OLG economies, and the agents can avert shortages either by intergenerational transfers or by borrowing.

We will quickly suppress the agent index i and pass to a notation that refers only to agent type. Because the period subscript (0 or 1) will carry an important distinction, in this chapter we will distinguish the strategic variables and consumption bundles of farmers from those of prospectors by using carets over all prospector variables. The buy-sell game will define market clearing, with notation b for bids, q for quantities offered, and p for clearing prices. In models with markets for both gold and food, bids,

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[5]Originally the Cobb-Douglas function was introduced as a production function with two inputs: land and labor, and exponents for these inputs that sum to unity. It has become common to refer to the same functional form, when used in utilities, as a "Cobb-Douglas" form, to relax the restriction that the coefficients sum to unity (since the result of this relaxation is at most monotone transformation), and to include in the "Cobb-Douglas" appellation all cardinal utilities which share the same homothetic preferences as the original power-law Cobb-Douglas form.

quantities, and prices in the gold market will be explicitly subscripted b_G, q_G, and p_G.

The lifecycle of capital stock is chosen longer than that of agents, so that some generations of agents cannot directly finance the capital stock's construction or own shares in the services it provides. The important *catalytic* function of the constructed capital good is that it converts durable gold from a good with no inherent consumption value into an entity delivering a stream of services with direct utility of consumption. It is the episodic nature of this *conversion* opportunity that may leave some generations of agents with a surplus of gold and a deficit of services rendered by gold, while other generations encounter a lumped demand for gold which leads to under-consumption of other goods.

5.1.6 A comment on logarithmic utilities

In this chapter as in the preceding we use logarithmic utilities of consumption as minimal models. Logarithmic utility reflects homothetic preferences and leads to price elasticities of unity, which rule out modeling certain classes of price response to scarcity. In the models below, this simplification has the desirable feature of separating the types of agents and thus simplifying analysis and solution of models where our interest is in demonstrating the nature of the financing. Apart from these simplifications, the qualitative differences among market systems that we demonstrate should not depend sensitively on our use of logarithmic utility.

5.1.7 Population structure is not a strategic variable, but may be optimized by adaptive adjustment

The abstraction that stress on market systems and allocative efficiencies is created by a mismatch between the timescales and cycles of physical assets, and the needs of agents, entails the assumptions that agents cannot freely shift production in response to cyclical exogenous constraints. We simplify this abstraction into a minimal form by supposing that numbers n of farmers and n_0 of prospectors, in each of the two generations, are slowly changing variables even relative to the cycle of capital stock, so that stationary solutions to strategic market games can be computed treating these quantities as fixed parameters. We take the total number of agents $2(n + n_0)$ as a fully fixed constraint, and consider the adjustment of the allocation of labor

n/n_0 to be a slow process that equalizes some aggregation of utilities across the three generations of farmers with the same aggregation across the three generations of prospectors. (In the examples solved below, we will take this aggregation to be simply the arithmetic mean.) The process of adjusting the labor allocation is not modeled explicitly as the strategic variables are, and the selection of the utility-equalizing value using stationary solutions for the strategic variables is therefore akin to a problem in comparative statics. Informally, we consider this a proxy for slow processes of cultural adaptation that are outside the scope of our models.

5.2   The competitive rational expectations equilibrium allocation and utilities

The temporal structure of OLG models exposes the difficulties with extending the competitive equilibrium definition, because given the finiteness of expected life many contracts implicit in GE are ruled out although, as noted by Samuelson [9] the presence of money helps to restore some contracts. Furthermore the formulation of full dynamics calls for a treatment of initial and terminal conditions that introduce many degrees of freedom that have to be accounted for in well-defining the models. An easy, but not always satisfactory way in which the infinite horizons can be treated, consistently with the spirit of GE is to account for initial and terminal conditions by a "rational expectations assumption" which solves only for dynamic equilibrium, but leaves unanalyzed the influence of transient states.

For the examples below, we suppose that complete markets exist both within and across the two periods of any individual's life cycle and the three periods of existence of the capital stock. That is, young farmers purchase forward contracts for food in their old periods, at prices equal to the spot-market prices in those periods, in which both young and old prospectors also trade.

The assumption that complete contracts must include forward contracts requires that, in an OLG setting with a definite starting period, it is necessary to suppose that agents who are already old in that period have initial allocations of gold, or forward contracts for food, in amounts that are consistent with the rational expectations equilibrium values inferred for later periods.

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5.2.1 Consumption levels, population structure, and utility level

Suppose that the society is either large enough or small enough that, at the equilibrium allocations of labor, we are in the linear regime of either the high-yield or the low-yield production function. In a competitive equilibrium among agents with identical preferences, relative prices between food and the services from capital stock are the same for all agents, and only their budgets have the potential to distinguish them. However, we consider the labor allocation n_0/n also identified by the criterion that all agent utilities be identical (so that changing professions over long times is never advantageous to either type of agent), making the market value of all agents' endowments equal.[6] Therefore we do not notationally distinguish consumption levels A or S either across periods or between farmers and prospectors, and solve the equilibrium for the values which are common to all of these.

With total production na of food per period, all equilibrium consumption levels are

$$A_0^{(\tau)} = A_1^{(\tau)} = \frac{na}{2\,(n + n_0)}. \tag{5}$$

With production $n_0 e_0$ of gold per period, and a durable capital stock that consumes $3n_0 e_0$ units to produce, but then yields an equal level of services for *three* periods, the service consumption level per period per agent becomes

$$S_0^{(\tau)} = S_1^{(\tau)} = \frac{3 n_0 e_0}{2\,(n + n_0)}. \tag{6}$$

At these symmetric allocations, the utility of any agent given by Eq. (4), with high-yielding production, becomes

$$\mathcal{U}^{(\tau)} = 2\log\left(\frac{1}{1 + n_0/n}\right) + 2s\,(1 + \theta)\log\left(\frac{3}{2}\,\frac{n_0/n}{1 + n_0/n}\right). \tag{7}$$

Since the service stream delivered over an entire cycle, when measured in units of invested gold, is three times the actual quantity of the scarce resource (gold) produced by prospector labor, it is $S^{(\tau)}/3$, rather than $S^{(\tau)}$, which appears in marginal rates of substitution. (That is, the factor $3/2$ in the logarithm of Eq. (7) leads to a constant summand in utility for all agents,

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[6]This is equivalent to the reduction to a labor-equivalent metric of utility developed in Ref. [10].

which does not affect the optimization problem. The equilibrium price system defined by the marginal rates of substitution is therefore

$$p_{\tau,\text{CE}} = \frac{S^{(\tau)}/3}{s\left(1+\theta\right)A^{(\tau)}} = \frac{n_0 e_0}{s\left(1+\theta\right)na}. \tag{8}$$

The competitive-equilibrium labor allocation, obtained by varying Eq. (7) with respect to n_0/n becomes

$$\frac{n_0}{n} \xrightarrow{\text{CE}} s\left(1+\theta\right), \tag{9}$$

giving $p_{\tau,\text{CE}} = e_0/a$.

6 Three economic systems: informal intergenerational transfer, gold banking, and fiat with government as a reserve buyer of gold

We now consider the way that economies with this common OLG model of agent preferences and threshold-limited production,but different levels of institutional structure, meet the needs of distribution of food, concentration of gold for investment, and distribution of services. We will use total population as a control variable that determines the stress on the system as measured by the shadow price of the capacity constraint. For each model a non-cooperative equilibrium exists for populations at or above a (model-dependent) critical size N_C. At $N = N_C$, and the utility-equalizing allocation n_0/n of the non-cooperative equilibrium, the total prospector gold endowment over three generations meets the threshold for high-efficiency production: $3n_0 e_0 = \mathcal{C}$. All of these non-cooperative equilibria require critical populations N_C larger than the competitive-equilibrium value (9), which provides one measure of their inefficiency.

In each case, for $N < N_C$, the capacity constraint develops a nonzero shadow price, which quantifies the stress on the system. The rate at which the economy can allow $N/N_C < 1$ to decrease, relative to the rise of the shadow price, provides a measure of the robustness of its functions of distribution.

6.1 Synopses of the three cases

6.1.1 Case 1: food markets denominated in gold money, with intergenerational transfers of gold

We model an "informal" economy without financial support as one with trading-post markets for food denominated in gold, and individual bequests of gold by agents in their old period to furnish an initial endowment of gold for agents of the same type in the next generation, in their young period. These bequests are the only mechanism of intergenerational transfer (IGT). The prospector endowment of gold flows primarily through the food markets to accumulate in period $\tau = 0$ when it is needed for capital stock construction. The internalization of offspring utilities of consumption of services from capital stock is sufficient to produce interior solutions in which those farmers who require gold to purchase shares in period $\tau = 0$ receive nonzero IGTs from the previous generation. Stress from an insufficient population leads to asymmetric contribution from farmers and prospectors to meet the capacity constraint, and strong divergence of the utilities of different prospector generations despite producing only mild divergence of food prices if the overall budget share from gold remains small.

6.1.2 Case 2: an inside bank for gold in place of intergenerational transfers

The concentration of stress on a single period of agents, and its resulting impact on the investment levels and on divergent consumption bundles of agents born in different generations, can be mitigated by introducing an inside bank, which provides a repository and pass-through institution to redistribute gold. In place of IGTs that propagate gold forward in time through the capital cycle, bank loans couple consumption between the young and old periods *within each generation.* A limited intergenerational flow of gold occurs through the payment of interest on loans and deposits. However, the need of an inside bank to balance interest payments of debtors and creditors creates a complicated coupling among the roles of gold as an input to production, a medium of exchange, and an inter-period (and hence, indirectly through the food markets) inter-generational store of value. This coupling is expressed in a counterintuitive requirement for a nonzero interest rate (indeed the maximal rate in the model) in the unstressed non-cooperative equilibrium, and a dependence of prices on this interest rate that moves them away from the

competitive-equilibrium value and contributes part of the non-CE allocation of labor in the population.

6.1.3 Case 3: a central bank for fiat and a reserve buyer for gold

The role of gold as a durable consumable can be decoupled from the function of the money system by introducing fiat exchange, which places all producers on an equal footing with respect to purchasing and interest. In the particular model we present here, multiple functions must be introduced, because the replacement of gold as a medium of exchange requires the introduction of both a gold market and of a reserve buyer for gold in periods when there is no demand in the open market of non-cooperative agents. We combine these functions in a model of a policy-guided central bank, which provides loans in fiat money, collects interest, provides gold demand in slack periods and restores stored gold when it is needed for production. The introduction of a fiat exchange system creates multiple control parameters by which the central bank may influence policy objectives. We show, however, that no single value for these parameters is generally utility-improving or price-stabilizing; rather the parameters must generally be tuned to the particular configuration of population constraints and shadow prices faced by the society.

6.2 The markets used to allocate food, gold, and services in all models

6.2.1 A notation for agent-symmetric solutions within each type

We look for non-cooperative equilibria in which all agents of a given type make the same bids and offers, and have the same consumption levels. We denote the bid and offer variables for farmers by $\left(b_0^{(\tau)}, q^{(\tau)}\right)$ in the young period, and by $b_1^{(\tau)}$ in the old period. Their bids on gold in the young and old periods are denoted $\left(b_{G0}^{(\tau)}, b_{G1}^{(\tau)}\right)$. Their consumption levels of food and services are $\left(A_0^{(\tau)}, S_0^{(\tau)}\right)$ in the young period and $\left(A_1^{(\tau)}, S_1^{(\tau)}\right)$ in the old period. The clearing price for food in period τ is denoted p_τ.

The corresponding quantities for prospectors are $\left(\hat{b}_0^{(\tau)}, \hat{b}_1^{(\tau)}, \hat{b}_{G,0}^{(\tau)}, \hat{q}_G^{(\tau)}\hat{b}_{G,1}^{(\tau)}, \hat{A}_0^{(\tau)}, \hat{S}_0^{(\tau)}, \hat{A}_1^{(\tau)}, \hat{S}_1^{(\tau)}\right)$. The gold clearing price is $p_{G\tau}$. The prospector variables replace an offer q of food with an offer $\hat{q}_G$ of gold

in cases when gold markets exist.

6.2.2 Price formation and market clearing

The clearing price for food markets relates to bids of both types, and to offers, as

$$
p_\tau = \frac{nb_1^{(\tau-1)} + n_0 \left(\hat{b}_1^{(\tau-1)} + \hat{b}_0^{(\tau)} \right)}{nq^{(\tau)}}.
\tag{10}
$$

The comparable clearing price for gold (in models with a gold market) is expressed as

$$
p_{G,\tau} = \frac{nb_{G,1}^{(\tau-1)} + n_0 \left(\hat{b}_{G,1}^{(\tau-1)} + \hat{b}_{G,0}^{(\tau)} \right)}{n\hat{q}_G^{(\tau)}}.
\tag{11}
$$

The consumption level of food for farmers is

$$
\begin{aligned}
A_0^{(\tau)} &= a - q^{(\tau)} + \frac{b_0^{(\tau)}}{p_\tau} \\
A_1^{(\tau)} &= \frac{b_1^{(\tau)}}{p_{\tau+1}}.
\end{aligned}
\tag{12}
$$

and the level for prospectors is

$$
\begin{aligned}
\hat{A}_0^{(\tau)} &= \frac{\hat{b}_0^{(\tau)}}{p_\tau}. \\
\hat{A}_1^{(\tau)} &= \frac{\hat{b}_1^{(\tau)}}{p_{\tau+1}},
\end{aligned}
\tag{13}
$$

6.2.3 Private share rights and publicly-held utilities for services

In the following models, we will solve separately for the non-cooperative equilibria with investments in either low-yielding or high-yielding capital stock. Agents who are alive in the period when capital stock is built can purchase proportional shares in the stream of services. When these agents die, the part of the service stream un-purchased becomes a publicly held utility, which we distribute equally among the remaining agents.

Letting C stand for either $C_<$ or $C_>$ in Sec. 5.1.3 above, according to the production function under consideration, the service consumption levels

measured in units of invested gold, for the three generations of agents and their two periods, are given by

$$
S_0^{(0)} = S_1^{(0)} = \frac{\sigma_0}{\sum_i \sigma_i} C\left(\sum_i \sigma_i\right) ; \ \hat{S}_0^{(0)} = \hat{S}_1^{(0)} = \frac{\hat{\sigma}_0}{\sum_i \sigma_i} C\left(\sum_i \sigma_i\right)
$$

$$
S_0^{(1)} = \hat{S}_0^{(1)} = \frac{1}{n+n_0} \frac{\sum_i \sigma_i - (n\sigma_0 + n_0\hat{\sigma}_0)}{\sum_i \sigma_i} C\left(\sum_i \sigma_i\right) ; \ S_1^{(1)} = \hat{S}_1^{(1)} = \frac{1}{N} C\left(\sum_i \sigma_i\right)
$$

$$
S_0^{(2)} = \hat{S}_0^{(2)} = \frac{1}{N} C\left(\sum_i \sigma_i\right) ; \ S_1^{(2)} = \frac{\sigma_1}{\sum_i \sigma_i} C\left(\sum_i \sigma_i\right) ; \ \hat{S}_1^{(2)} = \frac{\hat{\sigma}_1}{\sum_i \sigma_i} C\left(\sum_i \sigma_i\right) .
$$

$$(14)$$

In the first and third lines, for generations $\tau = 0$ and $\tau = 2$, the factors $\sigma_0, \hat{\sigma}_0$ and $\sigma_1, \hat{\sigma}_1$ represent explicit ownership rights of the focal agent, from investments in the $\tau = 0$ period. The expressions for $S_0^{(1)}, \hat{S}_0^{(1)}$ in the second line represent the proportional publicly-held service allocation of all explicit (farmer and prospector) rights from $\tau = 2$ agents who have died. The important feature of the sum $\sum_i \sigma_i - (n\sigma_0 + n_0\hat{\sigma}_0)$ in the numerator is that it contains only decision variables from the generation $\tau = 2$. Therefore, while it affects the consumption levels of $\tau = 1$, which appear in the utilities $\mathcal{U}_i^{(0)}$, those consumption levels do not depend on the $\tau = 0$ decision variables except at higher order in $1/N$ (hence, on finite replicates), which terms we omit. Note that in the linear ranges of either production function, the factors of $C(\sum_i \sigma_i)$ and $\sum_i \sigma_i$ cancel, leaving only simple linear functions of σ_0 or σ_1.

6.2.4 Budget conditions and budget constraints

If we take $\mu_0^{(\tau)}$ to be the initial budget from exogenous variables, whether endowment or inter-generational transfers, then a general Lagrangian for the budget constraint that can encompass all three models may be written

$$
\mathcal{L}_{\text{Common}}^{(\tau)} = \mathcal{U}^{(\tau)} + \eta_0^{(\tau)} \left(\mu_0^{(\tau)} + g_0^{(\tau)} - \sigma_0 \delta_{\tau,0} - b_0^{(\tau)} - b_{G,0}^{(\tau)} \right)
$$
$$
+ \eta_1^{(\tau)} \left(\mu_1^{(\tau)} - g_1^{(\tau)} - \sigma_1 \delta_{\tau,2} - b_1^{(\tau)} - \kappa^\tau \right) , \quad (15)
$$

with hatted variables used if the agent is a prospector. Here g are amounts borrowed from a bank (in either gold or fiat), with negative values representing lending to the bank. In models where the institutions entailed by

the use of some variable are not included, that variable is set to zero. We provide the general form here so that we may write in one place the relations between marginal utility of consumption, and the K-T multipliers $\eta_0^{(\tau)}, \eta_1^{(\tau)}$ for the budget constraint, in those first-order conditions that are common to all agent types in all models.

The second-period initial budget for the two types is given in terms of the first period budget, expenditures, and proceeds of sale for farmers and prospectors respectively by

$$
\begin{aligned}
\mu_1^{(\tau)} &= \mu_0^{(\tau)} + g_0^{(\tau)} - \sigma_0 \delta_{\tau,0} - b_0^{(\tau)} - b_{G,0}^{(\tau)} + q^{(\tau)} p_\tau \\
\hat{\mu}_1^{(\tau)} &= \hat{\mu}_0^{(\tau)} + \hat{g}_0^{(\tau)} - \sigma_0 \delta_{\tau,0} - \hat{b}_0^{(\tau)} - \hat{b}_{G,0}^{(\tau)} + \hat{q}_G^{(\tau)} p_{G,\tau}.
\end{aligned}
\tag{16}
$$

6.3 Representation of a jointly-binding capacity constraint with an effective Kuhn-Tucker multiplier and associated shadow price

Constraints on individual strategic variables are readily implemented with Kuhn-Tucker multipliers, which have the interpretation of shadow prices. The jointly-binding capacity constraint, that investment must reach a threshold before high-yielding production is possible, is not in general similarly representable by such a multiplier, because the threshold is not under the strategic control of any single agent. Often in such cases an additional market would be required in reality, to propagate real price signals to individuals.

In these models, we exploit a property of logarithmic utilities that permits us to represent the jointly-binding capacity constraint in terms of a single Kuhn-Tucker multiplier Λ which is common to the optimization problems of all agents, which which has the interpretation of a shadow price. Our approach is to first regularize the threshold behavior of the production function to a smooth but non-convex and sharply curved function. The strongly non-linear dependence of the derivative of this function, on the investment level in a neighborhood of the capacity threshold value, together with logarithmic dependence of utility on the consumption level, permits us to treat the log-derivative of the non-convex production function as a K-T multiplier. The detailed construction is presented in App. B.

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6.4 Common first-order conditions and their consequences for consumption

6.4.1 Farmer bid and offer variables

The common terms in the first-order condition for farmers in all models are

$$
\begin{aligned}
0 = \delta\mathcal{L}^{(\tau)} \;=\; & \left[\frac{1}{A_0^{(\tau)}p_\tau} - \eta_1^{(\tau)}\right]\left(\delta b_0^{(\tau)} - p_\tau\delta q^{(\tau)}\right) - \eta_0^{(\tau)}\delta b_0^{(\tau)} + \left[\frac{1}{b_1^{(\tau)}} - \eta_1^{(\tau)}\right]\delta b_1^{(\tau)} \\
& + \left[\frac{2s}{\sigma_0} + 2\left(1+\theta\right)\Lambda - \left(\eta_0^{(\tau)} + \eta_1^{(\tau)}\right)\right]\delta\sigma_0\delta_{\tau,0} \\
& + \left[\frac{s}{\sigma_1} + \left(1+2\theta\right)\Lambda - \eta_1^{(\tau)}\right]\delta\sigma_1\delta_{\tau,2}.
\end{aligned}
$$

(17)

Where Λ is the effective K-T multiplier for the capacity constraining on high-yielding production, derived in App. B.

(In Eq. (17) and all subsequent equations, $\delta_{\tau,0}$ is the Kronecker δ-function, which takes value 1 when $\tau = 0$ and zero otherwise, and $\delta_{\tau,2}$ is the corresponding Kronecker δ-function with respect to $\tau = 2$.) Because the utility (4) does not saturate, farmers will always consume a part of their food, and the first and third conditions in Eq. (17) therefore give

$$
A_0^{(\tau)}p_\tau = b_1^{(\tau)}.
$$

(18)

To provide a complete analysis we must consider the possibility that farmers engage in wash selling in their young period, as we did for models in previous chapters. The equation that determines the value of $\eta_0^{(\tau)}$, which excludes wash sales if it is positive in Eq. (17), is

$$
\frac{2s}{\sigma_0} + 2\left(1+\theta\right)\Lambda - \frac{1}{b_1^{(0)}} = \eta_0^{(0)}.
$$

(19)

We provide a systematic analysis covering all cases in App. C. There we show that wash selling may be excluded from all solutions derived below, either because it cannot occur or because it cannot affect prices or allocations, so that any solution with wash sales can be replaced by an equivalent solution without them.

6.4.2 Prospector bids on food

Prospectors bid on food in both periods, and generations ($\tau = 2$) or ($\tau = 0$) will invest in capital stock in period $\tau = 0$. The common terms in their first-order conditions including budget constraints are

$$
\begin{aligned}
0 = \delta\mathcal{L}^{(\tau)} \;=\; & \left[\frac{1}{\hat{b}_0^{(\tau)}} - \left(\hat{\eta}_0^{(\tau)} + \hat{\eta}_1^{(\tau)} \right) \right] \delta\hat{b}_0^{(\tau)} + \left[\frac{1}{\hat{b}_1^{(\tau)}} - \hat{\eta}_1^{(\tau)} \right] \delta\hat{b}_1^{(\tau)} \\
& + \left[\frac{2s}{\hat{\sigma}_0} + 2\left(1+\theta\right)\Lambda - \left(\hat{\eta}_0^{(\tau)} + \hat{\eta}_1^{(\tau)} \right) \right] \delta\hat{\sigma}_0 \delta_{\tau,0} \\
& + \left[\frac{s}{\hat{\sigma}_1} + \left(1+2\theta\right)\Lambda - \hat{\eta}_1^{(\tau)} \right] \delta\hat{\sigma}_1 \delta_{\tau,2}.
\end{aligned}
\tag{20}
$$

The general relation between bids in the two periods is

$$
\frac{1}{\hat{b}_0^{(\tau)}} - \frac{1}{\hat{b}_1^{(\tau)}} = \hat{\eta}_0^{(\tau)}.
\tag{21}
$$

Situations in which prospectors carry gold over between periods, requiring $\hat{\eta}_0^{(\tau)} = 0$, are distinguished from those in which the periods are related through lending at interest, requiring $\hat{\eta}_0^{(\tau)} \rho \hat{\eta}_1^{(\tau)} = \rho / \hat{b}_1^{(\tau)}$, and the relation between bids and total budget is then given by Eq. (21) in either case.

6.5 The model with intergenerational transfers and without banking

In this model, the Lagrangian $\mathcal{L}^{(\tau)} = \mathcal{L}_{\mathrm{Common}}^{(\tau)}$ of Eq. (15). The defining feature of the model is the introduction of intergenerational transfers (IGTs) $\kappa^{(\tau)}$ from each generation (τ) of farmers to their successor generation. The IGTs appear in through the budget conditions as

$$
\begin{aligned}
\kappa^{(\tau-1)} &= \mu_0^{(\tau)} \\
\mu_0^{(\tau)} - b_0^{(\tau)} - \sigma_0 \delta_{\tau,0} &= \mu_1^{(\tau)} \\
\mu_1^{(\tau)} + q^{(\tau)} p_\tau &= b_1^{(\tau)} + \sigma_1 \delta_{\tau,2} + \kappa^{(\tau)}.
\end{aligned}
\tag{22}
$$

The first-order conditions and their consequences in this economy are derived in detail in App. D. We summarize the major properties of the model here in a series of figures.

Analytic results for the labor allocation in the unstressed equilibrium give $n_0/n \to s(1+\theta)/6$. Pinning $3n_0e_0 = \mathcal{C}$ at the lower limit of this equilibrium, we arrive at a critical lower population for $N = 2(n+n_0)$ of

$$N_C \equiv \frac{2\mathcal{C}}{3e_0}\left(1 + \frac{n}{n_0}\right) = \frac{2\mathcal{C}}{3e_0}\left(1 + \frac{6}{s(1+\theta)}\right), \tag{23}$$

roughly six times larger for $s \ll 1$ than the corresponding CE critical population.

The stress level on the price system, represented as the shadow-price value of the capacity constraint, Λe_0, is shown versus the population below its threshold level N/N_C in Fig. 4.
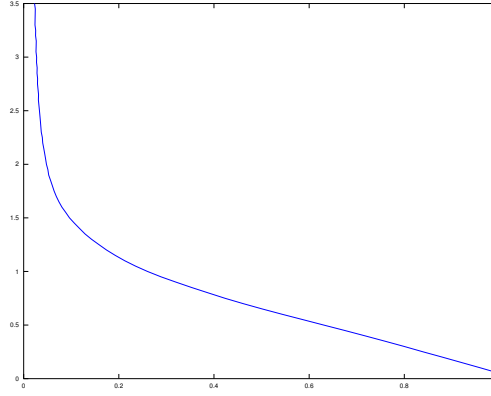

Figure 4: Stress level as measured by the shadow price $\Lambda e_0 \in [0, 3.5]$, as a function of $N/N_C \le 1$.

The response of food prices to $N/N_C < 1$ is shown, both in absolute terms and with the three-period mean subtracted, in Fig. 5. In the unstressed equilibrium, prices approximate the CE value $ap_\tau/e_0 \to 1$ to $\mathcal{O}(s)$.

The relation of investment to the market value of food in the competitive equilibrium is just that of the optimal labor allocation: $n_0e_0/(nap_{\text{CE}} + n_0e_0) = n_0/(n+n_0) \approx s(1+\theta)$ for $s \ll 1$. In the simulations we show $\theta = 1/2$, leading to a CE investment level of $3se_0/2$. In the unstressed noncooperative equilibrium, prospector investment levels $\hat{\sigma}_0 \approx se_0$, and $\hat{\sigma}_1 \approx se_0/2$ for $s \ll 1$. In the farmer sector, where generation ($\tau = 0$) depends on IGTs from
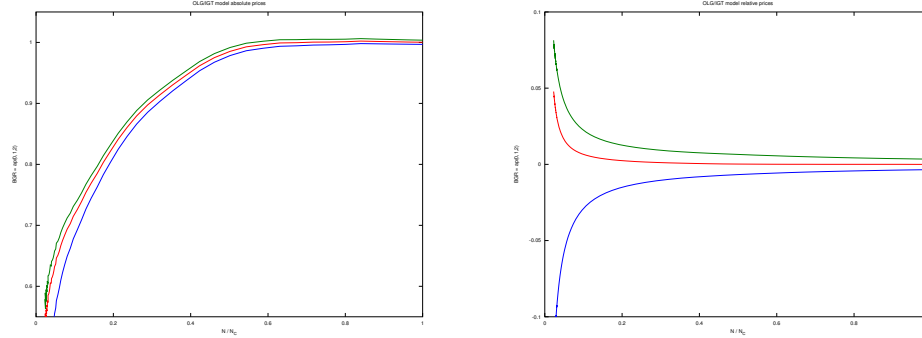
Figure 5: BGR are absolute prices ap_τ/e_0 for periods 0, 1, 2. Left panel is absolute; right panel is relative to mean: $\left(ap_\tau - \sum_{\tau'} ap_{\tau'}/3\right)/e_0$ for periods 0, 1, 2.

the ($\tau = 2$) generation, which are discounted by θ, the unstressed equilibrium investment levels are $\sigma_1 = e_0 s/2$ and $\sigma_0 = \kappa^{(2)} = e_0 s\theta$.

The distribution of investments needed to meet the constraint $n\left(\sigma_0 + \sigma_1\right) + n_0\left(\hat{\sigma}_0 + \hat{\sigma}_1\right) = \mathcal{C} = 3n_0 e_0$ in noncooperative equilibria with shadow prices is shown in Fig. 6. With decrease of $N/N_C < 1$, an increasing fraction of investment is met by prospectors.

While three-period average utility has been set equal for farmers and prospectors as the condition that determines the labor allocation n_0/n, the utilities within individual periods may still differ. Measures of this difference such as variance over the three generations may be used as a measure of the failure of allocative efficiency by the IGT mechanism, or the minimum single-generation utility may be used as a fragility threshold in a coalitional-form solution concept: if it falls below the noncooperative equilibrium-utility for low-efficiency production, this generation has no incentive to remain within the coalition that cooperates to produce high-yielding capital stock.

Fig. 7 shows the absolute utility levels versus $N/N_C \leq 1$. The utility levels of the CE with high-yielding and with low-yielding production are shown for comparison. The generation ($\tau = 0$)-prospectors suffer utilities far below autarchy for $N/N_C < 0.4$. The lower limit for population shown is $N/N_C \sim 1/50$, so still twice the sacrifice in total rate of capital services encountered in going from high-yielding production to autarchy.
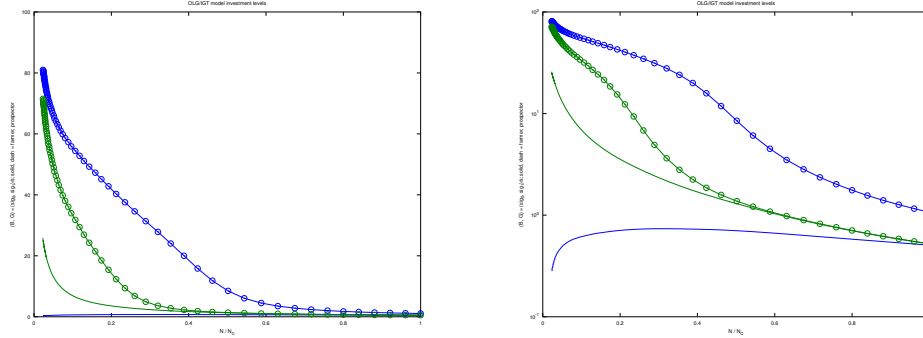
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Figure 6: Investment levels: or who is contributing most to meet the capacity constraint for the jointly-constructed good. Values σ are normalized by se_0. BG is period 0, 2. Solid is farmers, symbols are prospectors. Left panel is linear; right panel the same data on log scale. I think the reason both-period farmers group with $\tau = 2$ prospectors in the unstressed equilibrium is that the farmer σ_0 is limited by farmer $\kappa_{(2)}$, and in the $\tau = 2$ valuation of offspring utilities, the factor $\theta = 1/2$ here cancels the factor of 2 in the service stream.

6.6 An inside bank as a pass-through entity for gold, in place of intergenerational transfers

In the banking model agents of both types borrow or lend gold at a fixed rate of interest ρ to couple the expenditures between their own young and old periods, rather than across generations as in the IGT economy. Since the purpose behind banking is to optimize utilities, the bank will return all gold to the market economy within each cycle of generations. Since it is an inside bank, it has no other source of gold, and thus cannot return more than it receives. The condition that gold flow balance within each production cycle therefore closes the system.

The bank has no remaining degrees of freedom if it is constrained to balance gold flows. However, unlike Ref. [10], we are not modeling the costs of banking, so the bank is not required to make a profit in order to function. In this sense the bank is a strategic dummy, implicitly representing a banker who is equivalent to an unpaid bureaucrat.

The model introduces loan variables $\left(g_0^{(\tau)}, g_1^{(\tau)} \right)$ for farmers, and $\left(\hat{g}_0^{(\tau)}, \hat{g}_1^{(\tau)} \right)$
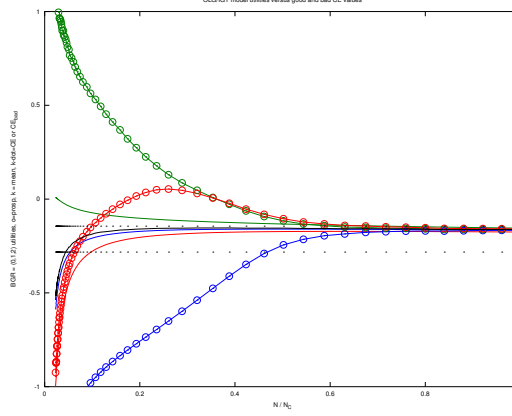
Figure 7: BGR are utilities for generations 0, 1, 2. Symbols mark prospectors, un-symboled are farmers. Solid black is the average, which is equal for both types. Upper black dotted is the CE utility, and lower black dotted is the CE minus the correction for autarchic production $2(1 + \theta) \log(0.01)$.

for prospectors, in each period, along with their associated KT-multipliers $\Lambda_B^{(\tau)}$ and $\hat{\Lambda}_B^{(\tau)}$. The Λ_B values are varied on a finite interval to implement the linear bankruptcy constraint of previous sections. We give values for these shadow prices of the bankruptcy constraint at interior solutions, but will assume that the penalty is strong enough that strategic default is always excluded, and will therefore not introduce a separate notation for the strength of the penalty. The sign convention will be that positive g denotes borrowing by agents from the bank, while negative $\hat{g}$ denotes lending to the bank.

The Lagrangians for farmers are then expansions of the common terms from Eq. (15) of the form

$$\mathcal{L}^{(\tau)} = \mathcal{L}_{\text{Common}}^{(\tau)} - \Lambda_B \left[(1 + \rho) g_0^{(\tau)} + g_1^{(\tau)} \right], \tag{24}$$

with an equivalent with hatted variables for prospectors. The same constraint term works for both borrowing and lending, ensuring that the amount withdrawn can never exceed the discounted negative of the amount deposited.

The corresponding budget sequences for farmers and prospectors, respec-

tively, become

$$0 = \mu_0^{(\tau)}$$
$$\mu_0^{(\tau)} + g_0^{(\tau)} - b_0^{(\tau)} - \sigma_0 \delta_{\tau,0} = \mu_1^{(\tau)}$$
$$\mu_1^{(\tau)} + q^{(\tau)} p_\tau = b_1^{(\tau)} + \sigma_1 \delta_{\tau,2} + g_1^{(\tau)} \qquad (25)$$
$$e_0 = \hat{\mu}_0^{(\tau)}$$
$$\hat{\mu}_0^{(\tau)} - \hat{b}_0^{(\tau)} - \hat{\sigma}_0 \delta_{\tau,0} - \left(-\hat{g}_0^{(\tau)}\right) = \hat{\mu}_1^{(\tau)}$$
$$\hat{\mu}_1^{(\tau)} + \hat{g}_1^{(\tau)} = \hat{b}_1^{(\tau)} + \hat{\sigma}_1 \delta_{\tau,2}. \qquad (26)$$

6.6.1 Solution properties, and a review of the problem of achieving high-yielding production below the critical population size

A full analysis of the first-order conditions and budget conditions, and their solutions, is given in App. E. There we show that the full return of gold from the banks to the individual sector is not compatible with the first-order conditions governing farmer investment, at arbitrary combinations of interest rate ρ and population composition n_0/n. The two conditions can be met only for a function ρ which approaches unity at $\Lambda e_0 = 0$, and *decreases* with increasing Λe_0, shown in Fig. 10. To understand the meaning and consequence of this interest rate, we review the problem of achieving high-yielding production in a society whose population N is below the critical value N_C required to invest the required gold without a shadow price.

The unstressed, noncooperative equilibrium with low-yield production is always a joint solution (autarchy) to this strategic market game at all populations $N \leq N_C$. If we consider the comparative statics of this class of solutions with fixed n_0/n as population is increased, we find a class of solutions homogeneous of order one in population size, bids, and investment levels, with a fixed interest rate $\rho = 1$. At $N = N_C$, the utility level undergoes a discontinuous jump $2(1 + \theta) \log(1/\epsilon)$ as low-yield production is replaced with high-yield production at the same investment level.

For $N \geq 2\mathcal{C}/3e_0$, however, agents may also lower the interest rate, reallocate labor until $n_0 = 2\mathcal{C}/3e_0$, invest at a higher absolute rate with most investments made by the expanded prospector sector, and adopt a noncooperative equilibrium in which the marginal utility of scarce food and uneven consumption of services is balanced by the shadow price of the capacity con-

straint. As $N \to N_C$ from below, n_0 remains fixed at $2\mathcal{C}/3e_0$, the excess gold per capita declines, prospector deposits and interest payments on them decline as well, and farmer investment levels for services in the young and old periods are driven nearly equal by cash-flow constraints. This solution – $\sigma_0 = \sigma_1 + \mathcal{O}(s)$ – is also a property of the competitive equilibrium, but not of any of the noncooperative equilibria we construct in this chapter, because the returns on investment and hence the first-order conditions differ for σ_0 and σ_1. In order to make the optimal investment levels converge, the discount factor to interest payments $1 + \rho$, must approach 2, returning the economy to the value $\rho = 1$ of the unstressed equilibrium. This sequence provides a continuous interpolating path between low-yield and high-yield production, as the population size is increased to its critical value.

6.6.2 Consumption, prices, and critical population size

For $s \ll 1$, utility levels are dominated by food consumption. Because prospectors borrow in both periods to buy food, their consumption becomes uneven (by the factor $1 + \rho$) in the old versus the young periods. Their young-period consumption is governed by their endowment and remains high, while their old-period consumption is further augmented by interest that they *earn*.

Since farmers always borrow and prospectors always lend, food prices ap_τ/e_0 must compensate the farmers for the this interest factor $(1 + \rho)$ by which the gold-value of prospector consumption is inflated. For logarithmic utilities of the form (4), this compensation gives the scaling of prices

$$\frac{ap_\tau}{e_0} \to \sqrt{1 + \rho}. \tag{27}$$

for $s \ll 1$. Therefore in the unstressed equilibrium $ap_\tau/e_0 \to \sqrt{2} + \mathcal{O}(s)$, a large distortion from the CE value of unity.

The corresponding critical population size, derived in App. E, becomes

$$N_C \equiv \frac{2\mathcal{C}}{3e_0}\left(1 + \frac{n}{n_0}\right) = \frac{2\mathcal{C}}{3e_0}\left(1 + \frac{3}{\sqrt{2}s}\right). \tag{28}$$

Numerical solutions for absolute and relative price levels, investment levels by each type and period, stress level Λe_0 and interest rate ρ, and single-period absolute utility levels, for comparison to those in the IGT economy without banking, are shown in Figures 8–11.
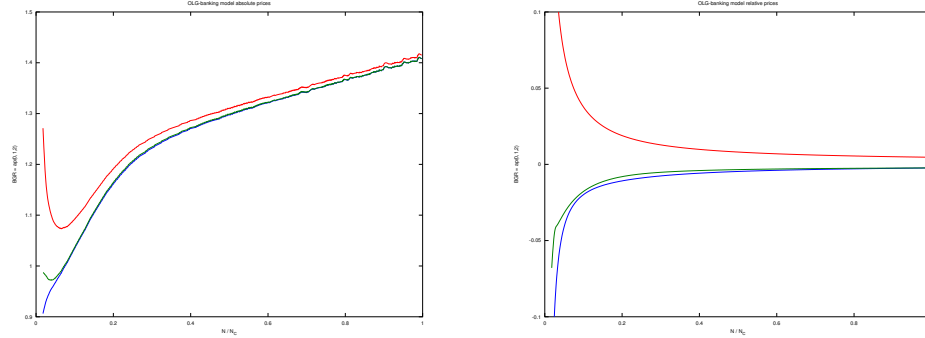
Figure 8: BGR are absolute prices ap_τ/e_0 for periods 0, 1, 2. Left panel is absolute; right panel is relative to mean: $\left(ap_\tau - \sum_{\tau'} ap_{\tau'}/3\right)/e_0$ for periods 0, 1, 2. Relative prices shown on the same scale as for Fig. 5 (absolute prices approach $\sqrt{2}$ in the unstressed equilibrium, rather than unity as in the IGT economy or the competitive equilibrium). Note that for banking $ap_2 > ap_1 > ap_0$, and we can confirm with the analytic expressions (90) that this must be the case. For IGTs the order was $ap_1 > ap_2 > ap_0$

6.6.3 Gold requirements of the bank

The gold requirements of the bank in order to meet net withdrawals with a minimum balance of zero are shown in Fig. 12. To interpret the per-capita version refer to Fig. 8 showing that at the lower end of the N/N_C range, $ap_\tau/e_0 \sim 1$. A (gold stock) $/ (n + n_0) e_0 \sim 0.3$ means that gold stored in the bank to buffer prices equals almost $1/3$ the total value of the food markets. (At the lower limit shown here, $n/n_0 \sim 5$, so $\sim 5/6$ of the economy's value is still represented by the food market. However this condition is still stressed relative to the unstressed equilibrium value $n/n_0 \sim 50$.)

The combination of price distortions, and the requirement that significant amounts of gold be taken maintained out of circulation, show the main weaknesses of banking with a commodity money that is also an input to production, and motivate the features of a fiat-banking model considered next.
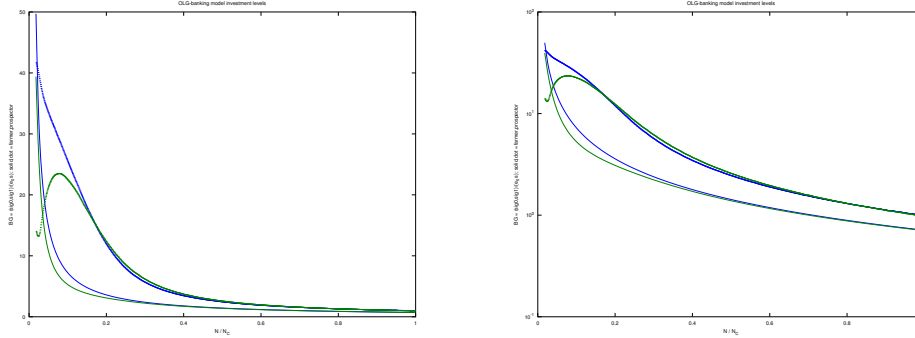
Figure 9: Investment levels: or who is providing the public good. Values σ are normalized by se_0. BG is period 0, 2. Solid is farmers, dashed is prospectors. Left panel is linear; right panel the same data on log scale. The pairings in the unstressed equilibrium are now different than for the IGT economy with both-period farmers grouping together and both-period prospectors grouping together. Here the cause should be that the interest rate $\rho \to 1$ in the unstressed equilibrium causes $1 + \rho$ to cancel the factor-of-2 differences that would otherwise distinguish σ_0 from σ_1.

7 An economy with a central bank that lends fiat and serves as a reserve buyer for gold

Our third OLG model changes the means of payment from gold to fiat money issued by a central bank.

7.1 Institutional structure of a fiat OLG economy

The removal of gold as a means of payment requires the introduction of a gold market, which we make a buy-sell market equivalent to the food market. This move immediately converts prospectors into a class of *producers* on an equal economic footing with farmers, and leads to prices in the unstressed noncooperative equilibrium that converge to those in the CE. Fiat is introduced into the economy through loans which are recollected with interest at rate ρ.

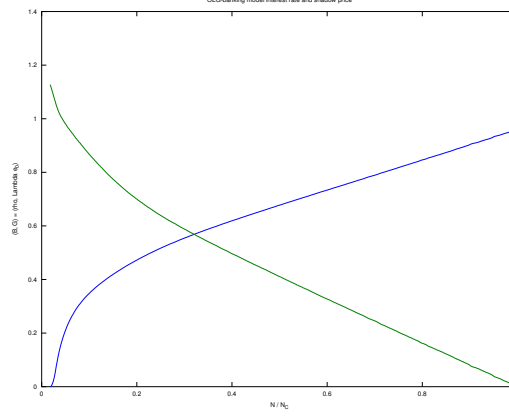The removal of gold as a means of payment also eliminates a form of

Figure 10: Interest rate (b), and stress level (g) as measured by the shadow price Λe_0, as a function of $N/N_C \leq 1$. Note that now $\Lambda e_0 \in [0, \sim 1]$, so banking permits a lower shadow price for the capacity constraint at comparable N/N_C.

cross-generation transmission through the food markets, removing demand for gold from the open markets in the two periods when it cannot be used to invest in capital stock. Therefore in addition to being a lender of fiat, the central bank must become a buyer of gold in two periods, and a net supplier in period $\tau = 0$. Its role as a gold buyer defines the numéraire for fiat. At the same time, the government's freedom to dictate offer prices for gold in periods when there is no open-market demand, together with the interest rate, provide control degrees of freedom that may be set to achieve policy objectives of the central bank. These degrees of freedom are not all independent. As in the inside-gold-banking model of the last section, we suppose that the central bank returns all gold to the private sector within each production cycle, and we also assume that it balances the flow of fiat to stabilize its price. The result of these constraints is that the central bank is left with two independent control variables that it may set freely within finite intervals.
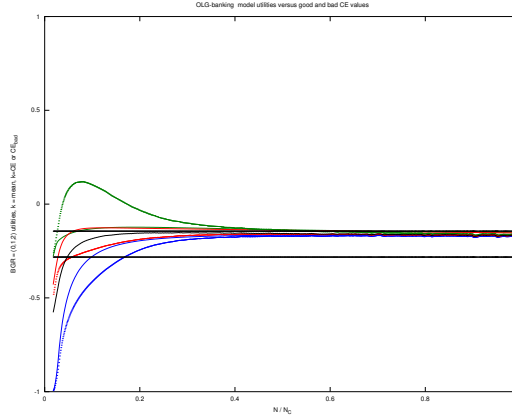
Figure 11: BGR are utilities for generations 0, 1, 2. Solid is farmer; dotted is prospector. (The greater demands of the numerics for resolution permitted the dotted line to show here.) Solid black is the average, again set equal for the two types. Upper black dotted is the CE utility, and lower black dotted is the CE minus the correction for autarchic production $2(1+\theta)\log(0.01)$. Axes are the same as those in Fig. 7 for comparison

7.1.1 The period structure for a model with gold markets and fiat money

The natural use of the existing OLG period structure makes gold markets parallel to food markets in their operation. Bids and offers are placed in one period; purchases and proceeds are distributed as the initial values of goods and money for the next period. Under this market structure, $(\tau = 0)$-farmers – who were not yet born in the previous period $\tau = 2$ – have no way to secure delivery of gold in time for them to invest in capital stock. Since all investors will still be alive in their young period, these farmers also have no publicly-held service stream from which to draw. This situation results in zero consumption and singular $(-\infty)$ utilities, unless we re-introduce intergenerational transfers, which are now no longer money but simply durable property. The re-introduction of IGTs restores the relation between investments and IGTs in the farmer sector which are the first two lines of Eq. (61), except for a factor $(1 + \rho)$ because the money must be borrowed to buy gold.
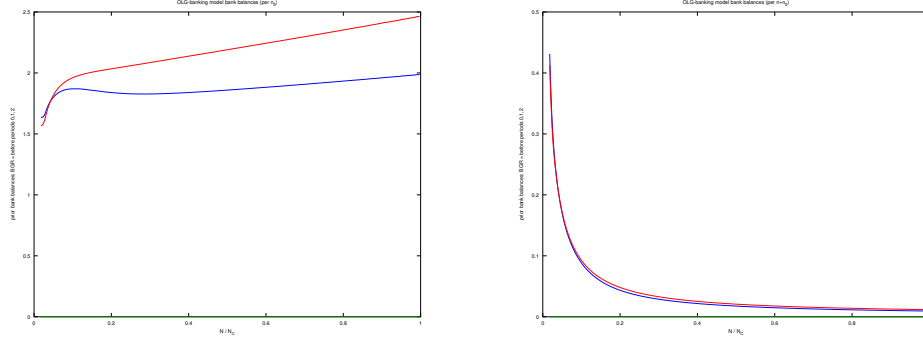
36

Figure 12: Prior gold balances of the bank in the three periods, normalized by $n_0 e_0$ (left) or by $(n + n_0) e_0$ (right), as a function of N/N_C. (Recall that n_0 is fixed by the capacity constraint $\mathcal{C} = 3 n_0 e_0$.) BGR are prior balances coming into periods $\tau = 0, 1, 2$. (Prior balance for $\tau = 1$ is zero, and may be hard to see in this figure.)

7.1.2 The use of control variables in the fiat economy

The fiat economy admits a regular $\rho \to 0$ limit of solutions for the unstressed equilibrium.

Maintaining $\rho \ll 1$ for all values of N/N_C is not optimal policy for a benevolent central bank. If interest rates are set to zero for populations below the critical size, the model solution becomes identical to that of the pre-financial IGT economy shown in Fig. 7. The stresses lead to price reductions and severe dispersion of utilities for agents in different generations. The following sections show that a policy objective of minimizing dispersion of utility between the generations of prospectors leads to a schedule for interest rates and payments that remains closer to the competitive equilibrium for smaller populations $N < N_C$ than either of the previous two models.

7.2 Farmer budgets, preferences, and first-order conditions

Because fiat rather than gold is the money in this economy, two separate budgets coexist, for gold and for money. Parallel to the notations $\left(\mu_0^{(\tau)}, \mu_1^{(\tau)} \right)$ for beginning young-period and old-period money budgets, we introduce no-

tations $\left(\gamma_0^{(\tau)}, \gamma_1^{(\tau)} \right)$ for beginning young-period and old-period gold stocks. Intergenerational transfers (which in most periods may be zero) provide the young-period endowment of gold for farmers, and the old-period gold stock may be augmented with gold purchases. Gold in either period may be used for investment and in the old period part of it may also provide an IGT to the next generation. The equations for gold stocks of farmers are

$$
\begin{aligned}
\kappa^{(\tau-1)} &= \gamma_0^{(\tau)} \\
\gamma_0^{(\tau)} - \sigma_0 \delta_{\tau,0} &= \gamma_1^{(\tau)} \geq 0 \\
\gamma_1^{(\tau)} + \frac{b_{G0}^{(\tau)}}{p_{G\tau}} - \sigma_1 \delta_{\tau,2} - \kappa^{(\tau)} &\geq 0.
\end{aligned}
\tag{29}
$$

The Lagrangian for farmers contains the common terms (15) as well as a constraint term for repayment of young-period borrowing (now denominated in fiat), as well as two new constraint terms for gold stocks, which we enforce with K-T multipliers $\eta_{G0}^{(\tau)}$ and $\eta_{G1}^{(\tau)}$, both on the interval $[0, \infty)$:

$$
\begin{aligned}
\mathcal{L}^{(\tau)} &= \mathcal{L}_{\text{Common}}^{(\tau)} - \Lambda_B^{(\tau)} \left[(1+\rho) g_0^{(\tau)} + g_1^{(\tau)} \right] \\
&\quad + \left(\gamma_0^{(\tau)} - \sigma_0 \delta_{\tau,0} \right) \eta_{G0}^{(\tau)} + \left(\gamma_1^{(\tau)} + \frac{b_{G0}^{(\tau)}}{p_{G\tau}} - \sigma_1 \delta_{\tau,2} - \kappa^{(\tau)} \right) \eta_{G1}^{(\tau)}.
\end{aligned}
\tag{30}
$$

The loans, bids, and repayments that determine the forms of the money balances in the budget terms of the common Lagrangian (15), for the farmers in this economy, become

$$
\begin{aligned}
0 &= \mu_0^{(\tau)} \\
\mu_0^{(\tau)} + g_0^{(\tau)} - b_0^{(\tau)} - b_{G0}^{(\tau)} &= \mu_1^{(\tau)} \geq 0 \\
\mu_1^{(\tau)} + q^{(\tau)} p_\tau - b_1^{(\tau)} + g_1^{(\tau)} &\geq 0.
\end{aligned}
\tag{31}
$$

Bids may now be made for gold as well as for food, but investments in capital stock are now made from the gold supply (29) rather than from the money budget (31).

The terms in the farmer first-order conditions that follow from these preference and budget expressions (suppressing the variations in K-T multipliers

which simply enforce the inequality constraints), are then

$$
\begin{aligned}
0 = \delta\mathcal{L}^{(\tau)} \;=\; & \left[\frac{1}{A_0^{(\tau)}p_\tau} - \eta_1^{(\tau)}\right]\left(\delta b_0^{(\tau)} - p_\tau \delta q^{(\tau)}\right) - \eta_0^{(\tau)}\delta b_0^{(\tau)} + \left[\frac{1}{b_1^{(\tau)}} - \eta_1^{(\tau)}\right]\delta b_1^{(\tau)} \\
& + \left[\eta_0^{(\tau)} + \eta_1^{(\tau)} - (1+\rho)\Lambda_B^{(\tau)}\right]\delta g_0^{(\tau)} + \left[\eta_1^{(\tau)} - \Lambda_B^{(\tau)}\right]\delta g_1^{(\tau)} \\
& + \left[\frac{2s}{\sigma_0} + 2(1+\theta)\Lambda - \left(\eta_{G0}^{(\tau)} + \eta_{G1}^{(\tau)}\right)\right]\delta\sigma_0\delta_{\tau,0} \\
& + \left[\frac{s}{\sigma_1} + (1+2\theta)\Lambda - \eta_{G1}^{(\tau)}\right]\delta\sigma_1\delta_{\tau,2} \\
& + \left[\frac{2\theta s}{\kappa^{(\tau)}} + 2\theta\Lambda - \eta_{G1}^{(\tau)}\right]\delta\kappa^{(\tau)}\delta_{\tau,2} \\
& + \left[\frac{\eta_{G1}^{(\tau)}}{p_{G\tau}} - \left(\eta_0^{(\tau)} + \eta_1^{(\tau)}\right)\right]\delta b_{G0}^{(\tau)} \tag{32}
\end{aligned}
$$

7.3 Prospector budgets, preferences, and first-order conditions

For prospectors, the endowment e_0 from their production function, rather than intergenerational transfers, furnishes their initial stock of gold, and they may offer part or all of this (a quantity $\hat{q}_G^{(\tau)}$) for sale on the gold market to provide money for food, as well as investing it in capital stock. The equations for the prospectors' gold stocks, denoted $\left(\hat{\gamma}_0^{(\tau)}, \hat{\gamma}_1^{(\tau)}\right)$, are then

$$
\begin{aligned}
e_0 \;&=\; \hat{\gamma}_0^{(\tau)} \\
\hat{\gamma}_0^{(\tau)} - \hat{\sigma}_0\delta_{\tau,0} - \hat{q}_G^{(\tau)} \;&=\; \hat{\gamma}_1^{(\tau)} \geq 0 \\
\hat{\gamma}_1^{(\tau)} - \sigma_1\delta_{\tau,2} \;&\geq\; 0. \tag{33}
\end{aligned}
$$

The Prospector Lagrangian is directly parallel to Eq. (30) for farmers (with hatted variables in $\mathcal{L}_{\text{Common}}^{(\tau)}$), and with modified gold-stock constraint terms enforced with K-T multipliers $\hat{\eta}_{G0}^{(\tau)}$ and $\hat{\eta}_{G0}^{(\tau)}$:

$$
\begin{aligned}
\hat{\mathcal{L}}^{(\tau)} \;=\; & \mathcal{L}_{\text{Common}}^{(\tau)} - \hat{\Lambda}_B^{(\tau)}\left[(1+\rho)\hat{g}_0^{(\tau)} + \hat{g}_1^{(\tau)}\right] \\
& + \left(\hat{\gamma}_0^{(\tau)} - \hat{\sigma}_0\delta_{\tau,0} - \hat{q}_G^{(\tau)}\right)\hat{\eta}_{G0}^{(\tau)} + \left(\hat{\gamma}_1^{(\tau)} - \hat{\sigma}_1\delta_{\tau,2}\right)\hat{\eta}_{G1}^{(\tau)}. \tag{34}
\end{aligned}
$$

The loans, bids, and repayments that determine the forms of the money balances in the budget terms of the common Lagrangian (15), for the prospectors in this economy, then become

$$
\begin{aligned}
0 &= \hat{\mu}_0^{(\tau)} \\
\hat{\mu}_0^{(\tau)} + \hat{g}_0^{(\tau)} - \hat{b}_0^{(\tau)} &= \hat{\mu}_1^{(\tau)} \geq 0 \\
\hat{\mu}_1^{(\tau)} + \hat{q}_G^{(\tau)} p_{G\tau} - \hat{b}_1^{(\tau)} + \hat{g}_1^{(\tau)} &\geq 0.
\end{aligned}
\tag{35}
$$

The terms in the prospector first-order conditions that follow from these preference and budget expressions (again suppressing the variations in K-T multipliers which enforce the inequality constraints), are then

$$
\begin{aligned}
0 = \delta\hat{\mathcal{L}}^{(\tau)} &= \left[\frac{1}{\hat{b}_0^{(\tau)}} - \left(\hat{\eta}_0^{(\tau)} + \hat{\eta}_1^{(\tau)}\right)\right]\delta\hat{b}_0^{(\tau)} + \left[\frac{1}{\hat{b}_1^{(\tau)}} - \hat{\eta}_1^{(\tau)}\right]\delta\hat{b}_1^{(\tau)} \\
&+ \left[\hat{\eta}_0^{(\tau)} + \hat{\eta}_1^{(\tau)} - (1+\rho)\,\hat{\Lambda}_B^{(\tau)}\right]\delta\hat{b}_0^{(\tau)} + \left[\hat{\eta}_1^{(\tau)} - \hat{\Lambda}_B^{(\tau)}\right]\delta\hat{b}_1^{(\tau)} \\
&+ \left[\frac{2s}{\hat{\sigma}_0} + 2(1+\theta)\Lambda - \left(\hat{\eta}_{G0}^{(\tau)} + \hat{\eta}_{G1}^{(\tau)}\right)\right]\delta\hat{\sigma}_0\delta_{\tau,0} \\
&+ \left[\frac{s}{\hat{\sigma}_1} + (1+2\theta)\Lambda - \hat{\eta}_{G1}^{(\tau)}\right]\delta\hat{\sigma}_1\delta_{\tau,2} \\
&+ \left[\hat{\eta}_1^{(\tau)}p_{G\tau} - \left(\hat{\eta}_{G0}^{(\tau)} + \hat{\eta}_{G1}^{(\tau)}\right)\right]\delta\hat{q}^{(\tau)}.
\end{aligned}
\tag{36}
$$

7.4 Price formation and the central bank as a reserve buyer for gold

Food clears according to the price-formation rule (10). If an analogous rule (11), which we write here as,

$$
p_{G,\tau}^{\text{private only}} = \frac{nb_{G,1}^{(\tau-1)} + n_0\left(\hat{b}_{G,1}^{(\tau-1)} + \hat{b}_{G,0}^{(\tau)}\right)}{nq_G^{(\tau)}},
\tag{37}
$$

receiving bids only from private individuals, then a price of zero would result in periods $\tau = 0, 1$, because in these generations no individual has a salvage value for gold in his old period, when it would be delivered from the markets. The $(\tau = 0, 1)$-prospectors would then have no source of income, and interior solutions would be unattainable.

The solution we adopt here to this market failure is to make the central bank a reserve buyer and seller of gold. Denoting the bank's bids and offers by $B_G^{(\tau)}$ and $Q_G^{(\tau)}$, respectively, we modify the price-formation rule (11) to the form

$$p_{G,\tau} = \frac{nb_{G,1}^{(\tau-1)} + n_0 \left(\hat{b}_{G,1}^{(\tau-1)} + \hat{b}_{G,0}^{(\tau)} \right) + B_G^{(\tau)}}{nq_G^{(\tau)} + Q_G^{(\tau)}}. \tag{38}$$

Participation by the central bank in the open gold markets does two things. First, it defines the numéraire of fiat, which would be left undetermined by the mere existence of a nonzero interest rate. Second, it gives the central bank several control variables, by which it may achieve policy objectives for distribution and welfare in the society.

To identify the control variables as well as to make a minimal model, we constrain the central bank's policies to be drawn from those which return all purchased gold to the economy within each three-period cycle, and which balance all payments of fiat aggregated over gold purchases and sales, and interest payments. The former constraint may be seen as a social-welfare condition: since gold has value as an input to production, and since the central bank (a strategic dummy) does not take profits or pay for labor, net extraction of gold would constitute waste of part of the endowment. The latter constraint ensures stable fiat prices for food and gold (no net fiat injected into the private economy) while enabling solutions without strategic default (no net fiat extraction required to avoid default).

We may finally require that the bank not engage in wash selling either as a buyer or a seller of gold. Any allocation that can be achieved by a solution with wash selling can be achieved by another solution without it, which differs only by additive constants in the accounts of gold and fiat held by the bank. Excluding wash selling results in bank moves that return all gold to the private economy in (at least) one round when the bank is a net seller, giving the lowest level of reserved gold in that class of solutions. (Solutions of the model will show that the bank is a buyer in two periods and a seller only in period $\tau = 2$.)

The four strategic parameters available to the central bank are the interest rate ρ, its two bid levels $B_G^{(0)}$ and $B_G^{(1)}$, and the offer level $Q_G^{(2)}$. Of these, the offer $Q_G^{(2)}$ is constrained by the requirement to recycle gold, and one combination of the bids and ρ is constrained by the requirement to recycle all fiat.

7.4.1 The central bank's policy objective: minimizing the cross-generation dispersion of prospector utilities

We demonstrate the use of the two remaining unconstrained variables as control parameters, by choosing the policy objective of the bank to be minimization of the cross-generation variance of prospector utilities,[7]

$$\text{var}\left(\left\{\mathcal{U}_{\text{Pros}}^{(\tau)}\right\}\right) \equiv \frac{1}{3}\sum_{\tau}\left(\mathcal{U}_{\text{Pros}}^{(\tau)}\right)^2 - \left(\frac{1}{3}\sum_{\tau}\mathcal{U}_{\text{Pros}}^{(\tau)}\right)^2. \tag{39}$$

Any feasible policy objective could, of course, be implemented by the central bank. The motivation to minimize the variance (39) is that it extends the presumptive region of validity of the joint high-production non-cooperative equilibrium. While we have not considered the problem of strategic labor-reallocation at the level of individuals, the justification for the joint non-cooperative equilibrium clearly becomes questionable if one generation of one profession systematically has very disparate and very low utility. To avoid secession of such a generation, a more fully-specified game might require other constraints against individual labor re-allocation.

In whatever domain the central bank can maintain all population utilities above the level of the joint non-cooperative equilibrium with autarchy, there is no utility improvement for any single generation or profession to secede, and the high-production non-cooperative equilibrium stands on its own without further qualification.

7.5 Properties of the fiat economy without and with active control from the central bank

The first-order conditions, budget conditions, price cycle, and consequences of central-bank purchases and sales are solved in App. F.

The availability of control in the fiat economy enables both higher mean utilities, and lower prospector-utility variance, than either the IGT or gold-banking economies at almost all values of n_0/n. The non-cooperative equilibrium with zero interest rate, as noted above, is identical to that for the pre-institutional IGT economy. Since the optimal interest rate $\rho \to 0$ as the shadow price on the capacity constraint $\Lambda e_0 \to 0$, the fiat economy has

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[7]Here we are not concerned with the best estimator for the variance, but simply with the second moment around the mean, so we normalize with $1/3$ rather than with $1/2$.

the same critical population size (23) as the pre-institutional IGT economy. With decreasing population size $N/N_C < 1$, increasing interest is required to redirect money flows, and most bids for gold are made in period $\tau = 0$, causing a rise in gold prices p_{G0}.

7.5.1 How interest rate and bid structure serve to reduce variance of prospector utilities

Fig. 13 illustrates the effects of interest rates, and the relative bids offered by the government when it is the sole buyer of gold, on utility levels. When $\rho = 0$, gold prices become constant over periods, the prospector gold offers and investment levels (102,103) become those of the IGT equilibrium, central-bank bids in fiat (108) substitute for prospector bids in the IGT gold-denominated markets, and farmer consumption and intergenerational transfers become those of the pre-institutional IGT model. In this limit the solution to the fiat model converges to that of the IGT model. In particular, under increasing shadow price Λe_0 with decreasing $N/N_C < 1$, uneven per-period demand for gold causes prospector utilities to diverge.

If instead parameters $\rho = 0.069$, $\alpha = -0.522$ are chosen, different-generation utilities become very different in the $\Lambda e_0 \to 0$ equilibrium, but at a particular value $N/N_C < 1$, they compensate for uneven gold demand to bring different generations' utilities together.

Fig. 14 illustrates the use of full control over interest rates and the offer prices in periods (1,2), by choosing values (ρ, θ) as functions of N/N_C to minimize the cross-generation variance of the prospector utilities at each value of $N/N_C < 1$. In the absence of control ($\rho = 0$), gold prices remain stable but food prices decline in all periods with decreasing N/N_C. Prospector utilities diverge sharply, making ($\tau = 0$) prospectors worse-off than they would be under autarchy for $N/N_C < 0.425$. If the government raises interest rates and bids appropriately, a rising price of gold in period $\tau = 0$ stabilizes all prospector utilities, and also food prices.

The variance-minimizing contour of ρ and θ is shown in Fig. 15, together with the shadow price that it generates as a function of N/N_C.[8]

---

[8]In all such solutions obtained numerically, discretization error requires control and smoothing. The relation between samples of ρ and θ on a grid that minimize utility variance, and smoothed parametric functions used for these variables to produce the "controlled" utility, price, and demand variables, is shown in Fig. 16.
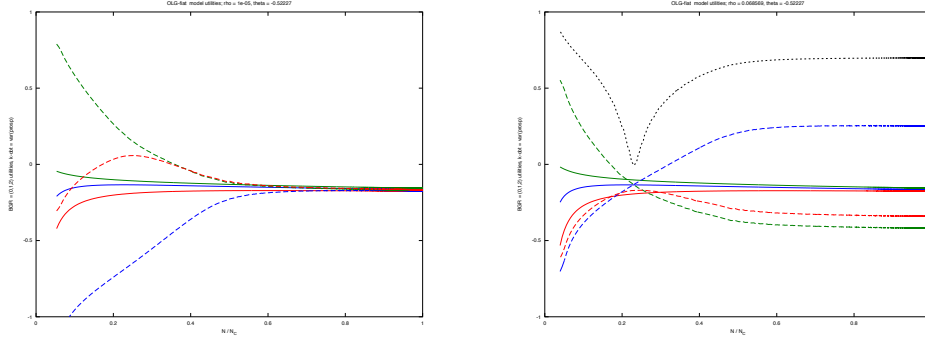
Figure 13: The use of interest to reduce variance in the prospector utilities. (Left) are generation-(0,1,2) utilities for farmers (solid) and prospectors (dashed) at $\rho = 0$ for a range of $N/N_C < 1$. Divergence of prospector utilities creates an incentive for some generations to leave the economy, because they are worse-off than they would be under autarchy. (Right) generation-(0,1,2) utilities the same range of $N/N_C < 1$ for values $\rho = 0.069$, $\alpha = -0.522$, which minimize variance of the three-generation prospector utilities at a particular value N/N_C. Black dotted curve is $1 + 8 \times \log(\text{variance})$, used for illustration to show the extent of reduction.

7.5.2 Properties of the variance-minimizing fiat economy

The remaining price and investment-level plots in the fiat economy, for comparison to the other cases, are shown in Fig. 17 and Fig. 18. Qualitatively, the investment levels closely resemble those of Fig. 6 for the IGT economy, even when interest rates become nonzero. Note that the largest value of ρ found in Fig. 15 is $\rho \sim 0.5$, only half of the interest rate at the unstressed noncooperative equilibrium of the gold-banking economy.

8 Summary and comparisons

The non-cooperative equilibria are inefficient relative to the competitive rational expectations equilibrium. This is expressed both in lower average utilities (true by the definition of efficiencies), and also in a larger critical size of the population required to meet the capacity constraint in an unstressed equilibrium (no shadow price for the capacity constraint).
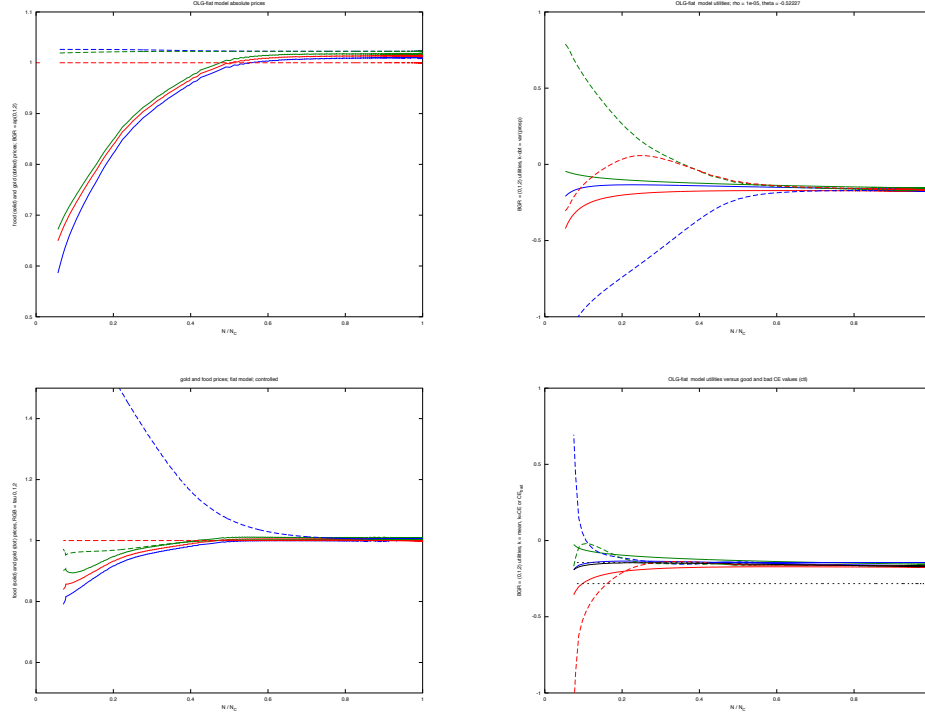
Figure 14: The use of tuned values of ρ and α to minimize variance of prospector utilities at each value of $N/N_C < 1$. (Left panels) food (solid) and gold (dashed) prices; (Right panels) single-generation farmer (solid) and prospector (dashed) utilities. (Top panels) with $\rho = 0$; (Bottom panels) ρ and α along the variance-minimizing contour.

The critical population size and the market value of the food endowment relative to the gold endowment are shown in Table 1.

Gold banking buffers prices and equalizes utilities for the three generations significantly more effectively than inter-generational transfers, and for $\theta \leq \sqrt{2} - 1/2 \approx 0.9$, it leads to lower critical population size. However, since the mean utility at the critical size is lower than the utility maintained with a nonzero shadow price, the lower value of N_C simply means that this inefficient noncooperative equilibrium attains for a larger range of absolute population size than for the other models. It is also a property of this model that, for small populations, a significant fraction of the wealth of the economy
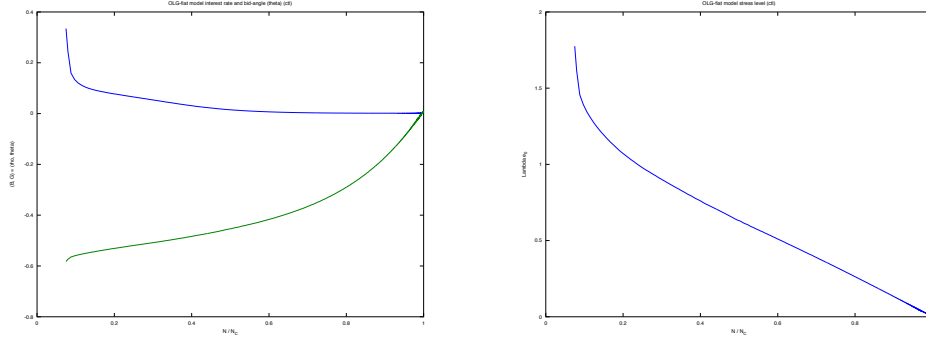
Figure 15: (Left) ρ and α values that minimize cross-generation variance of prospector utilities; (Right) stress level measured as the shadow-price value of the prospector endowment Λe_0, both as functions of $N/N_C < 1$. This control contour was used to produce the utilities and prices in the lower panels of Fig. 14.

is stored in the bank in some periods to buffer prices.

In a real society, the relation of a population to the critical value for its non-cooperative equilibrium with each money supply is not the relevant variable for comparison of financing systems. Rather, it is the absolute population size relative to the size of the gold supply required to meet the capacity constraint. We illustrate the interaction of N_C and utility levels by plotting mean utility versus N normalized relative to the critical population $(N_C)_{\text{CE}}$ at the *competitive equilibrium*, in Fig. 19. The fiat and IGT economies have the same critical sizes and unstressed noncooperative equilibria, but over the range $N < N_C$ the controlled fiat economy yields higher utility, closely approaching the CE utility when $N \sim (N_C)_{\text{CE}}$.

Mean utilities only present one aspect of social welfare. It must also be remembered, from Fig. 11 vs. Fig. 7, that the gold-banking economy maintains much lower cross-generation variance of utilities at comparable N/N_C, as well as having a lower critical population size N_C. Therefore, if the possibility of secession of some producers from the economy is considered, the superior average welfare of the gold-banking economy over the pre-institutional IGT economy for $N/(N_C)_{\text{CE}} < 1$ actually extends to a more robust solution to much larger values of N.
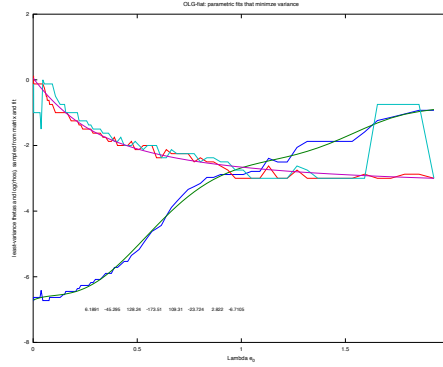
46

Figure 16: This is a working-figure for my notes, showing the relation between smoothed functions ρ and θ chosen parametrically, and the discretized versions obtained by minimizing variance on a grid of $(x, \Lambda e_0)$ values. The ρ curve is a 7th-order polynomial least-squares fit. The θ curve is a tangent which is a linear function of Λe_0.

9 Concluding Comments

Process models of the economy are by definition institutional because they require carriers of process that are abstractions of institutions. Basic game theory considerations tell us that the proliferation of reasonable models is hyperastronomical in size. However by adding the time structure of individuals and goods to a closed T period economy we can obtain enough special structure to build models that reflect many of the specific features that abound in an economy with physical assets and individuals existing on many different time scales. Many of these combinations call for the creation of the vast array of special financial instruments designed to cope with the timing and coordination. These mismatches must and can be overcome by a well designed financial system. The system however is open to its environment and requires considerable parametric specification, where each choice provides a somewhat institutionally different mechanism, fitted *ad hoc* to the micro-micro detail of the part of the economy under scrutiny. Yet all models obey the economic optimization structure reflecting Edgeworth's inaugural observations.

The transition from a general economic model illustrating static equilib-
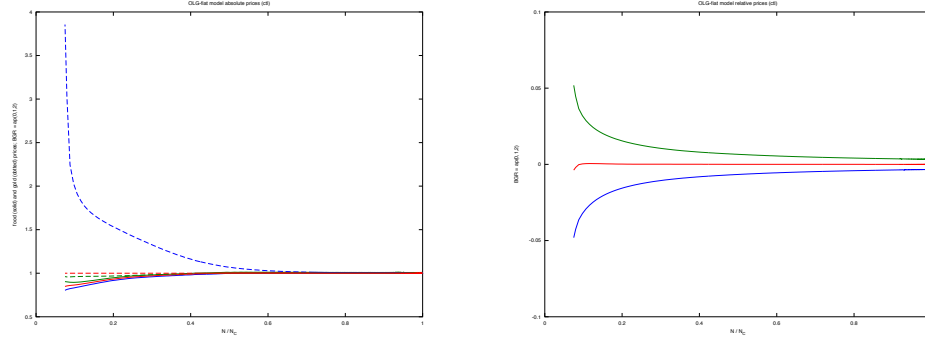
Figure 17: Absolute and relative food (solid) and gold (dashed) prices for the fiat model, using full control. Only food-price divergence is shown relative to the mean in the right panel to reduce clutter; gold-price divergence is dominated by period $\tau = 0$, as shown in the left.

rium to economic dynamics calls for the invention of financial institutions and instruments to guide the economy in motion. The optimization problem does not disappear, but it is manifested in economy and efficacy of the institutions and instruments called forth to provide an economic way to handle the needs of the economic dynamics.

Our models presented here were designed to provide illustrations of the physical richness of durable asset laden economy and some of financial arrangements called forth to supply the needed financial engineering. It is fairly evident on considering this relatively simple example that even elementary sensitivity analysis, let alone further complexity calls for simulation and computational methods beyond the type of analytical methods employed here and on other low dimensional economic models.
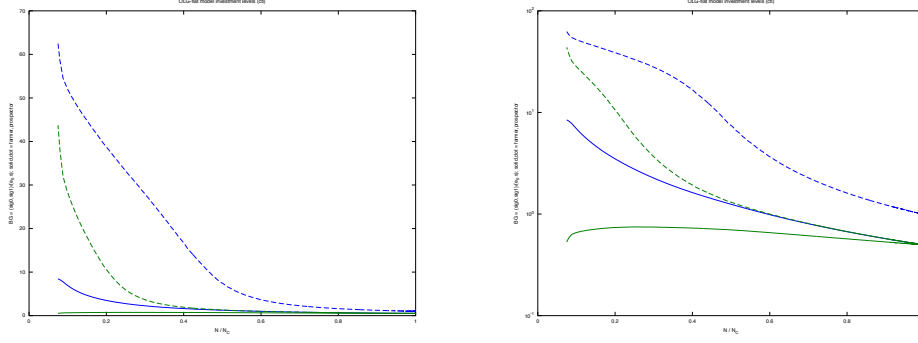
Figure 18: Investment levels for farmers (solid) and prospectors (dashed). BG is σ_0, σ_1.

	CE	Fiat	Bank	IGT
N_C	$\frac{\mathcal{C}}{e_0}\left(\frac{2}{3} + \frac{2}{3(1+\theta)s}\right)$	$\frac{\mathcal{C}}{e_0}\left(\frac{2}{3} + \frac{4}{(1+2\theta)s}\right)$	$\frac{\mathcal{C}}{e_0}\left(\frac{2}{3} + \frac{\sqrt{2}}{s}\right)$	$\frac{\mathcal{C}}{e_0}\left(\frac{2}{3} + \frac{4}{(1+2\theta)s}\right)$
$ap_\tau/e_0 p_{G2}$	1	1	$\sqrt{1+\rho} \to \sqrt{2}$	1

Table 1: N_C values and relative wealth values $ap_\tau/e_0 p_{G2}$ for the competitive equilibrium (CE), and the unstressed non-cooperative equilibria with banking and balanced interest payments (Bank), and intergenerational transfers without banking (IGT). (In all models except the fiat economy, gold may be taken as numéraire, in which case $p_{G2} \equiv 1$. In the fiat economy, this factor removes the numéraire dependence from ap_τ. p_τ values differ only by $\mathcal{O}(s)$ in this regime, so which τ is chosen does not matter. The CE price system is drawn from Eq. (8) and Eq. (9).

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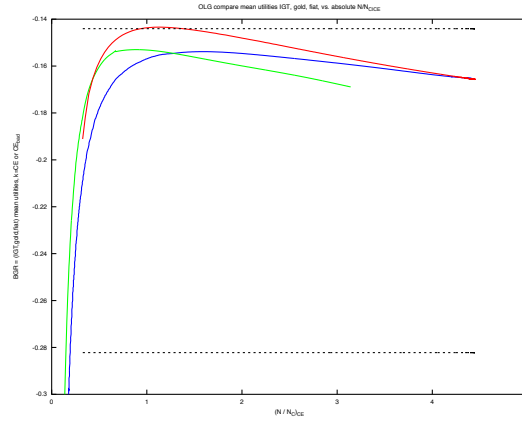
Figure 19: Cross-generation average utilities (by construction the same for farmers as for prospectors) in the three models, as a function of N now measured in absolute size relative to $(N_C)_{\mathrm{CE}}$ of the competitive equilibrium, rather than relative to the N_C specific to each economy. The fiat economy produces the highest maximal utility, converging to the CE value when the actual population is close to the CE critical value (1 on the abscissa).

A   Notations used in the chapter

n_0	number of prospectors of each generation
n	number of farmers of each generation
$N = 2\,(n_0 + n)$	total number of agents alive at any point
N_C	critical population size for high-yielding production w/o shadow price
$\mathcal{C}$	threshold of investment to achieve high-yielding production
ε	ratio of low-yielding to high-yielding production, per unit gold
τ	subscript indexing a period of time
(τ)	superscript indexing a generation of agents
ρ	interest rate on either gold or fiat money
θ	Edgeworth's "coefficient of concern"
a	food endowment to farmers
e_0	gold endowment to prospectors

Table 2: Parameters defining model properties, and parameters optimized outside the strategic context, either by adaptive adjustment or by optimizing policy objectives.

$\mathcal{U}^{(\tau)}$	consumption utility for an agent of generation (τ)
$\mathcal{L}^{(\tau)}$	Lagrangian for an agent of generation (τ)
$A_i^{(\tau)}$	food consumption of a farmer in (young/old) period i
$\hat{A}_i^{(\tau)}$	food consumption of a prospector in (young/old) period i
$S_i^{(\tau)}$	services consumed by a farmer in (young/old) period i
$\hat{S}_i^{(\tau)}$	services consumed by a prospector in (young/old) period i
$b_i^{(\tau)}$	bids by a farmer on food in (young/old) period i
$\hat{b}_i^{(\tau)}$	bids by a prospector on food in (young/old) period i
$q^{(\tau)}$	quantity of food offered by a young farmer in period τ
p_τ	price of food in period τ
$b_{Gi}^{(\tau)}$	bids by a farmer on gold in (young/old) period i
$\hat{q}_G^{(\tau)}$	quantity of gold offered by a young prospector in period τ
$Q_G^{(\tau)}$	quantity of gold offered by a central bank as seller
$B_G^{(\tau)}$	bids on gold by a central bank as buyer
α	parameter representing relation of $B_G^{(0)}$ to $B_G^{(1)}$
$p_{G,\tau}$	price of gold in period τ
σ_0	investment by young ($\tau = 0$) farmers
σ_1	investment by old ($\tau = 2$) farmers
$\hat{\sigma}_0$	investment by young ($\tau = 0$) prospectors
$\hat{\sigma}_1$	investment by old ($\tau = 2$) prospectors
$g_i^{(\tau)}$	loan or deposit (gold or fiat) by a farmer in (young/old) period i
$\hat{g}_i^{(\tau)}$	loan or deposit (gold or fiat) by a prospector in (young/old) period i
$\kappa^{(\tau)}$	intergenerational transfer by a farmer from generation (τ)
$\mu_i^{(\tau)}$	starting money budget of a farmer in (young/old) period i
$\hat{\mu}_i^{(\tau)}$	starting money budget of a prospector in (young/old) period i
$\eta_i^{(\tau)}$	K-T multiplier for farmer budget constraint in (young/old) period i
$\hat{\eta}_i^{(\tau)}$	K-T multiplier for prospector budget constraint in (young/old) period i
Λ	K-T multiplier for the capacity constraint on high-yielding production
$\gamma_i^{(\tau)}$	starting gold stock of a farmer in period i (fiat model)
$\hat{\gamma}_i^{(\tau)}$	starting gold stock of a prospector in period i (fiat model)
$\eta_{G,i}^{(\tau)}$	K-T multiplier for farmer gold-stock in period i (fiat model)
$\hat{\eta}_{G,i}^{(\tau)}$	K-T multiplier for prospector gold-stock in period i (fiat model)

Table 3: Strategic variables in the models

B Regularizing the threshold constraint on capacity with logarithmic utilities

Consider the optimization problem for an economy with one consumption good, the demand for which by each agent i we denote S_i, and which we think of as services from capital stock in the manner defined in Ref. [10]. Here, however, rather than considering the capital stock and its delivery of services disaggregated, we suppose that the capital stock is produced in a single package and that its services are distributed in proportions to shares in the package that agents own. We suppose that each agent i has a utility of demand

$$\mathcal{U}_i(S_i) = s \log S_i. \tag{40}$$

We will begin with the general notation for a strictly concave utility, to indicate its role in the calculation, and then use the logarithmic form to propose a specific simplified representation for the optimization problem.

Suppose that capital stock is built with contributions from agents, which we denote σ_i. The contributions must come entirely from endowments e_i that the agents receive, so that they maximize the Lagrangian

$$\mathcal{L}_i(\sigma_i, \eta_i) = \mathcal{U}_i(S_i) + \eta_i\left(e_i - \sigma_i\right). \tag{41}$$

in which $\eta_i \in [0, \infty]$ is the Kuhn-Tucker multiplier enforcing $e_i - \sigma_i \geq 0$.

We must now specify how the set of service streams $\{S_i\}$ is determined from the set of contributions $\{\sigma_i\}$. The integer programming problem that we wish to solve is a threshold problem: qualitatively, if the sum of investments $\sum_i \sigma_i$ exceeds some threshold capacity C, then the total output is $\sum_i \sigma_i$; otherwise it is zero. Because both the value and the derivative of this function are discontinuous at $\sum_i \sigma_i = C$, we replace the discontinuous function with a strongly non-convex but twice-differentiable function

$$c\left(\sum_i \sigma_i\right),$$

of the form indicated in Fig. 3.

The service streams are then allocated in the proportions of the buy-sell clearing rule for markets:

$$S_i = \frac{\sigma_i}{\sum_{i'} \sigma_{i'}} c\left(\sum_{i'} \sigma_{i'}\right). \tag{42}$$

The first-order condition for any agent's optimization problem is

$$\frac{d\mathcal{U}_i}{dS_i}\frac{dS_i}{d\sigma_i} - \eta_i = 0, \tag{43}$$

in which

$$\frac{dS_i}{d\sigma_i} = \frac{c(\sum_{i'}\sigma_{i'})}{\sum_{i'}\sigma_{i'}}\left(1 - \frac{\sigma_i}{\sum_{i'}\sigma_{i'}}\right) + S_i\left.\frac{d\log c(y)}{dy}\right|_{y=\sum_{i'}\sigma_{i'}} \tag{44}$$

The important property of $d\log c(y)/dy$ is that it is a function only of the total contributions by all agents, and it runs from a minimum of $1/C$ for $\sum_i \sigma_i > C$ to a maximum of ∞ for $\sum_i \sigma_i$ slightly less than C. By making the "corner" of the transition in the function c very sharp, we may compress the interval of this transition as much as desired.

In the special case (40) that the utility is logarithmic, the marginal utility $d\mathcal{U}_i/dS_i = s/S_i$, and we may rewrite the first-order condition (43) as

$$\frac{d\mathcal{U}_i(\sigma_i)}{d\sigma_i} + \Lambda - \eta_i \approx 0. \tag{45}$$

In Eq. (45) we have used $S_i \approx \sigma_i$ as an upper-semicontinuous function in Eq. (42) to replace the argument of $\mathcal{U}_i$, we have ignored terms at order $\sigma_i/\sum_{i'}\sigma_{i'}$ in Eq. (44) as a large-population approximation, and we have introduced a lumped representation

$$\begin{aligned}
\Lambda &\equiv \left.\frac{d\mathcal{U}_i}{dS_i}S_i\frac{d\log c(y)}{dy}\right|_{y=\sum_{i'}\sigma_{i'}} \\
&= \left.s\frac{d\log c(y)}{dy}\right|_{y=\sum_{i'}\sigma_{i'}}, \tag{46}
\end{aligned}$$

making use of the logarithmic utility. Note that Λ is a function only of the total contribution $\sum_{i'}\sigma_{i'}$ but not otherwise of σ_i.

The result is that we may solve the original optimization problem by maximizing (over $\{\sigma_i\}$) and minimizing (over $\{\eta_i\}$ and Λ) the expression

$$\mathcal{L} \equiv \sum_i [\mathcal{U}_i(\sigma_i) + \eta_i(e_i - \sigma_i)] + \Lambda\left(\sum_i \sigma_i - C\right), \tag{47}$$

treating Λ as a Kuhn-Tucker multiplier shared among all the agents.

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This solution is chosen for simplicity in the case of logarithmic utility, and it retains this form even when $\mathcal{U}_i$ has more complex dependence on σ_i, as occurs in the inheritance model of this chapter. The generalization of this method of regularizing threshold functions to more complicated utilities, or to utilities with different functional forms for different agents, is straightforward though the solution that results will no longer have the general form of a simple Kuhn-Tucker multiplier.

C   The exclusion of wash selling in the OLG models

It is possible simply to rule out wash selling from the rules of the game, but it is cleaner to model the economy as treating agent types symmetrically, so that all constraints that differentiate their types originate in their choices of production function. In these models it will be possible to show that wash selling does not arise, as a property of solutions.

The first line of Eq. (17) shows that $b_0^{(\tau)}$ will be zero if either $\eta_0^{(\tau)} > 0$, or the initial budget $\mu_0^{(\tau)} = 0$ (there is nothing to spend). Wash selling does not occur if either of these conditions can be established.

C.1   No wash selling in banking models

Borrowing at inter-period interest rate ρ from any kind of bank (gold or fiat) will produce a K-T multiplier from the default constraint that will always set

$$\eta_0^{(\tau)} = \rho \eta_1^{(\tau)} > 0, \tag{48}$$

so wash selling will not arise in models with banking.

C.2   No wash selling in models with intergenerational transfers and without banking

The case with inter-generational transfers (IGT) among farmers, but without banking, is more complicated but still tractable. ($\tau = 1$)-farmers cannot invest and their consumption level is not directly influenced by IGTs, so the ($\tau = 0$) first-order conditions set transfer $\kappa^{(0)} = 0$. Hence the ($\tau = 1$) budget $\mu_0^{(1)} = 0$ also and so $b_0^{(0)} = 0$.

C.2.1 No transfers from generation ($\tau = 0$)

For $\tau = 0$ and transfer $\kappa^{(2)}$ from the previous generation, it is possible to show that as long as $\theta < 1$, the transfer in the unstressed equilibrium is always smaller than the minimum for $\eta_0^{(0)}$ in Eq. (19), excluding wash sales in this limit. While we do not offer an analytic argument, the numerical solutions presented in this chapter show that as the capacity constraint binds, $\eta_0^{(0)}$ increases monotonically with the shadow price Λe_0, becoming linear at large Λe_0. (This result is not surprising, as a shadow price for the capacity constraint lowers all investment levels relative to their unstressed-equilibrium values, but does not similarly force food consumption downward.) Therefore wash sales are excluded from all solutions in the farmer ($\tau = 0$)-generation.

C.2.2 No transfers for a large range of shadow prices from generation ($\tau = 1$)

For ($\tau = 2$)-farmers, the first-order condition from the offspring-regarding utility at generation ($\tau = 1$), which determines whether transfers $\kappa^{(1)} > 0$, is complicated because these transfers may be partly used by ($\tau = 2$)-farmers for investment. It is again possible to show that for a very large range of stresses including the unstressed-equilibrium, the first-order condition never permits $\kappa^{(1)} > 0$, so the budget constraint sets $b_0^{(2)} = 0$. However, allocations consistent with $\kappa^{(1)} > 0$ do fall within the configuration space for sufficiently large Λe_0, and the fact that all investment for generation $\tau = 2$ occurs in the second period requires $\eta_0^{(2)} = 0$ if any non-binding level of wash selling is chosen. The required parameter range for $\kappa^{(1)} > 0$ is never encountered in the solutions we present, however, so for these wash selling is excluded by the budget.

C.2.3 Any solution at ($\tau = 2$) with wash sales may be replaced by an equivalent solution without wash sales

We note for completeness one precise sense in which the degree of wash sales has no effect on any macro-variables and thus does not matter. We consider the case that, whatever the degree of wash selling, it is the same for all agents.[9]

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[9]This assumption maintains the framework of type-symmetric solutions assumed for all other analyses of the monograph. The first-order conditions never exclude type-symmetric

The budget for $(\tau = 2)$-farmers in the IGT is

$$\kappa^{(1)} + \left(q^{(2)}p_2 - b_0^{(2)}\right) = b_1^{(2)} + \sigma_1 + \kappa^{(2)}. \tag{49}$$

The quantity in parentheses on the left-hand side does not depend on strategic variables of the $(\tau = 2)$-farmers, so the entire left-hand side is for them a boundary condition. The first-order conditions (given in Sec.D.2.2) then make the right-hand side a monotonic equation in $b_1^{(2)}$, which is solved uniquely for $\Lambda b_1^{(2)}$ in terms of $\Lambda \left[\kappa^{(1)} + \left(q^{(2)}p_2 - b_0^{(2)}\right)\right]$. Thus the $(\tau = 2)$-old-period expenditures are fixed, independent of wash selling.

But then, the first-order condition (18) may then be used to write the price level as

$$ap_2 = \left(q^{(2)}p_2 - b_0^{(2)}\right) + b_1^{(2)}. \tag{50}$$

The term in parentheses is an input, and $b_1^{(2)}$ is a wash-sale-independent function of the inputs, so the price p_2 is also independent of wash sales. At fixed prices, $q^{(2)}$ becomes an affine function of $b_0^{(2)}$ with coefficients determined only by $\kappa^{(1)} + \left(q^{(2)}p_2 - b_0^{(2)}\right)$, which then also leaves all consumption levels and budgets invariant.

Since nothing depends on wash selling, and since it is not dis-favored by the first-order conditions, we could exclude it by a secondary criterion of simplicity and identifiability by agents. It is one of two boundary cases ($b_0^{(2)} = 0$ or $b_0^{(2)} = \kappa^{(1)}$), and of these it the simpler because it does not depend on $\kappa^{(1)}$.[10]

D Analysis of the OLG model with intergenerational transfers but without banking

For the OLG model with intergenerational transfers but without banking, we analyze the prospector sector first because the logarithmic utility separates prospectors' decision variables from the rest of the economy except through

---

solutions, so at most we are failing to look for a high-dimensional set of non-symmetric solutions which we haven't proved do not exist.

[10]This analysis has been for the infinite-replicates limit. We note, however, that in other chapters with finite rather than infinite replicates in same buy-sell game, wash selling was *weakly but explicitly* excluded (at $\mathcal{O}(1/N)$) by the first-oder conditions.

the shadow price of the capacity constraint and the labor allocation required to equalize three-period average utilities between farmers and prospectors.

D.1 Prospector sector

The prospector first-order conditions are given in Eq. (20).

The prospectors get all endowments in the young period and must spend in both periods, so they must have carry-forward, which means that their $\eta_0^{(\tau)} \equiv 0$, and they are governed by Eq. (51) with $k = 2$. If we disallow IGTs for them, then their total wealth is just the endowment e_0 for every generation, and the results become very simple.

The solutions to a variety of first-order conditions of the form (20) are given in terms of the net worths of individual endowments by solutions to a closely-related set of quadratic equations. Therefore we introduce the notation for a function

$$\varphi_z^{(k)}(s;y) \equiv \frac{k + zs + (z + 2\theta)\, y}{2\,(z + 2\theta)\, y} \left[1 - \sqrt{1 - \frac{4k\,(z + 2\theta)\, y}{(k + zs + (z + 2\theta)\, y)^2}} \right]. \quad (51)$$

The sign convention for the square roots is that $\varphi_0^{(k)}(s;y) \equiv 1, \ \forall k, s, y$. $y \to 0$ will define the unstressed non-cooperative equilibrium for each model ($\Lambda e_0 \to 0$, or non-binding capacity constraint), so we note that

$$\varphi_z^{(k)}(s;0) = \frac{k}{k + zs}. \quad (52)$$

For the case of prospectors in the no-banking IGT model, their bids on food are given by
$$\hat{b}_0^{(\tau)} = \hat{b}_1^{(\tau)} = \frac{e_0}{2}\, \varphi_{\tau-1}^{(2)}(s; \Lambda e_0). \quad (53)$$

The investment levels $\hat{\sigma}_0$, $\hat{\sigma}_1$ are then given by the general budget relation

$$e_0 = \hat{b}_0^{(\tau)} + \hat{b}_1^{(\tau)} + \hat{\sigma}_0 \delta_{\tau,0} + \hat{\sigma}_1 \delta_{\tau,2}. \quad (54)$$

D.2 Farmer sector

For the farmers we give forms for the marginal utilities and budget constraints individually for each generation, because the offspring-regarding term presents different difficulties of evaluation in each generation. The general

method of solution begins with recognizing that the lack of an investment opportunity for $\tau = 1$ gives $\kappa^{(0)} = 0$ directly from the ($\tau = 0$)-first-order conditions. This creates a break-point for IGTs in the cycle, from which we can recursively solve the remainder of the allocation variables from any sequence of prices (p_0, p_1, p_2). The prices are determined by matching gold inflow to markets with farmer investment by which gold exits the system.

D.2.1 $\quad \tau = 0$

The extension of the consumption-first-order condition (17) to include the IGT variable $\kappa^{(0)}$ is

$$
\begin{aligned}
0 = \delta \mathcal{L}^{(0)} \quad = \quad & \left[\frac{1}{A_0^{(0)} p_0} - \eta_1^{(0)} \right] \left(\delta b_0^{(0)} - p_0 \delta q^{(0)} \right) - \eta_0^{(0)} \delta b_0^{(0)} + \left[\frac{1}{b_1^{(0)}} - \eta_1^{(0)} \right] \delta b_1^{(0)} \\
& + \left[\frac{2s}{\sigma_0} + 2 \left(1 + \theta \right) \Lambda - \left(\eta_0^{(0)} + \eta_1^{(0)} \right) \right] \delta \sigma_0 - \eta_1^{(0)} \delta \kappa^{(0)}. \quad (55)
\end{aligned}
$$

Start with the ansatz $\eta_0^{(0)} > 0$, which we will return and verify at the end. This requires both $b_0^{(0)} = 0$ and $\sigma_0 = \kappa^{(2)}$ from the IGT cycle. It also sets $\kappa_0^{(0)} = 0$, making the bid and offer variables

$$
\Lambda b_1^{(0)} \quad = \quad \frac{\Lambda a p_0}{2}, \quad (56)
$$

$$
\frac{q^{(0)}}{a} \quad = \quad \frac{1}{2}. \quad (57)
$$

D.2.2 $\quad \tau = 2$

Working backward in time, the condition $\sigma_0 = \kappa^{(2)}$ enables ($\tau = 2$)-farmers to evaluate all first-order conditions, for their own consumption and the offspring-regarding term for ($\tau = 0$)-farmers. These take the form

$$
\begin{aligned}
0 = \delta \mathcal{L}^{(2)} \quad = \quad & \left[\frac{1}{A_0^{(2)} p_2} - \eta_1^{(2)} \right] \left(\delta b_0^{(2)} - p_2 \delta q^{(2)} \right) - \eta_0^{(2)} \delta b_0^{(2)} + \left[\frac{1}{b_1^{(2)}} - \eta_1^{(2)} \right] \delta b_1^{(2)} \\
& + \left[\frac{s}{\sigma_1} + \left(1 + 2\theta \right) \Lambda - \eta_1^{(2)} \right] \delta \sigma_1 + \left[\frac{2\theta s}{\kappa^{(2)}} + 2\theta \Lambda - \eta_1^{(2)} \right] \delta \kappa^{(2)}.
\end{aligned}
$$

$$
(58)
$$

(Note in passing that for the value $\theta = 1/2$ used in numerical solutions, the first-order conditions for σ_1 and $\kappa^{(2)}$ in the unstressed equilibrium are the same. Therefore $(\tau = 2)$-farmers invest the same amount as they transfer to their $(\tau = 0)$-offspring for investment. In the $s \ll 1$ limit, where prices $ap_\tau \to 1$, this is equal to the investment level of $(\tau = 2)$-prospectors, and half that of $(\tau = 0)$-prospectors. These relations may be seen in Fig. 6 in the main text.)

From the first-order conditions and the budget constraint, the relation between any received transfers $\kappa^{(1)}$, the market value of the endowment ap_2, and the expenditures is

$$\kappa^{(1)} + ap_2 = 2b_1^{(2)} + \sigma_1 + \kappa^{(2)}. \tag{59}$$

We may express this as a relation between the worth of the farmer endowment and the received transfers scaled by the shadow price of the capacity constraint, $\Lambda ap_2 + \Lambda \kappa^{(1)}$, and the similarly-scaled bids $\Lambda b_1^{(2)}$, in the form of a cubic equation in $\Lambda b_1^{(2)}$:

$$\Lambda b_1^{(2)} \left[2 + \frac{s}{1 - (1 + 2\theta)\Lambda b_1^{(2)}} + \frac{2\theta s}{1 - 2\theta \Lambda b_1^{(2)}} \right] = \Lambda ap_2 + \Lambda \kappa^{(1)}. \tag{60}$$

For numerical solution in the IGT model and also the banking model (presented next), we treat $\Lambda b_1^{(2)}$ as an implicit function of prices and transfers through Eq. (60), meaning that we will take $\Lambda b_1^{(2)}$ as an independent variable, and express other quantities as functions of it, leaving the inverse functions implicit. The key to a constructive solution – a consequence of the separation between farmer and prospector sectors provided by logarithmic utilities – is that prospector bids and investment are functions of the absolute stress level in the economy Λe_0 only, while price levels, and farmer bids and investments, are matched to these scales through their dependence on the bid level $\Lambda b_1^{(2)}$, on population n_0/n, and on explicit prospector bids.

For use in all following sections, introduce the notation $x \equiv \Lambda b_1^{(2)}$. If we can simultaneously solve for x and $\Lambda \kappa^{(1)}$ in terms of prices, then we will have the $(\tau = 2)$-bid, and we can compute the following cascade of offer,

investment, and transfer variables:

$$\Lambda \sigma_1 = \frac{sx}{1 - (1 + 2\theta) x}$$

$$\Lambda \sigma_0 = \Lambda \kappa^{(2)} = \frac{2\theta s x}{1 - 2\theta x}$$

$$\frac{q^{(2)}}{a} (\Lambda a p_2) = x \left[1 + \frac{s}{1 - (1 + 2\theta) x} + \frac{2\theta s}{1 - 2\theta x} \right] - \Lambda \kappa^{(1)}. \qquad (61)$$

We will require a simultaneous solution in general, because x and $\Lambda \kappa^{(1)}$ are not independently identified. To resolve their dependence, we must analyze the previous period.

D.2.3 $\tau = 1$

Because $(\tau = 2)$-farmers may now make investments that depend on $\kappa^{(1)}$ if $\kappa^{(1)} > 0$, but because their level of use is no longer expressed in a simple constraint, the first-order condition for the offspring-regarding terms of $(\tau = 1)$-farmers are defined implicitly in terms of this partial use. The full first-order conditions, including this implicit dependence, may be written

$$
\begin{aligned}
0 = \delta \mathcal{L}^{(1)} =\ & \left[\frac{1}{A_0^{(1)} p_1} - \eta_1^{(1)} \right] \left(\delta b_0^{(1)} - p_1 \delta q^{(1)} \right) - \eta_0^{(1)} \delta b_0^{(1)} + \left[\frac{1}{b_1^{(1)}} - \eta_1^{(1)} \right] \delta b_1^{(1)} \\
& + \left[\left(\frac{\theta s}{\sigma_1} + \theta \Lambda \right) \frac{d\sigma_1}{d\kappa^{(1)}} - \eta_1^{(1)} \right] \delta \kappa^{(1)},
\end{aligned}
$$
$$(62)$$

Before considering the value of $\kappa^{(1)}$, the result $\kappa^{(0)} = 0$ that we already have from $\tau = 0$ gives the bid and offer relations to prices

$$\Lambda b_1^{(1)} = \frac{1}{2} \left(\Lambda a p_1 - \Lambda \kappa^{(1)} \right),$$

$$\frac{q^{(1)}}{a} (\Lambda a p_1) = \frac{1}{2} \left(\Lambda a p_1 + \Lambda \kappa^{(1)} \right). \qquad (63)$$

Now return to $\kappa^{(1)}$. If, on the first-order condition for $b_1^{(1)}$, the resulting coefficient

$$\left(\frac{\theta s}{\sigma_1} + \theta \Lambda \right) \frac{d\sigma_1}{d\kappa^{(1)}} - \frac{2}{a p_1 - \kappa^{(1)}}$$

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of $\delta\kappa^{(1)}$ in Eq. (62) cannot be made non-negative, then $\kappa^{(1)} \to 0$. Otherwise, vanishing of this coefficient determines the interior value for $\kappa^{(1)}$. The alternative between these two cases can be written

$$\Lambda\kappa^{(1)} = \max\left\{0, \Lambda ap_1 - \frac{2}{\theta}\left[\left(\frac{s}{\Lambda\sigma_1} + 1\right)\frac{d\sigma_1}{d\kappa^{(1)}}\right]^{-1}\right\}.\tag{64}$$

The expression (64) can be evaluated, because in the range where $\kappa^{(1)} > 0$ an analytic form can be written for the sensitivity

$$\left(\frac{s}{\Lambda\sigma_1} + 1\right)\frac{d\sigma_1}{d\kappa^{(1)}} = \frac{s\,(1 - 2\theta x)\,/x}{s + 2[1 - (1 + 2\theta)\,x]^2\,[1 + \theta s/(1 - 2\theta x)^2]}.\tag{65}$$

This is a non-monotonic function that starts at ∞ (when $x = 0$), rapidly decreases to $\mathcal{O}(s)$, and then increases (passing through a weak interior maximum) to value 1 at $x \to 1/(1 + 2\theta)$. The small-x branch of solutions is never the relevant one to Equations (60,64), and the large-x branch will only make Eq. (64) positive for $x \sim 1/(1 + 2\theta) - \mathcal{O}(s)$. From the second term in brackets on the left-hand side of Eq. (60), we may recognize that this is the extreme range of stressing where σ_1 accounts for a fraction of $ap_2 + \kappa^{(1)}$ comparable to the fraction from $b_1^{(2)}$.

Once the value of $x = \Lambda b_1^{(2)}$ is found from any pair of values $(\Lambda ap_2, \Lambda ap_1)$, jointly with the value of $\Lambda\kappa^{(1)}$ from Eq. (64), then all other quantities in the three periods are determined. Included in these (if we also supply Λp_0) is an expression for $\eta_0^{(0)}$, which we may verify is nonzero. Numerical checks show that the coefficient of $\delta\kappa^{(1)}/e_0$ in Eq. (62) remains within a few tenths of -2 over the range $\Lambda e_0 \in [0, 3.5]$.

Note that there is a strong non-equivalence in the roles of prices. Only the expression (56) for $b_1^{(0)}$ depends on p_0, and that through a constant relation. (Likewise $q^{(0)}$ is simply a constant.) All other quantities – even σ_0 – are explicitly functions only of the pair of prices $(\Lambda ap_2, \Lambda ap_1)$.

D.3   The price cycle

Once we have excluded wash selling, for any relative numbers n_0 of prospectors and n of farmers, the amount of gold bid in each of the markets is straightforward to express from Eq. (53):

$$q^{(\tau)}p_\tau = b_1^{(\tau-1)} + \frac{n_0}{n}\frac{e_0}{2}\left(\varphi_{\tau-1}^{(2)}(s; \Lambda e_0) + \varphi_{\tau-2}^{(2)}(s; \Lambda e_0)\right)\tag{66}$$

The first term is from old farmers, the second is from young prospectors, and the third is from old prospectors. (In the following equations, where no ambiguity results, we will adopt the shorthand φ_τ for $\varphi_z^{(2)}(s; \Lambda e_0)$ to improve readability.)

The expression (66) for inputs may be combined with the general budget relation for expenditures, which takes the form

$$\kappa^{(\tau-1)} + q^{(\tau)} p_\tau = b_1^{(\tau)} + \sigma_1 \delta_{\tau,2} + \sigma_0 \delta_{\tau,0} + \kappa^{(\tau)}. \qquad (67)$$

Adding the bids in all three periods from Eq. (66), canceling the explicit $b_1^{(\tau)}$ terms that appear on both sides, and using the budgets (67) to resolve the remaining $q^{(\tau)} p_\tau$ terms, we arrive at a relation between aggregate prospector bids and farmer investments:

$$
\begin{aligned}
\frac{n_0}{n} (\Lambda e_0)(1 + \varphi_1 + \varphi_2) &= \Lambda(\sigma_1 + \sigma_0) = \Lambda\left(\sigma_1 + \kappa^{(2)}\right) \\
&= \Lambda b_1^{(2)} \left[\frac{s}{1 - (1 + 2\theta)\Lambda b_1^{(2)}} + \frac{2\theta s}{1 - 2\theta \Lambda b_1^{(2)}} \right] \\
&= x \left[\frac{s}{1 - (1 + 2\theta)x} + \frac{2\theta s}{1 - 2\theta x} \right]. \qquad (68)
\end{aligned}
$$

(Recall, using Eq. (53), that prospectors invest $n_0 e_0 (2 - \varphi_1 - \varphi_2)$, resulting in a total of $3n_0 e_0$ over the economy.)

The following re-arrangements of the bid and budget equations then produce explicit relations among the three price levels:

$$
\begin{aligned}
\Lambda(ap_1 - ap_2) &= \frac{n_0}{n} \Lambda e_0 \varphi_2 \\
\Lambda\left(ap_2 + \kappa^{(1)}\right) &= \Lambda ap_0 + \frac{n_0}{n} \Lambda e_0 \\
\text{and recall Eq. (60)} &= x\left[2 + \frac{s}{1 - (1 + 2\theta)x} + \frac{2\theta s}{1 - 2\theta x} \right] \\
\text{now use Eq. (68)} &= 2x + \frac{n_0}{n}(\Lambda e_0)(1 + \varphi_1 + \varphi_2) \\
\text{subtract } l2 \text{ from } l4 & \\
\Lambda ap_0 &= 2x + \frac{n_0}{n}(\Lambda e_0)(\varphi_1 + \varphi_2). \qquad (69)
\end{aligned}
$$

The foregoing equations are solved numerically as follows: To any pair of values of $(x, \Lambda e_0)$, Eq. (68) assigns a value of n_0/n. From this, the first

line of Eq. (69) assigns the difference $\Lambda a p_2 - \Lambda a p_1$, and the last line assigns $\Lambda a p_0$. In the range where $\kappa^{(1)} = 0$, the third line assigns a value for $\Lambda a p_2$, fully determining the system. We need only search for the contour of equal utilities to specify the other variables as functions of n_0/n. We will show numerically that over the range $\Lambda e_0 \leq 3.5$, the first-order conditions remain far from permitting $\kappa^{(1)} > 0$, so the foregoing assignments are consistent and exact.

D.3.1 Utility differences

The utility differences have fewer terms than the absolute utility magnitudes, because the allocations of publicly-held services are the same to both types in the periods where they occur. Therefore we first identify the utility difference expressions that will select a particular contour of $(x, \Lambda e_0)$ values from the two-dimensional space.

The differences from the consumption utilities for food, using Equations (18,53), may be written as a general function of τ, as

$$\mathcal{U}_{\text{Farm,food}}^{(\tau)} - \mathcal{U}_{\text{Pros,food}}^{(\tau)} = 2 \log \left(\frac{2 b_1^{(\tau)}}{e_0 \varphi_{\tau-1}} \right) \tag{70}$$

The difference in utilities from consumption of services by the agents themselves is

$$\mathcal{U}_{\text{Farm,self}}^{(\tau)} - \mathcal{U}_{\text{Pros,self}}^{(\tau)} = (\tau - 1) \, s \log \left(\frac{\sigma_0 \delta_{\tau,0} + \sigma_1 \delta_{\tau,2}}{e_0 \left(1 - \varphi_{\tau-1} \right)} \right), \tag{71}$$

where all expressions $\tau - 1$ are understood to be evaluated cyclically.

The difference in utility of services consumed by offspring are cycled in τ and scaled by θ:

$$\mathcal{U}_{\text{Farm,offspr}}^{(\tau)} - \mathcal{U}_{\text{Pros,offspr}}^{(\tau)} = \theta \left(\mathcal{U}_{\text{Farm,self}}^{(\tau+1)} - \mathcal{U}_{\text{Pros,self}}^{(\tau+1)} \right). \tag{72}$$

It may be clearer to write these for each period independently, filling in evaluations for bid variables using the preceding relations. An evaluation

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that emphasizes the role of prices is

$$
\begin{aligned}
\mathcal{U}^{(0)}_{\text{Farm,food}} - \mathcal{U}^{(0)}_{\text{Pros,food}} &= 2 \log \left(\frac{ap_0}{e_0} \right) + 2 \log \left(\frac{1}{\varphi_2} \right) \\
\mathcal{U}^{(1)}_{\text{Farm,food}} - \mathcal{U}^{(1)}_{\text{Pros,food}} &= 2 \log \left(\frac{ap_1 - \kappa^{(1)}}{e_0} \right) \\
\mathcal{U}^{(2)}_{\text{Farm,food}} - \mathcal{U}^{(2)}_{\text{Pros,food}} &= 2 \log \left(\frac{ap_2 + \kappa^{(1)}}{e_0 \varphi_1} \right) - 2 \log \left(\frac{ap_2 + \kappa^{(1)}}{2b_1^{(2)}} \right) \\
&= 2 \log \left(\frac{ap_2 + \kappa^{(1)}}{e_0} \right) + 2 \log \left(\frac{1}{\varphi_1} \right) \\
&\quad - 2 \log \left(\frac{2 + \frac{s}{1-(1+2\theta)x} + \frac{2\theta s}{1-2\theta x}}{2} \right)
\end{aligned}
$$

(73)

The difference for services consumed by self become

$$
\begin{aligned}
\mathcal{U}^{(0)}_{\text{Farm,self}} - \mathcal{U}^{(0)}_{\text{Pros,self}} &= 2s \log \left(\frac{\sigma_0}{e_0 (1 - \varphi_2)} \right) = 2s \log \left(\frac{\kappa^{(2)}}{e_0 (1 - \varphi_2)} \right) \\
&= 2s \log \left(\frac{ap_2 + \kappa^{(1)}}{e_0} \right) + 2s \log \left(\frac{s}{1 - \varphi_2} \right) \\
&\quad - 2s \log \left(\frac{(1 - 2\theta x) \left[2 + \frac{s}{1-(1+2\theta)x} + \frac{2\theta s}{1-2\theta x} \right]}{2\theta} \right) \\
\mathcal{U}^{(1)}_{\text{Farm,self}} - \mathcal{U}^{(1)}_{\text{Pros,self}} &= 0 \\
\mathcal{U}^{(2)}_{\text{Farm,self}} - \mathcal{U}^{(2)}_{\text{Pros,self}} &= s \log \left(\frac{\sigma_1}{e_0 (1 - \varphi_1)} \right)
\end{aligned}
$$

(74)

A numerical analysis from the condition that the equally-weighted sum $\sum_\tau \mathcal{U}^{(2)}_{\text{Farm}} - \mathcal{U}^{(2)}_{\text{Pros}} = 0$ gives the shadow-price, food-price, and investment levels shown in the main text.

To convert the expressions for utility differences into absolute utility magnitudes, it is easiest to use the prospector variables. The prospector utility of food consumption is

$$
\mathcal{U}^{(\tau)}_{\text{Pros,food}} = \log \left(\frac{(e_0 \varphi_{\tau-1})^2}{ap_\tau \, ap_{\tau+1}} \right).
$$

(75)

65

The prospector investment levels are given in Eq. (54), and the service levels derived from them for the three periods are given in Eq. (14) in the main text. From these values for both the focal agent and the offspring generation, we obtain the utility levels plotted in Fig. 7 in the main text.

E   Analysis of the OLG model with an inside bank for gold

The first-order conditions for the banking model follow from the Lagrangian (24). The reason this model brings prices in the unstressed noncooperative equilibrium close to those in the competitive equilibrium is that now all agents face a similar relation between their bids and the market value of their endowments, apart from interest payments. Therefore solutions for farmers and prospectors are structurally more similar than in the IGT economy.

E.1   Farmer sector

The first-order conditions for farmers are

$$
\begin{aligned}
0 = \delta \mathcal{L}^{(\tau)} = & \left[\frac{1}{A_0^{(\tau)} p_\tau} - \eta_1^{(\tau)} \right] \left(\delta b_0^{(\tau)} - p_\tau \delta q^{(\tau)} \right) - \eta_0^{(\tau)} \delta b_0^{(\tau)} + \left[\frac{1}{b_1^{(\tau)}} - \eta_1^{(\tau)} \right] \delta b_1^{(\tau)} \\
& + \left[\frac{2s}{\sigma_0} + 2 \left(1 + \theta \right) \Lambda - \left(\eta_0^{(\tau)} + \eta_1^{(\tau)} \right) \right] \delta \sigma_0 \delta_{\tau,0} \\
& + \left[\frac{s}{\sigma_1} + \left(1 + 2\theta \right) \Lambda - \eta_1^{(\tau)} \right] \delta \sigma_1 \delta_{\tau,2} \\
& + \left[\left(\eta_0^{(\tau)} + \eta_1^{(\tau)} \right) - \Lambda_B^{(\tau)} \left(1 + \rho \right) \right] \delta g_0^{(\tau)} + \left[\Lambda_B^{(\tau)} - \eta_1^{(\tau)} \right] \left(-\delta g_1^{(\tau)} \right)
\end{aligned}
$$

$$(76)$$

The option to lend or borrow sets $\Lambda_B^{(\tau)} = \eta_1^{(\tau)} > 0$, and this in turn sets $\eta_0^{(\tau)} = \rho \eta_1^{(\tau)}$ for all τ. There is no wash selling ($b_0^{(\tau)} = 0$) and the remaining expenses satisfy the budget relations

$$
\begin{aligned}
g_0^{(\tau)} & = \sigma_0 \delta_{\tau,0} \\
A_0^{(\tau)} p_\tau & = b_1^{(\tau)} + \sigma_1 \delta_{\tau,2} + \left(1 + \rho \right) g_0^{(\tau)}.
\end{aligned}
$$

$$(77)$$

The other first-order condition we will use is the general relation Eq. (18).

66

(We will show below that the interest rate in the unstressed equilibrium goes to $\rho \to 1 + \mathcal{O}(s)$. Note that in Eq. (76) this value for ρ makes the first-order conditions for σ_0 and σ_1 the same, leading to the same investment level in both generations $\tau = 0, 2$. We will see a similar equality between generations for prospectors, but prospectors will invest at the higher level because they can lend rather than borrow. This difference (nominally a factor of 2) will be partly compensated (to a factor of $\sqrt{2}$) because the unstressed-equilibrium price levels $ap_\tau/e_0 \to \sqrt{2}$ rather than 1 as in the CE and the IGT economy. All these limits are met in numerical simulations in Fig. 9.)

The budget and first-order conditions lead to a relation between bids, investment, and prices of the form

$$ap_\tau = 2b_1^{(\tau)} + (1+\rho)\,\sigma_0\delta_{\tau,0} + \sigma_1\delta_{\tau,2}. \tag{78}$$

The first-order conditions between σ and b variables also lead to quadratic relations between b and ap levels comparable to those for prospectors in both the IGT and banking models. They are

$$
\begin{aligned}
b_1^{(0)} &= \frac{ap_0}{2}\varphi_2^{(2)}\left(s; \frac{\Lambda ap_0}{(1+\rho)}\right). \\
b_1^{(1)} &= \frac{ap_1}{2} \\
b_1^{(2)} &= \frac{ap_2}{2}\varphi_1^{(2)}(s; \Lambda ap_2).
\end{aligned}
\tag{79}
$$

E.2   Prospector sector

The first-order conditions for the prospectors are

$$
\begin{aligned}
0 = \delta\mathcal{L}^{(\tau)} &= \left[\frac{1}{b_0^{(\tau)}} - \left(\eta_0^{(\tau)} + \eta_1^{(\tau)}\right)\right]\delta b_0^{(\tau)} + \left[\frac{1}{b_1^{(\tau)}} - \eta_1^{(\tau)}\right]\delta b_1^{(\tau)} \\
&\quad + \left[\frac{2s}{\sigma_0} + 2\left(1+\theta\right)\Lambda - \left(\eta_0^{(\tau)} + \eta_1^{(\tau)}\right)\right]\delta\sigma_0\delta_{\tau,0} \\
&\quad + \left[\frac{s}{\sigma_1} + \left(1+2\theta\right)\Lambda - \eta_1^{(\tau)}\right]\delta\sigma_1\delta_{\tau,2} \\
&\quad + \left[\left(\eta_0^{(\tau)} + \eta_1^{(\tau)}\right) - \Lambda_B\left(1+\rho\right)\right]\delta g_0^{(\tau)} + \left[\Lambda_B - \eta_1^{(\tau)}\right]\left(-\delta g_1^{(\tau)}\right)
\end{aligned}
\tag{80}
$$

Two general conditions that follow immediately are $\eta_0^{(\tau)} = \rho \eta_1^{(\tau)}$ and $b_1^{(\tau)} = (1+\rho) b_0^{(\tau)}$.

The budget relation is

$$
\begin{aligned}
e_0 &= b_0^{(\tau)} - g_0^{(\tau)} + \sigma_0 \delta_{\tau,0} \\
-(1+\rho) g_0^{(\tau)} &= g_1^{(\tau)} = b_1^{(\tau)} + \sigma_1 \delta_{\tau,2},
\end{aligned}
\tag{81}
$$

which converts into an expression counterpart to Eq. (78), of the form

$$
e_0 = 2 b_0^{(\tau)} + \sigma_0 \delta_{\tau,0} + \frac{\sigma_1}{(1+\rho)} \delta_{\tau,2}.
\tag{82}
$$

From these, the same algebra used for the IGT economy produces three relations for the bids in terms of the endowment

$$
\begin{aligned}
b_0^{(0)} = \frac{b_1^{(0)}}{(1+\rho)} &= \frac{e_0}{2} \varphi_2^{(2)}(s; \Lambda e_0) \\
b_0^{(1)} = \frac{b_1^{(1)}}{(1+\rho)} &= \frac{e_0}{2} \\
b_0^{(2)} = \frac{b_1^{(2)}}{(1+\rho)} &= \frac{e_0}{2} \varphi_1^{(2)}(s; (1+\rho) \Lambda e_0).
\end{aligned}
\tag{83}
$$

These equations can be compared to Eq. (53) for the IGT economy. The relative forms are the same, but all positions of e_0 for prospectors are scaled by $(1+\rho)$ relative to their counterparts for ap_τ.

If we want a single expression that covers all three periods, we may write

$$
b_0^{(\tau)} = \frac{b_1^{(\tau)}}{(1+\rho)} = \frac{e_0}{2} \varphi_{\tau-1}^{(2)}(s; (1+\rho \delta_{\tau,2}) \Lambda e_0).
\tag{84}
$$

E.3   The price cycle

The three equations that relate bids to prices in the banking model, taking the place of Eq. (66) for the IGT model, are

$$
\begin{aligned}
(1+\rho) \sigma_0 + b_1^{(0)} &= q^{(0)} p_0 = b_1^{(2)} + \frac{n_0}{n} \frac{e_0}{2} (\varphi_2 + (1+\rho) \varphi_1) \\
b_1^{(1)} &= q^{(1)} p_1 = b_1^{(0)} + \frac{n_0}{n} \frac{e_0}{2} (1 + (1+\rho) \varphi_2) \\
\sigma_1 + b_1^{(2)} &= q^{(2)} p_2 = b_1^{(1)} + \frac{n_0}{n} \frac{e_0}{2} (\varphi_1 + (1+\rho)),
\end{aligned}
\tag{85}
$$

where all φs here are shorthand references to the prospector functions in Eq. (84).

Again adding all three equations, and canceling the b variables which appear on both sides, gives the relation of total prospector bids to the farmer level of investment, counterpart to Eq. (68) of the IGT economy:

$$(1 + \rho) \sigma_0 + \sigma_1 = \frac{n_0}{n} e_0 \left(1 + \frac{\rho}{2}\right) (1 + \varphi_1 + \varphi_2). \tag{86}$$

Price levels in this loop are fixed when we demand that interest payments on loans and deposits balance. Equality between total borrowing and total lending reads

$$\begin{aligned}
\sigma_0 &= \frac{n_0}{n} \frac{e_0}{2} \left[\varphi_2 + 1 + (2 - \varphi_1)\right] \\
&= \frac{n_0}{n} \frac{e_0}{2} \left(3 - \varphi_1 + \varphi_2\right),
\end{aligned} \tag{87}$$

where the expressions on the right-hand side in the first line refer to the prospector investments that draw interest in periods 0, 1, 2, respectively.

The condition (87) for balance of interest payments will not generally be compatible with the form for σ_0 from the first line of Eq. (79), at the price levels generated by balanced interest payments. Hence they will violate the first-order condition in the second line of Eq. (76). The contours of compatibility of these two conditions in $(x, \Lambda e_0)$ space can be determined numerically, and they are isoclines along which the utility differences between farmers and prospectors vary monotonically. Therefore, in numerical solutions, we must first assign compatible ρ values throughout this space, and then show how ρ varies as a function of $N/N_C \leq 1$ along the contour of equal utilities. Examples showing the contours associated with these two distinct constraints are given in Fig. 20.

Subtracting Eq. (87) from Eq. (86) gives the relation between $x \equiv \Lambda b_1^{(2)}$ and Λe_0 that specifies n_0/n:

$$\sigma_1 = \frac{sx}{1 - (1 + 2\theta) x} = \frac{n_0}{n} \frac{e_0}{2} \left[(1 + \varphi_1 + \varphi_2) - 2 (1 + \rho) (1 - \varphi_1)\right] \tag{88}$$

Note that, at any Λe_0, Eq. (88) sets a limit on the interest rates that could be compatible, of

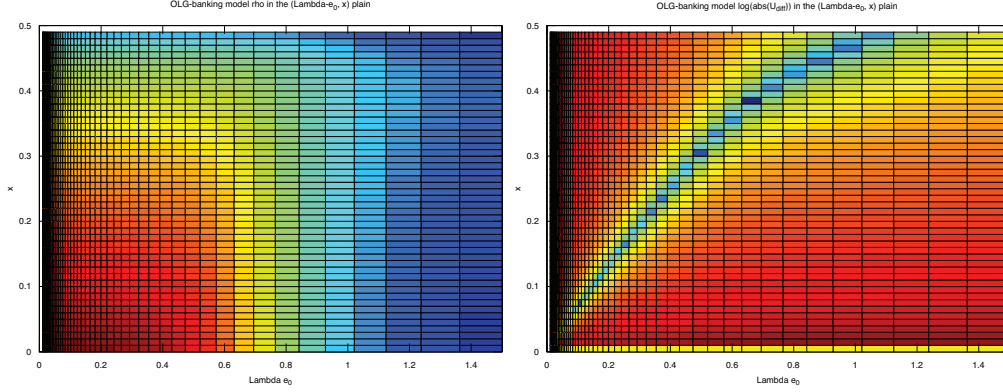$$\rho \leq \frac{3\varphi_1 + \varphi_2 - 1}{2 (1 - \varphi_1)}, \tag{89}$$

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Figure 20: pcolor plots of ρ (left) and $\mathcal{U}_{\mathrm{Farm}}^{(\tau)} - \mathcal{U}_{\mathrm{Pros}}^{(\tau)}$ (right). (Utility differences are plotted in log of absolute value so that the zero crossing shows as a sharp valley.) See that the iso-contours of ρ with the associated values of n_0/n that balance interest and satisfy the first-order conditions are circle-like arcs around the origin, while the iso-utility contours are rays from the origin that cut across these arcs. (Fine-resolution versions of these figures are available at about four time the filesize, in commented-out lines.)

and this in turn limits the range of stresses Λe_0 over which this model has stationary solutions with $\rho \geq 0$, to $3\varphi_1 + \varphi_2 \geq 1$.

From these conditions it is possible assign all price levels as functions of x and Λe_0, in the banking counterpart to the IGT-price cycle of Eq. (69),

$$
\begin{aligned}
\Lambda a p_2 &= 2\Lambda b_1^{(2)} + \Lambda \sigma_1 \\
&= x \left(2 + \frac{s}{1 - (1 + 2\theta)\, x} \right) \\
\Lambda a p_1 &= 2 \left(\Lambda b_1^{(2)} + \Lambda \sigma_1 \right) - \frac{n_0}{n} \left(\Lambda e_0 \right) \left[\varphi_1 + (1 + \rho) \right] \\
&= x \left(2 + \frac{2s}{1 - (1 + 2\theta)\, x} \right) - \frac{n_0}{n} \left(\Lambda e_0 \right) \left[\varphi_1 + (1 + \rho) \right] \\
\Lambda a p_0 &= \Lambda a p_2 - \frac{n_0}{n} \left(\Lambda e_0 \right) \left[1 + \frac{\rho}{2} \left(1 - \varphi_1 + \varphi_2 \right) \right]
\end{aligned}
\tag{90}
$$

From prices, the solutions (79) to the quadratic farmer first-order conditions then assign all remaining bid and investment levels.

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E.4 Equalization of utilities

The final contour of solutions parametrized by either Λe_0 or alternatively n_0/n is again given by equality of utilities. The banking-model utility differences from food consumption, corresponding to to Eq. (73) for the IGT economy, are given by[11]

$$
\begin{aligned}
\mathcal{U}^{(0)}_{\text{Farm,food}} - \mathcal{U}^{(0)}_{\text{Pros,food}} &= 2\log\left(\frac{ap_0}{e_0}\right) + 2\log\left(\frac{\varphi_2^{(2)}\left(s; \frac{\Lambda ap_0}{(1+\rho)}\right).}{\varphi_2^{(2)}(s; \Lambda e_0).}\right) - \log(1+\rho) \\
&= 2\log\left(\frac{\frac{\Lambda ap_0}{(1+\rho)}\,\varphi_2^{(2)}\left(s; \frac{\Lambda ap_0}{(1+\rho)}\right).}{\Lambda e_0\,\varphi_2^{(2)}(s; \Lambda e_0).}\right) + \log(1+\rho) \\
\mathcal{U}^{(1)}_{\text{Farm,food}} - \mathcal{U}^{(1)}_{\text{Pros,food}} &= 2\log\left(\frac{ap_1}{e_0}\right) - \log(1+\rho) \\
&= 2\log\left(\frac{\Lambda ap_1/\sqrt{1+\rho}}{\sqrt{1+\rho}\,\Lambda e_0}\right) + \log(1+\rho) \\
\mathcal{U}^{(2)}_{\text{Farm,food}} - \mathcal{U}^{(2)}_{\text{Pros,food}} &= 2\log\left(\frac{ap_2}{e_0}\right) + 2\log\left(\frac{\varphi_1^{(2)}(s; \Lambda ap_2).}{\varphi_1^{(2)}(s; (1+\rho)\,\Lambda e_0).}\right) - \log(1+\rho) \\
&= 2\log\left(\frac{\Lambda ap_2\,\varphi_1^{(2)}(s; \Lambda ap_2).}{(1+\rho)\,\Lambda e_0\,\varphi_1^{(2)}(s; (1+\rho)\,\Lambda e_0).}\right) + \log(1+\rho)
\end{aligned}
$$

$$(91)$$

The utility differences from an agent's own consumption of services, coun-

---

[11]In the second lines of each of the following equations, we group terms to emphasize the uniform way in which factors of $1+\rho$ interact with wealth levels measured either as Λap_τ (for farmers) or e_0 (for prospectors).

terpart to Eq. (74) for the IGT economy, are given by

$$
\begin{aligned}
\mathcal{U}^{(0)}_{\text{Farm,self}} - \mathcal{U}^{(0)}_{\text{Pros,self}} &= 2s\left[\log\left(\frac{ap_0}{e_0}\right) + \log\left(\frac{1 - \varphi_2^{(2)}\left(s; \frac{\Lambda ap_0}{(1+\rho)}\right).}{1 - \varphi_2^{(2)}(s; \Lambda e_0).}\right) - \log(1+\rho)\right] \\
&= 2s\log\left(\frac{\frac{\Lambda ap_0}{(1+\rho)}\left(1 - \varphi_2^{(2)}\left(s; \frac{\Lambda ap_0}{(1+\rho)}\right).\right)}{\Lambda e_0\left(1 - \varphi_2^{(2)}(s; \Lambda e_0).\right)}\right) \\
\mathcal{U}^{(1)}_{\text{Farm,self}} - \mathcal{U}^{(1)}_{\text{Pros,self}} &= 0 \\
\mathcal{U}^{(2)}_{\text{Farm,self}} - \mathcal{U}^{(2)}_{\text{Pros,self}} &= s\left[\log\left(\frac{ap_2}{e_0}\right) + \log\left(\frac{1 - \varphi_1^{(2)}(s; \Lambda ap_2).}{1 - \varphi_1^{(2)}(s; (1+\rho)\Lambda e_0).}\right) - \log(1+\rho)\right] \\
&= s\log\left(\frac{\Lambda ap_2\left[1 - \varphi_1^{(2)}(s; \Lambda ap_2)\right]}{(1+\rho)\Lambda e_0\left[1 - \varphi_1^{(2)}(s; (1+\rho)\Lambda e_0)\right]}\right)
\end{aligned}
$$

(92)

The offspring term continues to satisfy Eq. (72). Its role in the banking economy is much simpler than in the IGT economy, as it acts only through the shadow price of the capacity constraint.

Numerical solutions for for absolute and relative price levels, investment levels by each type and period, stress level Λe_0 and interest rate ρ, and single-period absolute utility levels, are shown in Figures 8–11 of the main text.

E.5 The distinctive response of interest rates to shadow price on the capacity constraint in the gold-banking model

An inside bank that returns all deposits is a highly-constrained entity, which may be required to adopt particular interest rates at zero and non-zero shadow prices in order to satisfy its constraints. In particular, the first-order conditions for consumption of services for agents who invest in generations $(\tau = 0)$ and $(\tau = 2)$, to be met, may require particular ratios of consumption levels σ_0/σ_1, or $\hat{\sigma}_0/\hat{\sigma}_1$, while the fact that these investments (for farmers) are limited by bidding levels for food may require other particular levels. The

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interest rate is the only device the bank has a available to bring these two conditions into compatibility and ensure its balance of deposits and withdrawals. We therefore consider here in detail the set of constraints that determine interest rates, and the way these depend on the shadow price of the capacity constraint.

E.5.1 Contribution from the first-order conditions

Comparison of the second and third lines in the first-order condition for farmers (76), together with the relation between the K-T multipliers $\eta_0^{(\tau)} = \rho \eta_1^{(\tau)}$ which will hold in any economy with lending at interest, lead us to expect the coarse scaling $\sigma_0 \sim 2\sigma_1 / (1 + \rho)$. The greater marginal utility of investment σ_0 in the young period is weighed against the requirement to borrow gold to supply it.

The full relations, obtained from Equations (78,79), are

$$
\begin{aligned}
(1 + \rho) \, \sigma_0 &= ap_0 \left[1 - \varphi_2^{(2)} \left(s; \frac{\Lambda ap_0}{(1 + \rho)} \right) \right] \to ap_0 s \, (1 + \mathcal{O}(s)) \\
\sigma_1 &= ap_2 \left[1 - \varphi_1^{(2)}(s; \Lambda ap_2) \right] \to \frac{ap_2 s}{2} \, (1 + \mathcal{O}(s)) .
\end{aligned}
\tag{93}
$$

The first expression on the right-hand side of each line is a closed form; the second expression is the leading small-s approximation when the corresponding $\Lambda ap_\tau \to 0$. At interest rates $\rho \ll 1$, the first-order conditions favor twice the investment level in σ_0 as in σ_1.

E.5.2 Accounting identities and balance of interest payments

The requirement to balance interest payments, in the context of the income/expense accounting relation for farmers, places an independent set of constraints on σ_0 and σ_1.

We may re-arrange the sum (86) of income and expenses for farmers into the young-period and old-period bids by prospectors, as

$$
(1 + \rho) \, \sigma_0 + \sigma_1 = \frac{n_0}{n} \frac{e_0}{2} \left[(1 + \varphi_1 + \varphi_2) + (1 + \rho) (1 + \varphi_1 + \varphi_2) \right],
\tag{94}
$$

in which the expressions φ_τ refer to the prospector allocations given in Eq. (83).

The equation (87) of total farmer loans with total prospector deposits, necessary to balance interest payments, may be similarly re-arranged as

$$\sigma_0 = \frac{n_0}{n} \frac{e_0}{2} \left[(1 + \varphi_1 + \varphi_2) + 2 (1 - \varphi_1) \right]. \tag{95}$$

The remainder of the prospector payments that support farmer investment, Eq. (88), which is simply the difference of the previous two equations, then becomes

$$\sigma_1 = \frac{n_0}{n} \frac{e_0}{2} \left[(1 + \varphi_1 + \varphi_2) - 2 (1 + \rho) (1 - \varphi_1) \right]. \tag{96}$$

By the prospector first-order conditions (80), their bids on food in the old and young periods always have the ratio $(1 + \rho)$. Moreover, in the unstressed $(\Lambda e_0 \to 0)$ equilibrium, almost all of the prospector endowment is used to buy food; that is, the factors $(1 - \varphi_1) \approx s/2 \ll 1$ in equations (95,96). This condition produces $\sigma_0 \sim \sigma_1$, so that σ_0 (with interest) is both paid for by *old*-period prospector bids, and in turn provides the interest stream to produce them, while σ_1 is paid for with *young*-period prospector bids, nearly equal in quantity to σ_0.

However, this condition, together with the property that when $s \ll 1$, inter-period fluctuations in food prices are $\mathcal{O}(s)$, then drives $(1 + \rho) \to 2$ in Eq. (93) giving the consequence of the first-order conditions.

E.5.3   Interest rates decrease from the unstressed-equilibrium value $\rho = 1$ with increasing shadow price

It is not obvious *a priori* whether interest rates should increase or decrease as the shadow price on the capacity constraint increases from zero due to insufficient population. Since the relative value of present to future consumption is inverse to the interest rate the way interest rates response to shadow prices will depend on the way payment streams respond. In the fiat-banking model considered next, interest rates will increase from a value $\rho = 0$ in the unstressed equilibrium, as Λe_0 increases. But in this model, interest rates *decrease* from their value $\rho = 1$ in the unstressed equilibrium, as Λe_0 increases. Here we summarize how that response is produced by the intersection of the first-order conditions and balance-of-payment constraints.

It is important to remember that, while the formation of a shadow price on the capacity constraint for capital stock represents a stress relative to the $\Lambda e_0 = 0$-noncooperative equilibrium, it corresponds to the situation $N < N_C$, in which there is a *surfeit* of gold per capita relative to the unstressed

equilibrium. Therefore, although all prospector bids on food decrease per capita as described by Eq. (94), as prospectors supply a larger relative share of investments, their deposits in the bank decrease *less quickly* than their bids on food, according to equations (95,96). At an interest rate ρ that balances total interest payments, the farmer investment σ_0 on which interest is paid therefore declines more slowly than the investment σ_1 on which interest is not paid. Moreover, the term $(1 - \varphi_1)$ responsible for the difference in bid levels increases with increasing Λe_0, while the common term $(1 + \varphi_1 + \varphi_2)$ decreases. When, at finite Λe_0, these two contributions pass through the ratio $2 : 1$, it is possible to satisfy both the first-order conditions and the accounting identity with $\rho \to 0$, even if food prices remain similar across periods.

E.6 Analytic solutions for limits in the unstressed non-cooperative equilibria

Once it is recognized that these constraints determine the dependence of ρ on Λe_0, closed-form solutions are easy to obtain at $s \ll 1$, for the population composition required for $\Lambda e_0 \to 0$, and hence for the critical population size for high-yielding production.

Utility levels are dominated by food consumption at $s \ll 1$, and at $\Lambda e_0 \to 0$, all φ terms cancel in Eq. (91), casting the condition of equal utility in the same functional form for all generations (τ). Therefore all gold-denominated food prices converge to the same value $ap_\tau/e_0 \to \sqrt{1 + \rho}$, reproduced in Eq. (27) in the text.

Using the general budget relation (78), and the particular evaluation from the first line of Eq. (79) to express σ_0 in terms of ap_0, the interest-balance condition (87) at $s \ll 1$ and $\Lambda e_0 \to 0$ gives the ratio of prospectors to farmers

$$\frac{n_0}{n} \to \frac{2s}{3(1+\rho)} \frac{ap_0}{e_0}. \tag{97}$$

As explained in the previous subsection, a similar but independent expression is obtained for σ_1 in terms of ap_2, using Eq. (78) and the second line of Eq. (79). If these two relations are inserted in the remaining accounting identity (88) between total bids and investment, together with the approximation at $s \ll 1$ that all ap_τ take the same value at $\Lambda e_0 \to 0$, we find that the required interest rate in the unstressed equilibrium is $\rho \to 1 + \mathcal{O}(s)$.

Eq. (27) then gives $ap_\tau/e_0 \to \sqrt{2}$, which in Eq. (97) goes to

$$\frac{n_0}{n} \to \frac{\sqrt{2}s}{3}. \tag{98}$$

The resulting critical population size for the unstressed equilibrium to support investment $3n_0e_0 = \mathcal{C}$ is given as Eq. (28) in the main text.

E.7 How much gold does the bank need?

If we require that the bank lend physical gold, we may track the amount that it must hold at the beginning of the period, in order to supply net demands by the agents. The details of required bank reserves can of course depend on the fine-structure of sequencing of deposits and withdrawals in each period. The simplest choice (and one consistent with the interpretation of periods as entire stages of life) is to not assume a detailed structure of lags between withdrawals and deposits of various kinds, and to model deposits and withdrawals within a period as being cleared in a single meeting of all agents with the bank in that period. This is always possible, because prospectors hold all gold they deposit from their endowments coming into the period, and farmers hold all gold they may owe from clearing of the previous-period's food market.

Total deposits in the bank, both lending by prospectors and repayment of principle and interest by farmers, may be computed in each period as

$$
\begin{aligned}
\frac{(\text{Deposits})_0}{n_0 e_0} &= \frac{1}{2}\varphi_2^{(2)}(s; \Lambda e_0) \\
\frac{(\text{Deposits})_1}{n_0 e_0} &= \frac{1}{2} + (1+\rho)\frac{\sigma_0}{e_0}\frac{n}{n_0} \\
\frac{(\text{Deposits})_2}{n_0 e_0} &= 1 - \frac{1}{2}\varphi_1^{(2)}(s; (1+\rho)\Lambda e_0).
\end{aligned}
\tag{99}
$$

Withdrawals in each period are

$$
\begin{aligned}
\frac{(\text{Withdrawals})_0}{n_0 e_0} &= (1+\rho)\left[1 - \frac{1}{2}\varphi_1^{(2)}(s; (1+\rho)\Lambda e_0)\right] + \frac{\sigma_0}{e_0}\frac{n}{n_0} \\
\frac{(\text{Withdrawals})_1}{n_0 e_0} &= (1+\rho)\frac{1}{2}\varphi_2^{(2)}(s; \Lambda e_0) \\
\frac{(\text{Withdrawals})_2}{n_0 e_0} &= (1+\rho)\frac{1}{2}
\end{aligned}
\tag{100}
$$

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The bank balance at the end of each period is the prior balance net of Deposits and Withdrawals, and we choose the maximum balance so that the minimum over the cycle is zero. Recall that the capacity constraint is $\mathcal{C} = 3n_0e_0$, so $(\text{Deposits})_\tau/\mathcal{C} = (1/3)\,(\text{Deposits})_\tau/n_0e_0$, and similarly for Withdrawals and balances. We will see from the solutions that in the unstressed equilibrium, the bank needs to hold approximately $(2.5/3)\,\mathcal{C}$ at the beginning of period $\tau = 2$, and in the small population limit where $\rho \to 0$, this requirement falls below $(2/3)\,\mathcal{C}$ for the value of $\theta = 1/2$.

Numerical solutions for the prior balance of the bank in all three periods are shown in Fig. 12 in the main text. The absolute gold reserve required diminishes weakly with N/N_C as interest rate diminishes, because it is determined mostly by the scale of capital stock. Bank reserves per capita therefore increase roughly as N_C/N.

F Solutions for the economy with fiat banking and control through purchase and sale of gold

F.1 Solutions to the first-order conditions

In the fiat economy, the relation of the gold endowment for investment becomes much more symmetrical with the role of the food endowment for single-period consumption than it is in an economy with gold money. As in the previous examples, logarithmic utility leads to convenient simplifications for the prospectors, so we begin with them. Because the numéraire of fiat is now an arbitrary choice of the central bank, all prices will be referenced to a single price that fixes this numéraire, which we take to be the $\tau = 2$-price for gold: p_{G2}.

F.1.1 Prospector sector

The accounting identity for the fiat-value of the prospector endowment of gold is

$$
\begin{aligned}
e_0 p_{G\tau} &= \hat{q}_G^{(\tau)} p_{G\tau} + (\hat{\sigma}_0 \delta_{\tau,0} + \hat{\sigma}_1 \delta_{\tau,2})\, p_{G\tau} \\
&= 2\hat{b}_1^{(\tau)} + (\hat{\sigma}_0 \delta_{\tau,0} + \hat{\sigma}_1 \delta_{\tau,2})\, p_{G\tau}.
\end{aligned} \tag{101}
$$

In the second line we have used the budget relations (36), along with the fact that the first-order conditions with lending at interest give $\hat{b}_1^{(\tau)} = (1 + \rho)\,\hat{b}_0^{(\tau)}$, as they did for the banking model in Eq. (84).

Then the first-order conditions for prospectors relate their bids on food to total gold price levels in each period as

$$(1 + \rho)\,\hat{b}_0^{(\tau)} = \hat{b}_1^{(\tau)} = \frac{e_0}{2} p_{G\tau} \varphi_{\tau-1}^{(2)}(s; \Lambda e_0)\,, \tag{102}$$

the investment levels are given by

$$\begin{aligned}
\hat{\sigma}_0 &= e_0\left(1 - \varphi_2^{(2)}(s; \Lambda e_0)\right) \\
\hat{\sigma}_1 &= e_0\left(1 - \varphi_1^{(2)}(s; \Lambda e_0)\right),
\end{aligned} \tag{103}$$

and therefore the offered quantities satisfy

$$\hat{q}_G^{(\tau)} = e_0 \varphi_{\tau-1}^{(2)}(s; \Lambda e_0)\,. \tag{104}$$

These inputs form the basis for all other price-formation rules, given any shadow-price valuation of the gold endowment Λe_0.

F.1.2 Farmer sector

The first-order conditions for investment relative to old-period bids on food take the same form as they did in the IGT economy,

$$\begin{aligned}
\Lambda \sigma_1 &= \frac{sx}{1 - (1 + 2\theta)\,x} \\
\Lambda \sigma_0 = \Lambda \kappa^{(2)} &= \frac{2\theta sx}{1 - 2\theta x},
\end{aligned} \tag{105}$$

in terms of a collection of variables that we abbreviate

$$x \equiv \frac{\Lambda b_1^{(2)}}{(1 + \rho)\,p_{G2}}\,. \tag{106}$$

(Definition (106) differs from the x in Eq. (61) in the IGT model by the factor $(1 + \rho)$, and by pricing specifically relative to period-$(\tau = 2)$ gold through the factor p_{G2}.)

F.1.3 Bank action

Only ($\tau = 2$) farmers in their young period can bid on gold in time for its delivery to serve either their own investments or their transfers to support the investments of their ($\tau = 0$)-offspring. Therefore we may write the gold-price-formation rule (38) more explicitly in the form

$$\frac{n}{n_0} b_{G0}^{(2)} \delta_{\tau,2} + \frac{B_G^{(\tau)}}{n_0} = \left(\hat{q}^{(\tau)} + \frac{Q_G^{(\tau)}}{n_0} \right) p_{G\tau}. \tag{107}$$

All other bids outside period $\tau = 2$ must come from the central bank. Using the expression (104) for offers of gold in these periods, and descaling with p_{G2} to remove the numéraire of fiat, we arrive at two of the central bank's control variables expressed in terms of their effect on inter-period ratios of gold prices, as

$$\frac{B_G^{(\tau)}}{n_0 e_0 p_{G2}} = \frac{p_{G\tau}}{p_{G2}} \varphi_{\tau-1}^{(2)}(s; \Lambda e_0), \tag{108}$$

for $\tau = 0, 1$.

The exclusion of wash selling means that offers are not made in these periods, so that the central bank offers gold only in period $\tau = 2$. Full return of gold then requires

$$\begin{aligned}
\frac{Q_G^{(2)}}{n_0} &= q^{(0)} + q^{(1)} \\
&= e_0 \left(\varphi_2^{(2)}(s; \Lambda e_0) + 1 \right)
\end{aligned} \tag{109}$$

Evaluating the price expression (107) at $\tau = 2$, and using Eq. (109) for $Q_G^{(2)}/n_0$ together with the fact that $B_G^{(\tau)} = 0$ by exclusion of wash sales, gives a relation between the total offered quantity (which depends only on e_0 and Λ), the labor allocation n_0/n, and the farmer bid level set by $b_{G0}^{(2)}$, in the form

$$\begin{aligned}
\frac{\sigma_1 + \sigma_0}{e_0} = \frac{\sigma_1 + \kappa^{(2)}}{e_0} = \frac{b_{G0}^{(2)}}{e_0 p_{G2}} &= \frac{n_0}{n} \left(1 + \varphi_1^{(2)}(s; \Lambda e_0) + \varphi_2^{(2)}(s; \Lambda e_0) \right) \\
&= \frac{sx}{\Lambda e_0} \left(\frac{1}{1 - (1 + 2\theta) x} + \frac{2\theta}{1 - 2\theta x} \right) \tag{110}
\end{aligned}$$

The expressions on the first line involve only relative quantities of gold, and ensure that the total investment from farmers and prospectors equals $3 n_0 e_0$

(by Eq. (103)). The relation to x – the farmer bid-level in $(\tau = 2)$-gold prices in the second line – follows from the farmer first-order conditions.

In numerical solutions, we will sweep the variables Λe_0 appearing in the first line and x appearing in the second line of Eq. (110), and for each pair, determine the consistent value for n_0/n to appear in later relations.

F.2   Price cycles

F.2.1   Cyclic accounting identities for fiat and gold

The farmer budget equation that cycles fiat among periods in the food markets is

$$
\begin{aligned}
q^{(\tau)} p_\tau &= b_1^{(\tau)} + (1 + \rho) \, b_{G0}^{(\tau)} \\
&= b_1^{(\tau-1)} + \frac{n_0}{n} \left(\hat{b}_1^{(\tau-1)} + \hat{b}_0^{(\tau)} \right) \\
&= b_1^{(\tau-1)} + \frac{n_0}{n} \left((1 + \rho) \, \hat{b}_0^{(\tau-1)} + \hat{b}_1^{(\tau)} \right)
\end{aligned}
\tag{111}
$$

The first line includes the bids (along with interest on bids from the first period) that farmers pay from proceeds of sales, and the second (and equivalently, third) line lists the source of money from the pricing rule (10).

Summing Eq. (111) over τ, and canceling the factors $b_1^{(\tau)}$ that appear on both sides of the equality, gives the accounting relation between farmer income and expenses

$$
(1 + \rho) \, \frac{n}{n_0} b_{G0}^{(2)} = (2 + \rho) \sum_\tau \hat{b}_0^{(\tau)}.
\tag{112}
$$

The only farmer expenses that are not recycled are the bid on gold in period $\tau = 2$, and the interest paid on it.

Summing, instead, the price-formation rule (107) over τ, and then using relations (104) and (102) for prospector gold offers and food bids, gives a relation between central-bank bids and offers, prospector food bids set by e_0 and Λ, and the farmer bidding scale set by $b_{G0}^{(2)}$,

$$
\begin{aligned}
\frac{n}{n_0} b_{G0}^{(2)} + \sum_\tau \frac{B_G^{(\tau)}}{n_0} &= \sum_\tau \hat{q}^{(\tau)} p_{G\tau} + \sum_\tau \frac{Q_G^{(\tau)} p_{G\tau}}{n_0} \\
&= (1 + \rho) \sum_\tau 2\hat{b}_0^{(2)} + \frac{Q_G^{(2)} p_{G2}}{n_0}.
\end{aligned}
\tag{113}
$$

The difference of Eq. (113) from Eq. (112), multiplied by n_0, is the cycle identity for fiat,

$$\rho \left(n b_{G0}^{(2)} + n_0 \sum_\tau \hat{b}_0^{(2)} \right) + Q_G^{(2)} p_{G2} = \sum_\tau B_G^{(\tau)}. \tag{114}$$

The left-hand side includes all expenditures by agents; the first term is interest on borrowing, the second is payment for gold in period $\tau = 2$. The right-hand side includes all fiat expenditures by the central bank, in bids for gold. This equation is predicated on the repayment of interest in full without strategic bankruptcy, which will imply one constraint on the interest rate in relation to the bid levels $B_G^{(\tau)}$, which is derived below.

F.2.2 The food price cycle

The relation between endowment and price for farmers in the food sector, parallel to Eq. (101) for prospectors in the gold sector, becomes

$$
\begin{aligned}
a p_\tau &= A_0^{(\tau)} p_\tau + q^{(\tau)} p_\tau \\
&= 2 b_1^{(\tau)} + (1 + \rho) b_{G0}^{(\tau)}, \tag{115}
\end{aligned}
$$

where one first-order condition is used to evaluate $A_0^{(\tau)} p_\tau$, and Eq. (111) is used to evaluate $q^{(\tau)} p_\tau$.

Combining these expense relations with Eq. (111) for inputs gives an expression for the food price cycle

$$
\begin{aligned}
a p_\tau + (1 + \rho) b_{G0}^{(\tau)} &= a p_{\tau-1} - (1 + \rho) b_{G0}^{(\tau-1)} + 2 \frac{n_0}{n} \left(\hat{b}_1^{(\tau-1)} + \frac{\hat{b}_1^{(\tau)}}{1 + \rho} \right) \\
&= a p_{\tau-1} - (1 + \rho) b_{G0}^{(\tau-1)} + \frac{n_0}{n} e_0 \left(p_{G\tau-1} \varphi_{\tau-2} + \frac{p_{G\tau} \varphi_{\tau-1}}{1 + \rho} \right). \tag{116}
\end{aligned}
$$

The starting level for the cycle may be taken as the value of $a p_2$ determined from x appearing in Eq. (110), Eq. (115), and the first-order conditions (105):

$$\frac{\Lambda a p_2}{(1 + \rho) p_{G2}} = x \left\{ 2 + \frac{s}{1 - (1 + 2\theta) x} + \frac{2\theta s}{1 - 2\theta x} \right\}. \tag{117}$$

F.2.3 Central-bank bid variables and control over gold prices

Eq. (110), together with the farmer income/expense balance (112) and the expression (102) for $\hat{b}_1^{(\tau)}$ gives a relation between inter-period ratios of gold prices (or equivalently, their control through central-bank bids) and interest rates, of

$$\sum_{\tau} \left(\frac{p_{G\tau}}{p_{G2}} - \frac{(1+\rho)^2}{(1+\rho/2)} \right) \varphi_{\tau-1}^{(2)}(s; \Lambda e_0) = 0. \tag{118}$$

Separating $\tau = 2$, in which in which the central bank does not bid, from the other two periods, we obtain a constraint on one linear combination of bids:

$$
\begin{aligned}
\rho \frac{3 + 2\rho}{2 + \rho} \varphi_1^{(2)}(s; \Lambda e_0) &= \sum_{\tau=0,1} \left(\frac{p_{G\tau}}{p_{G2}} - \frac{(1+\rho)^2}{(1+\rho/2)} \right) \varphi_{\tau-1}^{(2)}(s; \Lambda e_0) \\
&= \sum_{\tau=0,1} \left(\frac{B_G^{(\tau)}}{n_0 e_0 p_{G2}} - \frac{(1+\rho)^2}{(1+\rho/2)} \varphi_{\tau-1}^{(2)}(s; \Lambda e_0) \right) \tag{119}
\end{aligned}
$$

where the second line uses Eq. (108). The central bank cannot, thus, set all gold prices equal and at the same time require full repayment of fiat, except in the limit $\rho \to 0$.

We will introduce an "angle" α to express the two-period bids relative to the interest rate, as

$$
\begin{aligned}
\frac{B_G^{(0)}}{n_0 e_0 p_{G2}} - \frac{(1+\rho)^2}{(1+\rho/2)} \varphi_2^{(2)}(s; \Lambda e_0) &= \left(\frac{\cos \alpha}{\cos \alpha + \sin \alpha} \right) \rho \frac{3 + 2\rho}{2 + \rho} \varphi_1^{(2)}(s; \Lambda e_0) \\
\frac{B^{(1)}}{n_0 e_0 p_{G2}} - \frac{(1+\rho)^2}{(1+\rho/2)} &= \left(\frac{\sin \alpha}{\cos \alpha + \sin \alpha} \right) \rho \frac{3 + 2\rho}{2 + \rho} \varphi_1^{(2)}(s; \Lambda e_0).
\end{aligned}
\tag{120}
$$

Within the interior of the interval $\alpha \in [-\pi/4, 5\pi/4]$, either of the bids $B^{(0)}$, $B^{(1)}$ may be taken to zero. The range $\alpha < 0$ places maximum bids in $B^{(0)}$; the range $\alpha > \pi/2$ places maximum bids in $B^{(1)}$. In numerical solutions, we will sample over the variables ρ and α to minimize the prospector utility variance (39) at the values of x and Λe_0 that equate cycle-averaged utilities of farmers and prospectors, and are compatible with a given value of n_0/n through Eq. (110).

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